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A PSEUDOCONCAVE GENERALIZATION OF GRAUERT'S DIRECT IMAGE THEOREM: I

by YUM-TONG SIU (*)

Table of Contents

§ 0. Introduction.

A. In [2] Grauert proves the following direct image theorem.

THEOREM *G.* Suppose $\pi : X \longrightarrow Y$ is a proper holomorphic map of (not necessarily reduced) complex spaces and $\mathcal{F}$ is a coherent analytic sheaf on X. Then the l^{th} direct image $\pi_l(\mathcal{F})$ of $\mathcal{F}$ under π is a coherent analytic sheaf on Y for all $l \geqq 0$. (A simplified treatment of a key point of the proof for a special case is given in [3] to illustrate the idea of the proof. In [5] Knorr gives an amplified version of Grauert's original proof.)

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When the dimension of the complex space Y in Theorem G is zero, Theorem G is reduced to the following finiteness theorem of Cartan-Serre.

THEOREM *C-S*. Suppose X is a compact complex space and $\mathcal{F}$ is a coherent analytic sheaf on X. Then the dimension of $H^l(X, \mathcal{F})$ is finite for all $l \geqq 0$.

Theorem G can be regarded as a Theorem *C-S* with parameters (and the parameter space is the complex space Y).

In [1] Andreotti and Grauert generalize Theorem *C-S* to the following finiteness theorem for pseudoconvex and pseudoconcave spaces.

THEOREM *A-G*. Suppose X is a complex space and φ is a proper C^∞ map from X to (a, b), where $a \in \{-\infty\} \cup \mathbf{R}$ and $b \in \mathbf{R} \cup \{\infty\}$. Suppose $a < a' < b' < b$ and φ is strictly p-convex on $\{\varphi > b'\}$ and strictly q-convex on $\{\varphi < a'\}$. If $\mathcal{F}$ is a coherent analytic sheaf on X and codh $\mathcal{F} \geqq r$ on $\{\varphi < a'\}$, then the dimension of $H^l(X, \mathcal{F})$ is finite for $p \leqq l < r - q$.

It is natural to conjecture that Theorem G can be generalized to a Theorem *A G* with parameters.

CONJECTURE. Suppose $\pi : X \longrightarrow Y$ is a holomorphic map of complex spaces and φ is a C^∞ map from X to (a, b), where $a \in \{-\infty\} \cup \mathbf{R}$ and $b \in \mathbf{R} \cup \{\infty\}$ such that the restriction of π to $\{a^* \leqq \varphi \leqq b^*\}$ is proper for $a < a^* < b^* < b$. Suppose $a < a' < b' < b$ and, for every $y \in Y$, φ is strictly p-convex on $\pi^{-1}(y) \cap \{\varphi > b'\}$ and strictly q-convex on $\pi^{-1}(y) \cap \{\varphi < a'\}$. If $\mathcal{F}$ is a coherent analytic sheaf on X and codh $\mathcal{F} \geqq r$ on $\{\varphi < a'\}$, then $\pi_l(\mathcal{F})$ is coherent on Y for $p \leqq l < r - q - \dim Y$.

This conjecture is closely linked up with the theory of coherent analytic sheaf extension. In [7] coherent analytic sheaves are extended by proving that under special circumstances $\pi_1(\mathcal{F})_y$ is finitely generated over the local ring at y for $y \in Y$.

Not much has been done in the direction of this conjecture. In private correspondence Knorr told me that he could prove the following one-parameter version.

THEOREM *K*. Suppose X is a perfect complex space, S is a Riemann surface (with reduced complex structure), and $\pi : X \longrightarrow S$ is a holomorphic map. Suppose φ is a C^∞ map from X to (a, b), where $a \in \{-\infty\} \cup \mathbf{R}$ and $b \in \mathbf{R} \cup \{\infty\}$, such that the restriction of π to $\{a^* \leqq \varphi \leqq b^*\}$ is proper for $a < a^* < b^* < b$. Suppose $a < a' < b' < b$ and φ is strictly p-convex on $\{\varphi < b'\}$ and strictly q-convex on $\{\varphi < a'\}$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on X such that codh $\mathcal{F} \geqq r$ on $\{\varphi < a'\}$ and $\mathcal{F} | \{\varphi < a'\}$ is $(\pi | \{\varphi < a'\})$ — flat. Then $\pi_l(\mathcal{F})$ is coherent on S for $p \leqq l < r - q - 2$.

In this paper we prove a pseudoconcave version of the conjecture with a parameter space of any dimension which is a manifold. This gives a pseudoconcave generalization of Grauert's direct image theorem. Before we state the main result, we introduce a definition.

DEFINITION. Suppose q is a natural number and $\pi : X \longrightarrow Y$ is a holomorphic map of complex spaces. π is called *q-concave* if there exists a C^∞ map φ from X to (c_*, ∞), where $c_* \in \{-\infty\} \cup \mathbf{R}$, and there exists $c_* < < c_\# < \infty$ such that (i) for $c_* < c < \infty$, the restriction of π to $\{\varphi \geqq c\}$ is proper, and (ii) φ is strictly q-convex on $\{\varphi < c_\#\}$. We call φ an *exhaustion function* and call $c_*, c_\#$ *concavity bounds* for the q-concave holomorphic map π.

Our main result is the following.

MAIN THEOREM. Suppose X is a complex space, M is an n dimensional complex manifold (with reduced complex structure), and $\pi : X \longrightarrow M$ is a q-concave holomorphic map with exhaustion function φ and concavity bounds $c_*, c_\#$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on X such that codh $\mathcal{F} \geqq r$ on $\{\varphi < c_\#\}$ and $\mathcal{F} | \{\varphi < c_\#\}$ is $(\pi | \{\varphi < c_\#\})$ — flat. Then the l^{th} direct image $\pi_l(\mathcal{F})$ of $\mathcal{F}$ under π is a coherent analytic sheaf on M for $l < r - q - 2n$.

When $X = \{\varphi \geqq c_\#\}$, the Main Theorem gives Theorem G. The Main Theorem may still be valid if we replace $l < r - q - 2n$ by $l < r - q - n$ and drop the $(\pi | \{\varphi < c_\#\})$ — flatness of $\mathcal{F} | \{\varphi < c_\#\}$, but I cannot prove this sharpened version.

B. To understand the difficulty involved in any attempt to prove the conjecture, we analyze very briefly Grauert's proof of Theorem G. Clearly for the proof, we can assume that Y is the n-dimensional unit polydisc K. (We allow K to take on unreduced complex structures also.) Grauert's proof hinges on what Grauert calls the Hauptlemma ([2], p, 47).

Denote by $K(\varrho)$ the n-dimensional polydisc with polyradii equal to the n-tuple ϱ of positive number. Suppose $\mathfrak{U}$ and $\mathfrak{V}$ are finite Stein open coverings of X (i. e. nembers of the coverings are Stein open subsets of X). Denote by $\mathfrak{U}(\varrho)$ the covering obtained by replacing every number of $\mathfrak{U}$ by its intersection with $\pi^{-1}(K(\varrho))$. $\mathfrak{V}(\varrho)$ has a similar meaning. A simplified version of Grauert's Hauptlemma can be roughly described as follows.

$(0.1)_{n,l}$ $\left\{\right.$ Suppose $\mathfrak{U}$ and $\mathfrak{V}$ are suitably chosen and $\mathfrak{V}$ refines $\mathfrak{U}$ in a suitable manner. Then there exist $\xi_1, \ldots, \xi_k \in Z^l(\mathfrak{U}, \mathcal{F})$ such that for ϱ sufficiently small (in a suitably defined sense) the following is satisfied. For $\xi \in Z^l(\mathfrak{U}(\varrho), \mathcal{F})$, there exist holomorphic functions a_i on $K(\varrho)$ and $\eta \in C^{l-1}(\mathfrak{V}(\varrho), \mathcal{F})$ such that $\xi = \Sigma\, a_i \xi_i + \delta\eta$ when restricted to $\mathfrak{V}(\varrho)$ and some (suitably defined) norms of a_i and η are dominated by the product of some (suitably defined) norm of ξ and a fixed constant depending on ϱ.

In Grauert's proof $(0.1)_{n,l}$ is proved by double induction consisting of an ascending induction on n and a descending induction on l. When $n = 0$, $(0.1)_{n,l}$ follows from Theorem C-S and the open mapping theorem for Fréchet spaces. For l very large, $(0.1)_{n,l}$ is vacuous, because $\mathfrak{U}$ is finite. To prove the general $(0.1)_{n,l}$, the idea is to use «power series» expansion to go down to $(0.1)_{n-1,l}$. We expand ξ into «power series» in one of the coordinates of K. The trouble is that the coefficients of the «power series» may not be a cocycle. Calculation can show that, since ξ is a cocycle, even though the coefficients of the «power series» may not be a cocycle, the coboundary of any coefficient of the «power series» is small with respect to the norm under consideration. Now use $(0.1)_{n,l+1}$ coupled with other things to show that the coboundary of every coefficient of the «power series» is equal to the coboundary of an l-cochain which is small. So every coefficient can be approximated by an l-cocycle. By applying $(0.1)_{n-1,l}$ to the approximating l-cocycles, we can find a_i and η so that $\Sigma\, a_i \xi_i + \delta\eta$ approximates ξ. By taking limits, we have $(0.1)_{n,l}$. The step of taking limits involves a lot of technical details. The reason is the following. When we have an approximation, the covering is shrunk from $\mathfrak{U}(\varrho)$ to $\mathfrak{V}(\varrho)$. Unless we have some way to enlarge the covering from $\mathfrak{V}(\varrho)$ back to $\mathfrak{U}(\varrho)$, we may end up with nothing. Grauert overcomes this difficulty by proving a Leray's isomorphism theorem with bounds whose proof depends on a Cartan's theorem B with bounds.

The conjecture is, loosely speaking, a combination of a pseudoconvex generalization and a pseudoconcave generalization of Grauert's direct image theorem. Let us look at the pseudoconvex case and the pseudoconcave case separately.

For the pseudoconvex case, naturally we would only expect to have $(0.1)_{n,l}$ for $l \geqq l_0$, where l_0 is a fixed number. The ascending induction on n and the descending induction on l still work. For the step enlarging the covering from $\mathfrak{V}(\varrho)$ to $\mathfrak{U}(\varrho)$ we can use the techniques of [1] in addition to Leray's isomorphism theorem with bounds. Things seem to work out smoothly. However, in Grauert's proof, there is an isomorphism lemma used after the establishment of $(0.1)_{n,l}$ which poses great difficulties for the pseudoconvex case when $n>1$. We shall explain more about this isomorphism lemma in a short while.

For the pseudoconcave case, naturally we could only expect $(0.1)_{n,l}$ to hold for $l \leqq l_0$ where l_0 is a fixed non-negative integer. The descending induction on l obviously fails. However, for the case $n=1$ this difficulty can easily be circumvented in the following way. Obviously we have $(0.1)_{0,l}$ for $l \leqq l_0$, because of Theorem A-G and the open mapping theorem for Fréchet spaces. From $(0.1)_{0,l}$, by the techniques used in the proof of Proposition 14, [7], we can prove a weakened version of $(0.1)_{1,l}$ which can replace $(0.1)_{1,l}$ in proving the coherence when $n=1$. The weakened version of $(0.1)_{1,l}$ differs from $(0.1)_{1,l}$ in that, instead of requiring a_i to be defined on $K(\varrho)$ and η to be defined for $\mathfrak{V}(\varrho)$, we only require a_i to be defined on $K(\varrho^1)$ and η to be defined for $\mathfrak{V}(\varrho^1)$ for some smaller ϱ^1. The reason why this circumvention works only for $n=1$ is that from the weakened version of $(0.1)_{1,l}$ we cannot derive the weakened version of $(0.1)_{2,l}$. To derive the weakened version of $(0,1)_{2,l}$ we need the original version of $(0.1)_{1,l}$ which we do not have.

After establishing $(0.1)_{n,l}$ Grauert's proof employs diagram-chasing and other simpler techniques. Coherence is proved by ascending induction on n. At one point the following isomorphism lemma is used.

$$(0.2)_{n,l}\quad \left\{\begin{array}{l}\text{If } \pi(\operatorname{Supp}\mathcal{F}) \text{ is contained in a submanifold of codimension 1} \\ \text{in } K, \text{ then the canonical homomorphism from } H^l(X,\mathcal{F}) \text{ to} \\ \Gamma(K,\pi_l(\mathcal{F})) \text{ is an isomorphism.}\end{array}\right.$$

In Grauert's proof, $(0.2)_{n,l}$ follows from the induction hypothesis and the general statement that, if $\sigma: V \dashrightarrow W$ is a holomorphic map of complex spaces and $\mathcal{R}$ is a coherent analytic sheaf on V such that W is Stein and $\sigma_\nu(\mathcal{R})$ is coherent for $\nu<l$, then $H^l(V,\mathcal{R}) \approx \Gamma(W,\pi_l(\mathcal{R}))$. For the pseudoconcave generalization, $(0.2)_{n,l}$ offers no problem. It can be dealt with in the same way. However, for the pseudoconvex generalization, $(0.2)_{n,l}$ poses great difficulties except for the case $n=1$. In the case $n=1$, $(0.2)_{n,l}$ is trivially true.

C. In this paper we prove the pseudoconcave generalization with parameter manifolds of arbitrary dimensions by overcoming the difficulty concerning the pseudoconcave case of $(0.1)_{n,l}$. We derive a technique which enables us to derive $(0.1)_{n,l}$ from $(0.1)_{n-1,l}$ and $(0.1)_{n-1,l+1}$. Because instead of using only $(0.1)_{n-1,l}$ we use both $(0.1)_{n-1,l}$ and $(0.1)_{n-1,l+1}$ to prove

$(0.1)_{n,l}$, we could only prove the coherence of $\pi_l(\mathcal{F})$ for $l < r - q - 2n$ instead of $l < r - q - n$ which we believe should be the sharpest possible for this kind of situation.

One of the most messy parts of Grauert's proof is the part concerning measure charts and measure coverings ([2], § 4). The measure coverings would naturally be much more messy for the pseudoconcave case. To avoid undesired complications, we introduce a neater form of treatment. This is made possible by two techniques. One is to define for holomorphic functions on compact subsets of unreduced complex spaces a semi-norm which is practically independent of the local embeddings used to define it (§ 6). Another is to use Richberg's result on the extension of plurisubharmonic functions [6] to approximate Stein open subsets of a subvariety embedded in a Stein domain in a number space by Stein open subsets of the number space (§ 7).

Needles to say, most ideas in this paper evolve from the ideas of Grauert presented in [2]. Without the ingenious pioneering ideas of Grauert, any proof of the coherence of direct images of sheaves would not be possible. A considerable part of the development here parallels the development in [2]. Unfortunately this cannot be avoided by simply quoting intermediate results of Grauert in [2], because we use a new treatment for measure coverings and because we need slightly more general versions of the results.

As far as the proof of the Main Theorem goes, this paper is self contained, except that some simple statements are quoted from the earlier part of [7]. Hence our proof of the Main Theorem gives as a by-product another version of the proof of Theorem G.

In this presentation we try to separate « soft » analysis and « hard » analysis. « Soft » analysis is dealt with in the earlier part of this paper, whereas « hard » analysis is postponed to the latter part.

D. The following conventions and notations will be used in this paper. Additional ones will be introduced later on when needed.

Unless specified otherwise, all complex spaces and complex subspaces in this paper are in the sense of Grauert (i. e. their structure-sheaves may have non-zero nilpotent elements).

A holomorphic function on a complex space $(X, \mathcal{O})$ means an element of $\Gamma(X, \mathcal{O})$.

Suppose $\mathcal{F}$ is an analytic sheaf on a complex space X and $x \in X$. Then $\mathcal{F}_x$ denotes the stalk of $\mathcal{F}$ at x. If $f \in \Gamma(X, \mathcal{F})$, then f_x denotes the germ of f at x. If $\mathfrak{U} = \{U_i\}_{i \in I}$ is a collection of open subsets of X, then $U_{i_0 \dots i_p}$ denotes $U_{i_0} \cap \dots \cap U_{i_p}$ and $|\mathfrak{U}|$ denotes $\bigcup_{i \in I} U_i$. If Y is a subset of X, then $\mathfrak{U} \cap Y$ denotes $\{U_i \cap Y\}_{i \in I}$. If $g \in C^p(\mathfrak{U}, \mathcal{F})$, then $g_{i_0 \dots i_p} \in \Gamma(U_{i_0 \dots i_p}, \mathcal{F})$

denotes the value of g at the simplex $(i_0, \dots, i_p)$ of the nerve of $\mathfrak{U}$. Suppose $\mathfrak{V} = \{V_j\}_{j \in J}$ is a collection of open subsets of X. If there is a map $\tau : J \to I$ such that $V_j \subset U_{\tau(j)}$, then we write $\mathfrak{V} < \mathfrak{U}$. If $\mathfrak{V} < \mathfrak{U}$ and $V_j \subset\subset U_{\tau(j)}$, then we write $\mathfrak{V} << \mathfrak{U}$. Suppose $\mathfrak{V} < \mathfrak{U}$. We have a map $\tau^* : C^p(\mathfrak{U}, \mathcal{F}) \to C^p(\mathfrak{V}, \mathcal{F})$ induced by τ. If $g \in C^p(\mathfrak{U}, \mathcal{F})$, we call $\tau^*(g)$ the restriction of g to $\mathfrak{V}$ or on $\mathfrak{V}$ and denote it by $g \mid \mathfrak{V}$. If $g' \in C^p(\mathfrak{U}, \mathcal{F})$ such that $\tau^*(g) = \tau^*(g')$, then we say that $g = g'$ on $\mathfrak{V}$. If $\|\cdot\|$ is a norm on $C^p(\mathfrak{V}, \mathcal{F})$, then $\| \tau^*(g) \|$ is also denoted simply by $\| g \|$. If every U_i is Stein, we say that $\mathfrak{U}$ is a Stein open covering of $|\mathfrak{U}|$.

A holomorphic map Φ from a complex space $(X, \mathcal{O})$ to a complex space $(X', \mathcal{O}')$ means a morphism of ringed spaces. That is, $\Phi = (\varphi_0, \varphi_1)$, where φ_0 is a continuous map from X to Y and φ_1 is a continuous map from $\{(x, s) \mid x \in X, s \in \mathcal{O}'_{\varphi_0(x)}\}$ to $\mathcal{O}$. Sometimes, for the sake of notational simplicity, we suppress φ_0 and φ_1. In that case, we use Φ to represent also the continuous map $\varphi_0 : X \to Y$ and, for $h \in \Gamma(X', \mathcal{O}')$, we denote $\varphi_1(h) \in \Gamma(X, \mathcal{O})$ by $h \circ \Phi$. If no confusion can arise, we sometimes denote both h and $h \circ \Phi$ by h.

The q^{th} direct image of an analytic sheaf $\mathcal{F}$ under a holomorphic map Φ is denoted by $\Phi_q(\mathcal{F})$.

${}_m\mathcal{O}$ denotes the structure sheaf of $\mathbf{C}^m$.

n will denote a non-negative integer. It occupies a special position in this paper and does not simply represent a general non-negative integer. $t_1, \dots, t_n$ will denote the coordinate functions of $\mathbf{C}^n$.

$\mathbf{R}_+ = \{c \in \mathbf{R} \mid c > 0\}$. $\mathbf{N} =$ the set of all positive integers. $\mathbf{N}_* = \mathbf{N} \cup \{0\}$. If $a \in \mathbf{R}^m$, then $a_1, \dots, a_m$ denote the components of a. Suppose $a, b \in \mathbf{R}^m$. $a \leqq b$ means $a_i \leqq b_i$ for $1 \leqq i \leqq m$. $a < b$ means $a_i < b_i$ for $1 \leqq i \leqq m$. Suppose $\alpha, \beta \in \mathbf{N}_*^m$. $\alpha + \beta$ means $(\alpha_1 + \beta_1, \dots, \alpha_m + \beta_m)$. If $\beta \leqq \alpha$, then $\binom{\alpha}{\beta}$ means $\binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_m}{\beta_m}$. $|\alpha|$ means $\alpha_1 + \dots + \alpha_m$. If h is a holomorphic function on an open subset of $\mathbf{C}^m$, then $D^\alpha h$ or $D_z^\alpha h$ denotes $\dfrac{\partial^{|\alpha|} h}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_m} z_m}$, where $z_1, \dots, z_m$ are the coordinate functions of $\mathbf{C}^m$.

