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Semi-primal clusters


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A primal cluster is essentially a class $\mathcal{U}$ of universal algebras of the same species, where each $\mathcal{U}_i$ is primal (= strictly functionally complete), and such that every finite subset of $\mathcal{U}_i$ is independent. The concept of independence is essentially a generalization to universal algebras of the Chinese Remainder Theorem in number theory. Primal algebras themselves are further subsumed by the broader class of semi-primal algebras, and a structure theory for these algebras was recently established by Foster and Pixley [5] and Astromoff [1]. This theory subsumes and substantially generalizes well-known results for Boolean rings, p-rings, and Post algebras.

In order to expand the domain of applications of this extended Boolean theory, we should attempt to discover semi-primal clusters which, preferably, are as comprehensive as possible. In this paper we prove that certain large classes of semi-primal algebras form semi-primal clusters. Indeed, we show that the class of all two-fold surjective singular subprimal algebras which are pairwise non-isomorphic and in which each finite subset is co coupled forms a semi-primal cluster. A similar result is also shown to hold for regular subprimal algebras with pairwise non-isomorphic cores. Moreover, we prove that the class of all pairwise non-isomorphic s-couples, as well as the class of all $r$-frames with pairwise non-isomorphic cores, and even the union of these two classes, forms a semi-primal cluster. Finally, we construct classes of s-couples and $r$-frames.

1. Fundamental Concepts and Lemmas.

In this section we recall the following basic concepts of [2]-[5]. Let $\mathcal{U} = (A ; \Omega)$ be a universal algebra of species $S = (n_1, n_2, ...)$, where the
\( n_i \) are non-negative integers, and let \( \mathcal{O} = (O_1, O_2, \ldots) \) denote the primitive operation symbols of \( S \). Here, \( O_i = O_i(\xi_1, \ldots, \xi_{n_i}) \) is of rank \( n_i \). By an \( S \)-expression we mean any indeterminate symbol \( \xi, \eta, \ldots \) or any composition of these indeterminate symbols via the primitive operations \( O_i \). As usual, we use the same symbols \( O_i \) to denote the primitive operations of the algebras \( \mathcal{U}_1, \mathcal{U}_2, \ldots \) when these algebras are of species \( S \). We write \( \langle \mathcal{U} \rangle \) to mean that the \( S \)-expression \( \mathcal{P} \) is interpreted in the \( S \)-algebra \( \mathcal{U} \). This simply means that the primitive operation symbols are identified with the corresponding primitive operations of \( \mathcal{U} \), and the indeterminate symbols \( \xi, \ldots \) are now viewed as indeterminates over \( \mathcal{U} \). Moreover, \( \langle \mathcal{U} \rangle \) is called a strict \( \mathcal{U} \)-function. An identity between strict \( \mathcal{U} \)-functions \( \mathcal{P}, \Phi \) holding throughout \( \mathcal{U} \) is called a strict \( \mathcal{U} \)-identity, and is written as \( \mathcal{P}(\xi, \ldots) = \Phi(\xi, \ldots) \) (\( \mathcal{U} \)). We use \( Id(\mathcal{U}) \) to denote the family of all strict \( \mathcal{U} \)-identities. A finite algebra \( \mathcal{U} \) with more than one element is called categorical (respectively, semi-categorical) if every algebra, of the same species as \( \mathcal{U} \), which satisfies all the strict identities of \( \mathcal{U} \) is a subdirect power of \( \mathcal{U} \) (respectively, is a subdirect product of subalgebras of \( \mathcal{U} \)). A map \( f(\xi_1, \ldots, \xi_k) \) from \( A^k \) into \( A \) is \( S \)-expressible if there exists an \( S \)-expression \( \mathcal{P}(\xi_1, \ldots, \xi_k) \) such that \( f = \mathcal{P} \) for all \( \xi_1, \ldots, \xi_k \) in \( A \). A map \( f(\xi_1, \ldots, \xi_k) \) is conservative if for each subalgebra \( \mathcal{B} = (B ; \mathcal{O}) \) of \( \mathcal{U} \) and for all \( b_1, \ldots, b_k \) in \( B \), we have, \( f(b_1, \ldots, b_k) \in B \). An algebra \( \mathcal{U} \) is primal (respectively, semi-primal) if it is finite, with at least two elements, and every map from \( A \times \ldots \times A \) into \( A \) is \( S \)-expressible (respectively, every conservative mapping from \( A \times \ldots \times A \) into \( A \) is \( S \)-expressible). A semi-primal algebra \( \mathcal{U} \) which possesses exactly one subalgebra \( \mathcal{U}^* = (A^* ; \mathcal{O}) \) is called a \( \mathcal{U} \)-subalgebra. The subalgebra \( \mathcal{U}^* \) is called the core of \( \mathcal{U} \). If \( \mathcal{U}^* \) has exactly one element, \( \mathcal{U} \) is called a singular subprimal; otherwise it is called a regular subprimal. An algebra \( \mathcal{U} \) is said to be expressible if there exists an \( S \)-expression \( \Lambda_a(\xi) \) such that \( \Lambda_a(\xi) = a \) for each \( \xi \) in \( A \). An element \( a \) in \( A \) is said to be \( S \)-expressible if \( \Lambda_a^{A^*}(\xi) = a \) for each \( \xi \) in \( A \) (here, \( A \setminus A^* = \{ \xi \mid \xi \in A, \xi \notin A^* \} \)).

