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SEMINORMALITY AND PICARD GROUP

CARLO TRAVERSO

Introduction.

In this paper we investigate the structure of seminormal rings under noetherian assumptions and its connection with the Picard group.

The definitions of seminormality and seminormalization have arousen from a problem of classification ([1]), [2],). When one parametrizes algebraic objects (for example algebraic cycles of an algebraic variety) with an algebraic variety, one gets sometimes a variety which may depend, for example, from the choice of coordinates. If the transformation from one to another choice of coordinates acts birationally over the parameter variety, one knows that the normalization of the parameter variety is independent from that choice (for necessarily this birational map is bijective, so one can dominate it with the normalization). Of course the normalization may no longer parametrize the objects it was intended for, because one point of the variety may split in many points of the normalization. So one must « glue » together the different points of the normalization coming from one point of the variety.

This is the problem solved in [1], where the « weak normalization » is defined, in the standpoint of preschemes.

We study here a slightly different definition (which is the same of [1] whenever the prescheme is over a field of characteristic 0, for inseparability is then excluded).

We limit ourselves to affine schemes, i. e. to rings, although all results can be obtained for noetherian preschemes (proof by localization; see 2.2).

Let A be a ring, B an overring of A integral over A. We define

$$\overset{+}{_B}A = \{b \in B \mid \forall\, x \in \operatorname{spec} A,\ b_x \in A_x + R(B_x)\}$$

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where $R(B_x)$ is the radical of the ring B_x. ${}^{+}_{B}A$ is called the seminormalization of A in B, and if $A = {}^{+}_{B}A$, then A is called seminormal in B. If B is the normalizatiou of A (i. e. the integral closure of A in $Q(A)$, the total quotient ring of A) we put ${}^{+}A = {}^{+}_{B}A$, and A is called seminormal if $A = {}^{+}A$.

${}^{+}_{B}A$ is the greatest subring A' of B such that $A' \supseteq A$, and

i) $\forall\, x \in \operatorname{spec} A$, there is exactly one $x' \in \operatorname{spec} A'$ above x (i. e. such that $j_{x'} \cap A = j_x$)

ii) the canonical homomorphism $k(x) \rightarrow k(x')$ is an isomorphism.

For the proof, see section 1.

In Section 1 we prove some basic results on seminormalization and seminormal rings. We then restrict ourselves to noetherian rings A, with B an overring of A finite over A, and define the « glueings over a point $x \in \operatorname{spec} A$ » ; this is the local counterpart of obtaining ${}^{+}_{B}A$ from B, operating only over x; we obtain a new ring A' containing ${}^{+}_{B}A$ and seminormal in B, which equals B outside the closure of x, and has only one point x' above x, with the same residue field that A has in x.

In section 2 we investigate the structure of seminormal varieties. We prove the basic structure theorem 2.1 : if A is seminormal in B, (A noetherian, B finite over A), then A is obtained from B by a finite number of glueings.

By theorem 2.1 we then obtain, when A is reduced and B is the normalization of A, a kind of « purity theorem » (or Hartogs-like theorem, 2.3) which shall allow us to prove, in section 3, a generalization of a theorem of Endo, [6], using the methods of Bass and Murthy, [4].

Section 3 is dedicated to the proof of the main theorem of this paper :

Theorem 3.6. Suppose that A is a noetherian reduced ring and that the normalization of A is an overring of A of finite type ; let T be a finite set of indeterminates over A. Then the canonical homomorphism $\operatorname{Pic} A \rightarrow \operatorname{Pic} A[T]$ is an isomorphism if and only if A is seminormal.

The structure theorem of section 2 is not developped here in its full strenghth ; we can divide any glueing in two parts (a weakly normal part, for weak normality see [1], and a purely inseparable part) which would have allowed us to prove some results on products of seminormal rings (chiefly, under some freenes and finiteness assumptions, if A is seminormal in B, then $A \otimes C$ is seminormal in $B \otimes C$).

As the arguments for these generalizations are rather disjoint from this paper, while the proofs are in this present formulation considerably simpler, we have skipped the proofs.

We use standard notation and well-known results of commutative algebra (chiefly from [5] and [7]) without further mention.

We distinguish between $x \in \operatorname{spec} A$ and $j_x \subseteq A$ only if this does not lead to notational difficulties.

This work has been suggested to me by Andreotti, Bombieri and Salmon, whose works about weak normalization [1], [2] and Picard group [8], had led to identical examples.

