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KILLING FORMS IN A RIEMANNIAN MANIFOLD WITH BOUNDARY (*)

GRIGORIOS TSAGAS

1. Introduction.

Let M be a compact Riemannian manifold which is the closure of an open submanifold of an n -dimensional orientable Riemannian manifold V . The manifold M has a boundary $\partial M = B$, which is an $(n - 1)$ -dimensional compact orientable submanifold ([1]). We denote by $K_T^2(M, \mathbb{R})$ and $K_N^2(M, \mathbb{R})$ the Killing 2-forms on the manifold, which are tangential and normal to the boundary, respectively. We assume that the manifold M is negatively k -pinched, then the groups $K_T^2(M, \mathbb{R}), K_N^2(M, \mathbb{R})$ have some properties.

The aim of the present paper is to prove that if the number k is greater than a number μ and the second fundamental form on the boundary B satisfies some relations, then the two groups are trivial. These results are an extension of those given in ([7]).

2. A p -form $\alpha = (\alpha_{i_1 \dots i_p})$ is called Killing if it satisfies the relation, ([8], p. 66)

$$\nabla_X \alpha(Y, X_2, \dots, X_p) + \nabla_Y \alpha(X, X_2, \dots, X_p) = 0, X, Y, X_i \in T(M), i = 2, \dots, p,$$

which implies

$$(2.1) \quad \delta \alpha = 0.$$

For any p -form α , we have the formula, ([5], p. 4)

$$(2.2) \quad \frac{1}{2} \Delta(|\alpha|^2) = (\alpha, \Delta \alpha) - |\nabla \alpha|^2 + \frac{1}{2[(p-1)!]} Q_p(\alpha),$$

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where

$$(2.3) \quad |\nabla\alpha|^2 = \frac{1}{p!} \nabla^l \alpha^{i_1 \dots i_p} \nabla_l \alpha_{i_1 \dots i_p},$$

$$(2.4) \quad Q_p(\alpha) = (p-1) R_{ijkl} \alpha^{ij_3 \dots i_p} \alpha_{i_3 \dots i_p}^{kl} - 2R_{hl} \alpha^{hi_2 \dots i_p} \alpha_{i_2 \dots i_p}^l.$$

If α is a Killing p -form, then (2.2) takes the form ([7])

$$(2.5) \quad \frac{1}{2} \Delta(|\alpha|^2) = -|\nabla\alpha|^2 - \frac{1}{2(p!)} Q_p(\alpha).$$

We consider a point P on the boundary B . Let (u^1, \dots, u^{n-1}) be a local coordinate system of a neighborhood of the point P as a point of B and (v^1, \dots, v^n) another local coordinate system of a neighborhood of the same point considered as a point of V . The local representation of B is given by

$$(2.6) \quad v^i = f^i(u^1, \dots, u^{n-1}), \quad i = 1, \dots, n,$$

in $U(P) \cap M$, $U(P)$ being a coordinate neighborhood of V .

We denote by N the normal vector field to the boundary. We choose the local coordinate system (u^1, \dots, u^{n-1}) such that the vector fields $N, \partial/\partial u^1, \dots, \partial/\partial u^{n-1}$ form a positive sense of M with respect to vector fields $\partial/\partial v^1, \dots, \partial/\partial v^n$.

We assume that the mapping F of B into M defined by (2.6) is an isometric immersion, therefore the metric $h = (h_{\lambda\nu})$ on the manifold B is given by

$$h_{\lambda\nu} = \partial v^i / \partial u^\lambda \partial v^j / \partial u^\nu g_{ij},$$

where (g_{ij}) is the metric on the manifold M .

We denote by g and h the determinants of the metrics (g_{ij}) and $(h_{\lambda\nu})$, respectively.

If ω is any $(n-1)$ -form on the manifold M , then Stoke's theorem can be stated as follows

$$\int_M d\omega = \int_B \omega,$$

from which we obtain

$$(2.7) \quad \int_M \delta\gamma \eta = - \int_B (N, \gamma) \bar{\eta},$$

for any vector field $\gamma = (\gamma_i)$ on M and $\eta, \bar{\eta}$ are the volume elements of M, B , respectively, defined by

$$\eta = \sqrt{g} dv^1 \wedge \dots \wedge dv^n, \quad \bar{\eta} = \sqrt{h} du^1 \wedge \dots \wedge du^{n-1}.$$

The relation (2.7) is valid, if we define the codifferentiation of a p -form $\alpha = (\alpha_{i_1 \dots i_p})$ as follows

$$(2.8) \quad (\delta\alpha)_{i_2 \dots i_p} = -\nabla_l \alpha_{i_2 \dots i_p}^l.$$