We now proceed to define the concept of independence. Let \( \{\mathcal{U}_i \} = \{\mathcal{U}_1, \ldots, \mathcal{U}_r \} \) be a finite set of algebras of species \( S \). We say that \( \{\mathcal{U}_i \} \) is independent if corresponding to each set \( \mathcal{P}_i, \ldots, \mathcal{P}_r \) of \( S \)-expressions there exists a single expression \( \mathcal{P} \) such that \( \mathcal{P} = \mathcal{P}_i(\mathcal{U}_i), i = 1, \ldots, r \) (or equivalently, if there exists an \( r \)-ary \( S \)-expression \( \mathcal{P} \) such that \( \mathcal{P}(\xi_1, \ldots, \xi_r) = \xi_i(\mathcal{U}_i), i = 1, \ldots, r \)). A primal (respectively, subprimal, semi-primal) cluster of species \( S \) is defined to be a class \( \mathcal{U}_i = \{ \mathcal{U}_1, \ldots, \mathcal{U}_r \} \) of primal (respectively, subprimal, semi-primal) algebras of species \( S \), any finite subset of which is independent.
We are now in a position to state the following lemmas, the proofs of which have already been given in [2; 5].

**Lemma 1.** A primal algebra is categorical and simple.

**Lemma 2.** A semi-primal algebra is semi-categorical and simple.

**Lemma 3.** Let \( \mathcal{B} = (B; \Omega) \) be a subprimal algebra of species \( S \). Then,

(a) the core \( \mathcal{B}^* = (B^*; \Omega) \) is primal or else is a one-element subalgebra;
(b) each \( b \) in \( B^* \) is \( S \)-expressible;
(c) each \( b \) in \( B \setminus B^* \) is ex-expressible.

2. Semi-primal Clusters.

In this section some semi-primal clusters will be found. The methods of proof are similar to those of Foster [4] and O'Keefe [6]. We will be concerned, mainly, with co-coupled families of subprimal algebras, in the sense of the following

**Definition 1.** A family \( \mathcal{U}_i = (A_i; \Omega) \), \( i \in I \), of subprimal algebras of species \( S \), with cores \( \mathcal{U}^*_i = (A_i^*; \Omega) \), \( i \in I \), respectively, is said to be co-coupled if there exist two binary \( S \)-expressions \( \xi \times \eta \) \( (= \xi \cdot \eta = \xi \eta) \) and \( \xi T \eta \), and elements \( 0_i, 1_i \) in \( A_i (0_i \neq 1_i) \), for each \( i \in I \), such that

(a) if \( \mathcal{U}_i \) is a singular subprimal, then

\[
\{0_i\} = A_i^*;
\]

(b) if \( \mathcal{U}_i \) is a regular subprimal, then (2) and (3) hold in addition to

\[
\{0_i, 1_i\} \subseteq A_i^*.
\]

**Definition 2.** A family \( \mathcal{U}_i = (A_i; \Omega) \), \( i \in I \), of regular subprimal algebras of species \( S \) is said to be co-framed if there exist \( S \)-expressions \( \xi \times \eta \) \( (= \xi \cdot \eta = \xi \eta) \) and \( \xi 0 \), and elements \( 0_i, 1_i \) in \( A_i (0_i \neq 1_i) \), for each \( i \in I \), such that (2) and (4) hold in addition to

\[
\eta \text{ is a permutation of } A_i \text{ with } 0_i^\eta = 1_i.
\]
REMARK 1. If $\mathcal{U}_i, i \in I$, is a co-framal family of regular subprimal algebras, then by letting $\xi T \eta \overset{\text{def}}{=} (\xi^n \times \eta^n)^{\cup}$, where $\xi^n$ denotes the inverse of $\xi^n$, it follows that (3) holds and therefore the family is also co-coupled.

THEOREM 1. Let $\mathcal{U}_i = (A_i; \Omega), i = 1, \ldots, n$, be co-coupled subprimal algebras of species $S$. Then, if the $\mathcal{U}_i$ are pairwise independent, they are independent.

PROOF. Assume that $\mathcal{U}_1, \ldots, \mathcal{U}_n$ are pairwise independent. Then, for any two algebras $\mathcal{U}_i, \mathcal{U}_j$ (where $i \neq j$), there exists an $S$-expression $\Phi(\xi, \eta)$ such that

(6) \[ \Phi(\xi, \eta) = \begin{cases} \xi \in \mathcal{U}_i & \text{if } i \in A_j \setminus \{0_i\} \\ \eta \in \mathcal{U}_j & \text{if } j \in A_i \setminus \{0_i\} \end{cases} \]

Let $\mathcal{U}_i = (A_i; \Omega)$ denote the core of $\mathcal{U}_i$. If $\mathcal{U}_i$ is a regular subprimal (respectively, a singular subprimal), then $1_i \in A_i^*$ (respectively, $1_i \in A_i \setminus A_i^*$), and according to Lemma 3 it is expressible (respectively, ex-expressible). In either case, there exists a unary $S$-expression $A_i(\xi)$ such that

