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# ON THE RIESZ MEANS OF THE SOLUTIONS OF THE SCHRÖDINGER EQUATION

by SIGRID SJÖSTRAND

## 0. Introduction.

Consider the solution  $u = G(t)f$  of the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = i \Delta u, & \text{if } t > 0, \\ u = f, & \text{if } t = 0. \end{cases} \quad \left( \Delta = \sum_1^n \frac{\partial^2}{\partial x_1^2} \right),$$

At least formally we have

$$\mathcal{F} u(\xi) = \mathcal{F} G(t)f(\xi) = \exp(it|\xi|^2) \mathcal{F} f(\xi),$$

where  $\mathcal{F}$  denotes the Fourier transform. From this it is easily seen that  $G(t)$  is a bounded, even unitary, operator in  $L^2 = L^2(\mathbb{R}^n)$ . We also have the group property

$$\begin{aligned} G(t+s) &= G(t)G(s), \\ G(0) &= \text{identity}. \end{aligned}$$

Thus we have a unitary group of operators. In  $L^p = L^p(\mathbb{R}^n)$ ,  $p \neq 2$ ,  $G(t)$  is not bounded. See Hörmander [2] and Lanconelli [3]. See also Littman-McCarthy-Riviere [4]. A possible substitute for this, motivated by the theory of distribution (semi) groups, is that at least the Riesz means

$$(0.1) \quad I^k G(t)f = kt^{-k} \int_0^t (t-s)^{k-1} G(s)f ds,$$

of sufficiently large order  $k$  are bounded in  $L^p$ . See Peetre [7].

More generally we consider the operators  $G(t)$ ,  $t \geq 0$ , defined by

$$(0.2) \quad \mathcal{F}G(t)f(\xi) = \exp(itH(\xi))\mathcal{F}f(\xi),$$

where  $H$  is a positive homogeneous function of degree  $m > 0$ , and  $H$  is infinitely differentiable for  $\xi \neq 0$ . We will show that  $I^k G(t)$  is bounded in  $L^p$ , if  $k > n|1/p - 1/2|$ .

In particular we consider the case  $H(\xi) = |\xi|^m$ . In this case we show that the bound  $n|1/p - 1/2|$  is the best possible if  $m \neq 1$ , but can be improved to  $(n-1)|1/p - 1/2|$  but not more if  $m = 1$ . (If  $m = 2$ , our result easily follows from Lanconelli [3], th. 1, with the aid of lemma 2.1 below.)

The plan of the paper is as follows. Section 1 contains some preliminary theorems, mostly on Fourier multipliers. In section 2 we show that our problem is equivalent to the following one: For which  $k$  is the function

$$\Phi(H(\xi))H(\xi)^{-k}\exp(iH(\xi))$$

a Fourier multiplier on  $L^p$ . Here  $\Phi$  is infinitely differentiable and  $\Phi(t) = 0$  for  $t < 1/2$  and  $\Phi(t) = 1$  for  $t > 1$ . In section 3 we prove that  $k > n|1/p - 1/2|$  implies that  $I^k G(t)$  is bounded in  $L^p$ . Section 4 and 5, finally, treat the special cases  $H(\xi) = |\xi|^m$  for  $m \neq 1$  and  $m = 1$ , respectively.

The problem treated in this paper was suggested to me by professor Jaak Peetre. I thank him for valuable advice and great interest in my work.

## 1. Preliminaries on Fourier multipliers and asymptotic expansion.

By  $\hat{f}$  or  $\mathcal{F}f$  we denote the Fourier transform of  $f$  and by  $\check{f}$  or  $\mathcal{F}^{-1}f$  the inverse Fourier transform of  $f$ . Thus formally

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} f(x) dx$$

and

$$\check{g}(x) = \mathcal{F}^{-1}g(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} g(\xi) d\xi.$$

Let  $M_p = M_p(\mathbb{R}^n)$  denote the space of Fourier multipliers on  $L^p = L^p(\mathbb{R}^n)$ , i. e.  $\hat{a} \in M_p$  if and only if  $\hat{a}$  is a tempered distribution and

$$\|\hat{a}\|_{M_p} = \sup_{\|f\|_{L^p}=1} \|(\hat{a}f)^\vee\|_{L^p} < \infty.$$

Then

$$M_p = M_{p'}, \quad 1/p + 1/p' = 1.$$

In the sequel we may therefore consider only the case  $1 \leq p \leq 2$ . We also have

$$(1.1) \quad \mathcal{F}L^1 \subset M_1 \subset M_p$$

for every  $p$ , and

$$(1.2) \quad M_2 = L^\infty.$$

If  $\hat{a}_1 \in M_p$  and  $\hat{a}_2 \in M_p$ , then  $\hat{a}_1 \hat{a}_2 \in M_p$  and

$$(1.3) \quad \|\hat{a}_1 \hat{a}_2\|_{M_p} \leq \|\hat{a}_1\|_{M_p} \|\hat{a}_2\|_{M_p}.$$

Further  $M_p$  is invariant for homotheties, i. e. for a constant  $t$

$$(1.4) \quad \|\hat{a}(t\xi)\|_{M_p} = \|\hat{a}(\xi)\|_{M_p}.$$

For proofs and details see Hörmander [2].