If $a \in \mathbf{R}_+^m$, then $K^m(a)$ denotes $\{(z_1, \dots, z_m) \in \mathbf{C}^m \mid |z_1| < a_1, \dots, |z_m| < a_m\}$. $K^n(a)$ is simply denoted by $K(a)$. When $a = (1, \dots, 1)$, $K(a)$ is simply denoted by K.

In a collection of open subsets, members which are empty sets are ignored. Two collections of open subsets are considered being the same if they are identical after dropping all members which are empty sets.

If E is a subset of a topological space, ∂E denotes the boundary of E.

Norms in this paper are allowed to take on the value $+\infty$.

§ 1. A Criterion for Coherence.

We derive a criterion for coherence which will be used to prove the coherence of direct images of sheaves.

Suppose $(X, \mathcal{O})$ is a complex space and $\mathcal{F}$ is an analytic sheaf on X. For $x \in X$, $\mathfrak{m}(x)$ denotes the maximum ideal-sheaf on X for the subvariety $\{x\}$.

PROPOSITION 1.1. $\mathcal{F}$ is coherent at a point x_0 of X if and only if there exist Stein open neighborhoods $W \subset \widetilde{W}$ of x_0 and $\xi_1, \dots, \xi_k \in \Gamma(\widetilde{W}, \mathcal{F})$ satisfying the following.

(i) $\mathcal{F} = \Sigma_{i=1}^k \mathcal{O}\,\xi_i$ on W.

(ii) For every $x \in W$ there exists a function $p(x, \cdot) : \mathbf{N} \to \mathbf{N}_*$ such that

(a) $\lim\limits_{d \to \infty} p(x, d) = \infty$, and

(b) for every $\eta \in \Sigma_{i=1}^k \Gamma(\widetilde{W}, \mathcal{O})\,\xi_i$ with $\eta_x \in (\mathfrak{m}(x)^d\,\mathcal{F})_x$, there exist $\alpha_1, \dots, \alpha_k \in \Gamma(W, \mathfrak{m}(x)^{p(x,d)})$ with $\eta = \Sigma_{i=1}^k \alpha_i\,\xi_i$ on W.

PROOF:

I. « Only if » part.

We can find a Stein open neighborhood $\widetilde{W}$ of x_0 in X and $\xi_1, \dots, \xi_k \in \Gamma(\widetilde{W}, \mathcal{F})$ such that $\mathcal{F} = \Sigma_{i=1}^k \mathcal{O}\,\xi_i$ on $\widetilde{W}$ and $\mathcal{F}$ is coherent on $\widetilde{W}$. Thus we have (i). Let $W = \widetilde{W}$ and $p(x, d) = d$ for $x \in W$. We have a sheaf-epimorphism $\mathcal{O}^k \to \mathcal{F}$ on $\widetilde{W}$ defined by $\xi_1, \dots, \xi_k$. This sheaf-epimorphism induces a sheaf-epimorphism $\mathfrak{m}(x)^d\,\mathcal{O}^k \to \mathfrak{m}(x)^d\,\mathcal{F}$ on $\widetilde{W}$ for $x \in \widetilde{W}$. Since $\widetilde{W}$ is Stein, $\Gamma(\widetilde{W}, \mathfrak{m}(x)^d\,\mathcal{O}^k) \to \Gamma(\widetilde{W}, \mathfrak{m}(x)^d\,\mathcal{F})$ is surjective. (ii) follows.

II. « If » part.

Let $\mathcal{R} \subset \mathcal{O}^k \mid \widetilde{W}$ be the relation-sheaf of $\xi_1, \dots, \xi_k$. We need only prove that $\mathcal{R} \mid W$ is generated by global sections.

For $x \in W$. Let T be the $\mathcal{O}_x$ submodule of $\mathcal{R}_x$ generated by global sections of $\mathcal{R} \mid W$. Suppose U is an open neighborhood of x in W and $(\alpha_1, \dots, \alpha_k) \in \Gamma(U, \mathcal{R})$.

Fix $\nu \in \mathbf{N}$. There exists $d \in \mathbf{N}$ such that $p(x, d) \geqq \nu$ and $d \geqq \nu$. Since $\widetilde{W}$ is Stein, the map

$$\Gamma(\widetilde{W}, \mathcal{O}) \xrightarrow{\varphi} \Gamma(\widetilde{W}, \mathcal{O}/\mathfrak{m}(x)^d) \approx (\mathcal{O}/\mathfrak{m}(x)^d)_x$$

induced by the quotient map $\psi : \mathcal{O} \to \mathcal{O}/\mathfrak{m}(x)^d$ is surjective. There exists $\beta_i \in \Gamma(\widetilde{W}, \mathcal{O})$ such that $\varphi(\beta_i)_x = \psi((\alpha_i)_x)$. Hence $(\beta_i - \alpha_i)_x \in (\mathfrak{m}(x)^d)_x$.

$$(\Sigma_{i=1}^k \beta_i\,\xi_i)_x = (\Sigma_{i=1}^k \beta_i\,\xi_i - \Sigma_{i=1}^k \alpha_i\,\xi_i)_x = \Sigma_{i=1}^k (\beta_i - \alpha_i)_x\,(\xi_i)_x \in (\mathfrak{m}(x)^d\,\mathcal{F})_x .$$

By (ii) (*b*) there exist $\gamma_1, \dots, \gamma_k \in \Gamma(W, \mathfrak{m}(x)^{p(x,d)})$ such that $\Sigma_{i=1}^k \gamma_i \xi_i = \Sigma_{i=1}^k \beta_i \xi_i$ on W. $(\beta_1 - \gamma_1, \dots, \beta_k - \gamma_k) \in \Gamma(W, \mathcal{R})$.

$$(\beta_1 - \gamma_1, \dots, \beta_k - \gamma_k)_x - (\alpha_1, \dots, \alpha_k)_x$$

$$= (\beta_1 - \alpha_1, \dots, \beta_k - \alpha_k)_x - (\gamma_1, \dots, \gamma_k)_x \in (\mathfrak{m}(x)^\nu \mathcal{O}^k)_x .$$

Hence $\mathcal{R}_x \subset T + (\mathfrak{m}(x)^\nu \mathcal{O}^k)_x$. Since ν is arbitrarily fixed, $\mathcal{R}_x \subset \bigcap_{\nu \in \mathbf{N}} (T + (\mathfrak{m}(x)^\nu \mathcal{O}^k)_x) = T$. $\mathcal{R}_x = T$. q. e. d.

§ 2 Direct-finite $\mathcal{O}_{\mathfrak{U}}$ systems.

We prove in this section some preparatory propositions which are essentially algebraic in nature. These propositions deal with properties of certain direct systems of modules over rings of local holomorphic functions. In later sections these propositions will be applied to the defining presheaves for direct images of sheaves.

Suppose $(X, \mathcal{O})$ is a complex space and $x \in X$. $\mathfrak{U}$ denotes the directed set of all open neighborhoods of x in X. For $U \in \mathfrak{U}$, $\mathcal{O}_U^k$ denotes $\Gamma(U, \mathcal{O}^k)$. 1_U denotes the element of $\mathcal{O}_U$ whose germ at every point y of U is the unit of the local ring $\mathcal{O}_y$. $e_i^{U,k}$ denotes the element $(0, \dots, 0, 1_U, 0, \dots, 0)$ of $\mathcal{O}_U^k$, where 1_U is in the i^{th} place. For $U' \subset U$ in $\mathfrak{U}$, $\alpha_{U'U}^k : \mathcal{O}_U^k \to \mathcal{O}_{U'}^k$ denotes the restriction map and $\alpha_U^k : \mathcal{O}_U^k \to \mathcal{O}_x^k$ denotes the natural map.

Definition. A direct system $R_{\mathfrak{U}} = \{R_U, \varrho_{U'U}\}$ indexed by the directed set $\mathfrak{U}$ is called an $\mathcal{O}_{\mathfrak{U}}$*-system* if

(i) R_U is an $\mathcal{O}_U$-module, and

(ii) $\varrho_{U'U}$ is an $\mathcal{O}_U$-homomorphism from the $\mathcal{O}_U$-module R_U to the $\mathcal{O}_{U'}$-module $R_{U'}$ which is naturally regarded as an $\mathcal{O}_U$-module.

We denote the direct limit of $R_{\mathfrak{U}}$ by R_*. $\varrho_U : R_U \to R_*$ denotes the natural map.

We need some more notations. Suppose for some fixed $U \in \mathfrak{U}$, $\varphi_U : \mathcal{O}_U^k \to R_U$ is an $\mathcal{O}_U$-homomorphism. For $U' \subset U$ in $\mathfrak{U}$, denote by $\varphi_{U'}$ the $\mathcal{O}_{U'}$-homomorphism from $\mathcal{O}_{U'}^k$ to $R_{U'}$ defined by $\varphi_{U'}(e_i^{U',k}) = \varrho_{U'U}\varphi_U(e_i^{U,k})$ for $1 \leq i \leq k$. We say that $\varphi_{U'}$ is induced by φ_U. $\{\varphi_{U'}\}$ induces an $\mathcal{O}_x$-homomorphism from $\mathcal{O}_x^k$ to R_* which we denote by φ_*. We say that φ_* is induced by φ_U. These notations will be applied, in particular, to the case where $R_{\mathfrak{U}} = \{\mathcal{O}_U^l, \alpha_{U'U}^l\}$.

LEMMA 2.1. Suppose $U \in \mathfrak{U}$ and $\mathcal{M}$ is a coherent analytic subsheaf of a coherent analytic sheaf $\mathcal{N}$ on U. Then there exists $U' \subset U$ in $\mathfrak{U}$ such that, if $\xi \in \Gamma(U, \mathcal{N})$ and $\xi_x \in \mathcal{M}_x$, then $\xi_y \in \mathcal{M}_y$ for $y \in U'$.

PROOF. By replacing U by a smaller member of $\mathfrak{U}$, we can assume without loss of generality that U is Stein. Let $\tilde{\mathcal{M}}$ be the subsheaf of $\mathcal{N}$ on U generated by all elements $\eta \in \Gamma(U, \mathcal{N})$ satisfying $\eta_x \in \mathcal{M}_x$. By applying Cartan's theorem A to the coherent sheaf $\mathcal{M} \mid U$ on the Stein space U, we conclude that $\tilde{\mathcal{M}}_x = \mathcal{M}_x$. Being a subsheaf of coherent analytic sheaf and being generated by global sections, and $\tilde{\mathcal{M}}$ is coherent. Since the two coherent subsheaves $\mathcal{M}$ and $\tilde{\mathcal{M}}$ agree at x, $\mathcal{M}$ and $\tilde{\mathcal{M}}$ agree on some open neighborhood U' of x in U. We claim that U' satisfies the requirement.

Suppose $\xi \in \Gamma(U, \mathcal{M})$ and $\xi_x \in \mathcal{M}_x$. Foom the definition of $\tilde{\mathcal{M}}$, we conclude that $\xi \in \Gamma(U, \tilde{\mathcal{M}})$. Since $\mathcal{M}$ and $\tilde{\mathcal{M}}$ agree on U', $\xi \mid U' \in \Gamma(U', \mathcal{M})$. q.e.d.

LEMMA 2.2. Suppose $U \in \mathfrak{U}$ and $\varphi_U : \mathcal{O}_U^p \longrightarrow \mathcal{O}_U^q$ is an $\mathcal{O}_U$ homomorphism. Then there exists $U' \subset U$ in $\mathfrak{U}$ such that $(\operatorname{Im} \alpha^q_{U' U}) \cap (\alpha^q_{U'})^{-1} (\operatorname{Im} \varphi_*) \subset \operatorname{Im} \varphi_{U'}$.

PROOF. Let $\mathcal{M}$ be the subsheaf of $\mathcal{O}^q$ on U generated by $\operatorname{Im} \varphi_U$. $\mathcal{M}$ is coherent. By Lemma 2.1 there exists $U' \subset U$ in $\mathfrak{U}$ such that, if $\xi \in \mathcal{O}_U^q$ and $\xi_x \in \mathcal{M}_x$, then $\xi \mid U' \in \Gamma(U', \mathcal{M})$. By shrinking U', we can assume that U' is Stein. We clain that U' satisfies the requirement.

Take $\eta \in (\operatorname{Im} \alpha^q_{U' U}) \cap (\alpha^q_{U'})^{-1} (\operatorname{Im} \varphi_*)$. Then $\eta = \xi \mid U'$ for some $\xi \in \mathcal{O}_U^q$ and $\alpha^q_{U'}(\eta) \in \operatorname{Im} \varphi_*$. It is clear tat $\operatorname{Im} \varphi_* = \mathcal{M}_x$. Hence $\xi_x = \alpha^q_{U'}(\eta) \in \mathcal{M}_x$. $\xi \mid U' \in \Gamma(U', \mathcal{M})$. Since $\operatorname{Im} \varphi_{U'}$ generates $\mathcal{M} \mid U'$ and U' is Stein, by Cartan's theorem B we conclude that $\xi \mid U' \in \operatorname{Im} \varphi_{U'}$. q.e.d.

LEMMA 2.3. Suppose $U \in \mathfrak{U}$ and A is a subset of $\mathcal{O}_U^k$. Then there exists $U' \subset U$ in $\mathfrak{U}$ such that the $\mathcal{O}_{U'}$-submodule of $\mathcal{O}_{U'}^k$ generated by $\alpha^k_{U' U}(A)$ is finitely generated over $\mathcal{O}_{U'}$.

PROOF. Let T be the $\mathcal{O}_x$-submodule of $\mathcal{O}_x^k$ generated by A. Then there exist $\xi_1, \dots, \xi_l \in A$ such that $(\xi_1)_x, \dots, (\xi_l)_x$ generate T. Let $\mathcal{M} = \Sigma_{i=1}^l \mathcal{O}\, \xi_i$ on U. $\mathcal{M}$ is coherent.

By Lemma 2.1 there exists $U' \subset U$ in $\mathfrak{U}$ such that, if $\xi \in \mathcal{O}_U^k$ and $\xi_x \in \mathcal{M}_x$, then $\xi \mid U' \in \Gamma(U', \mathcal{M})$. By shrinking U', we can assume that U' is Stein. We claim that U' satisfies the requirement.

We need only check that $\alpha^k_{U'U}(A) \subset \Sigma^l_{i=1} \mathcal{O}_{U'} \alpha^k_{U'U}(\xi_i)$. Take $\xi \in A$. Then $\xi_x \in \mathcal{M}_x$. $\xi \mid U' \in \Gamma(U', \mathcal{M})$. Since U' is Stein, Cartan's theorem B implies that $\xi = \Sigma^l_{i=1} a_i \xi_i$ for some $a_i \in \mathcal{O}_{U'}$. q.e.d.

DEFINITION. Suppose $R_{\mathfrak{U}} = \{R_U, \varrho_{U'U}\}$ is an $\mathcal{O}_{\mathfrak{U}}$-system. $R_{\mathfrak{U}}$ is said to be *direct-finite* if

(i) R_* is finitely generated over $\mathcal{O}_x$, and

(ii) for every $U \in \mathfrak{U}$ there exists $U' \subset U$ in $\mathfrak{U}$ such that the $\mathcal{O}_{U'}$-submodule generated by $\operatorname{Im} \varrho_{U'U}$ is finitely generated over $\mathcal{O}_{U'}$.

LEMMA 2.4. In the preceding definition, (ii) is equivalent to each of the following two statements.

(ii)′ for every $U \in \mathfrak{U}$ there exist an $\mathcal{O}_U$ homomorphism $\varphi_U : \mathcal{O}^k_U \to R_U$ and $U' \subset U$ in $\mathfrak{U}$ such that $\operatorname{Im} \varrho_{U'U} \subset \operatorname{Im} \varphi_{U'}$.

(ii)″ for every $U \in \mathfrak{U}$ there exist $U' \subset U$ in $\mathfrak{U}$ and an $\mathcal{O}_{U'}$-homomorphism $\varphi_{U'} : \mathcal{O}^k_{U'} \to R_{U'}$ such that $\operatorname{Im} \varrho_{U'U} \subset \operatorname{Im} \varphi_{U'}$.

PROOF. It is clear that (ii) is equivalent to (ii)′. It is clear that (ii)′ implies (ii)″. We are going to prove that (ii)″ implies (ii).

Take $U \in \mathfrak{U}$. By (ii)″ there exists $U' \subset U$ in $\mathfrak{U}$ and an $\mathcal{O}_{U'}$-homomorphism $\varphi_{U'} : \mathcal{O}^k_{U'} \to R_{U'}$ such that $\operatorname{Im} \varrho_{U'U} \subset \operatorname{Im} \varphi_{U'}$. Consider the subset $A = \varphi^{-1}_{U'}(\operatorname{Im} \varrho_{U'U})$ of $\mathcal{O}^k_{U'}$. By Lemma 2.3 there exists $U'' \subset U'$ in $\mathfrak{U}$ such that the $\mathcal{O}_{U''}$-submodule of $\mathcal{O}^k_{U''}$ generated by $\alpha^k_{U''U'}(A)$ is finitely generated ove $\mathcal{O}_{U''}$. Hence the $\mathcal{O}_{U''}$-submodule of $R_{U''}$ generated by $\varphi_{U''}(\alpha^k_{U''U'}(A))$ is finitely generated over $\mathcal{O}_{U''}$.

Since $\operatorname{Im} \varrho_{U'U} \subset \operatorname{Im} \varphi_{U'}$, $\varphi_{U'}(A) = \varphi_{U'}(\varphi^{-1}_{U'}(\operatorname{Im} \varrho_{U'U})) = \varrho_{U'U}$. We have $\varphi_{U''}(\alpha^k_{U''U'}(A)) = \varrho_{U''U'}(\varphi_{U'}(A)) = \varrho_{U''U'}(\operatorname{Im} \varrho_{U'U}) = \operatorname{Im} \varrho_{U''U'}$. Hence the $\mathcal{O}_{U''}$-submodule generated by $\operatorname{Im} \varrho_{U''U}$ is finitely generated over $\mathcal{O}_{U''}$. q. e. d.

PROPOSITION 2.1. Suppose $R_{\mathfrak{U}} = \{R_U, \varrho_{U'U}\}$ is a direct-finite $\mathcal{O}_{\mathfrak{U}}$-system. Then for every $U \in \mathfrak{U}$ there exists $U' \subset U$ in $\mathfrak{U}$ such that $\operatorname{Ker} \varrho_U \subset \operatorname{Ker} \varrho_{U'U}$.

PROOF. Fix $U \in \mathfrak{U}$. There exist $U_1 \subset U$ in $\mathfrak{U}$ and an $\mathcal{O}_{U_1}$-homomorphism $\varphi_{U_1} : \mathcal{O}^k_{U_1} \to R_{U_1}$ such that $\operatorname{Im} \varrho_{U_1 U} \subset \operatorname{Im} \varphi_{U_1}$.

There exists an $\mathcal{O}_x$-homomorphism $\psi_* : \mathcal{O}^m_x \to \mathcal{O}^k_x$ such that $\operatorname{Im} \psi_* = \operatorname{Ker} \varphi_*$. There exists $U_2 \subset U_1$ in $\mathfrak{U}$ such that ψ_* is induced by some $\mathcal{O}_{U_2}$-homomorphism $\psi_{U_2} : \mathcal{O}^m_{U_2} \to \mathcal{O}^k_{U_2}$ and $\varphi_{U_2} \psi_{U_2} = 0$.

By Lemma 2.2 there exists $U' \subset U_2$ in $\mathfrak{U}$ such that $(\operatorname{Im} \alpha^k_{U'U_2}) \cap (\alpha^k_{U'})^{-1}(\operatorname{Im} \psi_*) \subset \operatorname{Im} \psi_{U'}$. We are going to prove that $\operatorname{Ker} \varrho_U \subset \operatorname{Ker} \varrho_{U'U}$.