(7) \[ A_i(\xi) = 1_i \quad (\xi \in A_i \setminus \{0_i\}). \]

From (6), (7), and the fact that $0_j \in A_j$ is $S$-expressible, say by $A_{0j}(\xi)$, it follows immediately that

(8) \[ \Pi_{ij}(\xi) = \Phi(A_{0j}(\xi), A_{ij}(\xi)) = \begin{cases} 1_i \quad (\xi \in A_i \setminus \{0_i\}) \\ 0_j \quad (\mathcal{U}_j) \end{cases} \quad (i \neq j). \]

Define, now, a unary $S$ expression $\Psi_i(\xi), 1 \leq i \leq n$, by

(8) \[ \Psi_i(\xi) = \Pi_{i1}(\xi) \times \ldots \times \Pi_{in}(\xi) = \begin{cases} 1_i \quad (\xi \in A_i \setminus \{0_i\}) \\ 0_j \quad (\mathcal{U}_j) \quad \text{(all } j \neq i) \end{cases} \]

where ^ denotes deletion and the $\Pi_{ij}(\xi)$ are associated in some fixed manner. Using (8) and the co-coupling binary $S$-expression $\xi T \eta$, define an $n$-ary $S$-expression $\Phi(\xi_1, \ldots, \xi_n)$ by

(8) \[ \Phi(\xi_1, \ldots, \xi_n) = [\Psi_1(\xi_1) \times \xi_1] T \ldots [\Psi_n(\xi_n) \times \xi_n], \]

the $T$-factors being associated in some fixed manner. It is easily checked that $\Phi(\xi_1, \ldots, \xi_n) = \xi_1(\mathcal{U}_1), 1 \leq i \leq n$. This proves the theorem.
Because of Theorem 1, it is important to discuss the pairwise independence of subprimal algebras. To do this we impose a surjectivity property on the primitive operations.

**Definition 3.** A subprimal algebra \( \mathcal{U} \) with core \( \mathcal{U}^* \) is said to be two-fold surjective if each primitive operation of \( \mathcal{U} \) is surjective on both \( \mathcal{U} \) and \( \mathcal{U}^* \).

We now show that subprimal algebras which are two-fold surjective satisfy a certain factorization property (compare with [6]).

**Theorem 2.** Let \( \mathcal{U} = (A; \Omega) \) be a two-fold surjective subprimal algebra of species \( S \). Then, for each unary \( S \)-expression, \( \Gamma(\xi) \), and each primitive operation \( O_i \) (of rank \( n_i \)) of \( \mathcal{U} \), there exist unary \( S \)-expressions \( \Psi_1(\xi), \ldots, \Psi_{n_i}(\xi) \) such that

\[
O_i(\Psi_1(\xi), \ldots, \Psi_{n_i}(\xi)) = \Gamma(\xi)(\mathcal{U}).
\]

**Proof.** Let \( A = \{a_1, \ldots, a_m, a_{m+1}, \ldots, a_t\} \) where \( A^* = \{a_1, \ldots, a_m\} \) is the base set of \( \mathcal{U}^* \) (= core of \( \mathcal{U} \)) Clearly. \( \Gamma(a_j) \in A^* \) for \( 1 \leq j \leq m \). Because of two-fold surjectivity, there exist elements \( a_{jk} (1 \leq j \leq t, 1 \leq k \leq n_i) \) of \( A \), with \( a_{jk} \) in \( A^* \) when \( 1 \leq j \leq m \), such that

\[
O_i(a_{j1}, \ldots, a_{jn_i}) = \Gamma(a_j).
\]

Now let unary functions \( g_1(\xi), \ldots, g_{n_i}(\xi) \) be defined on \( A \) by

\[
g_k(a_j) = a_{jk} (1 \leq k \leq n_i, 1 \leq j \leq t).
\]

Since \( g_k(a_j) \in A^* \), \( 1 \leq j \leq m \), each \( g_k \) is conservative and hence is \( S \)-expressible, say by \( \Psi_k(\xi) \). It follows that

\[
O_i(\Psi_1(a_j), \ldots, \Psi_{n_i}(a_j)) = O_i(g_1(a_j), \ldots, g_{n_i}(a_j)) = O_i(a_{j1}, \ldots, a_{jn_i}) = \Gamma(a_j)
\]

for \( 1 \leq j \leq t \) and (9) is verified.

From [6; Lemma 2.3] and Theorem 2 we immediately obtain the following factorization property.

**Theorem 3.** Let \( \mathcal{U} \) be a two-fold surjective subprimal algebra of species \( S \). Then, for each expression \( \Sigma(\xi_1, \ldots, \xi_p) \), and each expression \( \Theta(\xi_1, \ldots, \xi_p) \) in which no indeterminate \( \xi_i, 1 \leq i \leq p \), occurs twice in \( \Theta \), there exist expressions \( \Psi_1, \ldots, \Psi_p \) such that

\[
\Theta(\Psi_1, \ldots, \Psi_p) = \Sigma(\mathcal{U}).
\]
The pairwise independence of any two universal algebras \( \mathcal{U}, \mathcal{B} \) of species \( S \) assures that any two subalgebras of \( \mathcal{U}, \mathcal{B} \), of more than one element each, are non-isomorphic. In establishing independence, therefore, this must be taken as a minimal assumption.