**THEOREM 1.1.** If  $1 < p < 2$  and  $\theta = 2(1 - 1/p)$ , then

$$\|\hat{a}\|_{M_p} \leq \|\hat{a}\|_{M_1}^{1-\theta} \|\hat{a}\|_{M_2}^\theta.$$

**PROOF:** Apply the Riesz-Thorin convexity theorem.

By  $D^N \hat{a}$  we denote the set of all derivatives of  $\hat{a}$  of order  $N$  and by  $\|D^N \hat{a}\|_{L^2}$  the maximum of the  $L^2$ -norms of these derivatives.

**THEOREM 1.2.** (th. of Bernstein; cf. Peetre [6], chap. 1, th. 2.1). If  $N > n/2$ , then there is a constant  $C$ , so that

$$\|\hat{a}\|_{M_1} \leq C \|\hat{a}\|_{L^2}^{1-n/2N} \|D^N \hat{a}\|_{L^2}^{n/2N}.$$

**PROOF:** Using (1.1), Parseval's formula and the Cauchy-Schwarz inequality we get

$$\begin{aligned} \|\hat{a}\|_{M_1} &\leq \int |a(x)| \, dx = \int_{|x| \leq r} |a(x)| \, dx + \int_{|x| \geq r} |x|^{-N} |x|^N |a(x)| \, dx \leq \\ &\leq C(r^{n/2} \|\hat{a}\|_{L^2} + r^{-N+n/2} \|D^N \hat{a}\|_{L^2}). \end{aligned}$$

If  $\|\hat{a}\|_{L^2} \neq 0$  we choose

$$r = \|\hat{a}\|_{L^2}^{-1/N} \|D^N \hat{a}\|_{L^2}^{1/N}$$

to obtain the desired inequality. If  $\|\hat{a}\|_{L^2} = 0$ , the inequality is of course trivial.

**THEOREM 1.3.** Assume that  $F$  is infinitely differentiable on  $0 < u < \infty$  and that

$$|F(u) - F(0)| \leq C_0 u^\alpha, \quad 0 < u < 1,$$

$$|F(u)| \leq C_0 u^{-\beta}, \quad 1 < u < \infty,$$

$$|D^J F(u)| \leq C_J \min(u^{\alpha-J}, u^{-\beta-J}), \quad 0 < u < \infty,$$

where  $J \geq 1$  and  $\alpha, \beta > 0$ . Let  $H$  be a positive homogeneous function as in the introduction. Then

$$F(H(\xi)) \in \mathcal{FL}^1.$$

**PROOF :** See Löfström [5], lemma 1.4.

Next we want to compute the Fourier transform of a function which has a similar behaviour near the point  $x_0$  as the function  $f$  defined by

$$f: x \rightarrow (x - x_0)_+^\alpha = \begin{cases} (x - x_0)^\alpha, & \text{if } x > x_0, \\ 0, & \text{if } x \leq x_0. \end{cases}$$

For  $\xi > 0$  and  $\alpha \neq -1, -2, \dots$

$$(1.5) \quad \hat{f}(\xi) = K_\alpha \xi^{-\alpha-1} e^{-ix_0\xi},$$

where

$$K_\alpha = -i \Gamma(\alpha + 1) e^{-ia\frac{\pi}{2}}.$$

See Gelfand-Schilow [1], p. 169.

**THEOREM 1.4.** Suppose that

1°  $f$  is infinitely differentiable for  $x \neq x_0$  and has compact support.

2°  $D^M f(x) = O_M (x - x_0)_+^{\alpha-M} + O(|x - x_0|^{\beta-M}), x \rightarrow x_0$  for  $0 \leq M \leq N$ ,

where

$$\alpha \neq -1, -2, \dots, \beta > \alpha \quad \text{and} \quad N > \beta + 1.$$

Then

$$\hat{f}(\xi) = C\xi^{-a-1} e^{-i\alpha\xi} + 0 (\xi^{-\beta-1}), \xi \rightarrow \infty.$$

PROOF: By (1.5) we may suppose that  $C = 0$  and  $x_0 = 0$ . Here and on several occasions in the sequel we shall make use of the following construction. We let  $\Psi$  be a fixed function such that  $0 \leq \Psi \in C_0^\infty(\mathbb{R})$  and  $\text{supp } \Psi \subset \{x \mid 1/2 < |x| < 2\}$ . We write

$$\Psi_\nu(x) = \Psi(2^{-\nu}x)$$

and assume

$$\sum_{-\infty}^{\infty} \Psi_\nu(x) = 1, \quad x \neq 0.$$

(For the existence of such a function, see Hörmander [2].)