A p -form $\alpha = (\alpha_{i_1 \dots i_p})$ on the manifold M is tangential to B , if satisfies the relations ([10], p. 431)

$$\alpha^{i_1 \dots i_p} = \partial v^{i_1} / \partial u^{j_1} \dots \partial v^{i_p} / \partial u^{j_p} \bar{\alpha}_{j_1 \dots j_p},$$

or

$$\alpha^{hi_2 \dots i_p} N_h = 0,$$

where $\bar{\alpha} = (\bar{\alpha}_{j_1 \dots j_p})$ is a p -form defined over B , which imply, ([10], p. 434)

$$(2.9) \quad (\nabla_h \alpha_{j_1 i_2 \dots i_p}) (\alpha^{j_1 i_2 \dots i_p}) N^h = -H_{j_1} \bar{\alpha}_{i_2 \dots i_p}^j \bar{\alpha}^{i_2 \dots i_p} + \\ (\nabla_j \alpha_{hi_2 \dots i_p} + \nabla_h \alpha_{ji_2 \dots i_p}) \alpha^{j_1 i_2 \dots i_p} N^h.$$

We consider a p -form $\alpha = (\alpha_{i_1 \dots i_p})$ on the manifold M . This form is normal to the boundary B , if we have the relation, ([10], p. 432)

$$\alpha_{i_1 \dots i_p} \partial v^{i_1} / \partial u^{j_1} \dots \partial v^{i_p} / \partial u^{j_p} = 0,$$

from which, we obtain ([10], p. 435)

$$(2.10) \quad (\nabla_h \alpha_{j_1 i_2 \dots i_p}) \alpha^{j_1 i_2 \dots i_p} N^h = p (\nabla_h \alpha_{i_2 \dots i_p}^h) \alpha^{j_1 i_2 \dots i_p} N_j + \\ p H_i \bar{\alpha}_{i_2 \dots i_p} \bar{\alpha}^{i_2 \dots i_p} - p(p-1) H_{j_1} \bar{\alpha}_{i_2 \dots i_p}^j \bar{\alpha}^{i_2 \dots i_p},$$

where $\bar{\alpha} = (\bar{\alpha}_{i_2 \dots i_p})$ is a $(p-1)$ -form on the manifold B defined by

$$\alpha_{hi_2 \dots i_p} N^h = \bar{\alpha}_{j_1 \dots j_p} \partial v^{j_1} / \partial u^{i_1} \dots \partial v^{j_p} / \partial u^{i_p}.$$

3. We assume that the dimension of the manifold M is odd $n = 2m + 1$ and admits a metric which is negatively k -pinched. We also assume that α

is a Killing 2-form and consider the $2m$ -form β

$$\beta = \frac{1}{m!} \alpha \wedge \dots \wedge \alpha, \quad (m \text{ times}).$$

Let P be any point of the manifold M . There is a special base of the vector space M_P^* , such that the following inequalities hold at the point P , ([7]),

$$(3.1) \quad \frac{1}{2} Q_2(\alpha) \geq 2(2m-1)k|\alpha|^2 - \frac{8}{3}(1-k)\delta,$$

$$(3.2) \quad \frac{1}{2[(2m-1)!]} Q_{2m}(\beta) \geq 2mk|\beta|^2,$$

where

$$|\alpha|^2 = \alpha_{12}^2 + \alpha_{34}^2 + \dots + \alpha_{2m-1, 2m}^2, \quad \beta = \alpha_{12} \alpha_{34} \dots \alpha_{2m-1, 2m},$$

$$\delta = \alpha_{12} \alpha_{34} + \dots + \alpha_{12} \alpha_{2m-1, 2m} + \dots + \alpha_{2m-3, 2m-2} \alpha_{2m-1, 2m},$$

where $\alpha_{12}, \alpha_{34}, \dots, \alpha_{2m-1, 2m}$ the components of the Killing 2-form α with respect to the special base of the vector space M_P^* .

The formula (2.5) for $p=2$ becomes

$$\frac{1}{2} \Delta(|\alpha|^2) = -|\nabla\alpha|^2 - \frac{1}{4} Q_2(\alpha),$$

or

$$\frac{1}{2} \int_M |\alpha|^{2m-2} \Delta(|\alpha|^2) \eta = \int_M [-|\alpha|^{2m-2} |\nabla\alpha|^2 - \frac{1}{4} |\alpha|^{2m-2} Q_2(\alpha)] \eta,$$

which by means of (3.1) becomes

$$(3.3) \quad \frac{1}{2} \int_M |\alpha|^{2m-2} \Delta(|\alpha|^2) \eta \leq \int_M -[|\alpha|^{2m-2} |\nabla\alpha|^2 - \frac{4}{3}(1-k)|\alpha|^{2m-2} \delta + (2m-1)k|\alpha|^{2m}] \eta.$$

