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THE CARTAN-THULLEN THEOREM FOR BANACH SPACES

by SEÁN DINEEN

The object of this paper is to give a characterization of domains of holomorphy which is similar to that given by Cartan-Thullen for finite dimensional spaces in [4]. In § 1 we recall some results from the theory of holomorphic functions on a Banach space, define domains of holomorphy and prove a number of fundamental lemmas. In § 2 we prove our main results. Our main reference for the theory of holomorphic functions on a Banach space is [16] and for domains of holomorphy in finite dimensions we refer to [2], [3], [4], [10], [13] and [15]. In [1], [2], [3], [6], [11] and [12] there are a number of interesting results concerning domains of holomorphy in infinite dimensions.

SECTION 1. E will represent a complex Banach space with unit ball B_1. For each positive integer m let $\mathcal{L}(^mE)$ denote the set of all continous m-linear mappings from $E^m = E \times E \times \dots \times E$ (m times) into C (the complex numbers). Let Δ_m denote the mapping from E into E^m which takes x into $(x, x, \dots, x)$ (m times). A continuous m-homogeneous polynomial is a mapping from E into C which is the composition of Δ_m and an element of $\mathcal{L}(^mE)$. We denote by $\mathcal{P}(^mE)$ the set of all continuous polynomials on E and we note that it forms a Banach space under the norm

$$\| P \| = \sup_{\substack{\|x\| \leq 1 \\ x \in E}} | P(x) |.$$

A complex valued function f defined on an open subset U of E is said to be holomorphic at $\xi \in U$ if there exists a sequence $(P_n(\xi))_{n=0}^{\infty}$ (where $P_n(\xi) \in$

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$\in \mathcal{P}(^nE)$ for each n) such that

(1) $$\limsup_{m \to \infty} \| P_m(\xi) \|^{1/m} < \infty$$

(2) $$f(x) = \Sigma_{m=0}^{\infty} P_m(\xi)(x - \xi)$$

for all x in some neighbourhood of ξ. f is said to be holomorphic on U if it is holomorphic at all points of U. Since the expansion in (2) is unique (see [16]) we write $\frac{\widehat{d}^m f(\xi)}{m!} = P_m(\xi)$ and call (2) the Taylor series expansion of f at ξ. For the remainder of this paper U will denote a connected open subset of E with a nonempty boundary δU (the questions we consider are trivial for $U = E$). $\complement A$ will denote the complement with respect to E of the subset A of E. If A_1 and A_2 are subsets of E we denote by $d(A_1, A_2)$ the distance between A_1 and A_2, i. e.

$$d(A_1, A_2) = \inf_{\substack{x \in A_1 \\ y \in A_2}} \| x - y \|$$

DEFINITION 1. $B \subset E$ is U-bounded if B is a bounded subset of E and $d(B, \complement U) > 0$. We note if E is finite dimensional that the closed U-bounded sets are exactly the compact subsets of U.

DEFINITION 2. (a) $\mathcal{H}(U)$ is the set of all complex-valued holomorphic functions on U.

(b) $\mathcal{H}_b(U)$ is the set of all holomorphic functions on U which are bounded on all U-bounded sets.

DEFINITION 3. Let $B \subset U$ then $\widehat{B}_U$ (resp. $\widehat{B}_{U,b}$) is the set of all $\zeta \in U$ such that

$$|f(\zeta)| \leq \sup_{x \in B} |f(x)|$$

for all $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$) and is called the U (resp. U_b)-holomorphic hull of B.

DEFINITION 4. A connected open subset U of E is a domain of holomorphy (resp. b domain of holomorphy) if there exists $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$) which cannot be extended to an element of $\mathcal{H}(U_1)$ (resp. $\mathcal{H}_b(U_1)$) where U_1 is a connected open subset of E which properly contains U.

For a subset V of $(\mathcal{P}(^mE))'$ (the dual of $\mathcal{P}(^mE)$) we let $\| P \|_V = \sup_{\Phi \in V} | \Phi(P) |$ for all $P \in \mathcal{P}(^mE)$.