1. Seminormality.

We prove that $\overset{+}{_B}A$ is the largest subring A' of B such that:

(1.1) i) $\forall\, x \in \operatorname{spec} A$, there is exactly one $x' \in \operatorname{spec} A'$ above x.

ii) the canonical homomorphism $k(x) \rightarrow k(x')$ is an isomorphism.

$\overset{+}{_B}A$ satisfies i) and ii): let $x \in \operatorname{spec} A$, and let x', x'' be points of $\operatorname{spec} \overset{+}{_B}A$ above x. We shall prove $j_{x'} = j_{x''}$. Let $b \in \overset{+}{_B}A$, $b \in j_{x'}$, and let y', y'' be points of $\operatorname{spec} B$ above x', x''. Then $b \in j_{y'}$. Let $b_x = \alpha + \gamma$, with $\alpha \in A_x$, $\gamma \in R(B_x)$. As $b \in j_{y'}$, we have $\alpha \in \mathfrak{m}_x \subseteq R(B_x)$, so we may assume $\alpha = 0$. Then $b_x \in R(B_x)$, so $b \in j_{y''} \cap \overset{+}{_B}A = j_{x''}$.

To prove ii), it is sufficient to remark that if y' is a point of $\operatorname{spec} B$ above x, and $b \in \overset{+}{_B}A$, $b_x = \alpha + \gamma$ with $\alpha \in A_x$, $\gamma \in R(B_x) \subseteq j_{y'}$, so $b(y') = \alpha(x) + \gamma(y') = \alpha(x) \in k(x)$ (identifying $k(x)$ with a subfield of $k(y')$).

Now we shall prove that if A' satisfies i) and ii), then $A' \subseteq \overset{+}{_B}A$. Let $b \in A'$, $b \notin \overset{+}{_B}A$; then there is x such that $b_x \notin A_x + R(A'_x)$. Since we have supposed that i) holds for A', A'_x is local and $A'_x = A'_{x'}$, so $R(A'_x) = \mathfrak{m}_{x'}$, $b_x \notin A_x + \mathfrak{m}_{x'}$. As $\mathfrak{m}_{x'}$ is the kernel of the homomorphism $A_{x'} \rightarrow k(x')$, this is equivalent to $k(x) \rightarrow k(x')$ not being surjective, contradicting ii).

From this characterization it is obvious that $\overset{+}{_B}A$ is seminormal in B, that if $A \subseteq C \subseteq B$ and A is seminormal in B, then A is seminormal in C too, and that if $A \subseteq B$, $A \subseteq C \subseteq \overset{+}{_B}A$, then C has properties (1.1), so $\overset{+}{_B}C = \overset{+}{_B}A$; in particular, $\overset{+}{_B}A$ has no proper subrings containing A and seminormal in B.

LEMMA 1.2. *Let $A \subseteq B \subseteq C$, C integral over A. If A is seminormal in B and B is seminormal in C, then A is seminormal in C.*

PROOF. Let $a \in \overset{+}{_C}A$, $x' \in \operatorname{spec} B$. Let $x \in \operatorname{spec} A$ such that $j_{x'} \cap A = j_x$. We have $a_x = \alpha + \gamma$, with $\alpha \in A_x$, $\gamma \in R(C_x)$. But $R(C_x)_{x'} \subseteq R(C_{x'})$, so $a_{x'} \in B_{x'} + R(C_{x'})$, hence $a \in \overset{+}{_C}B = B$.

$\overset{+}{_C}A$ is then a subring of B satisfying (1.1), so $\overset{+}{_C}A \subseteq \overset{+}{_B}A = A$.

LEMMA 1.3. *Let A be semiuormal in B, and let $\mathfrak{f}$ be the conductor of A in B. Then $\mathfrak{f}$ is equal to its radical in B.*

PROOF. We shall prove that A contains the radical $\mathfrak{f}'$ of $\mathfrak{f}$ in B; as $\mathfrak{f}$ is the greatest B-ideal contained in A, this implies $\mathfrak{f} = \mathfrak{f}'$. Let $b \in B$, $b^n \in \mathfrak{f}$, $x \in \operatorname{spec} A$; if $j_x \supseteq \mathfrak{f}$, we have $b_x^n \in \mathfrak{f}_x \subseteq \mathfrak{m}_x \subseteq R(B_x)$, so $b_x \in R(B_x)$; if $j_x \not\supseteq \mathfrak{f}$, we have $b_x \in B_x = A_x$; so we have $\forall\, x \in \operatorname{spec} A$, $b_x \in A_x + R(B_x)$, whence $b \in {}^{+}_{B}A = A$.