**Theorem 4.** Let \( \mathcal{U} = (A; \Omega) \) and \( \mathcal{B} = (B; \Omega) \) be subprimal algebras of species \( S \) with cores \( \mathcal{U}^* = (A^*; \Omega) \) and \( \mathcal{B}^* = (B^*; \Omega) \), respectively. Suppose that either of the following holds:

(i) \( \mathcal{B} \) is a regular subprimal and \( \mathcal{U}, \mathcal{B}^* \) are non-isomorphic;

(ii) \( \mathcal{B} \) is a singular subprimal and \( \mathcal{U}, \mathcal{B}^* \) are non-isomorphic.

Then, there exist elements \( d_1, d_2 \) in \( B \) (\( d_1 \neq d_2 \)) and unary expressions \( r_1(x), r_2(x) \) such that

\[
\begin{align*}
\Gamma_1(\xi) &= \Gamma_2(\xi) (\mathcal{U}); \\
\Gamma_1(\xi) &= d_1(\xi \in B \setminus B^*); \\
\Gamma_2(\xi) &= d_2(\xi \in B \setminus B^*).
\end{align*}
\]

Moreover, if (i) holds, then \( d_1, d_2 \in B^* \) and

\[
\begin{align*}
\Gamma_1(\xi) &= d_1(\mathcal{B}); \\
\Gamma_2(\xi) &= d_2(\mathcal{B}).
\end{align*}
\]

**Proof.** First, assume that (i) holds. Then \( \mathcal{B}^* \) is primal (Lemma 3) and hence categorical (Lemma 1). Therefore, if \( \text{Id}(\mathcal{U}) \supset \text{Id}(\mathcal{B}^*) \) then \( \mathcal{U} \cong \mathcal{B}^{*k} \) (= \( k \)-th subdirect power of \( \mathcal{B} \)) for some \( k \geq 1 \). Now \( k \neq 1 \) since \( \mathcal{U} \) has a subalgebra \( (\neq \mathcal{U}) \). But if \( k \geq 2 \), there exists an epimorphism \( \mathcal{U} \rightarrow \mathcal{B}^* \), contradicting the simplicity of \( \mathcal{U} \). Thus, \( \text{Id}(\mathcal{U}) \supset \text{Id}(\mathcal{U}^*) \). Similarly, if \( \text{Id}(\mathcal{B}^*) \supset \text{Id}(\mathcal{U}) \), then since \( \mathcal{U} \) is semi-categorical (Lemma 2), \( \mathcal{B}^* \cong \mathcal{U}^{*k_1} \times \mathcal{U}^{*k_2} \) (= \( k_1 \)-subdirect product of subdirect powers of \( \mathcal{U} \) and \( \mathcal{U}^* \)) for some \( k_1, k_2 \). Now \( k_1 + k_2 \neq 1 \) since \( \mathcal{B}^*, \mathcal{U}^* \) are non-isomorphic and by assumption \( \mathcal{B}^*, \mathcal{U}^* \) are non-isomorphic. Thus, \( k_1 + k_2 \geq 2 \). But then there exists an epimorphism from \( \mathcal{B}^* \) onto either \( \mathcal{U} \) or \( \mathcal{U}^* \), contradicting the simplicity of \( \mathcal{B}^* \). Thus \( \text{Id}(\mathcal{B}^*) \supset \text{Id}(\mathcal{U}) \). These two non-inclusions assure the existence of expressions \( \psi_1(\xi_1, \ldots, \xi_p) \) and \( \psi_2(\xi_1, \ldots, \xi_p) \) such that

\[
\psi_1 = \psi_2(\mathcal{U}) \text{ and } \psi_1 \neq \psi_2(\mathcal{B}^*).
\]

From (11) it follows that there exist elements \( \beta_1, \ldots, \beta_p \) of \( B^* \) for which

\[
\begin{align*}
d_1 &= \text{def} = \psi_1(\beta_1, \ldots, \beta_p) \neq \psi_2(\beta_1, \ldots, \beta_p) = \text{def} = d_2.
\end{align*}
\]
Clearly, \( d_1, d_2 \in B^* \). Since \( \beta_1, \ldots, \beta_p \in B^* \), there exist expressions \( \Lambda_1(\xi), \ldots, \Lambda_p(\xi) \) such that (see Lemma 3)

\[
\Lambda_i(\xi) = \beta_i(\xi), \ 1 \leq i \leq p.
\]

If \( \Gamma_j(\xi) \) is defined by

\[
\Gamma_j(\xi) = \Psi_j(\Lambda_1(\xi), \ldots, \Lambda_p(\xi)), \ 1 \leq j \leq 2,
\]

from (12)-(14) it follows that \( \Gamma_1(\xi), \Gamma_2(\xi) \) satisfy (1°), (4°), and (5°).

