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SEMI-PRIMAL CLUSTERS

D. JAMES SAMUELSON

A primal cluster is essentially a class $\{\mathcal{U}_i\}$ of universal algebras of the same species, where each \mathcal{U}_i is primal (= strictly functionally complete), and such that every finite subset of $\{\mathcal{U}_i\}$ is «independent». The concept of independence is essentially a generalization to universal algebras of the Chinese Remainder Theorem in number theory. Primal algebras themselves are further subsumed by the broader class of «semi-primal» algebras, and a structure theory for these algebras was recently established by Foster and Pixley [5] and Astromoff [1]. This theory subsumes and substantially generalizes well-known results for Boolean rings, p -rings, and Post algebras.

In order to expand the domain of applications of this extended «Boolean» theory, we should attempt to discover semi-primal clusters which, preferably, are as comprehensive as possible. In this paper we prove that certain large classes of semi-primal algebras form semi-primal clusters. Indeed, we show that the class of all two-fold surjective singular subprimal algebras which are pairwise non-isomorphic and in which each finite subset is co coupled forms a semi-primal cluster. A similar result is also shown to hold for regular subprimal algebras with pairwise non-isomorphic cores. Moreover, we prove that the class of all pairwise non-isomorphic s -couples, as well as the class of all r -frames with pairwise non-isomorphic cores, and even the union of these two classes, forms a semi-primal cluster. Finally, we construct classes of s -couples and r -frames.

1. Fundamental Concepts and Lemmas.

In this section we recall the following basic concepts of [2]-[5]. Let $\mathcal{U} = (A; \Omega)$ be a universal algebra of species $S = (n_1, n_2, \dots)$, where the

n_i are non-negative integers, and let $\Omega = (\mathcal{O}_1, \mathcal{O}_2, \dots)$ denote the primitive operation symbols of S . Here, $\mathcal{O}_i = \mathcal{O}_i(\xi_1, \dots, \xi_{n_i})$ is of rank n_i . By an S -expression we mean any indeterminate symbol ξ, η, \dots or any composition of these indeterminate symbols via the primitive operations \mathcal{O}_i . As usual, we use the same symbols \mathcal{O}_i to denote the primitive operations of the algebras $\mathcal{U}_1, \mathcal{U}_2, \dots$ when these algebras are of species S . We write « $\Psi(\xi, \dots)(\mathcal{U})$ » to mean that the S -expression Ψ is interpreted in the S -algebra \mathcal{U} . This simply means that the primitive operation symbols are identified with the corresponding primitive operations of \mathcal{U} , and the indeterminate symbols ξ, \dots are now viewed as indeterminates over \mathcal{U} . Moreover, « $\Psi(\xi, \dots)(\mathcal{U})$ » is called a *strict \mathcal{U} -function*. An identity between strict \mathcal{U} -functions Ψ, Φ holding throughout \mathcal{U} is called a *strict \mathcal{U} -identity*, and is written as $\Psi(\xi, \dots) = \Phi(\xi, \dots)(\mathcal{U})$. We use $Id(\mathcal{U})$ to denote the family of all strict \mathcal{U} -identities. A finite algebra \mathcal{U} with more than one element is called *categorical* (respectively, *semi-categorical*) if every algebra, of the same species as \mathcal{U} , which satisfies all the strict identities of \mathcal{U} is a subdirect power of \mathcal{U} (respectively, is a subdirect product of subalgebras of \mathcal{U}). A map $f(\xi_1, \dots, \xi_k)$ from A^k into A is S -expressible if there exists an S -expression $\Psi(\xi_1, \dots, \xi_k)$ such that $f = \Psi$ for all ξ_1, \dots, ξ_k in A . A map $f(\xi_1, \dots, \xi_k)$ is *conservative* if for each subalgebra $\mathcal{B} = (B; \Omega)$ of \mathcal{U} and for all b_1, \dots, b_k in B , we have, $f(b_1, \dots, b_k) \in B$. An algebra \mathcal{U} is *primal* (respectively, *semi-primal*) if it is finite, with at least two elements, and every map from $A \times \dots \times A$ into A is S -expressible (respectively, every conservative mapping from $A \times \dots \times A$ into A is S -expressible). A semi-primal algebra \mathcal{U} which possesses exactly one subalgebra $\mathcal{U}^* = (A^*; \Omega) (\neq \mathcal{U})$ is called a *subprimal algebra*. The subalgebra \mathcal{U}^* is called the *core* of \mathcal{U} . If \mathcal{U}^* has exactly one element, \mathcal{U} is called a *singular subprimal*; otherwise it is called a *regular subprimal*. An element a in A is said to be *expressible* if there exists a unary S -expression $\Delta_a(\xi)$ such that $\Delta_a(\xi) = a$ for each ξ in A . An element a in $A \setminus A^*$ is said to be *ex-expressible* provided there exists a unary S -expression $\Gamma_a(\xi)$ such that $\Gamma_a(\xi) = a$ for each ξ in $A \setminus A^*$ (here, $A \setminus A^* = \{\xi \mid \xi \in A, \xi \notin A^*\}$).