From now on in this section we shall always suppose that A is a noetherian ring and B an overring of A finite (i. e. integral of finite type) over A.

Let $x \in \operatorname{spec} A$, and let $x_1, \dots, x_n$ be the points of spec B above x. Let w_i be the canonical homomorphism $k(x) \longrightarrow k(x_i)$.

Let A' be the subring of B consisting of all $b \in B$ such that

(1.4) $\quad a)\ \ \forall\, i,\ b(x_i) \in w_i\, k(x)$

$\quad b)\ \ \forall\, i, l\ \ w_i^{-1}(b(x_i)) = w_l^{-1}(b(x_l))$

A' is a subring of B containing A and such that

(1.5) $\quad$ i) there is exactly one $x' \in \operatorname{spec} A'$ above x

$\quad$ ii) the canonical homomorphism $k(x) \longrightarrow k(x')$ is an isomorphism.

$a)$ implies ii) for any prime of A' above x, and $b)$ implies that if $b \in A'$ is contained in some prime above x, then it is contained in any prime above x.

A' is moreover the largest subring of B satisfying (1.5), as any element of one such A' shall satisfy $a)$ and $b)$; as by (1.1) ${}^{+}_{B}(A')$ has also these properties, it follows that A' is seminormal in B.

We say that A' is the ring obtained from B by glueing over x.

The conductor of A' in B contains $j_{x'}$, for if $a \in j_{x'}$, $b \in B$, then $ab(x_i) = a(x_i)\, b(x_i) = 0$.

Let S be a multiplicative system of A; if $S \cap j_x \neq \emptyset$, then $S^{-1} A' = S^{-1} B$, since the conductor contains $S^{-1} j_x = A'$; if not, $S^{-1} A'$ is obtained from $S^{-1} B$ by glueing over $S^{-1} j_x$; hence $S^{-1} A'$ is always seminormal in $S^{-1} B$.

LEMMA 1.6. *Let $y \in \operatorname{spec} A$; let A' be obtained from B by glueing over $x \in \operatorname{spec} A$. Suppose $j_y \not\supseteq j_x$. The primes of A' above y are in $1 - 1$ correspondence with the primes of B above y.*

PROOF. Let $\mathfrak{f}$ be the conductor of A' in B. As $\mathfrak{f} \supseteq j_x$, $\mathfrak{f} \cap A$ is not contained in j_y, so $A'_y = B_y$. As the maximal ideals of B_y (resp. A'_y) are in $1-1$ correspondence with the primes of B (resp. A') above y, the lemma is proved.

LEMMA 1.7. *Let $\mathfrak{f}$ be the conductor of A in B; suppose that $\mathfrak{f}$ is intersection of prime ideals of B. Let $x \in spec\, A$ be a prime of $\mathfrak{f}$, and suppose that there is exactly one $x' \in spec\, B$ above x. Then the canonical homomorphism $k(x) \longrightarrow k(x')$ cannot be surjective.*

PROOF. As B is finite over A, $\mathfrak{f}_x = \mathfrak{m}_x$ is the conductor of A_x in B_x (which implies $A_x \neq B_x$). $\mathfrak{f}_x$ is intersection of primes of B_x; but if a prime of B_x contains $\mathfrak{f}_x = \mathfrak{m}_x$, by Cohen-Seidenberg theorem it must be a maximal ideal of B_x. B_x is a local ring, since the maximal ideals of B_x are in $1-1$ correspondence with the primes of B above x; so $B_{x'} = B_x$, and $\mathfrak{m}_x = \mathfrak{f}_x$, intersection of maximal ideals, must be equal to $\mathfrak{m}_{x'}$. Suppose that $k(x) = k(x')$ (identifying canonically). Then $A_x/\mathfrak{m}_x = B_{x'}/\mathfrak{m}_{x'}$. Let $b \in B_{x'}$, $b \notin A_x$ and let $a \in A_x$ such that $a(x) = b(x')$. Then $b - a \in \mathfrak{m}_{x'} = \mathfrak{m}_x \subseteq A$, so $b \in A_x$, a contradiction.