Secondly, assume that (ii) holds. Using arguments similar to those above, it can be established that \( \text{Id}(\xi) \not\supset \text{Id}(\beta) \) and \( \text{Id}(\beta) \not\supset \text{Id}(\xi) \). Thus, there exist expressions \( \Psi_1(\xi_1, \ldots, \xi_p), \Psi_2(\xi_1, \ldots, \xi_p) \) such that \( \Psi_1 = \Psi_2(\xi) \) and \( \Psi_1 = \Psi_2(\xi) \). Let \( \beta_1, \ldots, \beta_p \) be elements of \( B \) for which (12) holds. Because of Lemma 3, there exist expressions \( \Lambda_1(\xi), \ldots, \Lambda_p(\xi) \) with

\[
\Lambda_i(\xi) = \beta_i(\xi), \ 1 \leq i \leq p.
\]

Let \( \Gamma_1(\xi), \Gamma_2(\xi) \) be defined as in (14). It is easy to verify that they have the desired properties (1°)-(3°).

Next, we prove the following theorems.

**Theorem 5.** Let \( \mathcal{U} = (A ; \Omega) \) and \( \mathcal{B} = (B ; \Omega) \) be subprimal algebras of species \( S \) satisfying either (i) or (ii) of Theorem 4. Then there exist expressions \( \Phi_1(\xi), \ldots, \Phi_p(\xi) \) such that \( \Phi_1(\xi) = \ldots = \Phi_p(\xi) \) and such that every conservative unary function on \( B \) is identical, in \( B \), to one of \( \Phi_1(\xi), \ldots, \Phi_p(\xi) \).

**Proof.** Let the conservative unary functions on \( B \) be enumerated as \( b_1(\xi), \ldots, b_p(\xi) \) and let \( d_1, d_2, \Gamma_1(\xi), \Gamma_2(\xi) \) be as in Theorem 4. Since \( \mathcal{B} \) is semi-primal, each conservative function on \( B \) is \( S \)-expressible. Hence, there exists an expression \( \Phi(\xi, \xi_1, \ldots, \xi_p) \) for which

\[
\Phi(\xi, d_1, \ldots, d_p, d_2) = b_i(\xi), \ 1 \leq i \leq p \ (\xi \text{ in } B).
\]

(This follows since the above equation is a conservative condition). Using \( \Phi \) as a skeleton, we now define \( \Phi_1(\xi), \ldots, \Phi_p(\xi) \) by

\[
\Phi_i(\xi) = \Phi(\xi, \Gamma_1(\xi), \ldots, \Gamma_1(\xi), \Gamma_2(\xi), \ldots, \Gamma_2(\xi)), \ 1 \leq i \leq p.
\]

From (1°) of Theorem 4 it follows that \( \Phi_i(\xi) = \Phi_j(\xi) \) for all \( 1 \leq i, j \leq p \). If (i) holds, then from (4°) and (5°) of Theorem 4, \( \Phi_i(\xi) = b_i(\xi) \) (\( \xi \) in \( B \)), \( 1 \leq i \leq p \). If (ii) holds, then (2°) and (3°) assure that \( \Phi_i(\xi) = b_i(\xi) \) (\( \xi \) in \( B \setminus B^* \)).
Moreover, in \( S \), \( \Phi_i(\xi) \) and \( b_i(\xi) \) are both conservative. Since \( B^* \) consists of exactly one element, say \( B^* = \{0\} \), it follows that \( \Phi_i(0) = b_i(0) = 0 \). Hence, in case (ii) we also have \( \Phi_i(\xi) = b_i(\xi) (\xi \in B) \).

**Theorem 6.** Let \( \mathcal{U} = (A; \Omega), \mathcal{V} = (B; \Omega) \) be subprimal algebras of species \( S \) (with cores \( \mathcal{U}^* = (A^*; \Omega), \mathcal{V}^* = (B^*; \Omega) \), respectively) satisfying either (i) or (ii) of Theorem 4. If \( \mathcal{B} \) is two-fold surjective, then for each \( \alpha \) in \( A^* \) and each unary expression \( \Psi(\xi) \) there exists an expression \( \Omega = \Omega(\xi) \) such that

\[
\Omega = \begin{cases} 
\alpha(\mathcal{U}) \\
\Psi(\mathcal{B}).
\end{cases}
\]

**Proof.** If \( \alpha \in A^* \), there exists a unary expression \( \Theta(\xi) \) for which \( \Theta = \alpha(\mathcal{U}) \). Let \( \Theta'(\xi_1, ..., \xi_p) \) be the \( S \)-expression derived from \( \Theta \) by distinguishing each occurrence of \( \xi \) in \( \Theta \). Thus, by definition, \( \Theta'(\xi, ..., \xi) = \Theta(\xi) \). From Theorem 3, there exist expressions \( \Psi_1(\xi), ..., \Psi_p(\xi) \) such that

\[
\Theta'(\Psi_1(\xi), ..., \Psi_p(\xi)) = \Psi(\xi)(\mathcal{B}).
\]

Since \( \Psi_1(\xi), ..., \Psi_p(\xi) \) are conservative in \( \mathcal{B} \), by Theorem 5, there exist expressions \( \Phi_1(\xi), ..., \Phi_p(\xi) \) such that

\[
\Phi_i(\xi) = \Phi_j(\xi)(\mathcal{U}), \ 1 \leq i, j \leq p;
\]

\[
\Phi_i(\xi) = \Psi_i(\xi)(\mathcal{B}), \ 1 \leq i \leq p.
\]

Let \( \Omega(\xi) = \Theta'(\Phi_1(\xi), ..., \Phi_p(\xi)) \). It is easily verified that \( \Omega \) has the desired property of the theorem.