Suppose $\xi \in \operatorname{Ker} \varrho_U$. Then there exists $\eta \in \mathcal{O}^k_{U_1}$ such that $\varphi_{U_1}(\eta) = \varrho_{U_1 U}(\xi)$. $\varphi_* \alpha^k_{U_1}(\eta) = \varrho_U(\xi) = 0$. Hence $\alpha^k_{U_1}(\eta) \in \operatorname{Im} \psi_*$.

$$\alpha^k_{U' U_1}(\eta) \in (\operatorname{Im} \alpha^k_{U' U_2}) \cap (\alpha^k_{U'})^{-1} (\operatorname{Im} \psi_*).$$

$\alpha^k_{U' U_1}(\eta) = \psi_{U'}(\zeta)$ for some $\zeta \in \mathcal{O}^m_{U'}$. Since $\varphi_{U_2} \psi_{U_2} = 0$, $\varphi_{U'} \psi_{U'} = 0$.

$$\varrho_{U' U}(\xi) = \varrho_{U' U_1} \varrho_{U_1 U}(\xi) = \varrho_{U' U_1} \varphi_{U_1}(\eta) = \varphi_{U'} \alpha^k_{U' U_1}(\eta) = \varphi_{U'} \psi_{U'}(\zeta) = 0. \text{ q.e.d.}$$

LEMMA 2.5. Suppose $S_{\mathfrak{U}} = \{S_U, \sigma_{U' U}\}$ and $R_{\mathfrak{U}} = \{R_U, \varrho_{U' U}\}$ are $\mathcal{O}_{\mathfrak{U}}$-systems. Suppose for every $U' \subset U$ in $\mathfrak{U}$, S_U is an $\mathcal{O}_U$-submodule of R_U and $\sigma_{U' U}$ is the restriction of $\varrho_{U' U}$ to S_U. If $R_{\mathfrak{U}}$ is direct-finite, then $S_{\mathfrak{U}}$ is direct-finite.

PROOF. Since S_* is an $\mathcal{O}_x$-submodule of R_*, S_* is finitely generated over $\mathcal{O}_x$.

Fix $U \in \mathfrak{U}$. There exist $U_1 \subset U$ in $\mathfrak{U}$ and an $\mathcal{O}_{U_1}$— homomorphism $\varphi_{U_1}: \mathcal{O}^k_{U_1} \longrightarrow R_{U_1}$ such that $\operatorname{Im} \varrho_{U_1 U} \subset \operatorname{Im} \varphi_{U_1}$. There exists an $\mathcal{O}_x$-holomorphism $\psi_* : \mathcal{O}^m_x \longrightarrow \mathcal{O}^k_x$ such that $\operatorname{Im} \psi_* = \varphi_*^{-1}(S_*)$. For some $U_2 \subset U_1$ in $\mathfrak{U}$, ψ_* is induced by some $\mathcal{O}_{U_2}$-homomorphism $\psi_{U_2}: \mathcal{O}^m_{U_2} \longrightarrow \mathcal{O}^k_{U_2}$ such that Im $(\varphi_{U_2} \psi_{U_2}) \subset S_{U_2}$. By Lemma 2.2 there exist $U' \subset U$ such that $(\operatorname{Im} \alpha^k_{U' U_2}) \cap (\alpha^k_{U'})^{-1} (\operatorname{Im} \psi_*) \subset \operatorname{Im} \psi_{U'}$. We are going to prove that $\operatorname{Im} \varphi_{U'} \psi_{U'} \cap \supset \operatorname{Im} \sigma_{U' U}$.

Take $\xi \in S_U$. $\sigma_{U_1 U}(\xi) = \varphi_{U_1}(\eta)$ for some $\eta \in \mathcal{O}^k_{U_1}$. $\varphi_* \alpha^k_{U_1}(\eta) = \varrho_{U_1} \varphi_{U_1}(\eta) = \varrho_{U_1} \sigma_{U_1 U}(\xi) = \sigma_U(\xi) \in S_*$. Hence $\alpha^k_{U_1}(\eta) \in \varphi_*^{-1}(S_*) = \operatorname{Im} \psi_*$. $\alpha^k_{U' U_1}(\eta) \in (\operatorname{Im} \alpha^k_{U' U_2}) \cap (\alpha^k_{U'})^{-1} (\operatorname{Im} \psi_*) \subset \operatorname{Im} \psi_{U'}$. For some $\zeta \in \mathcal{O}^m_{U'}$, $\alpha^k_{U' U_1}(\eta) = \psi_{U'}(\zeta)$. $\sigma_{U' U}(\xi) = \varrho_{U' U_1} \varphi_{U_1}(\eta) = \varphi_{U'} \alpha^k_{U' U_1}(\eta) = \varphi_{U'} \psi_{U'}(\zeta)$. Hence $\operatorname{Im} \sigma_{U' U} \subset \operatorname{Im} \varphi_{U'} \psi_{U'}$.

Since $\operatorname{Im} \varphi_{U_2} \psi_{U_2} \subset S_{U_2}$, $\operatorname{Im} \varphi_{U'} \psi_{U'} \subset S_{U'}$. Since U is arbitrary, by Lemma 2.4, $S_{\mathfrak{U}}$ is direct-finite. q. e. d.

PROPOSITION 2.2 Suppose $R'_{\mathfrak{U}} = \{R'_U \varrho'_{U' U}\}$, $R_{\mathfrak{U}} = \{R_U, \varrho_{U' U}\}$, and $R''_{\mathfrak{U}} = \{R''_U, \varrho''_{U' U}\}$ are $\mathcal{O}_{\mathfrak{U}}$-systems. Suppose for every $U' \subset U$ in $\mathfrak{U}$, the diagram

$$\begin{array}{ccccc} R'_U & \xrightarrow{\varphi_U} & R_U & \xrightarrow{\psi_U} & R''_U \\ \varrho'_{U'U}\big\downarrow & & \varrho'_{U'U}\big\downarrow & & \varrho''_{U'U}\big\downarrow \\ R'_{U'} & \xrightarrow{\varphi_{U'}} & R_{U'} & \xrightarrow{\psi_{U'}} & R''_{U'} \end{array}$$

is commutativ eand has exact rows, where φ_U and ψ_U are $\mathcal{O}_U$-homomorphisms and $\varphi_{U'}$ and $\psi_{U'}$ are $\mathcal{O}_{U'}$-homomorphisms. If both $R'_{\mathfrak{U}}$ and $R''_{\mathfrak{U}}$ are direct-finite, then $R_{\mathfrak{U}}$ is direct-finite.

PROOF. By using Lemma 2.5 and by replacing R''_U by $\operatorname{Im}\psi_U$, we can assume that ψ_U is surjective.

Since $R'_* \longrightarrow R_* \longrightarrow R''_* \longrightarrow 0$ is exact, R_* is finitely generated over $\mathcal{O}_x$.

Fix $U \in \mathfrak{U}$. There exist $U_1 \subset U$ in $\mathfrak{U}$ and an $\mathcal{O}_{U_1}$-homomorphism $\beta_{U_1}: \mathcal{O}^k_{U_1} \longrightarrow R''_{U_1}$ such that $\operatorname{Im}\ \varrho''_{U_1U} \subset \operatorname{Im}\beta_{U_1}$. Since ψ_{U_1} is surjective, there exists an $\mathcal{O}_{U_1}$-homomorphism ν_{U_1}: $\mathcal{O}^k_{U_1} \longrightarrow R_{U_1}$ such that $\psi_{U_1}\nu_{U_1} = \beta_{U_1}$.

There exist $U' \subset U_1$ in $\mathfrak{U}$ and an $\mathcal{O}_{U'}$-homomorphism $\sigma_{U'}: \mathcal{O}^m_{U'} \longrightarrow R'_{U'}$ such that $\operatorname{Im}\ \varrho'_{U'U_1} \subset \operatorname{Im}\sigma_{U'}$. Let $\tau_{U'}: \mathcal{O}^{m+k}_{U'} \longrightarrow R_{U'}$ be the $\mathcal{O}_{U'}$-homomorphism defined by: $\tau_{U'}(a \oplus b) = (\varphi_{U'}\sigma_{U'})(a) + \nu_{U'}(b)$ for $a \in \mathcal{O}^m_{U'}$ and $b \in \mathcal{O}^k_{U'}$. We are going to prove that $\operatorname{Im}\ \varrho_{U'U} \subset \operatorname{Im}\tau_{U'}$.

Take $\xi \in R_U$. Then for some $\eta \in \mathcal{O}^k_{U_1}$, $\varrho''_{U_1U}(\psi_U(\xi)) = \beta_{U_1}(\eta)$. $\psi_{U_1}(\varrho_{U_1U}(\xi) - \nu_{U_1}(\eta)) = \varrho''_{U_1U}(\psi_U(\xi)) - \beta_{U_1}(\eta) = 0$. $\varrho_{U_1U}(\xi) - \nu_{U_1}(\eta) = \varphi_{U_1}(\zeta)$ for some $\zeta \in R'_{U_1}$. For some $\theta \in \mathcal{O}^m_{U'}$, $\varrho'_{U'U_1}(\zeta) = \sigma_{U'}(\theta)$. $\varrho_{U'U_1}(\xi) - \nu_{U'}(\alpha^k_{U'U_1}(\eta)) = \varrho_{U'U_1}(\varrho_{U_1U}(\xi) - - \nu_{U_1}(\eta)) = \varrho_{U'U_1}\varphi_{U_1}(\zeta) = \varphi_{U'}\varrho'_{U'U_1}(\zeta) = \varphi_{U'}\sigma_{U'}(\theta)$. $\varrho_{U'U}(\xi) = \nu_{U'}(\alpha^k_{U'U_1}(\eta)) + + \varphi_{U'}\sigma_{U'}(\theta) \in \operatorname{Im}\tau_{U'}$. q. e. d.

PROPOSITION 2.3. Suppose $R_{\mathfrak{U}} = \{R_U, \varrho_{U'U}\}$ is a direct-finite $\mathcal{O}_{\mathfrak{U}}$-system. Then there exist $\tilde{U} \in \mathfrak{U}$ and an $\mathcal{O}_{\tilde{U}}$-homomorphism $\varphi_{\tilde{U}}: \mathcal{O}^k_{\tilde{U}} \longrightarrow R_{\tilde{U}}$ satisfyng the following. For any $U \subset \tilde{U}$ in $\mathfrak{U}$ there exists $U' \subset U$ in $\mathfrak{U}$ such that $\operatorname{Im}\ \varrho_{U'U} \subset \operatorname{Im}\varphi_{U'}$.

PROOF. Since R^* is finitely generated, there exist $\tilde{U} \in \mathfrak{U}$ and an $\mathcal{O}_{\tilde{U}}$-homomorphism $\varphi_{\tilde{U}}: \mathcal{O}^k_{\tilde{U}} \longrightarrow R_{\tilde{U}}$ such that $\varphi_*: \mathcal{O}^k_x \longrightarrow R_x$ is surjective.

Fix $U \subset \tilde{U}$ in $\mathfrak{U}$. There exist $U_1 \subset U$ in $\mathfrak{U}$ and an $\mathcal{O}_{U_1}$-homomorphism $\psi_{U_1}: \mathcal{O}^m_{U_1} \longrightarrow R_{U_1}$ such that $\operatorname{Im}\ \varrho_{U_1U} \subset \operatorname{Im}\psi_{U_1}$. Since φ_* is surjective, there exists an $\mathcal{O}_x$-homomorphism $\beta_*: \mathcal{O}^m_x \longrightarrow \mathcal{O}^k_x$ such that $\varphi_*\beta_* = \psi_*$. For some $U' \subset U$ in $\mathfrak{U}$, β_* is induced by an $\mathcal{O}_{U'}$-homomorphism $\beta_{U'}: \mathcal{O}^m_{U'} \longrightarrow \mathcal{O}^k_{U'}$ such that $\varphi_{U'}\beta_{U'} = \psi_{U'}$. Since $\operatorname{Im}\ \varrho_{U_1U} \subset \operatorname{Im}\psi_{U_1}$, $\operatorname{Im}\ \varrho_{U'U} = \operatorname{Im}\ \varrho_{U'U_1}\varrho_{U_1U} \subset \subset \operatorname{Im}\varrho_{U'U_1}\psi_{U_1} \subset \operatorname{Im}\psi_{U'} = \operatorname{Im}\varphi_{U'}\beta_{U'} \subset \operatorname{Im}\varphi_{U'}$. q. e. d.

§ 3. Reduction of the Problem.

In this section we reduce the proof of the coherence of direct images to the verification of a certain property which we call H^l-finiteness.

Suppose X is a complex space and $\mathcal{F}$ is a coherent analytic sheaf on X. Suppose $\pi: X \longrightarrow K$ is a holomorphic map, where the n-dimensional unit polydisc K is given the reduced complex structure. For $\varrho \in \mathbf{R}_+^n$ let $X(\varrho) = \pi^{-1}(K(\varrho))$.

For $t^0 \in K$, $\mathfrak{m}(t^0)$ denotes the maximum ideal-sheaf on K for the subvariety $\{t^0\}$. The holomorphic functions $t_i \circ \pi$ on X obtained by lifting the coordinate-functions t_i on K are also denoted by t_i for the sake of notational simplicity.

DEFINITION. For $t^0 \in K$ and $l \in \mathbf{N}_*$, $\mathcal{F}$ is said to be *H^l-finite at t^0 with respect to π* if the ${}_n\mathcal{O}_{\mathfrak{U}}$-system $\{H^l(\pi^{-1}(U), \mathcal{F}), r_{U'U}\}_{U', U \in \mathfrak{U}}$ is direct-finite, where $\mathfrak{U}$ is the directed set of all open neighborhoods of t^0 in K and, for $U' \subset U$, $r_{U'U}: H^l(\pi^{-1}(U), \mathcal{F}) \to H^l(\pi^{-1}(U'), \mathcal{F})$ is the restriction map. $\mathcal{F}$ is said to be *H^l-finite with respect to π* if $\mathcal{F}$ is H^l-finite at every point of K with respect to π.

LEMMA 3.1. Suppose $0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G}'' \longrightarrow 0$ is an exact sequence of coherent analytic sheaves on X. If $\mathcal{G}'$ and $\mathcal{G}''$ are H^l-finite at a point t^0 of K with respect to π, then $\mathcal{G}$ is H^l-finite at t^0 with respect to π.

PROOF. For every open neighborhood U of t^0 in K, the sequence $H^l(\pi^{-1}(U), \mathcal{G}') \longrightarrow H^l(\pi^{-1}(U), \mathcal{G}) \longrightarrow H^l(\pi^{-1}(U), \mathcal{G}'')$ is exact. The Lemma follows from Proposition 2.2. q. e. d.

LEMMA 3.2. Suppose $\varrho^1 < \varrho^0 \leqq (1, \dots, 1)$ in $\mathbf{R}_+^n$, $t^0 \in K(\varrho^1)$, and $d \in \mathbf{N}_*$. Suppose the following three conditions are satisfied.

(i) $t_n - t_n^0$ is not a zero-divisor for $(t_n - t_n^0)^d \mathcal{F}_x$ for $x \in X(\varrho^0)$.

(ii) t_n is not a zero-divisor for $t_n^d \pi_{l+1}(\mathcal{F})_0$.

(iii) We have $\operatorname{Ker} \beta \subset \operatorname{Ker} \alpha$ in

$$\pi_{l+1}(\mathcal{F})_0 \xleftarrow{\beta} H^{l+1}(X(\varrho^0), \mathcal{F}) \xrightarrow{\alpha} H^{l+1}(X(\varrho^1), \mathcal{F}).$$

For every $\nu \in \mathbf{N}_*$ let $\mathcal{F}^{(\nu)} = (t_n - t_n^0)^\nu \mathcal{F}$ and $\mathcal{F}_\nu = \mathcal{F}/\mathcal{F}^{(\nu)}$.

Then, for every $\nu \in \mathbf{N}_*$, we have $\operatorname{Im} \psi \subset \operatorname{Im} \varphi$ in

$$H^l(X(\varrho^1), \mathcal{F}) \xrightarrow{\varphi} H^l(X(\varrho^1), \mathcal{F}_\nu) \xleftarrow{\psi} H^l(X(\varrho^0), \mathcal{F}_{\nu+2d}),$$

where φ is induced by the quotient map $\mathcal{F} \longrightarrow \mathcal{F}_\nu$ and ψ is induced by the inclusion map $X(\varrho^1) \hookrightarrow X(\varrho^0)$ and the quotient map $\mathcal{F}_{\nu+2d} \longrightarrow \mathcal{F}_\nu$.

PROOF. Fix $\nu \in \mathbf{N}_*$. From the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}^{(\nu+2d)} \arrow[r] \arrow[d, hook] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{F}_{\nu+2d} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}^{(\nu)} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}_\nu \arrow[r] & 0
\end{tikzcd}$$

we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & H^l(X(\varrho^0), \mathcal{F}_{\nu+2d}) \arrow[r, "a"] \arrow[d, "\psi"'] & H^{l+1}(X(\varrho^0), \mathcal{F}^{(\nu+2d)}) \arrow[r, "c"] \arrow[d, "b"'] & H^{l+1}(X(\varrho^0), \mathcal{F}) \\
H^l(X(\varrho^1), \mathcal{F}) \arrow[r, "\varphi"] & H^l(X(\varrho^1), \mathcal{F}_\nu) \arrow[r] & H^{l+1}(X(\varrho^1), \mathcal{F}^{(\nu)}). &
\end{tikzcd}$$

We need only prove that $ba = 0$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathcal{F} \arrow[d, equal] & \mathcal{F}^{(\nu+2d)} \arrow[l, hook', "v"'] \arrow[r, hook, "u"] & \mathcal{F}^{(\nu)} \\
 & \mathcal{F}^{(d)} \arrow[u, "p"'] \arrow[d, hook, "w"] & \\
\mathcal{F} & \mathcal{F} \arrow[l, "s"'] \arrow[r, "q"] & \mathcal{F}_\nu \arrow[uu, "r"']
\end{tikzcd} \tag{3.1}$$

where p and s are defined by multiplication by $(t_n - t_n^0)^{\nu+d}$, q is defined by multiplication by $(t_n - t_n^0)^d$, and r is defined by multiplication by $(t_n - t_n^0)^\nu$.

Because of (i), p is a sheaf-isomorphism on $X(\varrho^0)$.

Applying $H^{l+1}(X(\varrho^i), \cdot)$ to the diagram (3.1), we obtain a diagram with maps denoted by p_i, q_i, r_i, s_i, u_i, v_i, and w_i $(i = 0, 1)$.

Applying $\pi_{l+1}(\cdot)_0$ to the diagram (3.1), we obtain a diagram with maps denoted by p', q', r', s', u', v', and w'.

We first prove the following:

$$\operatorname{Ker} s' \subset \operatorname{Ker} q'. \tag{3.2}$$

When $t_n^0 \neq 0$, s is a sheaf isomorphism on $\pi^{-1}(U)$ for some open neighborhood U of 0 in K. Hence s' is an isomorphism. $\operatorname{Ker} s' = 0 \subset \operatorname{Ker} q'$. When $t_n^0 = 0$, (3.2) follows from (ii).

Consider

$$\pi_{l+1}(\mathcal{F}^{(\nu+2d)})_0 \xleftarrow{\tau} H^{l+1}(X(\varrho^0), \mathcal{F}^{(\nu+2d)}) \xrightarrow{\sigma} H^{l+1}(X(\varrho^1), \mathcal{F}^{(\nu+2d)}),$$

where σ and τ are natural maps.