We now proceed to define the concept of independence. Let $\{\mathcal{U}_i\} = \{\mathcal{U}_1, \dots, \mathcal{U}_r\}$ be a finite set of algebras of species S . We say that $\{\mathcal{U}_i\}$ is *independent* if corresponding to each set Ψ_1, \dots, Ψ_r of S -expressions there exists a single expression Ψ such that $\Psi = \Psi_i(\mathcal{U}_i)$, $i = 1, \dots, r$ (or equivalently, if there exists an r -ary S -expression Ψ such that $\Psi(\xi_1, \dots, \xi_r) = \xi_i(\mathcal{U}_i)$, $i = 1, \dots, r$). A *primal* (respectively, *subprimal*, *semi-primal*) *cluster* of species S is defined to be a class $\tilde{\mathcal{U}} = \{\dots, \mathcal{U}_i, \dots\}$ of primal (respectively, subprimal, semi-primal) algebras of species S , any finite subset of which is independent.

We are now in a position to state the following lemmas, the proofs of which have already been given in [2; 5].

LEMMA 1. *A primal algebra is categorical and simple.*

LEMMA 2. *A semi-primal algebra is semi-categorical and simple.*

LEMMA 3. *Let $\mathcal{B} = (B; \Omega)$ be a subprimal algebra of species S . Then,*

- (a) *the core $\mathcal{B}^* = (B^*; \Omega)$ is primal or else is a one-element subalgebra;*
- (b) *each b in B^* is S -expressible;*
- (c) *each b in $B \setminus B^*$ is ex-expressible.*

2. Semi-primal Clusters.

In this section some semi-primal clusters will be found. The methods of proof are similar to those of Foster [4] and O'Keefe [6]. We will be concerned, mainly, with co-coupled families of subprimal algebras, in the sense of the following

DEFINITION 1. A family $\mathcal{U}_i = (A_i; \Omega)$ $i \in I$, of subprimal algebras of species S , with cores $\mathcal{U}_i^* = (A_i^*; \Omega)$, $i \in I$, respectively, is said to be *co-coupled* if there exist two binary S -expressions $\xi \times \eta (= \xi \cdot \eta = \xi\eta)$ and $\xi T \eta$, and elements $0_i, 1_i$ in A_i ($0_i \neq 1_i$), for each $i \in I$, such that

(a) if \mathcal{U}_i is a singular subprimal, then

- (1) $\{0_i\} = A_i^*$;
- (2) $0_i \times \xi = \xi \times 0_i = 0_i$; $1_i \times \xi = \xi \times 1_i = \xi$ (all ξ in A_i);
- (3) $0_i T \xi = \xi T 0_i = \xi$ (all ξ in A_i);

(b) if \mathcal{U}_i is a regular subprimal, then (2) and (3) hold in addition to

- (4) $\{0_i, 1_i\} \subseteq A_i^*$.

DEFINITION 2. A family $\mathcal{U}_i = (A_i; \Omega)$, $i \in I$, of regular subprimal algebras of species S is said to be *co-framal* if there exist S -expressions $\xi \times \eta (= \xi \cdot \eta = \xi\eta)$ and ξ^n , and elements $0_i, 1_i$ in A_i ($0_i \neq 1_i$), for each $i \in I$, such that (2) and (4) hold in addition to

- (5) n is a permutation of A_i with $0_i^n = 1_i$.

REMARK 1. If \mathcal{U}_i , $i \in I$, is a co-framal family of regular subprimal algebras, then by letting $\xi T \eta = \text{def} = (\xi^\cap \times \eta^\cap)^\cup$, where ξ^\cup denotes the inverse of ξ^\cap , it follows that (3) holds and therefore the family is also co-coupled.

THEOREM 1. Let $\mathcal{U}_i = (A_i; \Omega)$, $i = 1, \dots, n$, be co-coupled subprimal algebras of species S . Then, if the \mathcal{U}_i are pairwise independent, they are independent.