If \( F \) is a family of subprimal algebras of species \( S \) let us use \( F_s \) (respectively, \( F_r \)) to denote the subfamily of all singular subprimal (respectively, regular subprimal) members of \( F \).

**Theorem 7.** (Principal Theorem) Let \( F \) be a family of two-fold surjective subprimal algebras of species \( S \), each finite subset of which is co-coupled. If, further,

(a) the members of \( F_s \) are pairwise non-isomorphic,

(b) the members of \( F_r \) have pairwise non-isomorphic cores, then \( F \) is a subprimal cluster.
PROOF. Because of (a), (b), and Theorem 6, for any two members \( U, \beta \) of \( F \), there exist expressions \( \Omega_1(\xi), \Omega_2(\xi) \) for which

\[
\begin{align*}
\Omega_1(\xi) &= \begin{cases} 
\xi (U) \\ 0 (\beta) 
\end{cases} \\
\Omega_2(\xi) &= \begin{cases} 
0 (U) \\ \xi (\beta) 
\end{cases}
\end{align*}
\]

Since each finite subset of \( F \) is co-coupled, there exists a binary expression \( \xi \in \eta \) satisfying (3). Thus

\[
\Omega_1(\xi_1) \circ \Omega_2(\xi_2) = \begin{cases} 
\xi_1 (U) \\ \xi_2 (\beta) 
\end{cases}
\]

and therefore \( \mathcal{U}, \mathcal{B} \) are independent. From Theorem 1 it follows that each finite subset of \( F \) is independent, and the theorem is proved.

COROLLARY 1. Let \( F(=F') \) be a family of two-fold surjective regular subprimal algebras of species \( S \) satisfying (b) of Theorem 7. Suppose that each finite subset of \( F \) is co-framal. Then \( F \) is a regular subprimal cluster.

This follows from the above theorem, upon applying Remark 1.

We now consider special subclasses of co-coupled and co-framal subprimal algebras.

DEFINITION 4. An \( s \)-couple is a singular subprimal algebra \( \mathcal{U} = (A; \times, T) \) of species \( S = (2, 2) \) containing elements 0,1 (0 \( \neq \) 1) such that (1)-(3) hold. An \( r \)-frame is a regular subprimal algebra \( \mathcal{U} = (A; \times, 0) \) of species \( S = (2, 1) \) containing elements 0,1 (0 \( \neq \) 1) for which (2), (4), and (5) hold.

Examples of \( s \)-couples are plentiful. Two such examples are (see [5]):

1°) The « double groups » \( G = (C; \times, +) \) of finite order \( n \geq 2 \) in which \( (C; +) \) is a cyclic group with identity 0 and generator 1, \( (C \setminus \{0\}; \times) \) is a group with identity 1, and 0 \( \times \xi = \xi \times 0 = 0 \) (\( \xi \) in \( C \)); and

2°) the algebras \( G_p = (\mathbb{Z}/p; \times, +) \) of \( p \) elements 0,1, \ldots, \( p-1 \) (\( p \) a prime) in which \( \xi + \eta = \text{addition mod } p \), and \( \xi \times \eta = \min(\xi, \eta) \) in the ordering 0, 2, 3, \ldots, \( p-1, 1 \).

To establish other classes of \( r \)-frames and \( s \)-couples we need the following definitions and lemmas.

DEFINITION 5. A binary algebra is an algebra \( \mathcal{B} = (B; \times) \) of species \( S = (2) \) which possesses elements 0,1 (0 \( \neq \) 1) satisfying

\[
0 \times \xi = \xi \times 0 = 0; \quad 1 \times \xi = \xi \times 1 = \xi \quad \text{(all } \xi \text{ in } B).
\]

The element 0 is called the null of \( \mathcal{B} \); 1 is called the identity.
LEMMA 4. (Foster and Pixley [5]). An algebra \( \mathfrak{B} = (B; \Omega) \) of species \( S \) is a regular subprimal if and only if there exist elements \( 0, 1 \) in \( B \) \( (0 \neq 1) \) and functions \( \times \) (binary) and \( n \) (unary) defined in \( B \) such that (15) holds, in addition to

1° \( \mathfrak{B} \) is a finite algebra of at least three elements;
2° \( \mathfrak{B} \) possesses a unique subalgebra \( (\mathfrak{B}^*; \Omega) \), denoted by \( \mathfrak{B}^* = (B^*; \Omega) \) and \( B^* \) contains at least two elements;
3° the elements 0, 1 and the functions \( \times, n \) are each \( S \)-expressible;
4° \( n \) is a permutation of \( B \) in which \( 0 \, n = 1 \);
5° for each \( b \) in \( B \), the characteristic function \( \delta_b (\xi) \) (defined below) is \( S \)-expressible:
\[
\delta_b (\xi) = 1 \text{ if } \xi = b \text{ and } \delta_b (\xi) = 0 \text{ if } \xi \neq b \text{ (all } \xi \text{ in } B);
\]
6° there exists an element \( b_0 \) in \( B \setminus B^* \) which is ex-expressible.