To prove the theorem we write  $f_\nu = \Psi_\nu f$ . Then

$$|f_\nu(x)| \leq C'_0 2^{\nu\beta},$$

$$|D^N f_\nu(x)| \leq C'_N 2^{\nu(\beta-N)}$$

and thus

$$|\hat{f}_\nu(\xi)| \leq C \frac{2^{\nu(\beta+1)}}{1 + (|\xi| \cdot 2^\nu)^N}.$$

This implies

$$|\hat{f}(\xi)| \leq \sum_{-\infty}^{\infty} |\hat{f}_\nu(\xi)| \leq C |\xi|^{-\beta-1} \sum_{-\infty}^{\infty} \frac{(|\xi| \cdot 2^\nu)^{\beta+1}}{1 + (|\xi| \cdot 2^\nu)^N} \leq C' |\xi|^{-\beta-1}.$$

## 2. A Lemma.

Using (0.1) and (0.2) we get after Fourier transformation

$$\mathcal{F} I^k G(t)f(\xi) = (kt^{-k} \int_0^t (t-s)^{k-1} e^{isH(\xi)} ds) \mathcal{F} f(\xi).$$

Thus our problem is to find for which  $k$

$$\hat{a}_t(\xi) = kt^{-k} \int_0^t (t-s)^{k-1} e^{isH(\xi)} ds$$

is a Fourier multiplier on  $L^p$ , i. e. belongs to  $M_p$ . By (1.3) it suffices to consider the case  $t = 1$ , i. e. to find for which  $k$  the function

$$\hat{a}(\xi) = k \int_0^1 (1-s)^{k-1} e^{isH(\xi)} ds$$

belongs to the space  $M_p$ .

Now let  $\Phi \in C^\infty(\mathbb{R})$  be such that  $\Phi(t) = 0$  if  $t < 1/2$  and  $\Phi(t) = 1$  if  $t > 1$ .

LEMMA 2.1.  $\hat{a} \in M_p$  if and only if  $\hat{b} \in M_p$ , where

$$\hat{b}(\xi) = \Phi(H(\xi)) H(\xi)^{-k} e^{iH(\xi)}.$$

PROOF: We want to apply theorem 1.3 to the function  $F$ , where

$$F(H(\xi)) = \hat{a}(\xi) - C_k \hat{b}(\xi),$$

and the constant  $C_k$  is defined by

$$k \int_{-\infty}^1 (1-s)^{k-1} e^{isu} ds = C_k u^{-k} e^{iu}, \quad u > 0.$$

Such a constant  $C_k$  exists in view of (1.5). The integral is of course to be understood as a Fourier transform. For  $u > 0$  we get

$$\begin{aligned} F(u) &= k \int_0^1 (1-s)^{k-1} e^{isu} ds - C_k \Phi(u) u^{-k} e^{iu} = \\ &= C_k u^{-k} e^{iu} - k \int_{-\infty}^0 (1-s)^{k-1} e^{isu} ds - C_k \Phi(u) u^{-k} e^{iu}. \end{aligned}$$

Thus for  $u \geq 1$

$$F(u) = -k \int_{-\infty}^0 (1-s)^{k-1} e^{isu} ds.$$

Put

$$f_k(s) = \begin{cases} (1-s)^{k-1} & \text{if } s < 0, \\ 0 & \text{if } s \geq 0. \end{cases}$$

Then for  $u \geq 1$

$$D^N F(u) = C_{N,k} (s^N f_k(s))^\vee(u).$$

But

$$s^N f_k(s) = s_-^N (f_k(s) - 1) + s_-^N,$$

where  $s_- = (-s)_+$ .

The first term belongs to  $C^{N+1}$  in a neighbourhood of  $s = 0$  and the derivatives of it of sufficiently high order belong to  $L^1(s < -1)$ . This gives the desired estimate

$$|D^N F(u)| \leq C_N u^{-N-1}, \quad u \geq 1.$$

For  $0 \leq u \leq 1$  we get

$$|F(u) - F(0)| \leq k \int_0^1 (1 - s)^{k-1} |e^{isu} - 1| ds + |C_k| |\Phi(u) u^{-k}| \leq C_0 u.$$

We can now apply theorem 1.3 with  $\alpha = \beta = 1$  and get

$$F(H(\xi)) \in \mathcal{FL}^1.$$

### 3. The general case.

**THEOREM 3.1.**  $\hat{a} \in M_p$ , if  $k > n |1/p - 1/2|$ .

**PROOF:** According to lemma 2.1 it is sufficient to show the corresponding statement for  $\hat{b}$ , where

$$\hat{b}(\xi) = \Phi(H(\xi)) H(\xi)^{-k} e^{iH(\xi)}.$$

We may also assume  $1 \leq p \leq 2$ . Choose  $\Psi$  as in the proof of theorem 1.4 and put

$$\hat{b}_\nu(\xi) = \Psi_\nu(|\xi|) \hat{b}(\xi).$$

Then

$$\|\hat{b}_\nu\|_{L^2} \leq C \cdot 2^{-\nu mk} \cdot 2^{\nu n/2}.$$

Since

$$\hat{b}(\xi) = F(H(\xi)),$$

where

$$F(u) = \Phi(u) u^{-k} e^{iu},$$



and

$$|D^M F(u)| \leq C_M u^{-k}, \quad M \geq 0,$$

we get

$$|D^M \hat{b}(\xi)| \leq C'_M |\xi|^{-mk+M(m-1)}, \quad M \geq 0,$$

and thus

$$\|D^N \hat{b}_\nu\|_{L^2} \leq C''_N 2^{\nu(-mk+N(m-1))} \cdot 2^{\nu n/2}.$$

If we take  $N > n/2$ , theorem 1.2 gives

$$\|\hat{b}_\nu\|_{M_1} \leq C \cdot (2^{\nu(n/2-mk)})^{1-n/2N} \cdot (2^{\nu(N(m-1)-mk+n/2)})^{n/2N} = C 2^{\nu m(n/2-k)}.$$