PROOF. Assume that $\mathcal{U}_1, \dots, \mathcal{U}_n$ are pairwise independent. Then, for any two algebras $\mathcal{U}_i, \mathcal{U}_j$ (where $i \neq j$), there exists an S -expression $\Phi(\xi, \eta)$ such that

$$(6) \quad \Phi(\xi, \eta) = \begin{cases} \xi(\mathcal{U}_i) \\ \eta(\mathcal{U}_j). \end{cases}$$

Let $\mathcal{U}_i^* = (A_i^*; \Omega)$ denote the core of \mathcal{U}_i . If \mathcal{U}_i is a regular subprimal (respectively, a singular subprimal), then $1_i \in A_i^*$ (respectively, $1_i \in A_i \setminus A_i^*$), and according to Lemma 3 it is expressible (respectively, ex-expressible). In either case, there exists a unary S -expression $\Delta_{1_i}(\xi)$ such that

$$(7) \quad \Delta_{1_i}(\xi) = 1_i \quad (\xi \text{ in } A_i \setminus \{0_i\}).$$

From (6), (7), and the fact that $0_j \in A_j$ is S -expressible, say by $\Delta_{0_j}(\xi)$, it follows immediately that

$$\Pi_{ij}(\xi) = \text{def} = \Phi(\Delta_{1_i}(\xi), \Delta_{0_j}(\xi)) = \begin{cases} 1_i & (\xi \text{ in } A_i \setminus \{0_i\}) \\ 0_j & (\mathcal{U}_j) \end{cases} \quad (i \neq j).$$

Define, now, a unary S expression $\Psi_i(\xi)$, $1 \leq i \leq n$, by

$$(8) \quad \Psi_i(\xi) = \Pi_{i1}(\xi) \times \dots \times \widehat{\Pi_{ii}(\xi)} \times \dots \times \Pi_{in}(\xi) = \begin{cases} 1_i & (\xi \text{ in } A_i \setminus \{0_i\}) \\ 0_j & (\mathcal{U}_j) \text{ (all } j \neq i). \end{cases}$$

where $\widehat{}$ denotes deletion and the $\Pi_{ij}(\xi)$ are associated in some fixed manner. Using (8) and the co-coupling binary S -expression $\xi T \eta$, define an n -ary S -expression $\Phi(\xi_1, \dots, \xi_n)$ by

$$\Phi(\xi_1, \dots, \xi_n) = [\Psi_1(\xi_1) \times \xi_1] T \dots T [\Psi_n(\xi_n) \times \xi_n],$$

the T -factors being associated in some fixed manner. It is easily checked that $\Phi(\xi_1, \dots, \xi_n) = \xi_i(\mathcal{U}_i)$, $1 \leq i \leq n$. This proves the theorem.

Because of Theorem 1, it is important to discuss the pairwise independence of subprimal algebras. To do this we impose a surjectivity property on the primitive operations.

DEFINITION 3. A subprimal algebra \mathcal{U} , with core \mathcal{U}^* , is said to be *two fold surjective* if each primitive operation of \mathcal{U} is surjective on both \mathcal{U} and \mathcal{U}^* .

We now show that subprimal algebras which are two-fold surjective satisfy a certain factorization property (compare with [6]).

THEOREM 2. Let $\mathcal{U} = (A; \Omega)$ be a two-fold surjective subprimal algebra of species S . Then, for each unary S -expression, $\Gamma(\xi)$, and each primitive operation \mathcal{O}_i (of rank n_i) of \mathcal{U} , there exist unary S -expressions $\Psi_1(\xi), \dots, \Psi_{n_i}(\xi)$ such that

$$(9) \quad \mathcal{O}_i(\Psi_1(\xi), \dots, \Psi_{n_i}(\xi)) = \Gamma(\xi)(\mathcal{U}).$$

PROOF. Let $A = \{a_1, \dots, a_m, a_{m+1}, \dots, a_t\}$ where $A^* = \{a_1, \dots, a_m\}$ is the base set of \mathcal{U}^* (= core of \mathcal{U}) Clearly, $\Gamma(a_j) \in A^*$ for $1 \leq j \leq m$. Because of two-fold surjectivity, there exist elements a_{jk} ($1 \leq j \leq t, 1 \leq k \leq n_i$) of A , with a_{jk} in A^* when $1 \leq j \leq m$, such that

$$\mathcal{O}_i(a_{j1}, \dots, a_{jn_i}) = \Gamma(a_j).$$

Now let unary functions $g_1(\xi), \dots, g_{n_i}(\xi)$ be defined on A by

$$g_k(a_j) = a_{jk} \quad (1 \leq k \leq n_i, 1 \leq j \leq t).$$

Since $g_k(a_j) \in A^*$, $1 \leq j \leq m$, each g_k is conservative and hence is S -expressible, say by $\Psi_k(\xi)$. It follows that

$$\mathcal{O}_i(\Psi_1(a_j), \dots, \Psi_{n_i}(a_j)) = \mathcal{O}_i(g_1(a_j), \dots, g_{n_i}(a_j)) = \mathcal{O}_i(a_{j1}, \dots, a_{jn_i}) = \Gamma(a_j)$$

for $1 \leq j \leq t$ and (9) is verified.

From [6; Lemma 2.3] and Theorem 2 we immediately obtain the following generalized factorization property.