LEMMA 1. Let $B \subset U$ and $V \subset (\mathcal{P}(^mE))'$ then for all $\zeta \in \widehat{B}_U$ (resp. $\widehat{B}_{U,b}$) we have

$$\| \widehat{d}^m f(\zeta) \|_V \leq \sup_{x \in B} \| \widehat{d}^m f(x) \|_V$$

for each integer m and all $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$).

PROOF. Let $\Phi \in V$ and $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$) then by the definition of holomorphic hull

$$| \Phi(\widehat{d}^m f(\zeta) | \leq \sup_{x \in B} | \Phi(\widehat{d}^m f(x))| \leq \sup_{x \in B} \| \widehat{d}^m f(x) \|_V.$$

Hence

$$\sup_{\Phi \in V} | \Phi(\widehat{d}^m f(\zeta)) | \leq \sup_{x \in B} \| \widehat{d}^m f(x) \|_V$$

i. e.

$$\| \widehat{d}^m f(\zeta) \|_V \leq \sup_{x \in B} \| \widehat{d}^m f(x) \|_V.$$

LEMMA 2. Let $B \subset U$ and $\alpha > 0$ be such that $B + \alpha B_1 \subset U$. If $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$) is bounded on $B + \alpha B_1$ then f is holomorphic (by analytic continuation if necessary) on $\widehat{B}_U + \alpha B_1$ (resp. $\widehat{B}_{U,b} + \alpha B_1$) and

$$|f(x+y)| \leq \sup_{w \in B + \alpha B_1} |f(w)| \cdot \frac{\alpha}{\alpha - \| y \|}$$

where $x \in \widehat{B}_U$ (resp. $\widehat{B}_{U,b}$) and $\| y \| < \alpha$.

PROOF. By Cauchy's inequalities (see [16], p. 22) we get

$$\sup_{x \in B} \left\| \frac{\widehat{d}^m f(x)}{m!} \right\| \leq \frac{M}{\alpha^m} \text{ for } m = 0, 1, \ldots$$

where $M = \sup_{x \in B + \alpha B_1} |f(x)|$. By lemma 1 this implies (taking $V = B_1$)

$$\sup_{x \in \hat{B}_U (\text{resp. } \hat{B}_{U,b})} \left\| \frac{\widehat{d}^m f(x)}{m!} \right\| \leq \frac{M}{\alpha^m}.$$

By the Cauchy-Hadamard formula ([16], P. 10) f is holomorphic (by analytic continuation if necessary) on $\widehat{B}_U + \alpha B_1$ (resp. $\widehat{B}_{U,b} + \alpha B_1$) and for $x \in \widehat{B}_U$

(resp. $\widehat{B}_{U,b}$) and $\|y\| < \alpha$ we have

$$|f(x+y)| \leq \Sigma_{n=0}^{\infty} \left\| \frac{d^n f(x)}{n!} \right\| \cdot \|y\|^n \leq \sup_{x \in B+\alpha B_1} |f(x)| \cdot \frac{\alpha}{\alpha - \|y\|}.$$

If α is a positive real number and $A \subset E$ we say $A + \alpha B_1$ is an α neighbourhood of A.

LEMMA 3. Let U be a domain of holomorphy (resp. b-domain of holomorphy) and let B be U-bounded. If each $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$) is bounded on some $\alpha(f)$ neighbourhood of B then $\widehat{B}_U$ (resp. $\widehat{B}_{U,b}$) is U-bounded.

PROOF. It is easy to check that $\widehat{B}_U$ (resp. $\widehat{B}_{U,b}$) is always a bounded subset of E so there remains only to show that $d(\widehat{B}_U, \complement U) > 0$ (resp. $d(\widehat{B}_{U,b}, \complement U) > 0$). If not there exists a sequence, $(\xi_n)_{n=0}^{\infty}$, of elements belonging to B_U (resp. $B_{U,b}$) such that $d(\xi_n, \complement U) \to 0$ as $n \to \infty$. Lemma 2 implies that every $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$) has an extension to a holomorphic function in some $\alpha(f)$ neighbourhood of each ξ_n. Since U is a domain of holomorphy (resp. b-domain of holomorphy) and $d(\xi_n, \complement U) \to 0$ as $n \to \infty$ this is a contradiction.