To prove $ba = 0$, take $\xi \in \operatorname{Im} a$. Then $c(\xi) = 0$.

$$s'w'(p')^{-1}\tau(\xi) = v'\tau(\xi) = \beta c(\xi) = 0. \text{ By (3.2),}$$

$$q'w'(p')^{-1}\tau(\xi) = 0. \text{ Hence } \beta q_0 w_0 (p_0)^{-1}(\xi) = 0. \text{ By (iii),}$$

$$\alpha q_0 w_0 (p_0)^{-1}(\xi) = 0. \text{ Hence } q_1 w_1 (p_1)^{-1}\sigma(\xi) = 0.$$

$$b(\xi) = u_1\sigma(\xi) = r_1 q_1 w_1 (p_1)^{-1}\sigma(\xi) = 0. \qquad \text{q. e. d.}$$

LEMMA 3.3. (*a*) Suppose R is a Noetherian ring and M is a finitely generated R module. If $f \in R$, then there exists $\nu \in \mathbf{N}_*$ such that f is not a zero-divisor for $f^\nu M$.

(*b*) Suppose $\mathcal{G}$ is a coherent analytic sheaf on a complex space Y and Q is a relatively compact open subset of Y. If g is a holomorphic function on Y, then there exists $\nu \in \mathbf{N}_*$ such that g is not a zero-divisor for $g^\nu \mathcal{G}_x$ for $x \in Q$.

PROOF. (*a*) For $\nu \in \mathbf{N}_*$ let N_ν be the kernel of the R-homomorphism $M \to M$ defined by multiplication by f^ν. $\{N_\nu\}$ is a non-decreasing sequence of R submodules of M. Since M is finitely generated over a Noetherian ring, $N_\nu = N_{\nu+1}$ for some $\nu \in \mathbf{N}_*$. It is easily checked that this ν satisfies the requirement.

(*b*) For $\nu \in \mathbf{N}_*$ let $\mathcal{K}_\nu$ be the kernel of the sheaf-homomorphis $\mathcal{G} \to \mathcal{G}$ defined by multiplication by g^ν. $\{\mathcal{K}_\nu\}$ is a non decreasing sequence of coherent analytic subsheaves of $\mathcal{G}$. Since Q is relatively compact, there exists $\nu \in \mathbf{N}_*$ such that $\mathcal{K}_\nu = \mathcal{K}_{\nu+1}$. It is easily checked that this ν satisfies the requirement. q. e. d.

Observe that, in Lemma 3.3, for $\mu \geqq \nu$, f is not a zero-divisor for $f^\mu M$ and g is not a zero-divisor for $g^\mu \mathcal{G}_x$ for $x \in Q$.

The proof of the following Lemma is a trivial modification of the proof of Satz 5, [2].

LEMMA 3.4. Suppose $\sigma : Y \to Z$ is a holomorphic map of complex spaces, $\mathcal{G}$ is a coherent analytic sheaf on Y, and Z is Stein. Suppose $l \in \mathbf{N}_*$ and $\sigma_k(\mathcal{G})$ is coherent for $0 \leqq k < l$. Then the natural homomorphism $H^l(Y, \mathcal{G}) \to \Gamma(Z, \sigma_l(\mathcal{G}))$ is an isomorphism.

PROOF. The case $l = 0$ is trivially true. We can therefore assume that $l > 0$.

Let $0 \to \mathcal{G} \to \mathcal{S}_0 \xrightarrow{\varphi_0} \mathcal{S}_1 \xrightarrow{\varphi_1} \mathcal{S}_2 \to \ldots$ be a flabby sheaf resolution for $\mathcal{G}$. By taking the zeroth direct image under σ, we obtain the followind sequence which in general is not exact:

$$0 \longrightarrow \sigma_0(\mathcal{G}) \longrightarrow \sigma_0(\mathcal{S}_0) \xrightarrow{\varphi_0'} \sigma_0(\mathcal{S}_1) \xrightarrow{\varphi_1'} \sigma_0(\mathcal{S}_2) \longrightarrow \cdots.$$

Since $\sigma_0(\mathcal{S}_k)$ is flabby for $k \geqq 0$, $H^p(Z, \sigma_0(\mathcal{S}_k)) = 0$ for $p \geqq 1$ and $k \geqq 0$. Since Z is Stein and $\sigma_k(\mathcal{G})$ is coherent for $0 \leqq k < l$, $H^p(Z, \sigma_k(\mathcal{G})) = 0$ for $p \geqq 1$ and $0 \leqq k < l$.

By considering the following two exact sequences

$$0 \longrightarrow \operatorname{Ker} \varphi_k' \longrightarrow \sigma_0(\mathcal{S}_k) \longrightarrow \operatorname{Im} \varphi_k' \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Im} \varphi_{k-1}' \longrightarrow \operatorname{Ker} \varphi_k' \longrightarrow \sigma_k(\mathcal{G}) \longrightarrow 0 \quad (\text{where } \operatorname{Im} \varphi_{-1}' = 0)$$

and by using induction on k, we obtain $H^p(Z, \operatorname{Ker} \varphi_k') = H^p(Z, \operatorname{Im} \varphi_k') = 0$ for $p \geqq 1$ and $0 \leqq k < l$.

$H^1(Z, \operatorname{Ker} \varphi_{l-1}') = 0$ implies that $\Gamma(Z, \sigma_0(\mathcal{S}_{l-1})) \xrightarrow{\alpha} \Gamma(Z, \operatorname{Im} \varphi_{l-1}') \longrightarrow 0$ is exact. $H^1(Z, \operatorname{Im} \varphi_{l-1}') = 0$ implies that $\Gamma(Z, \operatorname{Im} \varphi_{l-1}') \xrightarrow{\beta} \Gamma(Z, \operatorname{Ker} \varphi_l') \longrightarrow$ $\longrightarrow \Gamma(Z, \sigma_l(\mathcal{G})) \longrightarrow 0$ is exact. Hence $\Gamma(Z, \sigma_l(\mathcal{G})) = \Gamma(Z, \operatorname{Ker} \varphi_l') / \operatorname{Im} \beta\alpha$. Since $H^l(Y, \mathcal{G}) \approx \operatorname{Ker}(\Gamma(Y, \mathcal{S}_l) \longrightarrow \Gamma(Y, \mathcal{S}_{l+1}))/\operatorname{Im}(\Gamma(Y, \mathcal{S}_{l-1}) \longrightarrow \Gamma(Y, \mathcal{S}_l))$, the Lemma follows from $\Gamma(Z, \operatorname{Ker} \varphi_l') \approx \operatorname{Ker}(\Gamma(Z, \sigma_0(\mathcal{S}_l)) \longrightarrow \Gamma(Z, \sigma_0(\mathcal{S}_{l+1}))) \approx$ $\approx \operatorname{Ker}(\Gamma(Y, \mathcal{S}_l) \longrightarrow \Gamma(Y, \mathcal{S}_{l+1}))$ and $\operatorname{Im} \beta\alpha \approx \operatorname{Im}(\Gamma(Z, \sigma_0(\mathcal{S}_{l-1})) \longrightarrow \Gamma(Z, \sigma_0(\mathcal{S}_l)) \approx$ $\approx \operatorname{Im}(\Gamma(Y, \mathcal{S}_{l-1}) \longrightarrow \Gamma(Y, \mathcal{S}_l))$. q. e. d.

PROPOSITION 3.2. Suppose $l \in \mathbf{N}_*$ and the following three conditions are satisfied.

(i) For every $t^0 \in K$ and $\nu \in \mathbf{N}_*$, $\pi_k(\mathcal{F}/(t_n - t_n^0)^\nu \mathcal{F})$ is coherent for $0 \leqq k \leqq l$.

(ii) $\mathcal{F}$ is H^l-finite and H^{l+1}-finite with respect to π.

(iii) For every $t^0 \in K$ and every relatively compact open subset U of K there exists $\nu \in \mathbf{N}_*$ such that $t_n - t_n^0$ is not a zero-divisor for $(t_n - t_n^0)^\nu \mathcal{F}_x$ for $x \in \pi^{-1}(U)$.

Then $\pi_l(\mathcal{F})$ is coherent on K.

PROOF. Fix $y \in K$. We need only prove the coherence of $\pi_l(\mathcal{F})$ at y. Without loss of generality we can assume that $y = 0$.

By applying Propositions 2.1 and 2.3 to the direct-finite ${}_n\mathcal{O}_{\mathfrak{U}}$-systems $\{H^i(\pi^{-1}(U), \mathcal{F}), r^i_{U'U}\}_{U', U \in \mathfrak{U}}$, $i = l, l+1$, where $\mathfrak{U}$ is the directed set of all open neighborhoods of 0 in K and $r^i_{U'U}$ are restriction maps, we can find $\varrho^2 < \varrho^1 < \varrho^0 < (1, \dots, 1)$ such that the following two conditions hold.

(*a*) We have $\operatorname{Ker} \beta' \subset \operatorname{Ker} \alpha'$ in $\pi_{l+1}(\mathcal{F})_0 \xleftarrow{\beta'} H^{l+1}(X(\varrho^0), \mathcal{F}) \xrightarrow{\alpha'}$ $H^{l+1}(X(\varrho^1), \mathcal{F})$.

(*b*) There exist $\xi_1, \dots, \xi_k \in H^l(X(\varrho^0), \mathcal{F})$ such that we have $\operatorname{Im} \beta \subset \subset \sum_{i=1}^k \Gamma(K(\varrho^2), {}_n\mathcal{O})\, \beta\alpha(\xi_i)$ in

$$H^l(X(\varrho^0), \mathcal{F}) \xrightarrow{\alpha} H^l(X(\varrho^1), \mathcal{F}) \xrightarrow{\beta} H^l(X(\varrho^2), \mathcal{F}).$$

Fix arbitrarily $t^0 \in K(\varrho^2)$. By Proposition 1.1, to prove the coherence of $\pi_l(\mathcal{F})$ at 0, we need only prove the following two statements.

(3.3) The images of $\xi_1, \dots, \xi_k$ in $\Gamma(K(\varrho^0), \pi_l(\mathcal{F}))$ generate $\pi_l(\mathcal{F})_{t^0}$

(3.4) $\begin{cases} \text{For } \nu \in \mathbf{N}_*, \text{ if } \xi \in \sum_{i=1}^k \Gamma(K(\varrho^0), {}_n\mathcal{O})\, \xi_i \text{ and the image of } \xi \\ \text{in } \pi_l(\mathcal{F})_{t^0} \text{ belongs to } (\mathfrak{m}(t^0)^\nu \pi_l(\mathcal{F}))_{t^0}, \text{ then } \beta\alpha(\xi) \in \sum_{i=1}^k \Gamma(K(\varrho^2), \\ \mathfrak{m}(t^0)^\nu)\, \beta\alpha(\xi_i). \end{cases}$

By (ii), $\pi_{l+1}(\mathcal{F})_{t^0}$ is finitely generated over ${}_n\mathcal{O}_{t^0}$. By Lemma 3.3 (a) and by (iii), there exists $d \in \mathbf{N}_*$ such that t_n is not a zero-divisor for $t_n^d\, \pi_{l+1}(\mathcal{F})_0$ and $t_n - t_n^0$ is not a zero-divisor for $(t_n - t_n^0)^d\, \mathcal{F}_x$ for $x \in X(\varrho^0)$.

For $\nu \in \mathbf{N}_*$ define $\mathcal{F}^{(\nu)} = (t_n - t_n^0)^\nu\, \mathcal{F}$ and $\mathcal{F}_\nu = \mathcal{F}/\mathcal{F}^{(\nu)}$.

I. By Lemma 3.2 we have $\operatorname{Im} \psi \subset \operatorname{Im} \varphi$ in $H^l(X(\varrho^1), \mathcal{F}) \xrightarrow{\varphi} H^l(X(\varrho^1), \mathcal{F}_{d+1}) \xleftarrow{\psi} H^l(X(\varrho^0), \mathcal{F}_{3d+1})$, where φ is induced by the quotient map $\mathcal{F} \longrightarrow \mathcal{F}_{d+1}$ and ψ is induced by the quotient map $\mathcal{F}_{2d+1} \longrightarrow \mathcal{F}_{d+1}$ and the inclusion map $X(\varrho^1) \hookrightarrow X(\varrho^0)$.

Consider $H^l(X(\varrho^1), \mathcal{F}) \xrightarrow{a} \pi_l(\mathcal{F}_{d+1})_{t_0} \xleftarrow{b} \pi_l(\mathcal{F})_{t_0}$, where a and b are induced by the quotient map $\mathcal{F} \longrightarrow \mathcal{F}_{d+1}$. We are going to prove

(3.5) $$\operatorname{Im} b \subset {}_n\mathcal{O}_{t^0} \operatorname{Im} a,$$

where ${}_n\mathcal{O}_{t^0} \operatorname{Im} a$ denotes the ${}_n\mathcal{O}_{t^0}$-submodule of $\pi_l(\mathcal{F}_{d+1})_{t^0}$ generated by $\operatorname{Im} a$.

Consider the following commutative diagram.

$$\begin{array}{ccccc}
H^l(X(\varrho^0), \mathcal{F}_{3d+1}) & \xrightarrow{\psi} & H^l(X(\varrho^1), \mathcal{F}_{d+1}) & \xleftarrow{\varphi} & H^l(X(\varrho^1), \mathcal{F}) \\
{\scriptstyle c}\downarrow & & {\scriptstyle f}\downarrow & & \| \\
\pi_l(\mathcal{F}_{3d+1})_{t^0} & \xrightarrow{h} & \pi_l(\mathcal{F}_{d+1})_{t^0} & \xleftarrow{a} & H^l(X(\varrho^1), \mathcal{F}) \\
{\scriptstyle g}\uparrow & & {\scriptstyle b}\uparrow & & \\
\pi_l(\mathcal{F})_{t^0} & = & \pi_l(\mathcal{F})_{t^0}, & &
\end{array}$$

where c and f are natural maps, g is induced by the quotient map $\mathcal{F} \longrightarrow \mathcal{F}_{3d+1}$, and h is induced by the quotient map $\mathcal{F}_{3d+1} \longrightarrow \mathcal{F}_{d+1}$.

By (i) and Lemma 3.4, $H^l(X(\varrho^0), \mathcal{F}_{3d+1}) \approx \Gamma(K(\varrho^0), \pi_l(\mathcal{F}_{3d+1}))$. Since $\pi_l(\mathcal{F}_{3d+1})$ is coherent, by Cartan's theorem A, $\operatorname{Im} c$ generates $\pi_l(\mathcal{F}_{3d+1})_{t^0}$ over ${}_n\mathcal{O}_{t^0}$. $\operatorname{Im} b = \operatorname{Im} hg \subset \operatorname{Im} h = h({}_n\mathcal{O}_{t^0} \operatorname{Im} c) = {}_n\mathcal{O}_{t^0} \operatorname{Im} hc = {}_n\mathcal{O}_{t_0} \operatorname{Im} f\psi \subset \subset {}_n\mathcal{O}_{t^0} \operatorname{Im} f\varphi = {}_n\mathcal{O}_{t^0} \operatorname{Im} a$. (3.5) is proved.

Consider the following commutative diagram:

$$\begin{tikzcd}
\pi_l(\mathcal{F}^{(d)})_{t^0} \arrow[r, "r"] \arrow[d, "p"'] & \pi_l(\mathcal{F})_{t^0} \arrow[d, "q"'] & & \\
\pi_l(\mathcal{F}^{(d+1)})_{t^0} \arrow[r, "s"] & \pi_l(\mathcal{F})_{t^0} \arrow[r, "b"] & \pi_l(\mathcal{F}_{d+1})_{t^0} \arrow[r, equal] & \pi_l(\mathcal{F}_{d+1})_{t^0} \\
& H^l(X(\varrho^0), \mathcal{F}) \arrow[u, "\gamma"] \arrow[r, "\alpha"] & H^l(X(\varrho^1), \mathcal{F}) \arrow[u, "a"] \arrow[r, "\beta"] & H^l(X(\varrho^2), \mathcal{F}), \arrow[u, "\theta"]
\end{tikzcd}$$

where r is induced by $\mathcal{F}^{(d)} \hookrightarrow \mathcal{F}$, s is induced by $\mathcal{F}^{(d+1)} \hookrightarrow \mathcal{F}$, p and q are defined by multiplication by $t_n - t_n^0$, γ is the natural map, and θ is induced by the quotient map $\mathcal{F} \longrightarrow \mathcal{F}_{d+1}$.

Since $t_n - t_n^0$ is not a zero divisor for $(t_n - t_n^0)^d \mathcal{F}_x$ for $x \in X(\varrho^0)$, the sheaf-homomorphism $\mathcal{F}^{(d)} \longrightarrow \mathcal{F}^{(d+1)}$ on $X(\varrho^0)$, defined by multiplication by $t_n - t_n^0$, is a sheaf-isomorphism. Hence p is an isomorphism.

From the exact sequence $0 \longrightarrow \mathcal{F}^{(d+1)} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_{d+1} \longrightarrow 0$, we conclude that $\operatorname{Im} s = \operatorname{Ker} b$.

$$\operatorname{Im} a = \operatorname{Im} \theta\beta = \theta(\operatorname{Im} \beta) \subset \theta\left(\Sigma_{i=1}^k \Gamma(K(\varrho^2), {}_n\mathcal{O})\, \beta\alpha(\xi_i)\right) \subset$$

$$\subset \Sigma_{i=1}^k {}_n\mathcal{O}_{t^0}\, \theta\beta\alpha(\xi_i) = \Sigma_{i=1}^k {}_n\mathcal{O}_{t^0}\, b\gamma(\xi_i).$$

$$\operatorname{Im} b \subset {}_n\mathcal{O}_{t^0} \operatorname{Im} a \subset \Sigma_{i=1}^k {}_n\mathcal{O}_{t^0}\, b\gamma(\xi_i).$$

Let $T = \Sigma_{i=1}^k {}_n\mathcal{O}_{t^0}\, \gamma(\xi_i)$. (3.3) will follow if we can prove $T = \pi_l(\mathcal{F})_{t^0}$. Since $\operatorname{Im} b \subset \Sigma_{i=1}^k {}_n\mathcal{O}_{t^0}\, b\,\gamma(\xi_i)$,

$$\pi_l(\mathcal{F})_{t^0} \subset T + \operatorname{Ker} b = T + \operatorname{Im} s = T + \operatorname{Im} qrp^{-1}$$

$$\subset T + \operatorname{Im} q = T + (t_n - t_n^0)\, \pi_l(\mathcal{F})_{t^0} \subset T + (\mathfrak{m}(t^0)\, \pi_l(\mathcal{F}))_{t^0}.$$

By Nakayama's lemma, $\pi_l(\mathcal{F})_{t^0} = T$. (3.3) is proved.

II. Fix arbitrarily $\nu \in \mathbf{N}_*$. Consider the following commutative diagram:

$$\begin{tikzcd}
H^l(X(\varrho^1), \mathcal{F}^{(d)}) \arrow[r, "u"] \arrow[d, "w"'] & H^l(X(\varrho^1), \mathcal{F}^{(\nu+d)}) \arrow[d, "\sigma"] & & \\
H^l(X(\varrho^1), \mathcal{F}) \arrow[r, "v"] & H^l(X(\varrho^1), \mathcal{F}) \arrow[d, "\tau"] & H^l(X(\varrho^0), \mathcal{F}) \arrow[l, "\alpha"'] \arrow[r, "\lambda"] \arrow[d, "\varepsilon"] & \pi_l(\mathcal{F})_{t^0} \arrow[d, "\varkappa"] \\
& H^l(X(\varrho^1), \mathcal{F}_{\nu+d}) & H^l(X(\varrho^0), \mathcal{F}_{\nu+3d}) \arrow[l, "\chi"'] \arrow[r, "\mu"] & \pi_l(\mathcal{F}_{\nu+3d})_{t^0},
\end{tikzcd}$$

where σ is induced by $\mathcal{F}^{(\nu+d)} \hookrightarrow \mathcal{F}$, w is induced by $\mathcal{F}^{(d)} \hookrightarrow \mathcal{F}$, u and v are induced by multiplication by $(t_n - t_n^0)^\nu$, $\varkappa$, ε, and χ are induced by quotient maps, and λ and μ are natural maps.

Since $t_n - t_n^0$ is not a zero-divisor for $(t_n - t_n^0)^d \mathcal{F}_x$ for $x \in X(\varrho^0)$, the sheaf-homomorphism $\mathcal{F}^{(d)} \to \mathcal{F}^{(\nu+d)}$ on $X(\varrho^0)$ defined by multiplication by $(t_n - t_n^0)^\nu$ is a sheaf-isomorphism. Hence u is an isomorphism.