THEOREM 3. Let \mathcal{U} be a two-fold surjective subprimal algebra of species S . Then, for each expression $\Sigma(\xi_1, \dots, \xi_q)$ and each expression $\Theta(\xi_1, \dots, \xi_p)$ in which no indeterminate $\xi_i, 1 \leq i \leq p$, occurs twice in Θ , there exist expressions Ψ_1, \dots, Ψ_p such that

$$(10) \quad \Theta(\Psi_1, \dots, \Psi_p) = \Sigma(\mathcal{U}).$$

The pairwise independence of any two universal algebras \mathcal{U}, \mathcal{B} of species S assures that any two subalgebras of \mathcal{U}, \mathcal{B} , of more than one element each, are non-isomorphic. In establishing independence, therefore, this must be taken as a minimal assumption.

THEOREM 4. *Let $\mathcal{U} = (A; \Omega)$ and $\mathcal{B} = (B; \Omega)$ be subprimal algebras of species S with cores $\mathcal{U}^* = (A^*; \Omega)$ and $\mathcal{B}^* = (B^*; \Omega)$, respectively. Suppose that either of the following holds:*

(i) \mathcal{B} is a regular subprimal and $\mathcal{U}^*, \mathcal{B}^*$ are non-isomorphic;

(ii) \mathcal{B} is a singular subprimal and \mathcal{U}, \mathcal{B} are non-isomorphic.

Then, there exist elements d_1, d_2 in B ($d_1 \neq d_2$) and unary expressions $\Gamma_1(\xi), \Gamma_2(\xi)$ such that

$$(1^0) \quad \Gamma_1(\xi) = \Gamma_2(\xi)(\mathcal{U});$$

$$(2^0) \quad \Gamma_1(\xi) = d_1(\xi \text{ in } B \setminus B^*);$$

$$(3^0) \quad \Gamma_2(\xi) = d_2(\xi \text{ in } B \setminus B^*).$$

Moreover, if (i) holds, then $d_1, d_2 \in B^$ and*

$$(4^0) \quad \Gamma_1(\xi) = d_1(\mathcal{B});$$

$$(5^0) \quad \Gamma_2(\xi) = d_2(\mathcal{B}).$$

PROOF. First, assume that (i) holds. Then \mathcal{B}^* is primal (Lemma 3) and hence categorical (Lemma 1). Therefore, if $Id(\mathcal{U}) \supseteq Id(\mathcal{B}^*)$ then $\mathcal{U} \cong \mathcal{B}^{*(k)}$ ($= k^{\text{th}}$ subdirect power of \mathcal{B}) for some $k \geq 1$. Now $k \neq 1$ since \mathcal{U} has a subalgebra ($\neq \mathcal{U}$). But if $k \geq 2$, there exists an epimorphism $\mathcal{U} \rightarrow \mathcal{B}^*$, contradicting the simplicity of \mathcal{U} . Thus, $Id(\mathcal{U}) \not\supseteq Id(\mathcal{B}^*)$. Similarly, if $Id(\mathcal{B}^*) \supseteq Id(\mathcal{U})$, then since \mathcal{U} is semi-categorical (Lemma 2), $\mathcal{B}^* \cong \mathcal{U}^{(k_1)} \times \mathcal{U}^{*(k_2)}$ ($=$ subdirect product of subdirect powers of \mathcal{U} and \mathcal{U}^*) for some k_1, k_2 . Now $k_1 + k_2 \neq 1$ since $\mathcal{B}^*, \mathcal{U}$ are non-isomorphic and by assumption $\mathcal{B}^*, \mathcal{U}^*$ are non-isomorphic. Thus, $k_1 + k_2 \geq 2$. But then there exists an epimorphism from \mathcal{B}^* onto either \mathcal{U} or \mathcal{U}^* , contradicting the simplicity of \mathcal{B}^* . Thus $Id(\mathcal{B}^*) \not\supseteq Id(\mathcal{U})$. These two non-inclusions assure the existence of expressions $\Psi_1(\xi_1, \dots, \xi_p)$ and $\Psi_2(\xi_1, \dots, \xi_p)$ such that

$$(11) \quad \Psi_1 = \Psi_2(\mathcal{U}) \text{ and } \Psi_1 \neq \Psi_2(\mathcal{B}^*).$$

From (11) it follows that there exist elements β_1, \dots, β_p of B^* for which

$$(12) \quad d_1 = \text{def} = \Psi_1(\beta_1, \dots, \beta_p) \neq \Psi_2(\beta_1, \dots, \beta_p) = \text{def} = d_2.$$

Clearly, $d_1, d_2 \in B^*$. Since $\beta_1, \dots, \beta_p \in B^*$, there exist expressions $A_1(\xi), \dots, A_p(\xi)$ such that (see Lemma 3)

$$(13) \quad A_i(\xi) = \beta_i(\mathcal{C}), \quad 1 \leq i \leq p.$$

If $\Gamma_j(\xi)$ is defined by

$$(14) \quad \Gamma_j(\xi) = \Psi_j(A_1(\xi), \dots, A_p(\xi)), \quad 1 \leq j \leq 2,$$

from (12)-(14) it follows that $\Gamma_1(\xi), \Gamma_2(\xi)$ satisfy (1⁰), (4⁰), and (5⁰).