Moreover, by (1.2),

$$\|\hat{b}_\nu\|_{M_2} = \|\hat{b}_\nu\|_{L^\infty} \leq C \cdot 2^{-\nu mk}.$$

Theorem 1.1 with  $\theta = 2(1 - 1/p)$  gives

$$\|\hat{b}_\nu\|_{M_p} \leq C \cdot 2^{\nu m(n/2-k)(1-\theta)} \cdot 2^{-\nu mk\theta} = C \cdot 2^{\nu m(n(1/p-1/2)-k)}.$$

Remembering that  $\hat{b}(\xi)$  vanishes in a neighbourhood of 0, we get for some  $\nu_0$

$$\|\hat{b}\|_{M_p} \leq \sum_{\nu_0}^{\infty} \|\hat{b}_\nu\|_{M_p} < \infty,$$

if  $k > n(1/p - 1/2)$ .

**REMARK 3.1.** Exactly in the same way we can prove the more general result: Let  $F$  be infinitely differentiable on  $R$  and vanish in a neighbourhood of 0. Assume that

$$|D^J F(u)| \leq C_J u^{-k}, \quad J = 1, \dots$$

Then  $F(H(\xi)) \in M_p$ , if  $k > n|1/p - 1/2|$ . The corresponding result for the torus  $T^n$  is proved in Löfström [5], section 10.

#### 4. The case $H(\xi) = |\xi|^m$ , $m \neq 1$ .

In this section we always take  $H(\xi) = |\xi|^m$  and  $m \neq 1$ .

**THEOREM 4.1.** If  $H(\xi) = |\xi|^m$ ,  $m \neq 1$ , then  $\hat{a} \in M_p$  implies  $k \geq n|1/p - 1/2|$ .

For the proof we need two lemmata.

LEMMA 4.1. If  $m \neq 1$  and  $n + m'(k - n/2) > 0$ , then

$$b(x) = C_1 |x|^{-(n+m'(k-n/2))} \exp(iC_2 |x|^{m'}) [1 + o(|x|^{-m'/2})] + b_m(|x|), \quad |x|^{m'} \rightarrow \infty,$$

where  $m' = m/(m - 1)$ ,  $C_1$  and  $C_2$  are constants  $\neq 0$ , and  $b_m$  is a continuous function if  $m < 1$  and identically zero if  $m > 1$ .

REMARK 4.1. The lemma was proved in the case  $m < 1$  by Wainger [8], Part II. We will prove it in the case  $m > 1$ .

LEMMA 4.2. Let  $\Phi$  be the function in section 2 and put

$$\hat{f}_\alpha(\xi) = \Phi(|\xi|^m) |\xi|^{-\alpha}, \quad \alpha > 0.$$

Then  $f_\alpha \in L^p$ , if  $\alpha > n/p'$ ,  $1 \leq p \leq \infty$ ,  $1/p + 1/p' = 1$ .

PROOF OF THEOREM 4.1.

If  $\hat{a} \in M_p$ , then  $\hat{b} \in M_p$ . Consider

$$f^\lambda(\xi) = \Phi(|\xi|^m) |\xi|^{-m(k+\lambda)} \exp(i|\xi|^m), \quad \lambda m > n/p'.$$

Then  $f \in L^p$  according to lemma 4.2. On the other hand lemma 4.1 gives

$$f(x) = C_1 |x|^{-(n+m'(k+\lambda-n/2))} \exp(iC_2 |x|^{m'}) [1 + o(1)] + b_m(|x|), \quad |x|^{m'} \rightarrow \infty,$$

if  $n + m'(k + \lambda - n/2) > 0$ .

Now choose  $k$  so that

$$p(n + m'(k + \lambda - n/2)) = n,$$

i. e.  $0 < m\lambda - n/p' = m(n(1/p - 1/2)) - k$ . Thus  $\hat{a} \in M_p$  leads to a contradiction if

$$k < n(1/p - 1/2).$$

REMARK 4.2. This method was used by Wainger [8], Part IV, to prove the analogous theorem for Fourier multipliers on the torus  $T^n$ .

PROOF OF LEMMA 4.1 IN THE CASE  $m > 1$ .

We will restrict ourselves to the case  $n \geq 2$ . The case  $n = 1$  can be treated in an analogous way.