COROLLARY 1.8. *The conductor $\mathfrak{f}$ of A in $\overset{+}{_B}A$ is not intersection of primes of $\overset{+}{_B}A$.*

PROOF. Suppose that $\mathfrak{f}$ is intersection of primes, and let us glue over a minimal prime j_x containing $\mathfrak{f}$; if the A' thus obtained were equal to $\overset{+}{_B}A$, j_x could not be a prime of $\mathfrak{f}$ for, being A' obtained by glueing over x, by 1.5 the hypothesis of lemma 1.7 were true and the thesis false. We should then have $A' \subsetneqq \overset{+}{_B}A$, and A' should be seminormal in $\overset{+}{_B}A$, which is impossible.

2. The structure of seminormal rings.

In this section we prove the two main structure theorems.

THEOREM 2.1. *Let A be a noetherian ring, B an overring of A, finite over A. If A is seminormal in B, there is a sequence $B = B_0 \supseteq B_1 \supseteq \dots$ $\dots \supseteq B_{n-1} \supseteq B_n = A$ such that B_{i+1} is obtained from B_i by glueing over a point $x \in spec\, A$.*

PROOF. Suppose that B_i has already been determined; if $B_i = A$, all is done; otherwise, let $\mathfrak{f}_i$ be the conductor of A in B_i; by 1.3, $\mathfrak{f}_i$ is intersection of prime ideals. Let $x \in \operatorname{spec} A$ be a prime of $\mathfrak{f}_i$ of least codimension (the assumption on the codimension is not needed, is only useful for other proofs). We define B_{i+1} as obtained from B_i by glueing over x. If $\mathfrak{f}_{i+1}$ is the conductor of A in B_{i+1}, we have $\mathfrak{f}_{i+1} \supseteq \mathfrak{f}_i$. As A is seminormal in B_{i+1}, we can apply 1.3; as B_{i+1} is obtained by glueing over x, by 1.5 and 1.7, j_x is not a prime of $\mathfrak{f}_{i+1}$, so $\mathfrak{f}_i \neq \mathfrak{f}_{i+1}$. By the noetherian hypothesis, we cannot have an infinite increasing chain of $\mathfrak{f}_i$, so there is n such that $B_n = A$, and the theorem is proved.

COROLLARY 2.2. *If A is seminormal in B and S is a multiplicative system of A, then $S^{-1} A$ is seminormal in $S^{-1} B$.*

PROOF. As B_{i+1} is obtained from B_i by glueing, $S^{-1} B_{i+1}$ is seminormal in $S^{-1} B_i$; then apply 1.2.

If B is the normalization of A, we will deduce from the sequence B_i of 2.1 another sequence $A(i)$ defined thus: $A(0) = B$; for $i \geq 1$, $A(i) = B_s$, if B_s is the largest B_j such that the conductor $\mathfrak{f}_j$ of A in B_j is contained in no prime of codimension $\leq i$. So we pass from $A(i)$ to $A(i+1)$ by a finite number of glueings over primes of codimension $i+1$.

The sequence $\{A(i)\}$ is independent of the particular sequence $\{B_j\}$, as $A(i)$ is the largest subring of B containing A and such that $A_x = (A(i))_x$ for each $x \in \operatorname{spec} A$, $\operatorname{cod}(x) \leq i$. Obviously, $(A(m))(n) = A(s)$, with $s = \min(m, n)$.

THEOREM 2.3. *Let A be a reduced noetherian seminormal ring with finite normalization; suppose $A = A(m)$, and let $a \in A$ be a regular element of A. Then any prime of aA has codimension $\leq m$.*

PROOF. Let B be the normalization of A; let C be a ring, $A \subseteq C \subseteq B$. Let $x \in \operatorname{spec} C$. We define $\operatorname{cod}(x; A) = \operatorname{cod}(j_x \cap A)$. By the Cohen-Seidenberg theorem, we have $\operatorname{cod}(x; A) \geq \operatorname{cod}(x)$, and if $\operatorname{cod}(x; A) > \operatorname{cod}(x)$, there exists $x' \in \operatorname{spec} C$ such that $j_x \cap A = j_{x'} \cap A$, $\operatorname{cod}(x'; A) = \operatorname{cod}(x')$.

As $Q(A) = Q(B) \supseteq Q(C)$, the minimal primes of C are in $1-1$ correspondence with the minimal primes of A, hence $\operatorname{cod}(x) = 0 \Longrightarrow \operatorname{cod}(x; A) = 0$.