It can be easily proved the formula

$$\Delta[(|\alpha|^2)^m] = m|\alpha|^{2m-2} \Delta(|\alpha|^2) - m(m-1)|\alpha|^{2m-4} (\Delta(|\alpha|^2))^2,$$

from which we obtain

$$(3.4) \quad \int_M \Delta[(|\alpha|^2)^m] \eta \leq \int_M m|\alpha|^{2m-2} \Delta(|\alpha|^2) \eta.$$

The relation (3.4) by virtue of (2.7) becomes

$$(3.5) \quad - \int_B (N, |\alpha|^{2m-2} d(|\alpha|^2)) \bar{\eta} \leq \int_M |\alpha|^{2m-2} \Delta(|\alpha|^2) \eta.$$

From (3.3) and (3.5) we obtain

$$(3.6) \quad \frac{1}{2} \int_B (N, |\alpha|^{2m-2} d(|\alpha|^2)) \bar{\eta} \geq \int_M \left[|\alpha|^{2m-2} |\nabla \alpha|^2 - \frac{4}{3} (1-k) |\alpha|^2 \delta + (2m-1) k |\alpha|^{2m} \right] \eta.$$

The formula (2.2) for the $2m$ -form β takes the form

$$\frac{1}{2} \Delta(|\beta|^2) = (\beta, \Delta\beta) - |\nabla \beta|^2 + \frac{1}{2[(2m-1)!]} Q_{2m}(\beta),$$

or

$$\frac{1}{2} \int_M \Delta(|\beta|^2) \eta = \int_M \left[(\beta, \Delta\beta) - |\nabla \beta|^2 + \frac{1}{2[(2m-1)!]} Q_{2m}(\beta) \right] \eta,$$

which by means of (2.7), (3.2) and the relation, ([4], p. 187)

$$\langle \beta, \Delta\beta \rangle = \|d\beta\|^2 + \|\delta\beta\|^2,$$

becomes

$$(3.7) \quad - \frac{1}{2} \int_B (N, d(|\beta|^2)) \bar{\eta} \geq \int_M [\|\delta\beta\|^2 + \|d\beta\|^2 - |\nabla \beta|^2 + 2mk |\beta|^2] \eta.$$

It has been proved the inequality, ([7])

$$(3.8) \quad |\nabla \beta|^2 \leq |\nabla \alpha|^2 |\alpha|^{2m-2} / m^{m-3}.$$

From (3.7) by means of (3.8) we obtain

$$(3.9) \quad - \frac{1}{2} \int_B (N, m^{m-3} d(|\beta|^2)) \bar{\eta} \geq \int_M [-|\nabla \alpha|^2 |\alpha|^{2m-2} + 2km^{m-2} |\beta|^2] \eta.$$

We add the (3.6), (3.9) and after some calculations we obtain

$$\frac{3}{2} \int_B (N, |\alpha|^{2m-2} d(|\alpha|^2) - m^{m-3} d(|\beta|^2)) \bar{\eta} \geq \int_M [3(2m-1)k|\alpha|^{2m} - 4(1-k)|\alpha|^{2m-2}\delta + 6km^{m-2}|\beta|^2] \eta,$$

which can be written

$$(3.10) \quad 3 \int_B [|\alpha|^{2m-2} (V_h \alpha_{j_i}) \alpha^{j_i} N^h - m^{m-3} (V_h \beta_{j_{i_2} \dots i_{2m}}) \beta^{j_{i_2} \dots i_{2m}} N^h] \bar{\eta} \geq \int_M [3(2m-1)k|\alpha|^{2m} - 4(1-k)|\alpha|^{2m-2}\delta + 6km^{m-2}|\beta|^2] \eta.$$

$$j < i_2 < \dots < i_{2m}.$$

The inequality (3.10), if the Killing 2-form α is tangential to B by means of (2.9) becomes

$$(3.11) \quad \int_B -3 [H_{j_i} (|\alpha|^{2m-2} \bar{\alpha}_{i_2}^j \bar{\alpha}^{i_2} - m^{m-3} \bar{\beta}_{i_2 \dots i_{2m}}^j \bar{\beta}^{i_2 \dots i_{2m}}) - (V_j \beta_{h i_2 \dots i_{2m}} + V_h \beta_{j i_2 \dots i_{2m}}) \bar{\beta}^{h i_2 \dots i_{2m}} N^j] \bar{\eta} \geq \int_M [3(2m-1)k|\alpha|^{2m} - 4(1-k)|\alpha|^{2m-2}\delta + 6km^{m-2}|\beta|^2] \eta.$$

$$i_2 < \dots < i_{2m}.$$

The second member of the (3.11) is positive, if k satisfies the inequality, ([7])