Let  $\Psi_\nu$  be the standard functions introduced in the proof of theorem 1.4. We may assume that

$$\Phi(t) = \sum_0^\infty \Psi_\nu(t) \quad \text{for } t \geq 0.$$

Then  $\hat{b}(\xi) = \sum_0^\infty \hat{b}_\nu(\xi)$ , where

$$\hat{b}_\nu(\xi) = \Psi_\nu(|\xi|^m) |\xi|^{-mk} \exp(i|\xi|^m).$$

This implies

$$b_\nu(x) = C \int e^{i\langle x, \xi \rangle} \Psi_\nu(|\xi|^m) |\xi|^{-mk} \exp(i|\xi|^m) d\xi.$$

Clearly  $b_\nu$  is a function of  $|x|$ , so we can assume that  $x = (|x|, 0, \dots, 0)$ . Then we make the transformation

$$\xi_j \rightarrow |x|^{\frac{1}{m-1}} \xi_j, \quad 1 \leq j \leq n,$$

and get

$$b_\nu(x) =$$

$$C |x|^{-m'k+n/(m-1)} \int \exp(i|x|^{m'}(\xi_1 + |\xi|^m)) \Psi_\nu(|x|^{m'}|\xi|^m) |\xi|^{-mk} d\xi =$$

$$C |x|^{-m'k+n/(m-1)}.$$

$$\cdot \int_{\substack{-\infty < \xi_1 < \infty \\ r \geq 0}} \exp[i|x|^{m'}(\xi_1 + (\xi_1^2 + r^2)^{m/2})] \Psi_\nu(|x|^{m'}(\xi_1^2 + r^2)^{m/2}) (\xi_1^2 + r^2)^{-mk/2} r^{n-2} d\xi_1 dr.$$

Finally we make variable transformation

$$\begin{cases} s = \xi_1 + (\xi_1^2 + r^2)^{m/2} \\ t = \xi_1 \end{cases}$$

or

$$\begin{cases} \xi_1 = t \\ r = ((s - t)^{2/m} - t^2)^{1/2}. \end{cases}$$

We get

$$b_\nu(x) = C |x|^{-m'k+n/(m-1)}.$$

$$\cdot \int_{\Omega} \exp(i|x|^{m'}s) \Psi_\nu(|x|^{m'}(s-t)) (s-t)^{-k+1-2/m} ((s-t)^{2/m} - t^2)^{(n-3)/2} ds dt,$$

where  $\Omega = \{(s, t) \mid s \geq s_0, t + |t|^m \leq s\}$  and  $s_0 = \inf (t + |t|^m)$ .

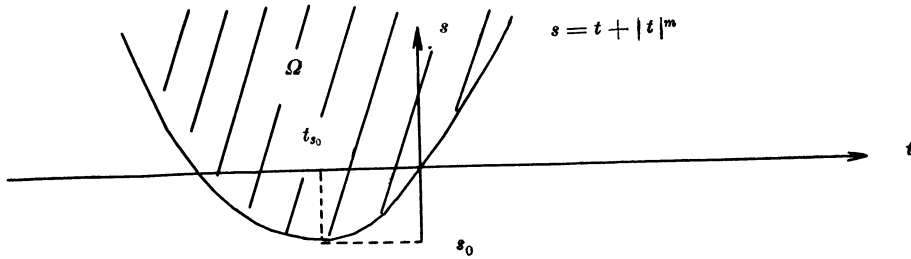


fig. 4.1.

Let  $t_{s_0}$  be defined by

$$s_0 = t_{s_0} + |t_{s_0}|^m.$$

We split the integral into two integrals

$$b_\nu(x) = C \cdot |x|^{-m'k+n/(m-1)} \left( \iint_{\Omega \cap \{t > t_{s_0}\}} + \iint_{\Omega \cap \{t < t_{s_0}\}} \right).$$

Both of them can be handled in the same way. Let us consider only the second one and call it  $b_\nu^*(x)$ . With  $t_s$  defined by

$$s = t_s + |t_s|^m, \quad t_s > t_{s_0}$$

we get

$$b_\nu^*(x) = C |x|^{-m'k+n/(m-1)} \int_{s_0}^{\infty} \exp(i|x|^{m'}s) C_\nu(|x|, s) ds,$$

where

$$C_\nu(|x|, s) = \int_{t_{s_0}}^{t_s} \Psi_\nu(|x|^{m'}(s-t)) (s-t)^{-k+1-2/m} ((s-t)^{2/m} - t^2)^{(n-3)/2} dt.$$

Since

$$\begin{aligned} g(|x|, s) &= \sum_0^\infty C_\nu(|x|, s) = \\ &= \int_{t_{s_0}}^{t_s} \Phi(|x|^{m'}(s-t)) (s-t)^{-k+1-2/m} ((s-t)^{2/m} - t^2)^{(n-3)/2} dt \end{aligned}$$

is uniformly convergent on compact subsets of  $\{s \mid s \geq s_0\}$  we get

$$b^*(x) = \sum_0^{\infty} b_v^*(x) = C |x|^{-m'k+n/(m-1)} \int \exp(is|x|^{m'}) g(|x|, s) ds,$$

where the integral, as in the rest of the proof, is to be understood as an inverse Fourier transform.

Now we write

$$g = g_0 + g_1 + g_2,$$

where

$$g_\nu(|x|, s) = \chi_\nu(s) g(|x|, s), \quad \nu = 0, 1, 2,$$

and  $\chi_0, \chi_1, \chi_2$  are as in fig. 4.2.