Secondly, assume that (ii) holds. Using arguments similar to those above, it can be established that $Id(\mathcal{C}) \not\cong Id(\mathcal{B})$ and $Id(\mathcal{B}) \not\cong Id(\mathcal{C})$. Thus, there exist expressions $\Psi_1(\xi_1, \dots, \xi_p), \Psi_2(\xi_1, \dots, \xi_p)$ such that $\Psi_1 = \Psi_2(\mathcal{C})$ and $\Psi_1 \neq \Psi_2(\mathcal{B})$. Let β_1, \dots, β_p be elements of B for which (12) holds. Because of Lemma 3, there exist expressions $A_1(\xi), \dots, A_p(\xi)$ with

$$A_i(\xi) = \beta_i(\text{in } B \setminus B^*), \quad 1 \leq i \leq p.$$

Let $\Gamma_1(\xi), \Gamma_2(\xi)$ be defined as in (14). It is easy to verify that they have the desired properties (1⁰)-(3⁰).

Next, we prove the following theorems.

THEOREM 5. *Let $\mathcal{C} = (A; \Omega)$ and $\mathcal{B} = (B; \Omega)$ be subprimal algebras of species S satisfying either (i) or (ii) of Theorem 4. Then there exist expressions $\Phi_1(\xi), \dots, \Phi_p(\xi)$ such that $\Phi_1(\xi) = \dots = \Phi_p(\xi)(\mathcal{C})$ and such that every conservative unary function on B is identical, in B , to one of $\Phi_1(\xi), \dots, \Phi_p(\xi)$.*

PROOF. Let the conservative unary functions on B be enumerated as $b_1(\xi), \dots, b_p(\xi)$ and let $d_1, d_2, \Gamma_1(\xi), \Gamma_2(\xi)$ be as in Theorem 4. Since \mathcal{C} is semi-primal, each conservative function on B is S -expressible. Hence, there exists an expression $\Phi = \Phi(\xi, \xi_1, \dots, \xi_p, \xi_{p+1})$ for which

$$\Phi(\xi, \underbrace{d_1, \dots, d_1}_{1 \text{ terms}}, d_2, \dots, d_2) = b_i(\xi), \quad 1 \leq i \leq p \quad (\xi \text{ in } B).$$

(This follows since the above equation is a conservative condition). Using Φ as a skeleton, we now define $\Phi_1(\xi), \dots, \Phi_p(\xi)$ by

$$\Phi_i(\xi) = \Phi(\xi, \underbrace{\Gamma_1(\xi), \dots, \Gamma_1(\xi)}_{1 \text{ terms}}, \Gamma_2(\xi), \dots, \Gamma_2(\xi)), \quad 1 \leq i \leq p.$$

From (1⁰) of Theorem 4 it follows that $\Phi_i(\xi) = \Phi_j(\xi)(\mathcal{C})$ for all $1 \leq i, j \leq p$. If (i) holds, then from (4⁰) and (5⁰) of Theorem 4, $\Phi_i(\xi) = b_i(\xi)$ (ξ in B), $1 \leq i \leq p$. If (ii) holds, then (2⁰) and (3⁰) assure that $\Phi_i(\xi) = b_i(\xi)$ (ξ in $B \setminus B^*$).

Moreover, in \mathcal{B} , $\Phi_i(\xi)$ and $b_i(\xi)$ are both conservative. Since B^* consists of exactly one element, say $B^* = \{0\}$, it follows that $\Phi_i(0) = b_i(0) = 0$. Hence, in case (ii) we also have $\Phi_i(\xi) = b_i(\xi)$ (ξ in B).

THEOREM 6. *Let $\mathcal{U} = (A; \Omega)$, $\mathcal{B} = (B; \Omega)$ be subprimal algebras of species S (with cores $\mathcal{U}^* = (A^*; \Omega)$, $\mathcal{B}^* = (B^*; \Omega)$, respectively) satisfying either (i) or (ii) of Theorem 4. If \mathcal{B} is two-fold surjective, then for each a in A^* and each unary expression $\Psi(\xi)$ there exists an expression $\Omega = \Omega(\xi)$ such that*

$$\Omega = \begin{cases} a(\mathcal{U}) \\ \Psi(\mathcal{B}). \end{cases}$$

PROOF. If $a \in A^*$, there exists a unary expression $\Theta(\xi)$ for which $\Theta = a(\mathcal{U})$. Let $\Theta'(\xi_1, \dots, \xi_p)$ be the S -expression derived from Θ by distinguishing each occurrence of ξ in Θ . Thus, by definition, $\Theta'(\xi, \dots, \xi) = \Theta(\xi)$. From Theorem 3, there exist expressions $\Psi_1(\xi), \dots, \Psi_p(\xi)$ such that