From the exact sequence $0 \to \mathcal{F}^{(\nu+d)} \to \mathcal{F} \to \mathcal{F}_{\nu+d} \to 0$ we conclude that $\operatorname{Im} \sigma = \operatorname{Ker} \tau$.

To prove (3.4), take $\xi \in \Sigma_{i=1}^k \Gamma(K(\varrho^0), {}_n\mathcal{O})\, \xi_i$ such that $\lambda(\xi) \in (\mathfrak{m}(t^0)^\nu \pi_l(\mathcal{F}))_{t^0}$, Then $\mu\varepsilon(\xi) \in (\mathfrak{m}(t^0)^\nu \pi_l(\mathcal{F}_{\nu+3d}))_{t^0}$.

By (i) and Lemma 3.4, the natural map $\eta : H^l(X(\varrho^0), \mathcal{F}_{\nu+3d}) \to \Gamma(K(\varrho^0).$ $\pi_l(\mathcal{F}_{\nu+3d}))$ is an isomorphism.

Since $\pi_l(\mathcal{F}_{\nu+3d})$ is coherent on K and $\eta\varepsilon(\xi) \in \Gamma(K(\varrho^0), \pi_l(\mathcal{F}_{\nu+3d}))$ and $\eta\varepsilon(\xi)_{t_0} = \mu\varepsilon(\xi) \in (\mathfrak{m}(t^0)^\nu \pi_l(\mathcal{F}_{\nu+3d}))_{t^0}$, by Part I of the proof of Proposition 1.1 we conclude that $\eta\varepsilon(\xi) \in \Gamma(K(\varrho_0), \mathfrak{m}(t^0)^\nu)\, \Gamma(K(\varrho_0), \pi_l(\mathcal{F}_{\nu+3d}))$. Hence

$$\varepsilon(\xi) \in \Gamma(K(\varrho^0), \mathfrak{m}(t_0)^\nu)\, H^l(X(\varrho^0), \mathcal{F}_{\nu+3d}).$$

$\chi\varepsilon(\xi) \in \Gamma(K(\varrho^1), \mathfrak{m}(t^0)^\nu) \operatorname{Im} \chi$, where $\Gamma(K(\varrho^1), \mathfrak{m}(t^0)^\nu) \operatorname{Im} \chi$ denotes $\{A_1 B_1 + \dots + A_j B_j \mid A_i \in \Gamma(K(\varrho^1), \mathfrak{m}(t^0)^\nu), B_i \in \operatorname{Im} \chi\}$. By Lemme 3.2, we have $\operatorname{Im} \chi \subset \operatorname{Im} \tau$. Hence $\chi\varepsilon(\xi) \in \Gamma(K(\varrho^1), \mathfrak{m}(t^0)^\nu) \operatorname{Im} \tau$.

$\tau\alpha(\xi) \in \Gamma(K(\varrho^1), \mathfrak{m}(t^0)^\nu) \operatorname{Im} \tau$. $\alpha(\xi) \in \Gamma(K(\varrho^1), \mathfrak{m}(t^0)^\nu)\, H^l(X(\varrho^1), \mathcal{F}) + \operatorname{Ker} \tau$. Since $\operatorname{Ker} \tau = \operatorname{Im} \sigma = \operatorname{Im} vwu^{-1} \subset \operatorname{Im} v = (t_n - t_n^0)^\nu H^l(X(\varrho^1), \mathcal{F}) \subset \Gamma(K(\varrho^1),$ $\mathfrak{m}(t^0)^\nu)\, H^l(X(\varrho^1), \mathcal{F})$, we have $\alpha(\xi) \in \Gamma(K(\varrho^1), \mathfrak{m}(t^0)^\nu)\, H^l(X(\varrho^1), \mathcal{F})$.

Since $\operatorname{Im} \beta \subset \Sigma_{i=1}^k \Gamma(K(\varrho^2), {}_n\mathcal{O})\, \beta\alpha(\xi_i)$, we have $'\beta\alpha(\xi) \in (\Gamma(K(\varrho^2), \mathfrak{m}(t^0)^\nu) \cdot$ $\cdot \operatorname{Im} \beta \subset \Sigma_{i=1}^k \Gamma(K(\varrho^2), \mathfrak{m}(t^0)^\nu)\, \beta\alpha(\xi_i)$. (3.4) is proved. q. e. d.

§ 4 Further Reduction of the Problem.

In this section we carry out another reduction for the proof of the Main Theorem. We show that, to prove the Main Theorem, we can replace X by $\{\varphi > c\}$, where $c_* < c < c_\#$.

A. Suppose $\mathcal{F}$ is a coherent analytic sheaf on a complex space X and $\pi : X \longrightarrow K(\varrho^0)$ is a holomorphic map, where $\varrho^0 \in \mathbf{R}^n_+$. Suppose $X_1 \subset X_2$ are open subsets of X and the restriction of π to $\overline{X_2}$ is proper. Let $\pi^i : X_i \longrightarrow K$ be the restriction of π to X_i, $i = 1, 2$. Suppose $t \in \mathbf{N}_*$. $\pi^i_l(\mathcal{F})$ denotes the l^{th} direct image of $\mathcal{F} \mid X_i$ under π^i.

PROPOSITION 4.1. Suppose $\pi^2_l(\mathcal{F})_0$ is finitetely generated over ${}_n\mathcal{O}_0$ and the map $\pi^2_l(\mathcal{F}/t^r_n\mathcal{F})_0 \longrightarrow \pi^1_l(\mathcal{F}/t^r_n\mathcal{F})$ is injective for every $r \in \mathbf{N}_*$. Then $\pi^2_l(\mathcal{F})_0 \longrightarrow \pi^1_l(\mathcal{F})_0$ is injective.

PROOF. By Lemma 3.3 (*b*), after shrinking ϱ^0 we can find $d \in \mathbf{N}_*$ such that t_n is not a zero-divisor for $t^d_n\, \mathcal{F}_x$ for $x \in X_2$.

Let $\mathcal{G}_r = \mathcal{F}/t^r_n\, \mathcal{F}$, $r \in \mathbf{N}$. For $r \geqq s$ in $\mathbf{N}$, consider the following commutative diagram:

$$\begin{array}{ccccc}
\pi^2_l(\mathcal{F})_0 & \xrightarrow{\sigma_r} & \pi^2_l(\mathcal{G}_r)_0 & \xrightarrow{\lambda^r_s} & \pi^2_l(\mathcal{G}_s)_0 \\
\Phi\downarrow & & \Phi_r\downarrow & & \Phi_s\downarrow \\
\pi^1_l(\mathcal{F})_0 & \xrightarrow{\tau_r} & \pi^1_l(\mathcal{G}_r)_0 & \xrightarrow{\mu^r_s} & \pi^1_l(\mathcal{G}_s)_0,
\end{array}$$

where Φ, Φ_r, Φ_s are induced by restriction maps and σ_r, τ_r, λ^r_s, μ^r_s are induced by quotient maps. Take arbitrarily $r \geqq d$. Since Φ_r is injective, $\operatorname{Ker} \Phi_r\, \sigma_r = \operatorname{Ker} \sigma_r$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
\pi^2_l(t^d_n\, \mathcal{F})_0 & \xrightarrow{b} & \pi^2_l(\mathcal{F})_0 & & \\
a\downarrow & & c\downarrow & & \\
\pi^2_l(t^r_n\, \mathcal{F})_0 & \xrightarrow{f} & \pi^2_l(\mathcal{F})_0 & \xrightarrow{\sigma_r} & \pi^2_l(\mathcal{G}_r)_0,
\end{array}$$

where the second row comes from the exact sequence $0 \longrightarrow t^r_n\, \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \longrightarrow \mathcal{G}_r \longrightarrow 0$, a comes from the sheaf-homomorphism $\tilde{a} : t^d_n\, \mathcal{F} \longrightarrow t^r_n\, \mathcal{F}$ defined by multiplication by t_n^{r-d}, b comes from $t_n^d \mathcal{F} \hookrightarrow \mathcal{F}$, and c is defined by multiplication by t_n^{r-d}.

Since t_n is not a zero-divisor for $t_n^d \mathcal{F}_x$ for $x \in X_2$, $\tilde{a}$ is a sheaf-isomorphism on X_2. a is therefore an isomorphism. Ker $\sigma_r = \operatorname{Im} f = \operatorname{Im} cba^{-1} \subset$ $\subset \operatorname{Im} c = t_n^{r-d}\, \pi_l^2 (\mathcal{F})_0$.
Ker $\Phi \subset$ Ker τ_r $\Phi =$ Ker $\Phi_r\, \sigma_r =$ Ker $\sigma_1 \subset t_n^{r-d}\, \pi_l^2(\mathcal{F})$. Since r is arbitrary, Ker $\Phi \subset \bigcap_{r \geqq d} t_n^{r-d}\, \pi_l^2 (\mathcal{F})_0$. Since $\pi_l^2 (\mathcal{F})_0$ is infinitely generated over ${}_n\mathcal{O}_0$, $\bigcap_{r \geqq d} t_n^{r-d}\, \pi_l^2 (\mathcal{F})_0 = 0$. q. e. d

B. LEMMA 4.1. Suppose $l \in \mathbf{N}_*$, X is a topological space, and $\mathcal{F}$ is a sheaf of abelian groups on X. Suppose $\{B_k\}_{k \in \mathbf{N}_*}$ is a non-decreasing sequence of open subsets of X whose union is X.

(*a*) If $H^l (B_{k+1}, \mathcal{F}) \to H^l (B_k, \mathcal{F})$ is surjective for $k \in \mathbf{N}_*$, then $H^l(X, \mathcal{F}) \to H^l (B_0, \mathcal{F})$ is surjective.

(*b*) If $H^l(B_{k+1}, \mathcal{F}) \to H^l(B_k, \mathcal{F})$ is injective and $H^{l-1}(B_{k+1}, \mathcal{F}) \to$ $\to H^{l-1}(B_k, \mathcal{F})$ is surjective for $k \in \mathbf{N}_*$, then $H^l(X, \mathcal{F}) \to H^l(B_0, \mathcal{F})$ is injective.

PROOF. When $l = 0$, both (*a*) and (*b*) are trivially true. Therefore we can assume that $l > 0$. Let $0 \to \mathcal{F} \to \mathcal{S}_0 \xrightarrow{\varphi_0} \mathcal{S}_1 \xrightarrow{\varphi_1} \mathcal{S}_2 \to \dots$ be a flabby sheaf resolution of $\mathcal{F}$ on X. Let $\mathcal{Z}_k = \operatorname{Ker} \varphi_k$ for $k \in \mathbf{N}_*$.

(*a*) Take $\xi^* \in H^l(B_0, \mathcal{F})$. ξ^* is represented by some $\xi_0 \in \Gamma(B_0, \mathcal{Z}_l)$.

We are going to construct, by induction on $k \in \mathbf{N}_*$, $\xi_k \in \Gamma(B_k, \mathcal{Z}_l)$ such that $\xi_{k+1} = \xi_k$ on B_k. We already have ξ_0. Suppose we have ξ_k for some $k \in \mathbf{N}_*$. Since $H^l(B_{k+1}, \mathcal{F}) \to H^l(B_k, \mathcal{F})$ is surjective, there exist $\xi'_{k+1} \in \Gamma(B_{k+1}, \mathcal{Z}_l)$ and $\eta \in \Gamma(B_k, \mathcal{S}_{l-1})$ such that $\xi'_{k+1} = \xi_k - \varphi_{l-1}(\eta)$ on B_v. Since $\mathcal{S}_{l-1}$ is flabby, η can be extended to $\eta' \in \Gamma(B_{k+1}, \mathcal{S}_{l-1})$. Set $\xi_{k+1} =$ $= \xi'_{k+1} + \varphi_{l-1}(\eta')$. The induction is complete. Define $\xi \in \Gamma(X, \mathcal{Z}_l)$ by $\xi | B_k = \xi_k$. Let $\tilde{\xi} \in H^l(X, \mathcal{F})$ be induced by ξ. Then $\tilde{\xi}$ is mapped to ξ^* under $H^l(X, \mathcal{F}) \to$ $\to H^l(B_0, \mathcal{F})$.

(*b*) Suppose $\xi^* \in H^l(X, \mathcal{F})$ is mapped to zero in $H^l(B_0, \mathcal{F})$. ξ^* is represented by some $\xi \in \Gamma(X, \mathcal{Z}_l)$.

Since $H^l(B_{k+1}, \mathcal{F}) \to H^l(B_k, \mathcal{F})$ is injective, ξ^* is mapped to zero in $H^l(B_k, \mathcal{F})$ as is seen by induction on k. $\xi = \varphi_{l-1}(\eta'_k)$ on B_k for some $\eta'_k \in \Gamma(B_k, \mathcal{S}_{l-1})$.

We are going to construct, by induction on $k \in \mathbf{N}_*$, $\eta_k \in \Gamma(B_k, \mathcal{S}_{l-1})$ such that $\xi = \varphi_{l-1}(\eta_k)$ on B_k and $\eta_{k+1} = \eta_k$ on B_k. Set $\eta_0 = \eta'_0$. Suppose we have η_k for some $k \in \mathbf{N}_*$. $\varphi_{l-1}(\eta'_{k+1} - \eta_k) = 0$ on B_k. $H^{l-1}(B_{k+1}, \mathcal{F}) \to$ $\to H^{l-1}(B_k, \mathcal{F})$ is surjective. When $l = 1$, $\eta'_{k+1} - \eta_k = \zeta$ on B_k for some $\zeta \in \Gamma(B_{k+1}, \mathcal{F})$ and we need only set $\eta_{k+1} = \eta'_{k+1} - \zeta$. When $l > 1$, $\eta'_{k+1} -$

$-\eta_k = \zeta + \varphi_{l-2}(\sigma)$ on B_k for some $\zeta \in \Gamma(B_{k+1}, \mathcal{Z}_{l-1})$ and $\sigma \in \Gamma(B_k, \mathcal{S}_{l-2})$. Since $\mathcal{S}_{l-2}$ is flabby, σ can be extended to some $\sigma' \in \Gamma(B_{k+1}, \mathcal{S}_{l-2})$. $\eta_{k+1} = \eta'_{k+1} - \zeta - \varphi_{l-2}(\sigma')$ satisfies the requirement. The induction is complete. Define $\eta \in \Gamma(X, \mathcal{S}_{l-1})$ by $\eta \mid B_k = \eta_k$. Then $\xi = \varphi_{l-1}(\eta)$ on X. $\xi^* = 0$ in $H^l(X, \mathcal{F})$. q. e. d.

LEMMA 4.2. Suppose X is a complex space and φ is a C^∞ map from X to (c_*, ∞) where $c_* \in \{-\infty\} \cup \mathbf{R}$, such that $\{\varphi \geqq c\}$ is compact for $c \in (c_*, \infty)$. Suppose $c_\# \in (c_*, \infty)$ and $\mathcal{F}$ is a coherent analytic sheaf on X such that $\operatorname{codh} \mathcal{F} \geqq r$ on $\{\varphi < c_\#\}$. For $c \in [c_*, \infty)$, let $X_c = \{\varphi > c\}$. If $c_* \leqq c' < c < c_\#$ and $0 \leqq l < r - q$, then $H^l(X_{c'}, \mathcal{F}) \rightarrow H^l(X_c, \mathcal{F})$ is bijective.

PROOF. Fix $c_* < c < c_\#$. We need only show that $H^l(X, \mathcal{F}) \xrightarrow{\approx} H^l(X_c, \mathcal{F})$ for $0 \leqq l < r - q$. For, if $c_* \leqq c' < c$, then $H^l(X_{c'}, \mathcal{F}) \xrightarrow{\approx} H^l(X_c, \mathcal{F})$ follows from $H^l(X, \mathcal{F}) \xrightarrow{\approx} H^l(X_c, \mathcal{F})$ and $H^l(X, \mathcal{F}) \xrightarrow{\approx} H^l(X_{c'}, \mathcal{F})$.

Let Γ be the set of $c' \in [c_*, c]$ such that $H^l(X_{c'}, \mathcal{F}) \xrightarrow{\approx} H^l(X_c, \mathcal{F})$ for $0 \leqq l < r - q$. Let $\tilde{c} = \inf \Gamma$. We claim that $\tilde{c} \in \Gamma$. Let $\{c_k\}_{k \in \mathbf{N}_*}$ be a sequence in I strictly decreasing to $\tilde{c}$ and $c_0 = c$. Since both $H^l(X_{c_{k+1}}, \mathcal{F})$ and $H^l(X_{c_k}, \mathcal{F})$ are isomorphic to $H^l(X_c, \mathcal{F})$ for $0 \leqq l < r - q$, $H^l(X_{c_{k+1}}, \mathcal{F}) \xrightarrow{\approx} H^l(X_{c_k}, \mathcal{F})$ for $0 \leqq l < r - q$. By Lemma 4.1, $H^l(X_{\tilde{c}}, \mathcal{F}) \xrightarrow{\approx} H^l(X_{c_0}, \mathcal{F})$ for $0 \leqq l < r - q$. Hence $\tilde{c} \in \Gamma$.

To finish the proof, we need only show that $\tilde{c} = c_*$. Suppose the contrary. Then $c_* < \tilde{c}$. By Proposition 17 on p. 239 of [1], there exists $c_\dagger \in (c_*, \tilde{c})$ such that $H^l(X_{c_\dagger}, \mathcal{F}) \xrightarrow{\approx} H^l(X_{\tilde{c}}, \mathcal{F})$ for $0 \leqq l < r - q$. Hence $\tilde{c}_\dagger \in \Gamma$, contradicting $c = \inf \Gamma$. q. e. d.

REMARK. Lemma 4 2 and its proof are implicitly contained in [1].

C. Suppose $(R, \mathfrak{m})$ is a local ring and M is an R module. A sequence of elements $a_1, \ldots, a_k$ of $\mathfrak{m}$ is called an *M-sequence* if a_i is not a zero divisor for $M/\Sigma_{j=1}^{i-1} a_j M$ for $1 \leqq i \leqq k$.

LEMMA 4.3. Suppose $\tau_1, \ldots, \tau_k \in \mathfrak{m}$ form an M-sequence. If $d_1, \ldots, d_k \in \mathbf{N}$ then $\tau_1^{d_1}, \ldots, \tau_k^{d_k}$ form an M-sequence.

PROOF. Prove by induction on k.

(i) $k = 1$. Since τ_1 is not a zero-divisor for M, $\tau_1^{d_1}$ is not a zero-divisor for M as seen by induction on d_1.

(ii) For the general case, assume $k > 1$. It is well-known that any rearrangement of an M-sequence is still an M-sequence. Since $\tau_1, \dots, \tau_k$ form an M-sequence, $\tau_2, \dots, \tau_k$ form an $(M/\tau_1 M)$-sequence. By induction hypothesis, $\tau_2^{d_2}, \dots, \tau_k^{d_k}$ form an $(M/\tau_1 M)$-sequence. $\tau_1, \tau_2^{d_2}, \dots, \tau_k^{d_k}$ form an M-sequence. τ_1 is not a zero divisor for $M/\Sigma_{i=2}^k \tau_i^{d_i} M$. Hence $\tau_1^{d_1}$ is not a zero-divisor for $M/\Sigma_{i=2}^k \tau_i^{d_i} M$. $\tau_1^{d_1}, \dots, \tau_k^{d_k}$ form an M-sequence. q. e. d.

LEMMA 4.4. Suppose $\tau_1, \dots, \tau_k, \tau \in \mathfrak{m}$. Let $N = M/\tau M$.

(*a*) If $d_1, \dots, d_k, d \in \mathbf{N}$, then there exists a natural R-homomorphism

$$\alpha : N/\Sigma_{i=1}^k \tau_i^{d_i} N \longrightarrow (\tau^{d-1} M + \Sigma_{i=1}^k \tau_i^{d_i} M)/(\tau^d M + \Sigma_{i=1}^k \tau_i^{d_i} M).$$

(*b*) If in addition $\tau_1, \dots, \tau_k, \tau$ form an M-sequence, then α is an isomorphism.