LEMMA 4. Let U be a connected open subset of E such that it is impossible to find two open connected subsets U_1 and U_2 of E with the following properties.

(1) $U \cap U_1 \supset U_2 \not\equiv \emptyset$ and $U_1 \not\subset U$

(2) For every $f \in \mathcal{H}_b(U)$ there exists an $f_1 \in \mathcal{H}_b(U_1)$ such that $f = f_1$ on U_2

then $\widehat{B}_{U,b}$ is U-bounded for each U-bounded set B and $d(\widehat{B}, \complement U) = d(B_{U,b}, \complement U)$.

PROOF. Since $B \subset \widehat{B}_{U,b}$ we get immediately that $d(B, \complement U) \geq d(\widehat{B}_{U,b}, \complement U)$. Suppose $d(B, \complement U) > d(\widehat{B}_{U,b}, \complement U)$. Choose $\xi_1 \in \widehat{B}_{U,b}$, $\xi_2 \in \complement U$ and $\alpha > 0$ such that $B + \underline{\xi_2 - \xi_1} + \alpha B_1 \subset U$ and is U-bounded where $\underline{\xi_2 - \xi_1}$ denotes the convex balanced hull of $\xi_2 - \xi_1$ (choose ξ_1, ξ_2 such that $\|\xi_2 - \zeta_1\| < d(B, \complement U)$ and take $0 < \alpha < 1/2\,[d(B, \complement U) - \|\xi_2 - \xi_1\|]$). For $z \in \widehat{B}_{U,b}$ we get by Cauchy's inequalities and lemma 1 that

$$\left\| \frac{\widehat{d^n f}(z)}{n!} \right\|_{\underline{\xi_2-\xi_1}+\alpha B_1} \leq \sup_{y \in B} \left\| \frac{\widehat{d^n f}(y)}{n!} \right\|_{\underline{\xi_2-\xi_1}+\alpha B_1} \leq \sup_{y \in B+\underline{\xi_2-\xi_1}+\alpha B_1} |f(y)|$$

(this is meaningful when we identify E with a subset of $(\mathcal{P}(^nE))'$ by means of the mapping $P \to P(x)$ for all $P \in \mathcal{P}(^nE)$). Lemma 2 implies that f is holomorphic on the set $\widehat{B}_{U,b} + \underline{\xi_2 - \xi_1} + \alpha B_1$ and for $\alpha_1 < \alpha$ we have

$$\sup_{y \in \hat{B}_{U,b} + \underline{\xi_2 - \xi_1} + \alpha_1 B_1} |f(y)| < \infty.$$

But $\xi_2 \in (\widehat{B}_{U,b} + \underline{\xi_2 - \xi_1}) \cap \complement U$ and this contradicts the hypothesis for U and hence proves the lemma.

The analogous lemma for $\mathcal{H}(U)$ can be proved in a similar fashion and we get

LEMMA 5. Let U be an open connected subset of E such that for each $\xi \in \delta U$ there exists $f \in \mathcal{H}(U)$ which cannot be extended to a holomorphic function in a neighbourhood of ξ. Then the holomorphic hull of each compact subset K of U, $\widehat{K}_U$, is a compact subset of U and $d(K, \complement U) = d(\widehat{K}_U, \complement U)$.

In the next two lemmas we concern ourselves with topologies on $\mathcal{H}(U)$ (see [5], [7] and [16]). Let T_0 denote the compact open topology on $\mathcal{H}(U)$ and let T denote the bornological topology associated with T_0 (see [7]). Since T_0 is complete (see [16]), T is also complete and barrelled.