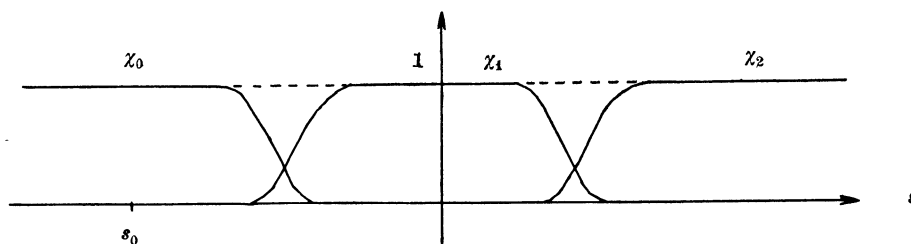


fig. 4.2.

Choosing  $|x|^{m'}$  sufficiently large, we can obtain that  $\Phi = 1$  in the integrals defining  $g_0$  and  $g_2$ .

We first consider  $g_0$ .

$$\begin{aligned} g_0(|x|, s) &= g_0(s) = \\ &= \chi_0(s) \int_{t_{s_0}}^{t_s} (s-t)^{-k+1-2/m} ((s-t)^{2/m} - t^2)^{(n-3)/2} dt = \\ &= (t_s - t_{s_0}) \chi_0(s) \int_0^1 (s-t(y))^{-k+1-2/m} ((s-t(y))^{2/m} - t(y)^2)^{(n-3)/2} dy, \end{aligned}$$

if

$$t(y) = t_{s_0} + (t_s - t_{s_0})y.$$

The integrand is a function  $f$  of  $y$  and  $(t_s - t_{s_0})$ . Elementary calculations give

$$f(y, t_s - t_{s_0}) = C(t_s - t_{s_0})^{n-3} (1 - y)^{(n-3)/2} \left(1 + \sum_{\substack{\nu \geq 1 \\ \mu \geq 0}} C_{\nu\mu} (t_s - t_{s_0})^\nu (1 - y)^\mu\right)$$

uniformly in  $\{0 \leq y \leq 1\} \times \{|t_s - t_{s_0}| < \varepsilon\}$  with  $C \neq 0$ .

But

$$t_s - t_{s_0} = h((s - s_0)^{1/2}),$$

where  $h$  is analytic in a neighbourhood of  $0$ , and  $h(0) = 0$ ,  $h'(0) \neq 0$ .

Hence  $g_0(s)$  fulfils the conditions of theorem 1.4 with  $\alpha = (n - 2)/2$  and  $\beta = (n - 1)/2$ . We thus get

$$\begin{aligned} & \int \exp(is|x|^{m'}) g_0(s) ds = \\ & = C|x|^{-m'((n-2)/2+1)} \exp(-is_0|x|^{m'}) [1 + o(|x|^{-m'/2})], |x|^{m'} \rightarrow \infty. \end{aligned}$$

We finally turn to  $g_1$  and  $g_2$ . One can show that for  $N$  sufficiently large

$$\|D_s^N g_1(|x|, s)\|_{L^1} \leq C|x|^{N/(m-1)+\text{const.}}$$

and

$$\|D_s^N g_2(s)\|_{L^1} < \infty.$$

This implies

$$\int \exp(is|x|^{m'}) g_\nu(|x|, s) ds = o(|x|^{-m'M}), |x|^{m'} \rightarrow \infty,$$

for every  $M \geq 1$   $\nu = 1, 2$ .

**PROOF OF LEMMA 4.2.**

At first we suppose  $n > \alpha > n/p'$ . Write

$$\hat{f}_\alpha(\xi) = |\xi|^{-\alpha} - (1 - \Phi(|\xi|^m)) |\xi|^{-\alpha}.$$

Thus

$$f_\alpha(x) = C_{\alpha,n} |x|^{\alpha-n} + g_\alpha(x),$$

where  $g_\alpha \in L^\infty$ .

By differentiation we get

$$f_\alpha(x) = o(|x|^{-N}), |x| \rightarrow \infty, \text{ for any } N.$$

Thus  $f_\alpha \in L^p$ , if

$$p(\alpha - n) > n,$$

i. e. if  $\alpha > n/p'$ .

Now the proof will be finished, if we show that

$$f_{\alpha_1} \in L^p \text{ and } \alpha_2 > \alpha_1$$

implies

$$f_{\alpha_2} \in L^p.$$

Write

$$f_{\alpha_2}(\xi) = [\Phi(|\xi|^m)^{1/2} |\xi|^{-(\alpha_2 - \alpha_1)}] \cdot [\Phi(|\xi|^m)^{1/2} |\xi|^{-\alpha_1}].$$

If  $\alpha_2 - \alpha_1 > 0$  the first factor belongs to  $M^p$  by theorem 1.3 The proof is finished.

### 5. The case $H(\xi) = |\xi|$ .

Now we turn to the case  $m = 1$ , i. e. in this section  $H(\xi) = |\xi|$ .