PROOF. (*a*) The second Dedekind-Noether isomorphism theorem for modules implies that there is a natural R-isomorphism

$$\beta : \tau^{d-1} M/(\tau^{d-1} M) \cap (\tau^d M + \Sigma_{i=1}^k \tau_i^{d_i} M) \longrightarrow$$
$$\longrightarrow (\tau^{d-1} M + \Sigma_{i=1}^k \tau_i^{d_i} M)/(\tau^d M + \Sigma_{i=1}^k \tau_i^{d_i} M).$$

Let $\eta : M \longrightarrow \tau^{d-1} M/(t^{d-1} M) \cap (\tau^d M + \Sigma_{i=1}^k \tau_i^{d_i} M)$ be the composite of the quotient homomorphism $\tau^{d-1} M \longrightarrow \tau^{d-1} M/(\tau^{d-1} M) \cap (\tau^d M + \Sigma_{i=1}^k \tau_i^{d_i} M)$ and the homonorphism $M \longrightarrow \tau^{d-1} M$ defined by multiplication by τ^{d-1}. Clearly $\tau M + \Sigma_{i=1}^k \tau_i^{d_i} M \subset \operatorname{Ker} \eta$. Hence η induces $\tilde{\eta} : M/(\tau M + \Sigma_{i=1}^k \tau_i^{d_i} M) \longrightarrow \tau^{d-1} M/(\tau^{d-1} M) \cap (\tau^d M + \Sigma_{i=1}^k \tau_i^{d_i} M)$.

Let $\gamma : N = M/\tau M \longrightarrow M/(\tau M + \Sigma_{i=1}^k \tau_i^{d_i} M)$ be the quotient homomorphism. It is clear that $\operatorname{Ker} \gamma = \Sigma_{i=1}^k \tau_i^{d_i} N$. Hence γ induces an isomorphism $\tilde{\gamma} : N/\Sigma_{i=1}^k \tau_i^{d_i} N \longrightarrow M/(\tau M + \Sigma_{i=1}^k \tau_t^{d_i} M)$.

$\alpha = \beta \tilde{\eta} \tilde{\gamma}$ is the homomorphism we look for.

(*b*) Now assume in addition that $\tau_1, \dots, \tau_k$, form an M-sequence. To show that α is an isomorphism, we need only show that $\operatorname{Ker} \eta \subset \tau M + \Sigma_{i=1}^k \tau_i^{d_i} M$.

Take $f \in \operatorname{Ker} \eta$. $\tau^{d-1} f = \tau^d g + \Sigma_{i=1}^k \tau_i^{d_i} h_i$ for some $g, h_i \in M$. $\tau^{d-1}(f - \tau g) = \Sigma_{i=1}^k \tau_i^{d_i} h_i$. Since τ^{d-1} is not a zero-divisor for $M/\Sigma_{i=1}^k \tau_l^{d_i} M$ according to Lemma 4.3, $f - \tau g = \Sigma_{i=1}^k \tau_i^{d_i} h'_i$ for some $h'_i \in M$. Hence $f \in \tau M + \Sigma_{i=1}^k \tau_i^{d_i} M$. q. e. d.

D. Suppose $\varrho^0 \in \mathbf{R}_+^n$ and $\pi : X \longrightarrow K(\varrho^0)$ is a q-concave holomorphic map with exhaustion function φ and concavity bounds $c_*, c_\#$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on X such that codh $\mathcal{F} \geqq r$ on $\}\varphi < c_\#\{$ and $t_1 - t_1^0, \dots, t_n - t_n^0$ form an $\mathcal{F}_x$-sequence for $t^0 \in K(\varrho^0)$ and $x \in \pi^{-1}(t^0) \cap \}\varphi < c_\#\{$.

For $c \in [c_*, \infty)$ let $X_c = \}\varphi > c\{$ and $\pi^c = \pi \mid X_c$. For $c \in [c_* \infty)$ and $l \in \mathrm{N}$ we denote by $\pi_l^c(\mathcal{F})$ the l^{th} direct image of $\mathcal{F} \mid X_c$ under π^c.

We introduce the following statement and shall prove that it implies the Main Theorem.

$$(4.1)_n \quad \begin{cases} \text{If } c \in (c_*, c_\#) \text{ and } 0 \leqq l < r - q - 2n, \text{ then} \\ \quad \text{(i) } \pi_l^c(\mathcal{F}) \text{ is coherent, and} \\ \quad \text{(ii) for } t^0 \in K(\varrho^0) \text{ there exists } c' \in (c_*, c) \text{ depending on } c \\ \text{and } t^0 \text{ such that } \pi_l^{c'}(\mathcal{F})_{t^0} \longrightarrow \pi_l^c(\mathcal{F})_{t^0} \text{ is surjective.} \end{cases}$$

Note that, when $X = \}\varphi \geqq c_\#\{$, $(4.1)_n$ implies Grauert's direct image theorem.

PROPOSITION 4.2. $(4.1)_n \Longrightarrow$ Main Theorem.

PROOF.

I. We are going to prove $(4.2)_k$ for $k \geqq 0$ by induction on k

$$(4.2)_k \quad \begin{cases} \text{If } d_{n-k+1}, \dots, d_n \in \mathrm{N},\ t^0 \in K(\varrho^0),\ 0 \leqq l < r - q - 2n + k, \text{ and} \\ c \in (c_*, c_\#), \text{ then } \pi_l^c(\mathcal{F}/\Sigma_{i=n-k+1}^n t_i^{d_i}\mathcal{F})_{t^0} \text{ is finitely generated} \\ \text{over } {}_n\mathcal{O}_{t^0}. \end{cases}$$

Note that, for every fixed k, $(4.2)_k$ makes sense only when $k \leqq n$. We shall restrict onrselves to this situation.

Fix $t^0 \in K(\varrho^0)$. When the i^{th} coordinate of t^0 is non-zero for some $n - k + 1 \leqq i \leqq n$, $\mathcal{F}/\Sigma_{i=n-k+1}^n t_i^{d_i}\mathcal{F} = 0$ on $\pi^{-1}(U)$ for some open neighborhood U of t^0 in $K(\varrho^0)$. Hence we can assume without loss of generality that $t^0 = 0$. Fix $c \in (c_*, c_\#)$.

When $k = 0$, $\mathcal{F}/\Sigma_{i=n-k+1}^n t_i^{d_i}\mathcal{F} = \mathcal{F}$ and $(4.2)_k$ follows from $(4.1)_n$.

For the general case, assume $0 < k \leqq n$ and further assume that $(4.2)_{k-1}$ is true (for all $n \geqq k - 1$). We are going to prove $(4.2)_k$ by induction on d_n. Let $\mathcal{G} = \mathcal{F}/t_n\mathcal{F}$.

When $d_n = 1$, $\mathcal{F}/\Sigma_{i=n-k+1}^n t_i^{d_i}\mathcal{F} \approx \mathcal{G}/\Sigma_{i=n-k+1}^{n-1} t_i^{d_i}\mathcal{G}$. codh $\mathcal{G} \geqq r - 1$ on $\{\varphi < c_\#\}$.

When X is replaced by $X \cap \{t_n = 0\}$ and $\mathcal{F}$ is replaced by $\mathcal{G} \mid X \cap \{t_n = 0\}$, $(4.2)_{k-1}$ implies that $\pi_l^c(\mathcal{G}/\Sigma_{i=n-k+1}^{n-1} t_i^{d_i}\mathcal{G})_0$ is finitely generated over ${}_n\mathcal{O}_0$ for $0 \leqq l < r - q - 2n + k$. $(4.2)_{k-1}$ is therefore proved for $d_n = 1$.

Suppose $d_n > 1$. For $\nu \in \mathrm{N}$ let $\mathcal{R}^{(\nu)} = \Sigma_{i=n-k+1}^{n-1} t_i^{d_i}\mathcal{F} + t_n^{\nu}\mathcal{F}$. The exact sequence $0 \longrightarrow \mathcal{R}^{(d_n-1)}/\mathcal{R}^{(d_n)} \longrightarrow \mathcal{F}/\mathcal{R}^{(d_n)} \longrightarrow \mathcal{F}/\mathcal{R}^{(d_n-1)} \longrightarrow 0$ gives rise to the exact sequence $\pi_l^c(\mathcal{R}^{(d_n-1)}/\mathcal{R}^{(d_n)})_0 \longrightarrow \pi_l^c(\mathcal{F}/\mathcal{R}^{(d_n-1)})_0$ for $l \in \mathrm{N}_*$. To complete the induction on d_n, we need only show that $\pi_l^c(\mathcal{R}^{(d_n-1)}/\mathcal{R}^{(d_n)})_0$ is finitely generated over ${}_n\mathcal{O}_0$ for $0 \leqq l < r - q - 2n + k$.

By Lemma 4.3 we a sheaf-homomorphism $\alpha: \mathcal{G}/\Sigma_{i=n-k+1}^{n-1} t_i^{d_i}\mathcal{G} \to \to \mathcal{R}^{(d_n-1)}/\mathcal{R}^{(d_n)}$ and α is a sheaf-isomorphism on $\{\varphi < c_\#\}$. $\operatorname{Supp}\operatorname{Ker}\alpha \subset \subset \{\varphi \geqq c_\#\}$ and $\operatorname{Supp}\operatorname{Coker}\alpha \subset \{\varphi \geqq c_\#\}$. The restriction of π to $\operatorname{Supp}\operatorname{Ker}\alpha$ and $\operatorname{Supp}\operatorname{Coker}\alpha$ are proper. Since $(4.1)_n$ implies Grauert's direct image theorem, $\pi_l^c(\operatorname{Ker}\alpha)_0$ and $\pi_l^c(\operatorname{Coker}\alpha)_0$ are finitely generated over ${}_n\mathcal{O}_0$ for $l \in \mathbf{N}_*$.

Since $\pi_l^c(\mathcal{G}/\Sigma_{i=n-k+1}^{n-1} t_i^{d_i}\mathcal{G})_0$ is finitely generated over ${}_n\mathcal{O}_0$ for $0 \leqq l < < r - q - 2n + k$, from the following two exact sequences

$$\pi_l^c(\mathcal{G}/\Sigma_{i=n-k+1}^{n-1} t_i^{d_i}\mathcal{G})_0 \to \pi_l^c(\operatorname{Im}\alpha)_0 \to \pi_{l+1}^c(\operatorname{Ker}\alpha)_0$$

$$\pi_l^c(\operatorname{Im}\alpha)_0 \to \pi_l^c(\mathcal{R}^{(d_n-1)}/\mathcal{R}^{(d_n)})_0 \to \pi_l^c(\operatorname{Coker}\alpha)_0$$

we conclude that $\pi_l^c(\mathcal{R}^{(d_n-1)}/\mathcal{R}^{(d_n)})_0$ is finitely generated over ${}_n\mathcal{O}_0$ for $0 \leqq \leqq l < r - q - 2n + k$. The induction on d_n is complete and $(4.2)_k$ is proved.

II. We are going to prove $(4.3)_k$ for $1 \leqq k \leqq n + 1$ by induction on k.

$$(4.3)_k \quad \begin{cases} \text{If } d_k, \dots, d_n \in \mathbf{N},\ c_* < c' < c < c_\#, \text{ and } 0 \leqq l < r - q - 2n + k, \text{ then} \\ \qquad \pi_l^{c'}(\mathcal{F}/\Sigma_{i=k}^n t_i^{d_i}\mathcal{F})_{t^0} \to \pi_l^c(\mathcal{F}/\Sigma_{i=k}^n t_i^{d_i}\mathcal{F})_{t^0} \\ \qquad \text{is injective for } t^0 \in K(\varrho^0). \end{cases}$$

Fix $t^0 \in K(\varrho^0)$, $d_k, \dots, d_n \in \mathbf{N}$, $c_* < c' < c < c_\#$, and $0 \leqq l < r - q - - 2n + k$. Without loss of generality we can assume that $t^0 = 0$.

When $k = 1$, by Lemma 4.3, $t_1^{d_1}, \dots, t_n^{d_n}$ form an $\mathcal{F}_x$-sequence for $x \in \in \pi^{-1}(0) \cap \{\varphi < c_\#\}$. On $\{\varphi < c_\#\}$ we have $\operatorname{codh}(\mathcal{F}/\Sigma_{i=1}^n t_i^{d_i}\mathcal{F}) \geqq r - n$. Since $\operatorname{Supp}(\mathcal{F}/\Sigma_{i=1}^n t_i^{d_i}\mathcal{F}) \subset \pi^{-1}(0)$, $(4.3)_1$ follows from applying Lemma 4.2 to the coherent analytic sheaf $\mathcal{F}/\Sigma_{i=1}^n t_i^{d_i}\mathcal{F}$ on $\pi^{-1}(0)$.

For the general case, assume $1 < k \leqq n + 1$. $(4.3)_k$ follows from $(4.3)_{k-1}$, $(4.2)_{n-k+1}$, and Proposition 4.1. The induction on k is complete and $(4.3)_k$ is proved.

III. We are going to prove the following:

$$(4.4) \quad \begin{cases} \text{If } \varrho^1 < \varrho^0 \text{ in } \mathbf{R}_+^n,\ 0 \leqq l < r - q - 2n, \text{ and } c_* < c < c_\#, \\ \text{then there exists } c_* < c' < c \text{ such that } \pi_l^{c'}(\mathcal{F}) \to \pi_l^c(\mathcal{F}) \\ \text{is bijective on } K(\varrho^1). \end{cases}$$

Fix $\varrho^1 < \varrho^0$ in $\mathbf{R}_+^n$, $c_* < c < c_\#$, and $0 \leqq l < r - q - 2n$. Take arbitrarily $t \in K(\varrho^1)^-$. $(4.1)_n$ implies that there exists $c_* < c'(t) < c$ such that $\pi_l^{c'(t)}(\mathcal{F})_t \to \pi_l^c(\mathcal{F})_t$ is surjective and hence is also bijective because of $(4.3)_{n+1}$.

Since $\pi_l^{c'(t)}(\mathcal{F})$ and $\pi_l^c(\mathcal{F})$ are both coherent on $K(\varrho^0)$, for some open neighborhood $D(t)$ of t in $K(\varrho^0)$, we have $\pi_l^{c'(t)}(\mathcal{F}) \xrightarrow{\approx} \pi_l^c(\mathcal{F})$ on $D(t)$. $(4.3)_{n+1}$ implies that $\pi_l^{c''}(\mathcal{F}) \xrightarrow{\approx} \pi_l^c(\mathcal{F})$ on $D(t)$ for $c'(t) \leqq c'' \leqq c$.

$K(\varrho^1)^- \subset \bigcup_{i=1}^k D(t^i)$ for some $t^1, \dots, t^k \in K(\varrho^0)$. Let $c' = \max\{c'(t^1), \dots, c'(t^k)\}$. Then $c' < c$ and $\pi_l^{c'}(\mathcal{F}) \xrightarrow{\approx} \pi_l^c(\mathcal{F})$ on $K(\varrho^1)$.

IV. We are going to prove the following:

$$(4.5)\quad \begin{cases} \text{If } 0 \leqq l < r - q - 2n \text{ and } c_* < c < c_\#, \text{ then } \pi_l(\mathcal{F}) \to \pi_l^c(\mathcal{F}) \\ \text{is bijective on } K(\varrho^0). \end{cases}$$

Fix $c_* < c < c_\#$. Take arbitrarily $\varrho^1 < \varrho^0$ in $\mathbf{R}_+^n$. Let Γ be the set of all $c' \in [c_*, c]$ such that $\pi_l^{c'}(\mathcal{F}) \to \pi_l^c(\mathcal{F})$ is bijective on $K(\varrho^1)$ for $0 \leqq l < r - q - 2n$. To prove (4.5), we need only show that $c_* \in \Gamma$.

$\Gamma \neq \Phi$, because $c \in \Gamma$. Let $\tilde{c} = \inf \Gamma$. First, we show that $\tilde{c} \in \Gamma$. Let $\{c_\nu\}_{\nu \in \mathbf{N}_*}$ be a sequence in Γ stringly decreasing to $\tilde{c}$ and $c_0 = c$.

Take an arbitrary Stein open subset U of $K(\varrho^1)$. For $c \in [c_*, \infty)$ let $U(c) = \pi^{-1}(U) \cap X_c$. Since $\pi_l^{c_{\nu+1}}(\mathcal{F}) \xrightarrow{\approx} \pi_l^{c_\nu}(\mathcal{F})$ on $K(\varrho^1)$ for $0 \leqq l < r - q - 2n$, we have $\Gamma(U, \pi^{c_{\nu+1}}(\mathcal{F})) \xrightarrow{\approx} \Gamma(U, \pi_l^{c_\nu}(\mathcal{F}))$ for $0 \leqq l < r - q - 2n$. By Lemma 3.4, $H^l(U(c_{\nu+1}), \mathcal{F}) \xrightarrow{\approx} H^l(U(c_\nu), \mathcal{F})$ for $0 \leqq l < r - q - 2n$. By Lemma 4.1, $H^l(U(\tilde{c}), \mathcal{F}) \xrightarrow{\approx} H^l(U(c_0), \mathcal{F})$. By letting U run through a neighborhood basis of any point t^0 of $K(\varrho^1)$ and taking direct limits, we have $\pi_l^{\tilde{c}}(\mathcal{F})_{t^0} \xrightarrow{\approx} \pi_l^{c_0}(\mathcal{F})_{t^0}$ for $0 \leqq l < r - q - 2n$.

Hence $\pi_l^{\tilde{c}}(\mathcal{F}) \xrightarrow{\approx} \pi_l^c(\mathcal{F})$ on $K(\varrho^1)$ for $0 \leqq l < r - q - 2n$. $\tilde{c} \in \Gamma$.

(4.4) implies that $\tilde{c}$ must be c_*, otherwise $\tilde{c} = \inf \Gamma$ is contradicted.
The proposition follows from (4.5). q. e. d.

§ 5. Bounded Sheaf Cocycles ou Domains of Number Spaces.

In this section we consider complex spaces of the form $K(\varrho^0) \times G$, where $\varrho^0 \in \mathbf{R}_+^n$ and G is an open subset of $\mathbf{C}^N$. After introducing necessary notations and defining norms for sheaf cochains, we will introduce Cartan's Theorem B with bounds for these spesial complex spaces and consider the change in norms when a sheaf section is divided by powers of t_n.

If $\mathfrak{U} = \{U_i\}_{i \in I}$ is a finite collection of open subsets of $\mathbf{C}^N$, we denote $\{K(\varrho) \times U_i\}_{i \in I}$ by $K(\varrho) \times \mathfrak{U}$.

If $f \in \Gamma(K(\varrho), {}_n\mathcal{O})$, then $|f|_\varrho$ denotes $\sup \{ |f_{\nu_1 \dots \nu_n}| \mid \nu_1, \dots, \nu_n \in \mathbf{N}_*\}$, where $f = \Sigma f_{\nu_1 \dots \nu_n} \left(\frac{t_1}{\varrho_1}\right)^{\nu_1} \dots \left(\frac{t_n}{\varrho_n}\right)^{\nu_n}$ is the Taylor series expansion.

If $f = (f_1, \dots, f_p) \in \Gamma(G, {}_N\mathcal{O}^p)$ for some open subset G of $\mathbf{C}^N$, then $|f|_G$ denotes $\sup \{ |f_i(x)| \mid x \in G, 1 \leqq i \leqq p\}$.

If $f \in \Gamma(K(\varrho) \times G, {}_{n+N}\mathcal{O}^p)$, then $|f|_{G, \varrho}$ denotes $\sup \{ |f_{\nu_1 \dots \nu_n}|_G \mid \nu_1 \dots, \nu_n \in \mathbf{N}_*\}$, where $f = \Sigma f_{\nu_1 \dots \nu_n} \left(\frac{t_1}{\varrho_1}\right)^{\nu_1} \dots \left(\frac{t_n}{\varrho_n}\right)^{\nu_n}$ is the Taylor series expansion of f in $t_1, \dots, t_n$.