LEMMA 6. Let $x_n \in U$ for $n = 1, 2, \dots$ and suppose $\sup_n |f(x_n)| < \infty$ for all $f \in \mathcal{H}(U)$ then $p(f) = \sup_n |f(x_n)|$ defines a continuous semi-norm on $(\mathcal{H}(U), T)$.

PROOF. Let $B_p = \{f;\ p(f) \leq 1\}$. Then B_p is absorbing (by hypothesis) and since $B_p = \bigcap_p \{f;\ |f(x_n)| \leq 1\}$ it is closed and convex. Since $(\mathcal{H}(U), T)$ is barrelled this completes the proof.

LEMMA 7. Let p be a continuous semi-norm on the space $(\mathcal{H}(U), T)$ and suppose $(V_n)_{n=1}^{\infty}$ is an increasing sequence of open subsets of U such that $\bigcup_{n=1}^{\infty} V_n = U$ then there exists a positive integer n_0 and $C > 0$ such that

$$p(f) \leq C \sup_{x \in V_{n_0}} |f(x)|.$$

PROOF. Suppose the result is not true. Then for each positive integer n we can choose $f_n \in \mathcal{H}(U)$ such that $p(f_n) \geq n$ and $\sup_{x \in V_n} |f_n(x)| \leq 1/n$.

The sequence $(f_n)_{n=1}^{\infty}$ is a bounded subset of $(\mathcal{H}(U), T_0)$ and consequently of $(\mathcal{H}(U), T)$ but $p(f_n) \to \infty$ as $n \to \infty$ which is a contradiction.

SECTION 2.

THEOREM (*Cartan-Thullen I*). *Let U be a connected open subset of the Banach space E. For the properties listed below we have* $(1) \Longrightarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Longleftrightarrow (5) \Longleftrightarrow (6)$. $(2) \Longrightarrow (1)$ *if E is separable but this does not hold for arbitrary E.*

(1) *U is a b-domain of holomorphy*

(2) *For each $\xi \in \delta U$ there exists $f \in \mathcal{H}_b(U)$ which cannot be extended analytically to a neighbourhood of ξ.*

(3) *It is impossible to find two open connected subsets U_1 and U_2 of E such that:*

(*a*) $U \cap U_1 \supset U_2 \neq \emptyset$ *and* $U \not\supset U_1$

(*b*) *For every $f \in \mathcal{H}_b(U)$ there exists an $f_1 \in \mathcal{H}_b(U_1)$ such that $f = f_1$ on U_2*

(4) *If B is U-bounded then $\widehat{B}_{U,b}$ is U-bounded and* $d(B, \complement U) = d(\widehat{B}_{U,b}, \complement U)$.

(5) *If B is U-bounded then $\widehat{B}_{U,b}$ is U-bounded.*

(6) *For each sequence $(\xi_n)_{n=1}^{\infty}$ of elements of U such that $\xi_n \to \xi \in \delta U$ as $n \to \infty$ there exists $f \in \mathcal{H}_b(U)$ such that* $\sup_n |f(\xi_n)| = \infty$.

PROOF. $(1) \Longrightarrow (2) \Longrightarrow (3)$, $(4) \Longrightarrow (5)$ and $(6) \Longrightarrow (2)$ are obvious.

Suppose (5) holds and (6) does not. Then there exists a sequence, $(\xi_n)_{n=0}^{\infty}$, of elements of U such that $\xi_n \to \xi \in \delta U$ as $n \to \infty$ and $\sup_n |f(\xi_n)| < \infty$ for all $f \in \mathcal{H}_b(U)$.

Now $\mathcal{H}_b(U)$ endowed with the topology of uniform convergence on U-bounded sets is a Frechet space and the mapping $f \to f(\xi_n)$ $(f \in \mathcal{H}_b(U))$ is continuous with respect to this topology for each n. Hence $p(f) = \sup_n |f(\xi_n)|$ defines a continuous semi-norm on $\mathcal{H}_b(U)$ and thus there exists $B \subset U$, U-bounded and $C > 0$ such that

$$\sup_n |f(\xi_n)| \leq C \sup_{x \in B} |f(x)|$$

for all $f \in \mathcal{H}_b(U)$. By using the fact that $\mathcal{H}_b(U)$ is an algebra we easily show that C can be taken equal to 1.