$$\Theta'(\Psi_1(\xi), \dots, \Psi_p(\xi)) = \Psi(\xi)(\mathcal{B}).$$

Since $\Psi_1(\xi), \dots, \Psi_p(\xi)$ are conservative in \mathcal{B} , by Theorem 5, there exist expressions $\Phi_1(\xi), \dots, \Phi_p(\xi)$ such that

$$\Phi_i(\xi) = \Phi_j(\xi)(\mathcal{U}), \quad 1 \leq i, j \leq p;$$

$$\Phi_i(\xi) = \Psi_i(\xi)(\mathcal{B}), \quad 1 \leq i \leq p.$$

Let $\Omega(\xi) = \Theta'(\Phi_1(\xi), \dots, \Phi_p(\xi))$. It is easily verified that Ω has the desired property of the theorem.

If F is a family of subprimal algebras of species S let us use F_s (respectively, F_r) to denote the subfamily of all singular subprimal (respectively, regular subprimal) members of F .

THEOREM 7. (Principal Theorem) *Let F be a family of two-fold surjective subprimal algebras of species S , each finite subset of which is co-coupled. If, further,*

(a) *the members of F_s are pairwise non-isomorphic,*

(b) *the members of F_r have pairwise non-isomorphic cores, then F is a subprimal cluster.*

PROOF. Because of (a), (b), and Theorem 6, for any two members \mathcal{U}, \mathcal{B} of F , there exist expressions $\Omega_1(\xi), \Omega_2(\xi)$ for which

$$\Omega_1(\xi) = \begin{cases} \xi^{(\mathcal{U})} \\ 0^{(\mathcal{B})} \end{cases}; \quad \Omega_2(\xi) = \begin{cases} 0^{(\mathcal{U})} \\ \xi^{(\mathcal{B})} \end{cases}.$$

Since each finite subset of F is co-coupled, there exists a binary S -expression $\xi T \eta$ satisfying (3). Thus

$$\Omega_1(\xi_1) T \Omega_2(\xi_2) = \begin{cases} \xi_1^{(\mathcal{U})} \\ \xi_2^{(\mathcal{B})} \end{cases}$$

and therefore \mathcal{U}, \mathcal{B} are independent. From Theorem 1 it follows that each finite subset of F is independent, and the theorem is proved.

COROLLARY 1. Let $F (= F_r)$ be a family of two-fold surjective regular subprimal algebras of species S satisfying (b) of Theorem 7. Suppose that each finite subset of F is co-framal. Then F is a regular subprimal cluster.

This follows from the above theorem, upon applying Remark 1.

We now consider special subclasses of co-coupled and co-framal subprimal algebras.

DEFINITION 4. An s -couple is a singular subprimal algebra $\mathcal{U} = (A; \times, T)$ of species $S = (2, 2)$ containing elements $0, 1$ ($0 \neq 1$) such that (1)-(3) hold. An r -frame is a regular subprimal algebra $\mathcal{U} = (A; \times, \cap)$ of species $S = (2, 1)$ containing elements $0, 1$ ($0 \neq 1$) for which (2), (4), and (5) hold.

Examples of s -couples are plentiful. Two such examples are (see [5]):

(1^o) The « double groups » $\mathcal{C} = (C; \times, +)$ of finite order $n \geq 2$ in which $(C; +)$ is a cyclic group with identity 0 and generator 1 , $(C \setminus \{0\}; \times)$ is a group with identity 1 , and $0 \times \xi = \xi \times 0 = 0$ (ξ in C); and

(2^o) the algebras $\mathcal{C}_p = (C_p; \times, +)$ of p elements $0, 1, \dots, p-1$ (p a prime) in which $\xi + \eta =$ addition mod p , and $\xi \times \eta = \min(\xi, \eta)$ in the ordering $0, 2, 3, \dots, p-1, 1$.

To establish other classes of r -frames and s -couples we need the following definitions and lemmas.

DEFINITION 5. A binary algebra is an algebra $\mathcal{B} = (B; \times)$ of species $S = (2)$ which possesses elements $0, 1$ ($0 \neq 1$) satisfying

$$(15) \quad 0 \times \xi = \xi \times 0 = 0; \quad 1 \times \xi = \xi \times 1 = \xi \text{ (all } \xi \text{ in } B).$$

The element 0 is called the *null* of \mathcal{B} ; 1 is called the *identity*.