Take in particular $C = B$; let $\mathfrak{f}$ be the conductor of A in B. Let $x \in \operatorname{spec} B$, such that $\operatorname{cod}(x; A) > \operatorname{cod}(x)$; suppose that if $y \in \operatorname{spec} B$, $y \neq x$ and $j_y \subseteq j_x$, then $j_y \not\supseteq \mathfrak{f}$, so in the chain $\{B_i\}$ we have no glueing over $j_y \cap A$. From 1.6 we get that in the chain $\{B_i\}$ we have a glueing over a prime $j_y \cap A$, $y \in \operatorname{spec} B$, such that $j_y \subseteq j_x$ (otherwise in each B_i, hence also in $B_n = A$, one would have at least two primes above $j_x \cap A$), so we have a glueing over x. As $A = A(m)$, any prime of A over which we glue has codimension $\leq m$, hence $\operatorname{cod}(x, A) \leq m$. As the assumptions are satisfied if $\operatorname{cod}(x) = 1$ ($\mathfrak{f}$ has no primes of codimension 0), we have $\operatorname{cod}(x) = 1 \Longrightarrow \Longrightarrow \operatorname{cod}(x\,;A) \leq m$.

Now we prove, by induction on the sequence $\{B_i\}$, that in B_i any prime x of aB_i is such that $\operatorname{cod}(x\,;A) \leq m$.

If $i = 0$, we have $B_0 = B$; as A is reduced, B is a direct sum of a finite number of integrally closed noetherian domains, for which it is well known that a prime of a principal ideal has codimension ≤ 1, so each prime y of aB has codimension 1, hence $\operatorname{cod}(y\,;A) \leq m$.

Suppose that the theorem is proved for B_i, and that B_{i+1} is obtained from B_i by glueing over $x \in \operatorname{spec} A$. Let x' be a prime of aB_{i+1}, and suppose $\operatorname{cod}(x'\,;A) > m$. As $j_{x'}$ is a prime of aB_{i+1}, we have $aB_{i+1} : j_{x'} \neq \neq aB_{i+1}$, so there is $\xi \notin aB_{i+1}$ such that $\xi j_{x'} \subseteq aB_{i+1}$. So $\xi j_{x'} B_i \subseteq aB_i$, and thus $\xi \in aB_i : j_{x'} B_i$.

Let $X_1, \ldots, X_j$ be the primes of $j_{x'} B_i$. We have $\operatorname{cod}(X_l\,;A) \geq \geq \operatorname{cod}(x'\,;A) > m$, so by induction hypothesis no j_{X_l} is contained in any prime of aB_i; thus $aB_i : j_{X_l} = aB_i$. As B_i is noetherian, $j_{x'} B_i$ contains a product of the j_{X_l}, so $aB_i : j_{x'} B_i = aB_i$. So we have proved $\xi \in aB_i$.

As a is regular, $f = \xi/a \in Q(A) \supseteq B_i$; $\xi \in aB_i$ implies $f \in B_i$. As $\xi \notin aB_{i+1}$, we have $f \notin B_{i+1}$; by the definition of B_{i+1}, (1.4), there is $x_q \in \operatorname{spec} B_i$ above x such that $f(x_q) \notin w_q\, k(x)$ (or there are $x_p, x_q \in \operatorname{spec} B_i$ above x such that $w_p^{-1} f(x_p) \neq w_q^{-1} f(x_q)$). Let us take $\zeta \in A$, $\zeta \in j_{x'}$, such that $\zeta(x) \neq 0$ (always possible, as $\zeta(x) \neq 0 \Longleftrightarrow \zeta \notin j_x$, and $\operatorname{cod}(x\,;A) \leq n < \operatorname{cod}(x'\,;A)$, so $j_{x'} \cap A \not\subseteq j_x$). Hence $(\zeta f)(x_q) = f(x_q)\, w_q\, \zeta(x) \notin w_q\, k(x)$ (or $w_p^{-1}(\zeta f)(x_p) = = \zeta(x)\, w_p^{-1} f(x_p) \neq \zeta(x)\, w_q^{-1} f(x_q) = w_q^{-1}(\zeta f)(x_q)$), so $\zeta f \notin B_{i+1}$ and $\zeta f a = \zeta \xi \notin \notin aB_{i+1}$ (a is regular), a contradiction, as we have assumed $\xi \in aB_{i+1} : j_{x'}$ and $\zeta \in j_{x'}$.

3. The Picard group.

Here we shall use freely definitions, notations and results of [4].