Therefore $\xi_n \in \widehat{B}_{U,b}$ for each n which contradicts the fact that $\widehat{B}_{U,b}$ is U-bounded. Hence $(5) \Longrightarrow (6)$. $(3) \Longrightarrow (4)$ by lemma 4.

In [12] there is an example of a Banach space whose unit ball is not a b-domain of holomorphy. This gives an example in which $(2) \not\Longrightarrow (1)$.

We complete the proof by showing $(4) \Longrightarrow (1)$ if E is separable. Since E is separable, U contains a countable dense subset M. Let $(\xi_n)_{n=2}^{\infty}$ be a sequence of elements in M containing each point in M infinitely often. For each ξ_n let A_n be the open ball with centre ξ_n and radius $d(\xi_n, \complement U)$. If (4) holds we can construct by induction a sequence $(B_n)_{n=2}^{\infty}$ of U-bounded sets and a sequence $(z_n)_{n=2}^{\infty}$ of points of U with the following properties:

1) $B_2 = \xi_2 = z_2$

2) $z_{n+1} \in A_n \cap \complement \widehat{B}_{n, U, b} \quad z_{n+1} \in B_{n+1}$

3) B_n is an increasing sequence and each U-bounded set is contained in some B_n. (This is possible since $d(A_n, \complement U) = 0$ and $d(\widehat{B}_{n, U, b}, \complement U) > 0$).

By construction we can choose $f_n \in \mathcal{H}_b(U)$ such that $\sup_{x \in B_n} |f_n(x)| \leq 1/2^n$ and $|f_n(z_{n+1})| > 2^n + |\Sigma_{i=1}^{n-1} f_i(z_{n+1})|$. Let $f = \Sigma_{n=2}^{\infty} f_n$. Since $f_n \in \mathcal{H}_b(U)$ for all n and $\Sigma_{n=m+1}^{\infty} \|f_n\|_{B_m} \leq \Sigma_{n=m+1}^{\infty} 1/2^n < \infty$ for each m, we have $f \in \mathcal{H}_b(U)$. Also

$$|f(z_n)| \geq |f_{n-1}(z_n)| - |\Sigma_{i=1}^{n-2} f_i(z_n)| - |\Sigma_{i=n}^{\infty} f_i(z_n)| \geq 2^{n-1} - \Sigma_{i=n}^{\infty} 1/2^i \geq 2^{n-}.$$

Hence $f(z_n) \longrightarrow \infty$ as $n \longrightarrow \infty$. This implies that if $\xi \in \delta U$ and $\varepsilon > 0$ is arbitrary

$$\sup_{x \in (\xi + \varepsilon B_1) \cap U} |f(x)| = \infty.$$

Thus f has no extension as a holomorphic function to a larger subset of E than U. This completes the proof.

As regards $\mathcal{H}(U)$ we do not know if the converse to lemma 5 is true even if E is separable. We now prove some results about $\mathcal{H}(U)$ similar to theorem 1.

THEOREM (*Cartan-Thullen* II).

Let U be an open connected subset of the separable Banach space E then the following are equivalent

(1) *U is a domain of holomorphy*

(2) *There exists an increasing sequence of U-bounded sets, $(B_n)_{n=2}^{\infty}$, such that $B_n = \widehat{B}_{n, U}$ and each compact set is contained in the interior of some B_n.*