LEMMA 4. (Foster and Pixley [5]). An algebra $\mathcal{B} = (B; \Omega)$ of species S is a regular subprimal if and only if there exist elements $0, 1$ in B ($0 \neq 1$) and functions \times (binary) and \circ (unary) defined in B such that (15) holds, in addition to

- (1⁰) \mathcal{B} is a finite algebra of at least three elements;
- (2⁰) \mathcal{B} possesses a unique subalgebra ($\neq \mathcal{B}$), denoted by $\mathcal{B}^* = (B^*; \Omega)$ and B^* contains at least two elements;
- (3⁰) the elements $0, 1$ and the functions \times, \circ are each S -expressible;
- (4⁰) \circ is a permutation of B in which $0 \circ = 1$;
- (5⁰) for each b in B , the characteristic function $\delta_b(\xi)$ (defined below) is S -expressible:

$$\delta_b(\xi) = 1 \text{ if } \xi = b \text{ and } \delta_b(\xi) = 0 \text{ if } \xi \neq b \text{ (all } \xi \text{ in } B);$$

- (6⁰) there exists an element b_0 in $B \setminus B^*$ which is ex -expressible.

LEMMA 5 (Foster and Pixley [5]). An algebra $\mathcal{B} = (B; \Omega)$ of species S is a singular subprimal if and only if there exist elements $0, 1, 1^0$ in B ($0 \neq 1$) and two binary functions \times, T defined in B such that (15) holds in addition to

- (1⁰) \mathcal{B} is a finite algebra of at least two elements;
- (2⁰) \mathcal{B} possesses exactly one one-element subalgebra $\mathcal{B}^* = (B^*; \Omega)$ and no other subalgebra ($\neq \mathcal{B}$);
- (3⁰) the element 0 and the functions \times, T are each S -expressible;
- (4⁰) $0 T \xi = \xi T 0 = \xi$ for each ξ in B and $1 T 1^0 = 1^0 T 1 = 0$;
- (5⁰) for each b in $B \setminus B^*$, the characteristic function $\delta_b(\xi)$ is S -expressible;
- (6⁰) there exists an element b_0 in $B \setminus B^*$ which is ex -expressible.

REMARK 2. If ξ° is a permutation on a set, we use ξ^u to denote its inverse. Moreover, for each positive integer s we define:

$$\xi^{n^s} = \text{def} = (\dots (\xi^\circ)^\circ \dots)^\circ \text{ (} s \text{ iterations).}$$

We define ξ^{u^s} similarly. Note that if ξ° is a permutation on a finite set, then there exists an integer s such that $\xi^{n^s} = \xi^u$. Hence, any (\circ, u) -expression is just a (\circ) -expression.

The following theorems provide large classes of r -frames and s -couples.

THEOREM 8. Let $\mathcal{B} = (P; \times, \circ)$ be a primal algebra for which

- (1⁰) $(P; \times)$ is a binary algebra (with null 0 and identity 1); and
- (2⁰) \circ is a cyclic permutation on P with $0^\circ = 1$. If $P_m = P \cup \{\lambda_1, \dots, \lambda_m\}$

where $\lambda_i \notin P$, $1 \leq i \leq m$, then the operations \times and n can be extended to P_m such that $\mathcal{B}_m = (P_m; \times, ^n)$ is an r -frame with core $\mathcal{C}\mathcal{B}$.

PROOF. Let $^n = (0, 1, \beta_3, \dots, \beta_n)$ in P . Because of primality, there exists a unary $(\times, ^n)$ -expression $\Delta(\xi)$ such that

$$\Delta(\xi) = \begin{cases} 1 & \text{if } \xi = 0 \\ 0 & \text{if } \xi \neq 0 \end{cases} \quad (\xi \text{ in } P).$$

We extend the definitions of \times and n to P_m as follows :

- (i) For ξ, η in P define $\xi \times \eta$ and ξ^n in P_m just as in P ;
- (ii) $\lambda_1^n = \lambda_2, \lambda_2^n = \lambda_3, \dots, \lambda_m^n = \lambda_1$;
- (iii) $\lambda_i \times \lambda_j = \lambda_{\min(i,j)}$ (if $i \neq j$) and $\lambda_i \times \lambda_i = 1$ (for each i);
- (iv) $0 \times \lambda_i = \lambda_i \times 0 = 0, 1 \times \lambda_i = \lambda_i \times 1 = \lambda_i$;
- (v) $\xi \times \lambda_i$ and $\lambda_i \times \xi$ (ξ in P) are defined arbitrarily for other ξ .

Each characteristic function $\delta_\tau(\xi), \tau \in P_m$, is $(\times, ^n)$ -expressible since the following identities hold in P_m (for a product of more than two terms, assume that the association is from the left) :

$$\delta_0(\xi) = \Delta(\xi) \Delta(\xi \times \xi), \delta_1(\xi) = \delta_0(\xi^U), \dots, \delta_{\beta_n}(\xi) = \delta_0(\xi^{U^{n-1}});$$

$$\delta_{\lambda_1}(\xi) = (\xi \cdot \xi^n \dots \xi^{n^{n-1}})^2 (((\xi \xi^n \dots \xi^{n^{m-1}})(\xi^n \xi^{n^2} \dots \xi^{n^{m-1}}))^U)^2$$

$$\delta_{\lambda_i}(\xi) = \delta_{\lambda_1}(\xi^U), \dots, \delta_{\lambda_m}(\xi) = \delta_{\lambda_1}(\xi^{U^{m-1}}).$$