Let A be a noetherian ring; $\operatorname{Pic} A$ is the group of isomorphism classes of projective A-modules of rank 1 with $\otimes_A$ as product, $U(A)$ the group of units of A. We can associate functorially to any ring homomorphism f a group $\operatorname{Pic} \Phi f$ (defined in [4]) such that, if

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B' \end{array}$$

is a commutative square, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} U(A) & \to & U(B) & \to & \mathrm{Pic}\,\Phi f & \to & \mathrm{Pic}\,A & \to & \mathrm{Pic}\,B \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ U(A') & \to & U(B') & \to & \mathrm{Pic}\,\Phi f' & \to & \mathrm{Pic}\,A' & \to & \mathrm{Pic}\,B'. \end{array} \tag{3.1}$$

If S is a multiplicative system of A composed of regular elements, $B = S^{-1}A$, and f is the canonical injection, Pic Φf is naturally isomorphic to $\mathrm{inv}(A, S)$, where $\mathrm{inv}(A, S)$ is the subgroup of the group of invertible fractionary A-ideals spanned by the integral invertible A-ideals whose intersection with S is not empty ([5], [4] prop. 5.3).

We recall the following result:

LEMMA 3.2. *Let A be a reduced ring, and let T be a finite set of indeterminates over A; let $\mathfrak{a}$ be an ideal of $A[T]$, invertible and such that $\mathfrak{a}_0 = \mathfrak{a} \cap A$ contains a regular element. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ be the primes of $\mathfrak{a}$, and let $\mathfrak{p}_i = \mathfrak{P}_i \cap A$. Then $\mathfrak{a} = \mathfrak{a}_0 A[T]$ if and only if each $\mathfrak{a}A_{\mathfrak{p}_i}[T]$ is principal.*

For the proof, see [4], lemma 5.6, page 33.

THEOREM 3.3. *Let S be a multiplicative system of A composed of regular elements; suppose that if $s \in S$ and $\mathfrak{p}$ is a prime of sA, then Pic $A_{\mathfrak{p}}[T] = 0$. Then the canonical homomorphism φ: $\mathrm{inv}(A, S) \to \mathrm{inv}(A[T], S)$ is an isomorphism.*

PROOF. The theorem and the following corollary are proved in [4], 5.5 and 5.7; as there the proof is only sketched, and the theorem is stated only in a weaker form, we prove them completly here.

It is obviously sufficient to prove that f is surjective. Let $\mathfrak{a}$ be an $A[T]$-ideal intersecting S. Let $\mathfrak{a}_0 = \mathfrak{a} \cap A$ and let $s \in S \cap \mathfrak{a} \subseteq \mathfrak{a}_0$. As s is regular, we can apply lemma 3.2.

Let $\mathfrak{P}_i$ be a prime of $\mathfrak{a}$, $\mathfrak{p}_i = \mathfrak{P}_i \cap A$. As $\mathfrak{P}_i$ is a prime of an invertible ideal, we have $\mathrm{depth}(\mathfrak{P}_i) = 1$, so $\mathrm{depth}(\mathfrak{p}_i) \leq 1$. As $s \in \mathfrak{p}_i$ is regular, $\mathrm{depth}(\mathfrak{p}_i) = 1$, and $\mathfrak{p}_i$ is then a prime of any invertible ideal it contains, in particular of sA; by hypothesis, we have then $\mathrm{Pic}(A_{\mathfrak{p}_i}[T]) = 0$, so $\mathfrak{a}A_{\mathfrak{p}_i}[T]$ is principal. By lemma 3.2, $\mathfrak{a} = \mathfrak{a}_0 A[T]$, and we need only prove that $\mathfrak{a}_0$ is invertible. We can make A an $A[T]$-algebra, by the isomorphism $A \cong A[T]/_{TA[T]}$, and we have $\mathfrak{a}_0 \cong \mathfrak{a}_0 A[T] \otimes_{A[T]} A \cong \mathfrak{a} \otimes_{A[T]} A$, so $\mathfrak{a}_0$ is invertible. As $\mathfrak{a}_0 \cap S \neq \varnothing$, we have $\mathfrak{a}_0 \in \mathrm{inv}(A, S)$, $\varphi(\mathfrak{a}_0) = \mathfrak{a}$.

COROLLARY 3.4. *Let S be as in theorem 3 3; if $S^{-1}A$ is semilocal, we have an exact sequence*

$$0 \to \mathrm{Pic}\,A \to \mathrm{Pic}\,A[T] \to \mathrm{Pic}\,S^{-1}A[T].$$

PROOF (as in [4], 7.12). We need only exactness in $\operatorname{Pic} A[T]$. We have $\operatorname{Pic} S^{-1} A = 0$, hence

$$\begin{array}{ccccc} \operatorname{inv}(A, S) & \longrightarrow & \operatorname{Pic} A & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \operatorname{inv}(A[T], S) & \longrightarrow & \operatorname{Pic} A[T] & \longrightarrow & \operatorname{Pic} S^{-1} A[T] \end{array}$$

has exact rows, and the first vertical arrow is an isomorphism. So the result is immediate.