**THEOREM 5.1.**  $\hat{a} \in M_p$  if  $H(\xi) = |\xi|$  and  $k > (n-1)|1/p - 1/2|$ .

**THEOREM 5.2.** If  $H(\xi) = |\xi|$ , then  $\hat{a} \in M_p$  implies  $k \geq (n-1)|1/p - 1/2|$ .

For the proofs we need the following

**LEMMA 5.1.** If  $b(\xi) = \Phi(|\xi|) |\xi|^{-k} e^{i|\xi|}$ , then

$$(5.1) \quad b \in L_{loc}^\infty(|x| \neq 1),$$

$$(5.2) \quad b(x) = b_1(x) + O(\operatorname{sgn}(1 - |x|) |1 - |x||^{k - (n-1)/2 - 1} (1 + \sigma(1)), |x| \rightarrow 1,$$

where  $b_1 \in L^\infty$ , if  $k - (n-1)/2 < 1$  and  $k - (n-1)/2$  is not an integer

$$(5.3) \quad b(x) = O(|x|^{-N}), |x| \rightarrow \infty, \text{ for any } N.$$

**PROOF OF THEOREM 5.1.**

As usual we prove the theorem for  $\hat{b}$  instead of  $\hat{a}$ . By theorem 1.3 it is obvious that if  $\hat{b} \in M_1$  for a certain  $k = k_0$ , then  $\hat{b} \in M_1$  for all  $k \geq k_0$ . By lemma 5.1 we get

$$b \in L^1, \text{ if } 0 < k - (n-1)/2 < 1,$$

and thus

$$\hat{b} \in M_1, \text{ if } 0 < k - (n - 1)/2.$$

To prove the theorem for  $p > 1$  we use again the standard functions  $\Psi_\nu$ , from the proof of theorem 1.4 and put

$$\hat{b}_\nu(\xi) = \Psi_\nu(|\xi|) \hat{b}(\xi).$$

Let  $k_0$  be a number  $> (n - 1)/2$ . Then

$$\begin{aligned} \hat{b}_\nu(\xi) &= \Psi(2^{-\nu}|\xi|) \Phi(|\xi|) |\xi|^{-k} e^{i|\xi|} = \\ &= \{ \Psi(2^{-\nu}|\xi|) | 2^{-\nu} \xi |^{k_0-k} \} \cdot \{ \Phi(|\xi|) |\xi|^{-k_0} e^{i|\xi|} \} \cdot 2^{\nu(k_0-k)}, \end{aligned}$$

which implies, in view of (1.2), (1.3) and (1.4),

$$\| \hat{b}_\nu \|_{M_1} \leq C \cdot 2^{\nu(k_0-k)}$$

and

$$\| \hat{b}_\nu \|_{M_2} \leq C 2^{-\nu k}.$$

Theorem 1.1 with  $\theta = 2(1 - 1/p)$  gives

$$\| \hat{b}_\nu \|_{M_p} \leq C 2^{\nu(k_0-k)(1-\theta) - \nu k \theta} = C 2^{\nu(k_0(1-\theta) - k)}.$$

Thus  $\hat{b} \in M_p$ , if  $k > k_0(1 - \theta)$ . But  $k_0$  is arbitrary  $> (n - 1)/2$  and hence  $\hat{b} \in M_p$ , if

$$k > ((n - 1)/2)(1 - \theta) = (n - 1)(1/p - 1/2).$$

**PROOF OF THEOREM 5.2.**

Let  $\mathbf{0}$  be the annulus  $\{x \mid 1/2 \leq |x| \leq 2\}$ . Then by (5.2)  $b \in L^p(\mathbf{0})$  implies

$$p(k - (n - 1)/2 - 1) > -1,$$

i. e.

$$k > (n + 1)/2 - 1/p.$$

Suppose now that

$$k = (n - 1)(1/p - 1/2) - \varepsilon, \quad \varepsilon > 0.$$

Put

$$\hat{g}(\xi) = \Phi(|\xi|) |\xi|^{-\lambda}, \quad \text{where } \lambda = n/p' + \varepsilon.$$



By lemma 4.2  $g \in L_p$ . But

$$\hat{b}(\xi) \cdot \hat{g}(\xi) = \Phi(|\xi|^2) |\xi|^{-(k+\lambda)} e^{i|\xi|},$$

where

$$k + \lambda = (n-1)(1/p - 1/2) - \varepsilon + n/p' + \varepsilon = (n+1)/2 - 1/p.$$

Thus  $(\hat{b}\hat{g})^\vee \notin L_p$ , which implies  $\hat{b} \notin M_p$ .

PROOF OF LEMMA 5.1.

By writing  $\hat{b}(\xi) = \sum_0^\infty \hat{b}_\nu(\xi)$  as in the proof of lemma 4.1 we can justify the following calculations:

$$\begin{aligned} b(x) &= C \int e^{i|x|\xi_1} \Phi(|\xi|) |\xi|^{-k} e^{i|\xi|} d\xi = \\ &= C \int \Phi(s) s^{-k+n-1} e^{is} \left( \int_{-1}^1 e^{i|x|st} (1-t^2)^{(n-3)/2} dt \right) ds. \end{aligned}$$

Write

$$g(|x|s) = (2\pi)^{-1} \int_{-1}^1 e^{i|x|st} (1-t^2)^{(n-3)/2} dt.$$

Then  $g$  is the inverse Fourier transform of the function

$$t \rightarrow (1-t^2)_+^{(n-3)/2} = (1-t)_+^{(n-3)/2} (1+t)_+^{(n-3)/2}.$$

Let

$$(1-t)_+^{(n-3)/2} = \sum_0^\infty c_\nu (1+t)^\nu$$

and

$$(1+t)_+^{(n-3)/2} = \sum_0^\infty d_\nu (1-t)^\nu$$

be the corresponding Taylor series.