PROOF. Suppose (1) is true; then there exists $f \in \mathcal{H}(U)$ which has U as its natural domain of existence. For each compact subset K of U choose $\alpha(K) > 0$ such that f is bounded on $K + 2\alpha(K) B_1$. The sets $K + \alpha(K) B_1$

cover U as K ranges over all compact subsets of U. Since E is separable we can choose a sequence $(K_n)_{n=2}^{\infty}$ of compact subsets of U such that $U = \bigcup_{n=2}^{\infty} \{K_n + \alpha(K_n) B_1\}$. Now let $\beta_n = \inf_{i \leq n} \alpha(K_n)$ and define $B_n = \bigcup_{i=2}^{n} \{K_n + \alpha(K_n) B_1\}$. Then B_n is an increasing sequence of U-bounded sets and f is bounded on $B_n + \beta_n B_1$. Lemma 2 implies that f is holomorphic (by analytic continuation if necessary) on $\widehat{B}_{n,U} + \beta_n B_1$. Since f has U as its natural domain of existence this implies $d(\widehat{B}_{n,U}, \complement U) \geq \beta_n > 0$. Since B_n is U-bounded for each n this means that $\widehat{B}_{n,U}$ is U-bounded for each n. Each compact subset of U is easily seen to be contained in the interior of some B_n. To complete the proof we note that the sequence $\widehat{B}_{n,U}$ has all the required properties.

(2) $\Longrightarrow$ (1) This is quite similar to the proof that (4) $\Longrightarrow$ (1) in the previous theorem. Let $(B_n)_{n=2}^{\infty}$ be the sequence of U-bounded given by hypothesis. Let M be a countable dense subset of U and take $(\xi_n)_{n=2}^{\infty}$ as a sequence of elements in M containing each point in M infinitely often. For each ξ_n let A_n be the open ball with centre ξ_n and radius $d(\xi_n, \complement U)$. Let $C_2 = B_2$ and choose $z_2 \in A_2 \cap \complement B_2$ (this is possible since $d(A_2, \complement U) = 0$ and B_2 is U-bounded).

Suppose $C_2, \ldots, C_n$ and $z_2, \ldots, z_n$ have been chosen. Choose k_{n+1} such that $B_{k_{n+1}}\ C_n$, $z_n \in B_{k_{n+1}}$. Let $C_{n+1} = B_{k_{n+1}}$ and choose $z_{n+1} \in A_{n+1} \cap \complement C_{n+1}$. For each n there exists $f_n \in \mathcal{H}(U)$ such that

$$\sup_{x \in B_n} |f_n(x)| < 1/2^n \text{ and } |f_n(z_n)| > 2^n + |\Sigma_{i=2}^{n} f_i(z_n)|.$$

The function $f = \Sigma_{n=2}^{\infty} f_n \in \mathcal{H}(U)$ and has U as its natural domain of existence. Hence (2) $\Longrightarrow$ (1).

The example quoted in theorem 1 also shows that the separability condition was essential in theorem II.

THEOREM (*Cartan-Thullen* III).

Let U be a connected open subset of a Banach space E then the following are equivalent:

(1) *For each $\xi \in \delta U$ there exists $f \in \mathcal{H}(U)$ which cannot be extended to a holomorphic function in a neighbourhood of ξ.*

(2) *For each sequence, $(\xi_n)_{n=1}^{\infty}$, of elements of U which converges to some point in δU there exists $f \in \mathcal{H}(U)$ such that $\sup_n |f(\xi_n)| = \infty$.*

PROOF. (2) $\Longrightarrow$ (1) is obvious. We now show if (2) is not true then (1) is not true.

If (2) is not true there exists a sequence, $(\xi_n)_{n=1}^{\infty}$, of elements of U which converges to $\xi \in \delta U$ and such that $\sup_n |f(\xi_n)| < \infty$ for all $f \in \mathcal{H}(U)$.

By lemma 6 $p(f) = \sup_n |f(\xi_n)|$ is a continuous semi-norm on $(\mathcal{H}(U), T)$. Let $f \in \mathcal{H}(U)$ be arbitrarily chosen. For each positive integer n let

$$U_n = \{x, |f(x)| < n\} \text{ and take}$$

$$V_n = \{x; x \in U_n \text{ and } d(x, \delta U_n) < 1/n\}.$$

$(V_n)_{n=1}^{\infty}$ is an increasing sequence of subsets of U and $\bigcup_{n=1}^{\infty} V_n = U$. By lemma 7 there exists n_0 and C such that

$$p(f) = \sup_n |f(\xi_n)| \leq C \sup_{x \in V_{n_0}} |f(x)|.$$

Since $\mathcal{H}(U)$ is an algebra we can take $C = 1$ and hence $\xi_n \in \widehat{V}_{n_0, U}$ for each n. By our choice of the V_n's, f is bounded on a $1/n_0$ neighbourhood of V_{n_0}.