$$(3.12) \quad k > \mu = 2m^2(m-1)/(8m-5)m^2 + 6.$$

We consider the following quadratic form

$$(3.13) \quad G(\alpha, \alpha) = H_{ij} (|\alpha|^{2m-2} \bar{\alpha}_{i_2}^j \bar{\alpha}^{i_2} - m^{m-3} \bar{\beta}_{i_2 \dots i_{2m}}^j \bar{\beta}^{i_2 \dots i_{2m}}) -$$

$$(V_j \beta_{h i_2 \dots i_{2m}} + V_h \beta_{j i_2 \dots i_{2m}}) \beta^{h i_2 \dots i_{2m}} N^j.$$

$$i_2 < \dots < i_{2m}.$$

From (3.11), (3.12) and (3.13) we have the theorem

THEOREM (I). *Let M be a compact negatively k -pinched Riemannian manifold of dimension $n = 2m + 1$ with a boundary B . If the number $k > \mu$, given by (3.12), and the quadratic form $G(\alpha, \alpha)$ is semipositive, then the group $K_T^2(M, \mathbb{R}) = 0$.*

We assume that the Killing 2-form α is normal to B , then (3.10) by means of (2.1), (2.10) and the relation, ([10], p. 436)

$$H_i^l \bar{\alpha}_{i_2} \bar{\alpha}^{i_2} - H_{j_i} \bar{\alpha}^j \bar{\alpha}^i = 0,$$

takes the form

$$(3.14) \quad \int_B 3m^{m-2} [(2m - 1) H_{j_i} \bar{\beta}_{i_2 \dots i_{2m}}^j \bar{\beta}^{i_2 \dots i_{2m}} - H_i^l \bar{\beta}_{i_2 \dots i_{2m}} \bar{\beta}^{i_2 \dots i_{2m}} - (V_h \beta_{i_2 \dots i_{2m}}^h) \beta^{i_2 \dots i_{2m}} N_l] \bar{\eta} \geq$$

$$i_2 < \dots < i_{2m}.$$

$$\int_M [3(2m - 1)k |\alpha|^{2m} - 4(1 - k) |\alpha|^{2m-2} \delta + 6km^{m-2} |\beta|^2] \eta.$$

Let $L(\alpha, \alpha)$ be a quadratic form defined by

$$(3.15) \quad L(\alpha, \alpha) = (2m - 1) H_{j_i} \bar{\beta}_{i_2 \dots i_{2m}}^j \bar{\beta}^{i_2 \dots i_{2m}} - H_i^l \bar{\beta}_{i_2 \dots i_{2m}} \bar{\beta}^{i_2 \dots i_{2m}} - (V_h \beta_{i_2 \dots i_{2m}}^h) \beta^{i_2 \dots i_{2m}} N_l.$$

$$i_2 < \dots < i_{2m}.$$

From (3.12), (3.14) and (3.15) we obtain the theorem :

THEOREM (II). *We consider a compact negatively k -pinched Riemannian manifold M of dimension $n = 2m + 1$ with boundary B . If $k > \mu$, given by (3.12), and the quadratic form $L(\alpha, \alpha)$ is semi-negative, then $K_N^2(M, \mathbb{R}) = 0$.*

BIBLIOGRAPHY

- [1] S. CHERN, *Lecture note on differential geometry I*. Chicago, 1955.
- [2] G. F. D. DUFF, *Differential forms in manifolds with boundary*, Ann. of Math. 56 (1952), pp. 115-127.
- [3] G. F. D. DUFF and D. C. SPENCER, *Harmonic tensors on Riemannian manifolds with boundary*, Ann. of Math. 56 (1952), pp. 128-156.
- [4] A. LICHNEROWICZ, *Théorie Globale des Connexions et des Groupes D'Holonomie*, Ed. Cremonese, Rome, 1955.
- [5] A. LICHNEROWICZ, *Géométrie des Groupes de Transformation*, Dunod Paris, 1958.
- [6] G. TSAGAS, *A relation between Killing tensor fields and negative pinched Riemannian manifolds*, Proc. Amer. Math. Soc. 22 (1969), pp. 476-478.
- [7] G. TSAGAS, *An improvement of the method of Yano on the Killing tensor fields and curvature of a compact manifold*, to appear in the Tensor.
- [8] K. YANO and S. BOCHNER, *Curvature and Betti numbers*, Ann. of Math. Studies, N° 32, Princ. Univ. Press, 1953.
- [9] K. YANO, *Harmonics and Killing vector fields in Riemannian spaces with boundary*. Ann. of Math. 69 (1959), pp. 588-597.
- [10] K. YANO, *Harmonic and Killing tensor fields in Riemannian spaces with boundary*, Journ. of the Math. Soc. Jap., 19 (1958), pp. 430-437.