If $g \in C^r(K(\varrho) \times \mathfrak{U}, {}_{n+N}\mathcal{O}^p)$, then $|g|_{\mathfrak{U}, \varrho}$ denotes

$$\sup \{ |g_{i_0 \dots i_r}|_{U_{i_0 \dots i_r}, \varrho} \mid i_0, \dots, i_r \in I\}.$$

Suppose $\mathcal{F}$ is a coherent analytic sheaf on $K(\varrho^0) \times \widetilde{G}$, where $\varrho^0 \in \mathbf{R}_+^n$ and $\widetilde{G}$ is an open subset of $\mathbf{C}^N$. Suppose H is a relatively compact open subset of a Stein open subset G' of $\widetilde{G}$ and $\varrho < \varrho^0$ in $\mathbf{R}_+^n$. We are going to define a norm $|f|_{H, \varrho}$ for $f \in \Gamma(K(\varrho) \times H, \mathcal{F})$.

By shrinking ϱ^0 and G', we can assume that we have a sheaf-epimorphism $\varphi : {}_{n+N}\mathcal{O}^p \to \mathcal{F}$ on $K(\varrho_0) \times G'$. For $f \in \Gamma(K(\varrho) \times H, \mathcal{F})$, define

$$|f|_{H, \varrho}^{\varphi} = \inf \{ |g|_{H, \varrho} \mid g \in \Gamma(K(\varrho) \times H, {}_{n+N}\mathcal{O}^p), \varphi(g) = f \}.$$

Suppose $\psi ; {}_{n+N}\mathcal{O}^p \to \mathcal{F}$ is another sheaf-epimorphism on $K(\varrho^0) \times G'$. Then we have a sheaf-homomorphism $\sigma : {}_{n+N}\mathcal{O}^p \to {}_{n+N}\mathcal{O}^q$ on $K(\varrho^0) \times G'$ such that $\psi \sigma = \varphi$. By Lemma 1(*a*) of [7], for $g \in \Gamma(K(\varrho) \times H, {}_{n+N}\mathcal{O}^p)$, $|\sigma(g)|_{H, \varrho} \leqq C |g|_{H, \varrho}$, where C is a constant independent of ϱ when $\varrho \leqq \varrho^1$ for a fixed $\varrho^1 < \varrho^0$. Hence, for $f \in \Gamma(K(\varrho) \times H, \mathcal{F})$, $|f|_{H, \varrho}^{\psi} \leqq C |f|_{H, \varrho}^{\varphi}$. Likewise, $|f|_{H, \varrho}^{\varphi} \leqq C' |f|_{H, \varrho}^{\psi}$ for some constant C' independent of ϱ when $\varrho \leqq \varrho^1$. The two norms $|\cdot|_{H, \varrho}^{\varphi}$ and $|\cdot|_{H, \varrho}^{\psi}$ are therefore equivalent. For the sake of notational simplicity, in what follows, whenever such a norm arises, we assume that we arbitrarily fix a sheaf-epimorphism and denote the norm by $|\cdot|_{H, \varrho}$. Whenever possible, we choose always the obviously most convenient sheaf-epimorphism.

When $\mathcal{F} = {}_{n+N}\mathcal{O}^p$, we can choose φ to be the identity sheaf-homomorphism and the norm agrees with the one defined earlier. Whrn $\mathcal{F} \subset {}_{n+N}\mathcal{O}^q$, we have two norms for $\Gamma(K(\varrho) \times H, \mathcal{F})$: one is $|\cdot|_{H, \varrho}^{\varphi}$ defined by a sheaf-epimorphism $\varphi : {}_{n+N}\mathcal{O}^p \to \mathcal{F}$ and another is induced from the norm on

$\Gamma(K(\varrho)\times H, {}_{n+N}\mathcal{O}^p)$. To distinguish the two, we denote the second one by $|\cdot|'_{H,\varrho}$.

Suppose $\mathfrak{U}=\{U_i\}_{i\in I}$ is a finite collection of open subsets of $\mathbf{C}^N$ and every U_i is a relatively compact subset of a Stein open subset $\widetilde{U}_i$ of $\widetilde{G}$. If $h\in C^r(K(\varrho)\times\mathfrak{U}, \mathcal{F})$, then we define $|h|_{\mathfrak{U},\varrho}=\sup_{i_0,\dots,i_r}|h_{i_0\dots i_r}|_{U_{i_0\dots i_r},\varrho}$. When $\mathcal{F}={}_{n+N}\mathcal{O}^p$ the norm $|\cdot|_{\mathfrak{U},\varrho}$ can be chosen to agree with the one defined earlier. If $\mathcal{F}\subset{}_{n+N}\mathcal{O}^q$, then we define

$$|h|'_{\mathfrak{U},\varrho}=\sup_{i_0,\dots,i_r}|h_{i_0\dots i_r}|'_{U_{i_0\dots i_r},\varrho}.$$

In [7], $|\cdot|_{\mathfrak{U},\varrho}$ is denoted by $\|\cdot\|^{*}_{\mathfrak{U},\varrho}$ and $\|\cdot\|'_{\mathfrak{U},\varrho}$ is denoted by $\|\cdot\|_{\mathfrak{U},\varrho}$.

DEFINITION. An n-tuple $\omega=(\omega_1\dots,\omega_n)$ is called an *echelon function of order n if*

(i) $\omega_1\in\mathbf{R}_+$, and

(ii) for $1<i\leqq n$, ω_i is a map from

$$\{(\varrho_1,\dots,\varrho_{i-1})\in\mathbf{R}_+^{i-1}\mid \varrho_1<\omega_1, \varrho_2<\omega_2(\varrho_1),\dots,\varrho_{i-1}<\omega_{i-1}(\varrho_1,\dots,\varrho_{i-2})\}$$

to $\mathbf{R}_+$.

$\Omega^{(n)}$ denotes the set of all echelon functions of order n. If $\omega\in\Omega^{(n)}$, then $\omega_1,\dots,\omega_n$ denote the components of ω.

Suppose $\omega\in\Omega^{(n)}$ and $\varrho\in\mathbf{R}_+^n$. We say that $\varrho<\omega$ if $\varrho_1<\omega_1$ and $\varrho_i<\omega_i(\varrho_1,\dots,\varrho_{i-1})$ for $1<i\leqq n$. Note that, $\varrho'<\varrho$ and $\varrho<\omega$ do *not* imply $\varrho'<\omega$. We say that $\omega<\varrho$ if $\omega_1<\varrho_1$ and for $1<i\leqq n$ we have

$$\sup\{\omega_i(\varrho'_1,\dots,\varrho'_{i-1})\mid(\varrho'_1,\dots,\varrho'_{i-1})\in\mathbf{R}_+^{i-1}, (\varrho'_1,\dots,\varrho'_{i-1})<(\omega_1,\dots,\omega_{i-1})\}<\varrho_i.$$

We order $\Omega^{(n)}$ by the following ordering. For $\omega,\omega'\in\Omega^{(n)}$, $\omega\leqq\omega'$ if anp only if

(i) $\omega_1\leqq\omega'_1$ and

(ii) $\omega_i(\varrho_1,\dots,\varrho_{i-1})\leqq\omega'_i(\varrho_1,\dots\varrho_{i-1})$ when

$$1<i\leqq n, (\varrho_1,\dots,\varrho_{i-1})\in\mathbf{R}_+^{i-1}, \text{ and } (\varrho_1,\dots,\varrho_{i-1})<(\omega_1,\dots,\omega_{i-1}).$$

We identify every $\varrho\in\mathbf{R}_+^n$ with an element ω of $\Omega^{(n)}$ as follows: (i) $\omega_1=\varrho_1$, and (ii) $\omega_i(\varrho'_1,\dots,\varrho'_{i-1})\equiv\varrho_i$ when $1<i\leqq n$, $(\varrho'_i,\dots,\varrho'_{i-1})\in\mathbf{R}_+^{i-1}$, and $(\varrho'_1,\dots,\varrho'_{i-1})<(\varrho_1,\dots,\varrho_{i-1})$.

After the identification of $\mathbf{R}_+^n$ as a subset of $\Omega^{(n)}$, in $\mathbf{R}_+^n$ the relation « $<$ » has three meanings and the relation « $\leqq$ » has two meanings. However the three meanings of « $<$ » are identical and the two meanings of « $\leqq$ » are also identical. Note that, for two general $\omega, \omega' \in \Omega^{(n)}$, we do *not* define $\omega < \omega'$.

A. The following three Propositions are proved as Propositions 2, 4, and 5 of [7]. Even though the last statement in each Proposition is not explicitly stated there, it is easily seen to be a consequence of the proofs in [7].

Suppose $\varrho^0 \in \mathbf{R}_+^n$, $\widetilde{G}$ is a Stein open subset of $\mathbf{C}^N$, and $\mathcal{F}$ is a coherent analytic sheaf on $K(\varrho^0) \times \widetilde{G}$. Suppose $\varphi : {}_{n+N}\mathcal{O}^p \longrightarrow {}_{n+N}\mathcal{O}^q$ is a sheaf-homomorphism $K(\varrho^0) \times \widetilde{G}$.

PROPOSITION 5.1. Suppose $H_1 \subset\subset H_2$ are open subset of $\widetilde{G}$ and H_2 is Stein. Then there exists $\omega \in \Omega^{(n)}$ satisfyng the following. If $\varrho < \omega$ and $f \in \Gamma(K(\varrho) \times H_2$, Im $\varphi)$ with $|f|'_{H_2, \varrho} = e < \infty$, then for some $g \in \Gamma(K(\varrho) \times \times H_1, {}_{n+N}\mathcal{O}^p)$, $\varphi(g) = f$ on $K(\varrho) \times H_1$ and $|g|_{H_1, \varrho} \leqq C_\varrho$ e, where C_ϱ is a constant depending only on ϱ. Moreover if t_n is not a zero-divisor for $(\text{Coker } \varphi)_x$ for $x \in K(\varrho^0) \times \widetilde{G}$, then C_ϱ can be chosen to be independent of ϱ_n.

PROPOSITION 5.2. Suppose $G_1 \subset\subset G_2$ are open supsets of $\widetilde{G}$ such that $G_2 \subset\subset \widetilde{G}$ and G_2 is Stein. Suppose $\mathfrak{U}_i$ is a finite Stein open covering of $G_i (i = 1,2)$ such that $\mathfrak{U}_1 << \mathfrak{U}_2$. Then for $l \geqq 1$ there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $f \in Z^l (K(\varrho) \times \mathfrak{U}_2, \mathcal{F})$ with $|f|_{\mathfrak{U}_2, \varrho} <$ e $< \infty$, then for some $g \in C^{l-1}(K(\varrho) \times \mathfrak{U}_1, \mathcal{F})$, $\delta g = f$ on $K(\varrho) \times \mathfrak{U}_1$ and $|g|_{\mathfrak{U}_1, \varrho} < C_\varrho$ e, where C_ϱ is a constant depending only on ϱ. Moreover, if t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in K(\varrho^0) \times G$, then C_ϱ can be chosen to be independent of ϱ_n.

PROPOSITION 5.3. Suppose $G_1 \subset\subset G_2$ are relatively compact Stein open subsets of $\widetilde{G}$ and $\mathfrak{U}_2$ is a finite Stein open covering of G_2. Then there exists $\omega \in \Omega^{(n)}$ satisfyng the following. If $\varrho < \omega$ and $f \in \Gamma(K(\varrho) \times G_2, \mathcal{F})$ such that $|\widehat{f}|_{\mathfrak{U}, \varrho} < e < \infty$, where $\widehat{f} \in Z^0 (K(\varrho) \times \mathfrak{U}_2, \mathcal{F})$ is induced by f, then $|f|_{G_1, \varrho} < C_\varrho$ is a constant depending only on ϱ. Moreover, if t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in K(\varrho^0) \times G$, then C_ϱ can be chosen to be independent of ϱ_n.

REMARK. It can be proved that, in Propositions 5.3, even if we do not assume that t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in K(\varrho^0) \times \widetilde{G}$, C_ϱ can still be chosen to be independent of ϱ_n. This more general statement is not needed in this paper. However, in Proposition 5.1, for C_ϱ to be independent of ϱ_n it is essential that t_n is not a zero-divisor for $(\mathrm{Coker}\ \varphi)_x$ for $x \in K(\varrho^0) \times \widetilde{G}$. This can easily be seen by taking the special case where $\varphi : {}_{n+N}\mathcal{O} \longrightarrow {}_{n+N}\mathcal{O}$ is defined by multiplication by t_n and $f = \dfrac{t_n}{\varrho_n}$.

B. PROPOSITION 5.4. Suppose $G_1 \subset\subset G_2 \subset\subset \widetilde{G}$ are Stein open subsets of $\mathbf{C}^N$. Suppose $\varrho^0 \in \mathbf{R}^n_+$ and $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a sheaf-homomorphism of coherent analytic sheaves on $K(\varrho^0) \times \widetilde{G}$. Then there exist $\omega \in \Omega^{(n)}$ satisfying the following. If $\varrho < \omega$ and $g \in \Gamma(K(\varrho) \times G_2, \mathrm{Im}\ \varphi)$ with $|g|_{G_2, \varrho} < e < \infty$, then for some $f \in \Gamma(K(\varrho) \times G_1, \mathcal{F})$, $\varphi(f) = g$ and $|f|_{G_1, \varrho} < C_\varrho\, e$, where C_ϱ is a constant depending only on ϱ.

PROOF. By shrinking ϱ^0 and $\widetilde{G}$, we can assume that we have the following commutative diagram of sheaf-homomorphisms with exact rows:

$$\begin{array}{ccccc}
{}_{n+N}\mathcal{O}^p & \xrightarrow{\alpha} & \mathcal{F} & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\varphi} & & \\
{}_{n+N}\mathcal{O}^q & \xrightarrow{\beta} & \mathcal{G} & \longrightarrow & 0
\end{array}$$

such that $\mathrm{Im}\ \psi = \beta^{-1}(\mathrm{Im}\ \varphi)$.

Take $\omega \in \Omega^{(n)}$ and we shall impose conditions on ω later. Take $\varrho < < \omega$ and $g \in \Gamma(K(\varrho) \times G_2, \mathrm{Im}\ \varphi)$ with $|g|_{G_2, \varrho} < e$. Then for some $g' \in \Gamma(K(\varrho) \times G_2, {}_{n+N}\mathcal{O}^q)$, $\beta(g') = g$ and $|g'|_{G_2, \varrho} < e$.
$g' \in \Gamma(K(\varrho) \times G_2, \mathrm{Im}\ \psi)$. By Proposition 5.3, if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then for some $f' \in \Gamma(K(\varrho) \times G_1, {}_{n+N}\mathcal{O}^p)$, $\psi(f') = g'$ and $|f'|_{G_1, \varrho} < C_\varrho\, e$, where C_ϱ is a constant depending only on ϱ.

Let $f = \alpha(f')$. Then $f \in \Gamma(K(\varrho) \times G_1, \mathcal{F})$. $\varphi(f) = g$ and $|f|_{G_1, \varrho} < C_\varrho\, e$.

q. e. d.

C. PROPOSITION 5.5. Suppose G is an open neighborhood of 0 in $\mathbf{C}^N$ and $\varrho^0 \in \mathbf{R}^n_+$. Suppose $\mathcal{M}$ is a coherent analytic subsheaf of ${}_{n+N}\mathcal{O}^q$ on $K(\varrho^0) \times G$ such that t_n is not a zero divisor for $({}_{n+N}\mathcal{O}/\mathcal{M})_x$ for $x \in K(\varrho^0) \times G$. Then there exist a Stein open neighborhood Δ of 0 in G and $\omega \in \Omega^{(n)}$ such that, if $f \in \Gamma(K(\varrho) \times \Delta, \mathcal{M} + t_n^l {}_{n+N}\mathcal{O}^q)$ for some $l \in \mathbf{N}_*$ and $|f|'_{\Delta,\varrho} = e < \infty$, then for some $g \in \Gamma(K(\varrho) \times \Delta, \mathcal{M})$ and $h \in \Gamma(K(\varrho) \times \Delta, {}_{n+N}\mathcal{O}^q)$, $f = g + \left(\frac{t_n}{\varrho_n}\right)^l h$ and $|g|'_{\Delta,\varrho} \leqq 3e$ and $|h|'_{\Delta,\varrho} \leqq 2e$.

PROOF. By shrinking ϱ^0 and G we can assume that $\mathcal{M}$ is the image of a sheaf-homomorphism $\varphi: {}_{n+N}\mathcal{O}^p \to {}_{n+N}\mathcal{O}^q$ on $K(\varrho^0) \times G$. We can also assume that φ is defined on some open neighborhood of $(K(\varrho^0) \times G)^-$.

Let φ be represented by the $q \times p$ matrix φ^* and let $\varphi^* = \Sigma_i \varphi_i^* t_n^i$ be the Taylor series expansion. Let $\psi: {}_{n-1+N}\mathcal{O}^p \to {}_{n-1+N}\mathcal{O}^q$ be the sheaf-homomorphism defined by φ_0^*.
By Proposition 1 of [7] we can find $\bar{\omega} \in \Omega^{(n-1)}$ and a Stein open neighborhood Δ of 0 in G such that, if $\bar{\varrho} \in \mathbf{R}_+^{n-1}$ and $\bar{\varrho} < \bar{\omega}$ and $f' \in \Gamma(K^{n-1}(\bar{\varrho}) \times \Delta, {}_{n-1+N}\mathcal{O}^q)$ with $f'_x \in (\operatorname{Im} \psi)_x$ for $x \in 0 \times \Delta$ and $|f'|'_{\Delta,\bar{\varrho}} = e < \infty$, then for some $g' \in \Gamma(K^{n-1}(\bar{\varrho}) \times \Delta, {}_{n-1+N}\mathcal{O}^p)$, $\psi(g') = f'$ and $|g'|'_{\Delta,\bar{\varrho}} \leqq C_{\bar{\varrho}}^{(1)} e$, where $C_{\bar{\varrho}}^{(1)}$ is a constant depending only on $\bar{\varrho}$.

Take $\varrho^1 < \varrho^0$ in $\mathbf{R}_+^n$. By Lemma 1 (*b*) of [7] there exists $C^{(2)} \in \mathbf{R}_+$ satisfying the following. If $\varrho \leqq \varrho^1$, $l \in \mathbf{N}$, $f' \in \Gamma(K(\varrho) \times \Delta, {}_{n+N}\mathcal{O}^q)$, $g' \in \Gamma(K(\varrho) \times \Delta, {}_{u+N}\mathcal{O}^p)$, and $h' \in \Gamma(K(\varrho) \times \Delta, {}_{n+N}\mathcal{O}^q)$ such that $f' - \varphi(g') = \left(\frac{t_n}{\varrho_n}\right)^l h'$ and the Taylor series expansion of g' in t_n has no power higher than t_n^{l-1}, then $|h'|'_{\Delta,\varrho} \leqq |f'|'_{\Delta,\varrho} + \varrho_n C^{(2)} |g'|'_{\Delta,\varrho}$.

Choose $\omega \in \Omega^{(n)}$ such that the following conditions hold: (i) $\omega \leqq \varrho^1$, (ii) $\omega_i(\varrho_1, \dots, \varrho_{i-1}) = \bar{\omega}_i(\varrho_1, \dots, \varrho_{i-1})$ for $1 \leqq i < n$, and (iii) $\omega_n(\bar{\varrho}) \leqq (1 + 2\, C^{(2)} C_{\bar{\varrho}}^{(1)})^{-1}$ for $\bar{\varrho} \in \mathbf{R}_+^{n-1}$. We claim that Δ and ω so obtained satisfy the requirement. To verify the claim, we are going to prove $(5.1)_l$ for $l \in \mathbf{N}_*$ by induction on l.

$$(5.1)^l \quad \left\{ \begin{array}{l} \text{If } \varrho < \omega \text{ and } f \in \Gamma(K(\varrho) \times \Delta, \mathcal{M} + t_n^l {}_{n+N}\mathcal{O}^q) \text{ such that } |f|'_{\Delta,\varrho} = \\ = e < \infty, \text{ then there exist } g^{(l)} \in \Gamma(K(\varrho) \times \Delta, {}_{n+N}\mathcal{O}^p) \text{ and } h^{(l)} \in \\ \in \Gamma(K(\varrho) \times \Delta, {}_{n+N}\mathcal{O}^q) \text{ such that } f = \varphi(g^{(l)}) + \left(\frac{t_n}{\varrho_n}\right)^l h^{(l)}, \ |g^{(l)}|'_{\Delta,\varrho} \leqq \\ \leqq 2 C_{\bar{\varrho}}^{(1)} e, \ |h^{(l)}|'_{\Delta,\varrho} \leqq 2e, \text{ and the Taylor series expansion of } g_l \text{ in} \\ t_n \text{ has no power higher than } t_n^{l-1}, \text{ where } \bar{\varrho} = (\varrho_1, \dots, \varrho_{n-1}). \end{array} \right.$$

$(5.1)_0$ is trivial, because we can set $g^{(0)} = 0$ and $h^{(0)} = f$. Suppose $l > 0$ and $(5.1)_{l-1}$ is true.