In the above, ξ^U denotes the inverse of ξ^n . Since P_m is finite, ξ^U is a $(^n)$ -expression. Moreover, $0, 1, \beta_3, \dots, \beta_n$ are $(\times, ^n)$ -expressible and $\lambda_1, \dots, \lambda_m$ are $(\times, ^n)$ -ex-expressible, since

$$0 = \delta_0(\xi) \times \delta_1(\xi), 1 = 0^n, \beta_3 = 0^{n^2}, \dots, \beta_n = 0^{n^{n-1}} (\xi \text{ in } P_m);$$

$$\lambda_1 = \xi \cdot \xi^n \dots \xi^{n^{m-1}}, \lambda_2 = \lambda_1^n, \dots, \lambda_m = \lambda_1^{n^{m-1}} (\xi \text{ in } P_m \setminus P).$$

Clearly, $\mathcal{C}\mathcal{B}$ is the unique proper subalgebra of \mathcal{B}_m . The conditions (1⁰)-(6⁰) of Lemma 4 are verified. Thus, \mathcal{B}_m is a regular subprimal algebra and, indeed, even an r -frame. The theorem is proved.

THEOREM 9. *Let $(B; \times)$ be a finite binary algebra. Then a binary operation $\xi T \eta$ can be defined on B such that $(B; \times, T)$ is an s -couple.*

PROOF. For the two-element binary algebra $(\{0, 1\}; \times)$ it is easily verified that conditions (1^0) - (6^0) of Lemma 5 hold if $\xi T \eta$ is defined by $0 T \xi = \xi T 0 = \xi$ and $1 T 1 = 0$. Let, then $B = \{0, 1, b_1, \dots, b_m\}$ be the base set of a binary algebra of order $m + 2$, where $m \geq 1$. Consider the cases (I) $m \geq 2$ and (II) $m = 1$. For (I), define T on B such that

$$(16) \quad 0 T \xi = \xi T 0 = \xi \quad (\text{each } \xi \text{ in } B);$$

$$(17) \quad 1 T 1 = b_1, b_1 T b_1 = b_2, \dots, b_m T b_m = 1;$$

$$1 T b_1 = 1, b_1 T b_2 = b_2 T b_3 = \dots = b_{m-1} T b_m = b_m T 1 = 1 T b_m = 0;$$

$\xi T \eta$ is defined arbitrarily for other ξ, η in B ;

hold, while for (II), define T on B such that (16) and (17) hold in addition to

$$1 T b_1 = b_1 T 1 = 0.$$

In either case (I) or (II), let $\xi^n = \xi T \xi$. In the characteristic function $\delta_1(\xi)$ is (\times, T) -expressible then $\delta_{b_1}(\xi), \dots, \delta_{b_m}(\xi), I_1(\xi)$, and 0 are (\times, T) -expressible since

$$\delta_{b_m}(\xi) = \delta_1(\xi^n), \delta_{b_{m-1}}(\xi) = \delta_1(\xi^{n^2}), \dots, \delta_{b_1}(\xi) = \delta_1(\xi^{n^m}) \quad (\xi \text{ in } B);$$

$$I_1(\xi) = \delta_1(\xi) T \delta_{b_1}(\xi) T \dots T \delta_{b_m}(\xi) = 1 \quad (\xi \text{ in } B \setminus \{0\});$$

$$0 = \delta_1(\xi) \times \delta_{b_1}(\xi) \quad (\xi \text{ in } B).$$

In case (I), $\delta_1(\xi) = \xi T \xi^n$, while in case (II), $\delta_1(\xi) = \xi^2, (\xi T \xi^2)^2$, or $\xi^n T (\xi \xi^n)$, according as $b_1^2 = 0, 1$, or b_1 , respectively. In each case, it is clear that $\{0\}$ is the unique subalgebra of $(B; \times, T)$. The conditions (1^0) - (6^0) of Lemma 5 are verified. Thus, $(B; \times, T)$ is a singular subprimal algebra and, in fact, an s -couple.

We conclude with the following easily proved corollaries of Theorem 7.

COROLLARY 2. *Any subfamily of the family F_{s_0} of all pairwise non-isomorphic s -couples forms a singular subprimal cluster.*

COROLLARY 3. *Any subfamily of the family F_{r_0} of all r -frames with pairwise non isomorphic cores forms a regular subprimal cluster.*

COROLLARY 4. *Any subfamily of the family $F_{s_0} \cup F_{r_0}$ is a subprimal cluster.*

The algebras given in Theorems 8 and 9 apply, of course, to these corollaries.

Note Added in Proof. Theorem 8 was obtained independently by A. L. Foster, *Monatshefte für Mathematik* 72 (1968), 315-324.