We recall another result of [4], theorem 7.2, page 41:

LEMMA 3.5. *Let B be a finite overring of A, and let j be the inclusion of A in B; let $\mathfrak{f}$ be the conductor of A in B, and let j' be the inclusion of $A' = A/\mathfrak{f}$ in $B' = B/\mathfrak{f}$. Then* $\operatorname{Pic} \Phi j \longrightarrow \operatorname{Pic} \Phi j'$ *is an isomorphism.*

We can now prove:

THEOREM 3.6. *Let A be a reduced noetherian ring with finite normalization. Let T be a finite set of indeterminates over A. The canonical homomorphism* $\operatorname{Pic} A \longrightarrow \operatorname{Pic} A[T]$ *is an isomorphism if and only if A is seminormal.*

PROOF: ($\Longleftarrow$) Suppose that A is seminormal. We prove the theorem, by induction on n, supposing $A = A(n)$. If $n = 0$, A is normal and the theorem is well-known. Suppose the theorem true for B if $B = B(n-1)$. We set $B = A(n-1)$; then $B(n-1) = B$.

Let $\mathfrak{f}$ be the conductor of A in B; as A is seminormal, if $A \neq B$, $\mathfrak{f}$ is intersection of prime ideals of A. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the primes of $\mathfrak{f}$ in A; as $A = A(n)$ and $B = A(n-1)$, the $\mathfrak{p}_i$ are all of codimension n.

Consider the multiplicative system S composed by those regular elements of A that are in no $\mathfrak{p}_i$. S is the complement of a finite number of primes (the $\mathfrak{p}_i$ and the primes of (0)), so $S^{-1} A$ is semilocal. Let $s \in S$, and let $\mathfrak{p}$ be a prime of sA. By theorem 2.3, $\operatorname{cod}(\mathfrak{p}) \leq n$; $\mathfrak{p}$ is not one of the $\mathfrak{p}_i$, as $s \notin \mathfrak{p}_i$, so $\mathfrak{p} \not\supseteq \mathfrak{p}_i$. As the $\mathfrak{p}_i$ are the minimal primes containing $\mathfrak{f}$, $\mathfrak{p} \not\supseteq \mathfrak{f}$, so $A_\mathfrak{p} = B_\mathfrak{p}$.

$A_\mathfrak{p} = B_\mathfrak{p}$ is seminormal, $B_\mathfrak{p}(n-1) = B_\mathfrak{p}$, hence the induction hypothesis holds, $\operatorname{Pic} A_\mathfrak{p}[T] = 0$. We can thus apply Corollary 3.4, so

$$0 \longrightarrow \operatorname{Pic} A \longrightarrow \operatorname{Pic} A[T] \longrightarrow \operatorname{Pic} S^{-1} A[T]$$

is exact. If we prove $\operatorname{Pic} S^{-1} A[T] = 0$, all is done.

Let us put $\bar{A} = S^{-1} A$, $\bar{B} = S^{-1} B$, $\bar{\mathfrak{f}} = S^{-1}\mathfrak{f}$. $\bar{A}$ is seminormal, $\bar{B}(n-1) = \bar{B}$. Any $S^{-1}\mathfrak{p}_i$ is a maximal ideal of $\bar{A}$, so $\bar{\mathfrak{f}}$, conductor of $\bar{A}$ in $\bar{B}$, contains the radical of $S^{-1} A$.

We take $A' = \bar{A}/\bar{\mathfrak{f}}$, $B' = \bar{B}/\bar{\mathfrak{f}}$. From lemma 1.3, B', A' are artinian reduced, hence they are direct sum of fields, so $\operatorname{Pic} A'[T] = \operatorname{Pic} B'[T] = 0$. By induction hypothesis, $\operatorname{Pic} \bar{B}[T] = 0$. We can consider the four exact sequences (with j, j', j_T, j'_T obvious inclusions)

$$\alpha) \qquad U(\bar{A}) \to U(\bar{B}) \to \operatorname{Pic} \Phi j \to 0 \to 0$$