Put

$$R_N(t) = (1-t^2)_+^{(n-3)/2} - \sum_0^{N-1} c_\nu (1+t)_+^{(n-3)/2+\nu} - \sum_0^{N-1} d_\nu (1-t)_+^{(n-3)/2+\nu}.$$

From

$$D^N R_N \in L_{loc}^1 \text{ and}$$

$$D^M R_N \in L^1(|t| \geq 2) \text{ if } M \text{ is large}$$

we conclude that

$$|\check{R}_N(|x|s)| \leq C_N(|x|s)^{-N}.$$

Thus by (1.5)

$$g(|x|s) = \sum_0^{N-1} c'_\nu(|x|s)^{-(n-3)/2-\nu-1} e^{-i|x|s} + \sum_0^{N-1} d'_\nu(|x|s)^{-(n-3)/2-\nu-1} e^{i|x|s} + \check{R}_N(|x|s)$$

and

$$b(x) = \sum_0^{N-1} c''_\nu |x|^{-(n-3)/2-\nu-1} f_\nu(1-|x|) + \sum_0^{N-1} d''_\nu |x|^{-(n-3)/2-\nu-1} f_\nu(1+|x|) + G_N(x)|x|^{-N},$$

where  $G_N \in L^\infty$  and  $f_\nu$  is defined by

$$\hat{f}_\nu(s) = \Phi(s) s^{-(k-(n-1)/2+\nu)}.$$

Now it is obvious that (5.3) is fulfilled. To obtain (5.2) we consider the function  $f$  defined by

$$(5.4) \quad \hat{f}(s) = \Phi(s) s^{-\alpha}.$$

If  $\alpha > 1$ , then  $f \in L^\infty$ .

If  $\alpha < 1$  is not an integer, then

$$\Phi(s) s^{-\alpha} = s_+^{-\alpha} - (1 - \Phi(s)) s_+^{-\alpha},$$

The inverse Fourier transformation gives

$$f(y) = C_1 y_+^{\alpha-1} + C_2 y_-^{\alpha-1} + g(y),$$

where  $g \in L^\infty$  and  $(y_-)^{\alpha-1} = (-y)_+^{\alpha-1}$ . (See Gelfand-Schilow [1], p. 169).

Using this with  $y = 1 - |x|$  and  $y = 1 + |x|$  and  $\alpha = k - (n - 1)/2 + \nu$  we obtain (5.2). To prove (5.1) it is now sufficient to consider  $|x| \leq 1/2$ . By changing the order of integration and with  $f$  defined by (5.4) and  $\alpha = k - n + 1$  we get

$$|b(x)| = \left| C \int_{-1}^1 (1-t^2)^{(n-3)/2} f(1+|x|t) dt \right| \leq \left| C \int_{-1}^1 (1-t^2)^{(n-3)/2} (1+|x|t)^{-N} dt \right|$$

for  $N$  sufficiently large. But

$$(1 + |x|t)^{-N} \leq (1 - 1/2)^{-N}, \text{ for } |x| \leq 1/2,$$

and thus

$$|b(x)| \leq \text{const. for } |x| \leq 1/2.$$

#### REFERENCES

- [1] I. M. GELFAND and G. E. SCHILOV, *Verallgemeinerte Funktionen (Distributionen) I*, VEB Deutscher Verlag der Wissenschaften, Berlin 1960.
- [2] L. HÖRMANDER, *Estimates for translation invariant operators in  $L_p$ -spaces*, Acta Math. 104 (1960), 93-140.
- [3] E. LANCONELLI, *Valutazioni in  $L_p(\mathbb{R}^n)$  della soluzione del problema di Cauchy per l'equazione di Schrödinger*, Boll. Un. Mat. Ital (1968), 591-607,
- [4] W. LITTMAN, C. MC CARTHY and N. RIVIERE, *The non-existence of  $L^2$ -estimates for certain translation-invariant operators*, Studie Math. 30. (1968), 219-229.
- [5] J. LÖFSTRÖM, *Besov spaces in the theory of approximation*, Ann. Mat. Pura Appl. 85 (1970), 93-184.
- [6] J. PEETRE, *Applications de la théorie des espaces d'interpolation dans l'Analyse Harmonique*, Ricerche Mat. 15 (1966), 3-36.
- [7] J. PEETRE, *Sur la théorie des semi-groupes distributions*, Collège de France, séminaires sur les équations aux dérivées partielles II, nov. 1963-mai 1964, 76-99.
- [8] S. WAINGER, *Special trigonometric series in  $k$ -dimensions*, Mem. Amer. Math. Soc 59 (1965).