LEMMA 5 (Foster and Pixley [5]). An algebra \( \mathfrak{B} = (B; \Omega) \) of species \( S \) is a singular subprimal if and only if there exist elements \( 0, 1, 1^o \) in \( B \) \( (0 \neq 1) \) and two binary functions \( \times, T \) defined in \( B \) such that (15) holds in addition to

1° \( \mathfrak{B} \) is a finite algebra of at least two elements;
2° \( \mathfrak{B} \) possesses exactly one one-element subalgebra \( \mathfrak{B}^* = (B^*; \Omega) \) and no other subalgebra \( (\neq \mathfrak{B}) \);
3° the element 0 and the functions \( \times, T \) are each \( S \)-expressible;
4° \( 0 \, T \, 0 = 1 \) for each \( \xi \) in \( B \) and \( 1 \, T \, 1 = 1 \);
5° for each \( b \) in \( B \setminus B^* \), the characteristic function \( \delta_b (\xi) \) is \( S \)-expressible;
6° there exists an element \( b_0 \) in \( B \setminus B^* \) which is ex-expressible.

REMARK 2. If \( \xi^n \) is a permutation on a set, we use \( \xi^u \) to denote its inverse. Moreover, for each positive integer \( s \) we define:
\[
\xi^{0s} = \text{def} = (\ldots (\xi^n)^s \ldots)^n \text{ (s iterations)}.
\]
We define \( \xi^{u} \) similarly. Note that if \( \xi^n \) is a permutation on a finite set, then there exists an integer \( s \) such that \( \xi^{0s} = \xi^u \). Hence, any \( (n, u) \)-expression is just a \( (n) \)-expression.

The following theorems provide large classes of \( r \)-frames and \( s \)-couples.

THEOREM 8. Let \( \mathfrak{B} = (P; \times, n) \) be a primal algebra for which
1° \( P; \times \) is a binary algebra (with null \( 0 \) and identity \( 1 \)); and
2° \( n \) is a cyclic permutation on \( P \) with \( 0^n = 1 \). If \( P_m = P \cup \{ \lambda_1, \ldots, \lambda_m \} \)
where \( \lambda_i \in P, 1 \leq i \leq m \), then the operations \( \times \) and \( \circ \) can be extended to \( P_m \) such that \( \mathcal{B}_m = (P_m; \times, \circ) \) is an r-frame with core \( \mathcal{C} \).

**Proof.** Let \( \mathcal{C} = (0, 1, \beta_2, \ldots, \beta_n) \in P \). Because of primality, there exists a unary \((\times, \circ)\)-expression \( A(\xi) \) such that

\[
A(\xi) = \begin{cases} 
1 & \text{if } \xi = 0 \\
0 & \text{if } \xi \neq 0
\end{cases} \quad (\xi \in P).
\]

We extend the definitions of \( \times \) and \( \circ \) to \( P_m \) as follows:

(i) For \( \xi, \eta \in P \) define \( \xi \times \eta \) and \( \xi \circ \eta \) in \( P_m \) just as in \( P \);

(ii) \( \lambda_0 = \lambda_2, \lambda_2 = \lambda_3, \ldots, \lambda_m = \lambda_1 \);

(iii) \( \lambda_i \times \lambda_j = \lambda_{\min(i,j)} \) (if \( i \neq j \)) and \( \lambda_i \times \lambda_i = 1 \) (for each \( i \));

(iv) \( 0 \times \lambda_i = \lambda_i \times 0 = 0, 1 \times \lambda_i = \lambda_i \times 1 = \lambda_i \);

(v) \( \xi \times \lambda_i \) and \( \lambda_i \times \xi \) (\( \xi \in P \)) are defined arbitrarily for other \( \xi \).

Each characteristic function \( \delta_i(\xi), \xi \in P_m \), is \((\times, \circ)\)-expressible since the following identities hold in \( P_m \) (for a product of more than two terms, assume that the association is from the left):

\[
d_0(\xi) = A(\xi) A(\xi \times \xi), \quad d_1(\xi) = d_0(\xi \circ \xi), \ldots, \quad d_{\beta_n}(\xi) = d_0(\xi \circ \xi^{n-1});
\]

\[
d_{\beta_1}(\xi) = (\xi \cdot \xi^n \cdot \xi^{n-1}) \beta_1 \beta_0 (\xi \cdot \xi^n \cdot \xi^{n-1}) \cdot \xi^{m-1};
\]

\[
d_{\beta_2}(\xi) = \delta_{\lambda_1}(\xi), \ldots, \quad d_{\beta_m}(\xi) = \delta_{\lambda_1}(\xi^{m-1}).
\]