$$\beta) \qquad U(A') \to U(B') \to \operatorname{Pic} \Phi j' \to 0 \to 0$$

$$\gamma) \qquad U(\bar{A}[T]) \to U(\bar{B}[T]) \to \operatorname{Pic} \Phi j_T \to \operatorname{Pic} \bar{A}[T] \to 0$$

$$\delta) \qquad U(A'[T]) \to U(B'[T]) \to \operatorname{Pic} \Phi j'_T \to 0 \to 0$$

with arrows connecting $\alpha \to \beta$, $\alpha \to \gamma$, $\beta \to \delta$, $\gamma \to \delta$ and thus making a big commutative diagram. As any ring in question is reduced, the first and the second arrows connecting $\alpha \to \gamma$, $\beta \to \delta$ are isomorphisms. Applying the five lemma to $\beta \to \delta$, we get that $\operatorname{Pic} \Phi j' \to \operatorname{Pic} \Phi j'_T$, is an isomorphism. By lemma 3.5, we have that $\operatorname{Pic} \Phi j \to \operatorname{Pic} \Phi j'$, $\operatorname{Pic} \Phi j_T \to \operatorname{Pic} \Phi j'_T$ are isomorphisms, so $\operatorname{Pic} \Phi j \to \operatorname{Pic} \Phi j_T$ is an isomorphism. Applying the five lemma to $\alpha \to \gamma$, we get $\operatorname{Pic} S^{-1} A[T] = 0$.

($\Longrightarrow$) We shall prove that if A is not seminormal, then $\operatorname{Pic} A \to \operatorname{Pic} A[T]$ is not an isomorphism. Let $B = A^+$, and let $\mathfrak{f}$ be the conductor of A in B. Let $A' = A/\mathfrak{f}$, $B' = B/\mathfrak{f}$, j, j', j_T, j'_T inclusions as in the first part of the proof. Consider the square:

$$\begin{array}{ccc} \operatorname{Pic} A & \to & \operatorname{Pic} A[T] \\ \downarrow f & & f' \downarrow \\ \operatorname{Pic} B & \to & \operatorname{Pic} B[T]. \end{array}$$

The bottom arrow is an isomorphism as $B = {}^+A$ is seminormal; if the top arrow were an isomorphism, one would have that the injection $\ker f \to \ker f'$ is an isomorphism, and we shall prove that this is false.

As A, B are reduced, $U(A) = U(A[T])$, $U(B) = U(B[T])$, so it is sufficient to prove that $\operatorname{Pic} \Phi j \to \operatorname{Pic} \Phi j_T$ is not an isomorphism.

From lemma 3.5, this is equivalent to $\operatorname{Pic} \Phi j' \to \operatorname{Pic} \Phi j'_T$ not being an isomorphism.

Consider the diagram with exact rows:

$$\begin{array}{ccccccc} & & U(B') & \xrightarrow{f_1} & \mathrm{Pic}\, \Phi j' & \xrightarrow{f_2} & \mathrm{Pic}\, A' \\ & & \downarrow \varphi & & \downarrow \chi & & \downarrow \psi \\ U(A'[T]) & \xrightarrow{g_1} & U(B'[T]) & \xrightarrow{g_2} & \mathrm{Pic}\, \Phi j' & \xrightarrow{g_3} & \mathrm{Pic}\, A'[T] \end{array}$$

φ and g_1 are inclusions, and ψ is injective.

Suppose that χ is an isomorphism, and let $\alpha \in U(B'[T])$. Then $g_2(\alpha) \in \in \ker g_3 = \chi \ker(f_2 \circ \psi) = \chi \ker f_2$ (as ψ is injective), so $\chi^{-1} g_2(\alpha) = f_1(\alpha')$ (with $\alpha' \in U(B')$, $\chi f_1(\alpha') = g_2(\alpha)$. As $\chi f_1 = g_2 \varphi$, we have $g_2(\alpha) = g_2 \varphi(\alpha')$, so $\alpha^{-1} \alpha' \in \ker g_2 = \mathrm{im}\, g_1$ (remember that in the group of units the product is the composition law), hence there is $\beta \in U(A'[T])$ such that $\alpha = \beta\alpha'$. We shall exibit an α such that this is false.

By corollary 1.8, $\mathfrak{f}$ does not coincide with its root in $B = {}^+A$, so there is an $a \in B$ such that $a \notin A$, $a^n \in \mathfrak{f}$ for some n. Hence there is in B' a nilpotent a' not contained in A.

Let $t \in T$; $1 + a't$ is an unit of $B'[T]$, and is not product of an unit of $A'[T]$ and one of B'. Hence the theorem is complete.

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