Lemma 2 implies f can be continued as a holomorphic function in a $1/n_0$ neighbourhood of ξ_n for each n and hence in some neighbourhood of ξ. This completes this proof.

If the closed bounded subsets of E' (the dual of E) are week*-sequentially compact condition (2) of the last theorem can be replaced by the following equivalent condition (see [8]).

(2′) For each sequence, $(\xi_n)_{n=1}^{\infty}$, of elements of U which has no limit point in U there exists $f \in \mathcal{H}(U)$ such that

$$\sup_n |f(\xi_n)| = \infty.$$

In particular (2′) $\Longleftrightarrow$ (2) if E is separable or reflexive.

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BIBLIOGRAPHY

[1] ALEXANDER, H., *Analytic Functions on Banach spaces,* Thesis, University of California at Berkeley, 1968.

[2] BREMERMANN, H J., *Holomorphic functionals and complex convexity in Banach spaces,* Pacific J. of Math., 7 (1957), 811-831.

[3] BREMERMANN, H. J., *Complex Convexity,* Trans, Amer. Math. Soc., 82 (1956) 17-51.

[4] CARTAN, H. and P. THULLEN, *Regularitäts — und Konvergenzbereiche,* Math. Ann. 106 (1932) 617-647.

[5] CHAE, S. B., *Holomorphic germs on Banach spaces,* University of Rochester, Thesis (1969).

[6] COEURÉ, G., *Fonction plurisousharmoniques sur les espaces vectoriels topologiques et applications à l'étude des fonction analytiques,* Thesis, Université de Nancy (1969).

[7] DINEEN, S., *Holomorphy types on a Banach space,* University of Maryland, Thesis, (1969) (to be published in Studia Math.).

[8] DINEEN, S., *Unbounded holomorphic functions on a Banach space,* (to be published in Jour. Lond. Math. Soc.).

[9] GUPTA, C. P., *Malgrange theorem for nuclearly entire functions of bounded type on a Banach space,* University of Rochester, Thesis (1966). Reproduced in Notas de Matematica Pura e Aplicada, Rio de Janeiro (1968).

[10] GUNNING, R. and H. ROSSI, *Analytic functions of several complex variables,* Prentice Hall, 1965.

[11] HIRSCHOWITZ, A., *Remarques sur les ouverts d'holomorphie d'un produit dénombrable de droites.,* Annales de l'Institut Fourier, 19 (1969) 219-229.

[12] HIRSCHOWITZ, A., *Sur le non-plongement des varietes analytiques banachiques Réelles.* C. R. Acad. Sc., Paris, 269 (844-846) 1969.

[13] HÖRMANDER, L., *An introduction to complex analysis in several variables,* Van Nostrand, 1966.

[14] HORVÁTH, J., *Topological vector spaces and distributions,* Vol. I, Addison-Wesley, 1966.

[15] NACHBIN, L., *Holomorphic functions, domains of holomorphy and local properties.* Lecture notes at the University of Rochester, 1967, Prepared by R. M. Aron.

[16] NACHBIN, L., *Thopology on spaces of holomorphic mappings.* Ergebnisse der Matematik und ihre grenzgebiete 47. Springer-Verlag, Berlin (1969).

[17] NACHBIN, L., *Concerning holomorphy types for Banach spaces,* Proceedings of the Colloquium on Nuclear Spacer and Ideals in Operator Algebras held in Warsaw, Poland, (1969) (to be published in Studia Mathematica).

[18] NOVERRAZ, P., *Fonctions plurisousharmonique et analytiques dans les espaces vectoriels topologiques complexes,* Thesis, Faculte des Sciences de Paris, 1969.

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