In the above, \( \xi^u \) denotes the inverse of \( \xi \). Since \( P_m \) is finite, \( \xi^u \) is a \((\circ)\)-expression. Moreover, \( 0, 1, \beta_2, \ldots, \beta_n \) are \((\times, \circ)\)-expressible and \( \lambda_1, \ldots, \lambda_m \) are \((\times, \circ)\)-expressible, since

\[
0 = d_0(\xi) \times d_1(\xi), 1 = 0^n, \beta_2 = 0^n, \ldots, \beta_n = \circ^{n-1}(\xi \in P_m);
\]

\[
\lambda_1 = \xi \cdot \xi^n \cdot \xi^{n-1}, \lambda_2 = \lambda_1^n, \ldots, \lambda_m = \lambda_1^{m-1}(\xi \in P_m \setminus P).
\]

Clearly, \( \mathcal{B} \) is the unique proper subalgebra of \( \mathcal{B}_m \). The conditions \((1^o)-(6^o)\) of Lemma 4 are verified. Thus, \( \mathcal{B}_m \) is a regular subprimal algebra and, indeed, even an r-frame. The theorem is proved.

**Theorem 9.** Let \( (B; \times) \) be a finite binary algebra. Then a binary operation \( \xi T \eta \) can be defined on \( B \) such that \( (B; \times, T) \) is an s-couple.
PROOF. For the two-element binary algebra \(((0, 1); \times)\) it is easily verified that conditions \((1^0)-(6^0)\) of Lemma 5 hold if \(\xi T \eta\) is defined by \(0 T \xi = \xi T 0 = \xi\) and \(1 T 1 = 0\). Let, then \(B = \{0, 1, b_1, \ldots, b_m\}\) be the base set of a binary algebra of order \(m + 2\), where \(m \geq 1\). Consider the cases (I) \(m \geq 2\) and (II) \(m = 1\). For (I), define \(T\) on \(B\) such that

\[
0 T \xi = \xi T 0 = \xi \quad \text{(each} \ \xi \ \text{in} \ \mathcal{B});
\]
\[
1 T 1 = b_1, b_1 T b_1 = b_2, \ldots, b_m T b_m = 1;
\]

\[
1 T b_1 = 1, b_1 T b_2 = b_2 T b_3 = \ldots = b_{m-1} T b_m = b_m T 1 = 1 T b_m = 0;
\]

\(\xi T \eta\) is defined arbitrarily for other \(\xi, \eta\) in \(B\);

hold, while for (II) define \(T\) on \(B\) such that (16) and (17) hold in addition to

\[
1 T b_1 = b_1 T 1 = 0.
\]

In either case (I) or (II), let \(\xi^n = \xi T \xi\). In the characteristic function \(\delta_i(\xi)\) is \((\times, T)\)-expressible then \(\delta_{b_1}(\xi), \ldots, \delta_{b_m}(\xi), I_1(\xi), \) and \(0\) are \((\times, T)\)-expressible since

\[
\delta_{b_m}(\xi) = \delta_i(\xi^n), \delta_{b_{m-1}}(\xi) = \delta_i(\xi^n T) \ldots, \delta_{b_1}(\xi) = \delta_i(\xi^n) \quad \text{(in} \ \mathcal{B});
\]

\[
I_1(\xi) = \delta_i(\xi) T \delta_{b_1}(\xi) T \ldots T \delta_{b_m}(\xi) = 1 \quad \text{(in} \ \mathcal{B} \ \setminus \ {0});
\]

\[
0 = \delta_i(\xi) \times \delta_{b_i}(\xi) \quad \text{(in} \ \mathcal{B}).
\]

In case (I), \(\delta_i(\xi) = \xi T \xi^n\), while in case (II), \(\delta_i(\xi) = \xi^n, \xi T \xi^n, \text{or} \xi^n T(\xi \xi^n)\), according as \(b_i^2 = 0, 1, \text{or} \ b_i\), respectively. In each case, it is clear that \(\{0\}\) is the unique subalgebra of \((B; \times, T)\). The conditions \((1^0)-(6^0)\) of Lemma 5 are verified. Thus, \((B; \times, T)\) is a singular subprimal algebra and, in fact, an \(s\)-couple.

We conclude with the following easily proved corollaries of Theorem 7.

COROLLARY 2. Any subfamily of the family \(F_{b_0}\) of all pairwise non-isomorphic \(s\)-couples forms a singular subprimal cluster.

COROLLARY 3. Any subfamily of the family \(F_{r_0}\) of all \(r\)-frames with pairwise non-isomorphic cores forms a regular subprimal cluster.

COROLLARY 4. Any subfamily of the family \(F_{r_0} \cup F_{r_0}\) is a subprimal cluster.

The algebras given in Theorems 8 and 9 apply, of course, to these corollaries.

Note Added in Proof. Theorem 8 was obtained independently by A. L. Foster, Monatshefte für Mathematik 72 (1968), 315-324.
REFERENCES

1. A. Astromoff, Some structure theorems for primal and categorical algebras, Math. Z. 87 (1965), 365-377.