Take $\varrho < \omega$ and $f \in \Gamma(K(\varrho) \times \Delta, \mathcal{M} + t_n^l {}_{n+N}\mathcal{O}^q)$ with $|f|'_{\Delta,\varrho} = e < \infty$. By $(5.1)_{l-1}$ there exist $g^{(l-1)}$ and $h^{(l-1)}$ satisfying the requirements in $(5.1)_{l-1}$. Let $h^{(l-1)} = \Sigma_i h_i^{(l-1)} \left(\frac{t_n}{\varrho_n}\right)^i$ be the Taylor series expansion.

Since t_n is not a zero-divisor for $({}_{n+N}\mathcal{O}^q/\mathcal{M})_x$ for $x \in K(\varrho^0) \times G$, $(h_0^{(l-1)})_x \in (\operatorname{Im} \psi)_x$ for $x \in 0 \times \Delta$. Hence there exists $a \in \Gamma(K^{n-1}(\bar{\varrho}) \times \Delta, {}_{n-1+N}\mathcal{O}^p)$ such that $\psi(a) = h_0^{(l-1)}$ and $|a|'_{\Delta, \bar{\varrho}} \leqq 2 C_{\bar{\varrho}}^{(1)} e$.

Let $g^{(l)} = g^{(l-1)} + a \left(\frac{t_n}{\varrho_n}\right)^{l-1}$. Then $|g^{(l)}|'_{\Delta, \varrho} \leqq 2 C_{\bar{\varrho}}^{(1)}$ e and the Taylor series expansion of $g^{(l)}$ in t_n has no power higher than t_n^{l-1}. $f - \varphi(g^{(l)}) = \left(\frac{t_n}{\varrho_n}\right)^l h^{(l)}$ for some $h^{(l)} \in \Gamma(K(\varrho) \times \Delta, {}_{n+N}\mathcal{O}^q)$. $|h^{(l)}|'_{\Delta, \varrho} \leqq (1 + 2\varrho_n C^{(2)} C_{\bar{\varrho}}^{(1)}) e \leqq \leqq 2e$. $(5.1)_l$ is proved.

The claim follows, because $|\varphi(g^{(l)})|'_{\Delta, \varrho} = \left| f - \left(\frac{t_n}{\varrho_n}\right)^l h^{(l)} \right|'_{\Delta, \varrho} \leqq |f|'_{\Delta, \varrho} + |h^{(l)}|'_{\Delta, \varrho} \leqq 3e$. q. e. d.

PROPOSITION 5.6. Suppose $G_1 \subset\subset G_2 \subset\subset \widetilde{G}$ are Stein open subsets of $\mathbf{C}^N$ and $\varrho^0 \in \mathbf{R}_+^n$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on $K(\varrho^0) \times \widetilde{G}$ such that t_n is not a zero-divisor for $\mathcal{F}_x$ for $x \in K(\varrho^0) \times \widetilde{G}$. Then there exists $\omega \in \Omega^{(n)}$ satisfying the following. If $f \in \Gamma(K(\varrho) \times G_2, \mathcal{F})$ and $\left| \left(\frac{t_n}{\varrho_n}\right)^l f \right|_{G_2, \varrho} < e < \infty$ for some $l \in \mathbf{N}_*$, then $|f|_{G_1, \varrho} < C_\varrho e$, where C_ϱ is a constant depending only on ϱ and is independent of l.

PROOF. By shrinking ϱ^0 and $\widetilde{G}$ we can assume that we have a sheaf-epimorphism $\varphi : {}_{n+N}\mathcal{O}^q \longrightarrow \mathcal{F}$ on $K(\varrho^0) \times \widetilde{G}$. Let $\mathcal{M} = \operatorname{Ker} \varphi$. Take a Stein open subset G_3 of $\mathbf{C}^N$ such that $G_1 \subset\subset G_3 \subset\subset G_2$.

Take $\omega \in \Omega^{(n)}$ and we shall impose conditions on ω later. Take $\varrho < \omega$ and $f \in \Gamma(K(\varrho) \times G_2, \mathcal{F})$ such that $\left| \left(\frac{t_n}{\varrho_n}\right)^l f \right|_{G_2, \varrho} < e < \infty$ for some $l \in \mathbf{N}_*$.

For some $f' \in \Gamma(K(\varrho) \times G_2, {}_{n+N}\mathcal{O}^q)$, $\varphi(f') = \left(\frac{t_n}{\varrho_n}\right)^l f$ and $|f'|_{G_2, \varrho} < e$. $f' \in \Gamma(K(\varrho) \times G_2, \mathcal{M} + t_n^l\, {}_{n+N}\mathcal{O}^q)$.

By Proposition 5.5, if $\omega \leqq \omega^1$ for a suitable $\omega^1 \in \Omega^{(n)}$ (and we assume this to be the case), then we can find Stein open subsets Δ_i of G_2 and $g_i \in \Gamma(K(\varrho) \times \Delta_i, \mathcal{M})$ and $h_i \in \Gamma(K(\varrho) \times \Delta_i, {}_{n+N}\mathcal{O}^q)$, $1 \leqq i \leqq k$, such that (i) $G_3 \subset \bigcup_{i=1}^k \Delta_i$, (ii) $f' = g_i + \left(\frac{t_n}{\varrho_n}\right)^l h_i$ on $K(\varrho) \times \Delta_i$, (iii) $|g_i|_{\Delta_i, \varrho} < 3e$, and (iv) $|h_i|_{\Delta_i, \varrho} < 2e$. Note that $\omega^{(1)}$ and Δ_i are all independent of l.

Let $\mathfrak{U} = \{\Delta_i \cap G_3\}$. Since $\varphi(h_i) = f$ on $K(\varrho) \times \Delta_i$, we have $|\widehat{f}|_{\mathfrak{U}, \varrho} < 2e$, where $\widehat{f} \in Z^0(K(\varrho) \times \mathfrak{U}, \mathcal{F})$ is induced by f. By Proposition 5.3, if $\omega \leqq \omega^2$ for a suitable $\omega^2 \in \Omega^{(n)}$ (and we assume this to be the case), then $|f|_{G_1, \varrho} < < C'_\varrho 2e$, where C'_ϱ is a constant depending only on ϱ. q. e. d.

§. 6. Semi-norms for Holomorphic Functions on Unreduced Spaces.

For a *reduced* complex space, we know that the set of all holomorphic functions has a natural Fréchet space structure whose semi-norms are the sup norms of holomorphic functions on compact subsets. When the space is not necessarily reduced, the set of holomorphic functions still has a natural Fréchet space structure, but to obtain the semi-norms we have to resort to local embeddings of the space into domains in complex number spaces. In our development later on, these local embeddings give rise to many complications. To avoid such complications, we seek in this section to define semi-norms on some unreduced spaces in a manner which suppresses the role of local embeddings.

A. Suppose V is a subvariety in an open subset G of $\mathbf{C}^N$ and $p \in \mathbf{N}_*$. We denote by $\mathcal{I}_V(p)$ the sheaf of germs of holomorphic functions on G whose derivatives of order $\leqq p$ vanish identically on V.

PROPOSITION 6.1. $\mathcal{I}_V(p)$ is a coherent ideal-sheaf on G.

PROOF. It is clear that $\mathcal{I}_V(p)$ is an ideal-sheaf on G.

To prove the coherence, we use induction on p. The case $p=0$ is the well-known theorem of Cartan-Oka. Suppose the statement is true when p is replaced by $p-1$. By shrinking G, we can assume that $\mathcal{I}_V(p-1) = \sum_{i=1}^k {}_N\mathcal{O} f_i$ on G for some $f_1, \dots, f_k \in \Gamma(G, {}_N\mathcal{O})$.

Let l be the number of $\alpha \in \mathbf{N}_*^N$ with $|\alpha| = p$. Define $\varphi : {}_N\mathcal{O}^k \to {}_N\mathcal{O}^l$ on G as follows. For $x \in G$ and

$$(a_1, \dots, a_k) \in {}_N\mathcal{O}_x^k, \ \varphi(a_1, \dots, a_k) = (\dots, \textstyle\sum_{i=1}^k a_i (D^\alpha f_i)_x, \dots),$$

where α runs through all elements of $\mathbf{N}_*^N$ with $|\alpha| = p$.

Let $\eta : {}_N\mathcal{O}^l \to {}_N\mathcal{O}^l / \mathcal{I}_V(0)_N\mathcal{O}^l$ be the natural sheaf-epimorphism. Let $\tilde{\varphi} = \eta\varphi$. Define ψ: ${}_N\mathcal{O}^k \to {}_N\mathcal{O}$ on G by $\psi(a_1, \dots, a_k) = \sum_{i=1}^k a_i (f_i)_x$ for $x \in G$ and $(a_1, \dots, a_k) \in {}_N\mathcal{O}_x^k$.

We are going to prove that $\mathcal{I}_V(p) = \psi(\operatorname{Ker} \tilde{\varphi})$. This will imply the coherence of $\mathcal{I}_V(p)$.

Since $\mathcal{I}_V(p-1) = \operatorname{Im} \psi$, we observe that both $\mathcal{I}_V(p)$ and $\psi(\operatorname{Ker} \tilde{\varphi})$ are subsheaves of $\mathcal{I}_V(p-1)$. Suppose $g \in \Gamma(U, \mathcal{I}_V(p-1))$ for some Stein open subset U of G. Then $g = \Sigma_i a_i f_i$ on U for some $a_1, \ldots, a_k \in \Gamma(U, {}_N\mathcal{O})$. Fix arbitrarily $|\alpha| = p$.

$$D^\alpha g = \Sigma_i D^\alpha (a_i f_i) = \Sigma_i \Sigma_{\beta \leqq \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} a_i)(D^\beta f_i) =$$

$$= \Sigma_i (a_i D^\alpha f_i + \Sigma_{\beta \leqq \alpha, |\beta| \leqq p-1} \binom{\alpha}{\beta} (D^{\alpha-\beta} a_i)(D^\beta f_i)).$$

Since $D^\beta f_i = 0$ on V for $|\beta| \leqq p-1$, we have $D^\alpha g = \Sigma_i a_i D^\alpha f_i$ on $V \cap U$. If follows that $g \in \Gamma(U, \mathcal{J}_V(p))$ if and only if $(a_1, \ldots, a_k) \in \Gamma(U, \operatorname{Ker} \widetilde{\varphi})$. Hence $\mathcal{J}_V(p) = \psi(\operatorname{Ker} \widetilde{\varphi})$. q. e. d.

Suppose U is an open subset of V and $f \in \Gamma(U, {}_N\mathcal{O}/\mathcal{J}_V(p))$. For $x \in U$, we can find an open neighborhood D of x in G and $\widetilde{f} \in \Gamma(D, {}_N\mathcal{O})$ such that $\widetilde{f}$ induces $f | D \cap V$. For $|\alpha| \leqq p$, define ${}^\alpha f(x) = (D^\alpha \widetilde{f})(x)$. Obviously ${}^\alpha f(x)$ is independent of the choices of D and $\widetilde{f}$. For any open subset E of U, define ${}_p\|f\|_E = \sup\{|{}^\alpha f(x)| \mid |\alpha| \leqq p,\ x \in E\}$.

Note that, for $g \in \Gamma(U, {}_N\mathcal{O}/\mathcal{J}_V(p))$, we do *not* in general have ${}_p\|fg\|_E \leqq ({}_p\|f\|_E)({}_p\|g\|_E)$, but we have the following weaker inequality: ${}_p\|fg\|_E \leqq 2^{pN}({}_p\|f\|_E)({}_p\|g\|_E)$, because ${}^\alpha(fg) = \Sigma_{\beta \leqq \alpha} \binom{\alpha}{\beta} ({}^\beta f)({}^{\alpha-\beta} g)$.

PROPOSITION 6.2. The semi-norms ${}_p\|\cdot\|_L$ on $\Gamma(V, {}_N\mathcal{O}/\mathcal{J}_V(p))$ define a Fréchet space structure when L runs through all relatively compact subsets of V.

PROOF. We use induction on p. The case $p = 0$ is well-known. Suppose the statement is true when p is replaced by $p-1$.

Let $\{f_\nu\}_{\nu \in \mathbb{N}}$ be a Cauchy sequence in $\Gamma(V, {}_N\mathcal{O}/\mathcal{J}_V(p))$. The proof will be complete if we can show that $\{f_\nu\}$ converges to some element in $\Gamma(V, {}_N\mathcal{O}/\mathcal{J}_V(p))$ with respect to the seminorms ${}_p\|\cdot\|_L$. We need only prove that every point of V admits an open neighborhood U in V such that $\{f_\nu | U\}$ converges to some element in $\Gamma(U, {}_N\mathcal{O}/\mathcal{J}_V(p))$ with respect to the seminorms ${}_p\|\cdot\|_Q$, where Q runs through all relatively compact subset of U. Hence we can assume without loss of generality that G is Stein and $\mathcal{J}_V(p-1) = \Sigma_{i=1}^k {}_N\mathcal{O}\, s_i$ for some $s_i \in \Gamma(G, {}_N\mathcal{O})$.

The maps in the following commutative diagram are all natural sheaf-homomorphisms :

$$\begin{array}{ccc} {}_N\mathcal{O} & \xrightarrow{\varphi} & {}_N\mathcal{O}/\mathcal{J}_V(p) \\ \| & & \downarrow{\scriptstyle \eta} \\ {}_N\mathcal{O} & \xrightarrow{\psi} & {}_N\mathcal{O}/\mathcal{J}_V(p-1). \end{array}$$

By induction hypothesis, $\{\eta(f_\nu)\}$ converges to some element $f' \in \Gamma(V, {}_N\mathcal{O}/\mathcal{I}_V(p-1))$ with respect to the semi-norms ${}_{p-1}\|\cdot\|_L$.

Since G is Stein, ψ induces a surjection $\widetilde{\psi}:\Gamma(G, {}_N\mathcal{O}) \to \Gamma(V, {}_N\mathcal{O}/\mathcal{I}_V(p-1))$. $\widetilde{\psi}$ is obviously continuous when $\Gamma(G, {}_N\mathcal{O})$ is given the natural Fréchet space structure and when $\Gamma(V, {}_N\mathcal{O}/\mathcal{I}_V(p-1))$ is given the Fréchet space structure defined by the semi-norms ${}_{p-1}\|\cdot\|_L$. By the open mapping theorem for Fréchet spaces, we can find g_ν, $g \in \Gamma(G, {}_N\mathcal{O})$ such that $\psi(g_\nu) = \eta(f_\nu)$, $\psi(g) = f'$, and $\{g_\nu\}$ converges to g.

If we can prove that $\{f_\nu - \varphi(g_\nu)\}$ converges to some element $f \in \Gamma(V, {}_N\mathcal{O}/\mathcal{I}_V(p))$ with respect to ${}_p\|\cdot\|_L$, then $\{f_\nu\}$ converges to $f + \varphi(g)$. Since $\eta(f_\nu - \varphi(g_\nu)) = \eta(f_\nu) - \psi(g_\nu) = 0$, by replacing $\{f_\nu\}$ by $\{f_\nu - \varphi(g_\nu)\}$, we can assume without loss of generality that $\eta(f_\nu) = 0$.

$f_\nu = \varphi(f_\nu^*)$ for some $f_\nu^* \in \Gamma(G, {}_N\mathcal{O})$. Since $\eta(f_\nu) = 0$, we have $f_\nu^* \in \Gamma(G, \mathcal{I}_V(p-1))$. $f_\nu^* = \Sigma_i a_{\nu i} s_i$ for some $a_{\nu i} \in \Gamma(G, {}_N\mathcal{O})$. For $|\alpha| = p$, $D^\alpha f_\nu^* = \Sigma_i a_{\nu i} D^\alpha s_i$ on V, because $D^\beta s_i = 0$ on V for $|\beta| \leqq p-1$.

Let l be the number of $\alpha \in \mathbf{N}_*^N$ with $|\alpha| = p$. For $1 \leqq i \leqq k$, let $u_i = (\ldots, (D^\alpha s_i \mid V), \ldots)$, where α runs through all elements of $\mathbf{N}_*^N$ with $|\alpha| = p$. $u_i \in \Gamma(V, ({}_N\mathcal{O}/\mathcal{I}_V(0))^l)$. Let $\mathcal{M}$ be the subsheaf of $({}_N\mathcal{O}/\mathcal{I}_V(0))^l$ generated by $\{u_i\}_{1 \leq i \leq k}$.

Let $h_\nu = (\ldots, (D^\alpha f_\nu^* \mid V), \ldots)$, where α runs through all elements of $\mathbf{N}_*^N$ with $|\alpha| = p$. Since $D^\alpha f_\nu^* = \Sigma_i a_{\nu i} D^\alpha s_i$ on V for $|\alpha| = p$, $h_\nu \in \Gamma(V, \mathcal{M})$.

$\{h_\nu\}$ is a Cauchy sequence in $\Gamma(V, \mathcal{M})$ with respect to the topology of uniform convergence on compact subsets. Since $\mathcal{M}$ is defined on the reduced space $(V, {}_N\mathcal{O}/\mathcal{I}_V(0))$, $\{h_\nu\}$ converges to some $h \in \Gamma(V, \mathcal{M})$ with respect to the topology of uniform convergence on compact subsets.

For some $b_i \in \Gamma(V, {}_N\mathcal{O}/\mathcal{I}_V(0))$, $h = \Sigma_i b_i u_i$. Since G is Stein, b_i is induced by some $a_i \in \Gamma(G, {}_N\mathcal{O})$. Let $f^* = \Sigma_i a_i s_i$. Then, for $|\alpha| = p$, $D^\alpha f^* = \Sigma_i b_i D^\alpha s_i$ on V. Let $f = \varphi(f^*)$.

$\{f_\nu\}$ converges to f in $\Gamma(V, {}_N\mathcal{O}/\mathcal{I}_V(p))$ with respect to the semi norms ${}_p\|\cdot\|_K$. The reason is the following. For $|\beta| \leqq p-1$, $D^\beta f_\nu^* = D^\beta f^* = 0$ on V. For $|\alpha| = p$, $D^\alpha f_\nu^* \mid V$ and $D^\alpha f^* \mid V$ are respectively the components of h_ν and h. q. e. d.

Now we have a Fréchet space topology T on $\Gamma(V, {}_N\mathcal{O}/\mathcal{I}_V(p))$ defined by the semi-norms ${}_p\|\cdot\|_L$. The quotient topology induced by $\Gamma(G, {}_N\mathcal{O}) \to \Gamma(V, {}_N\mathcal{O}/\mathcal{I}_V(p))$ is the usual Fréchet space topology T' on $\Gamma(V, {}_N\mathcal{O}/\mathcal{I}_V(p))$. T' is obviously finer than T, because any sequence in $\Gamma(V, {}_N\mathcal{O}/\mathcal{I}_V(p))$ which converges with respect to T' converges with respect to T. By the open mapping theorem for Fréchet spaces, the two topologies T and T' agree.

B. Suppose V is a subvariety of an open subset G of $\mathbf{C}^N$ and V' is a subvariety of an open subset G' of $\mathbf{C}^{N'}$. Suppose $\mathcal{I}$ is an ideal-sheaf on

G whose zero-set is V and $\mathcal{J}'$ is an ideal-sheaf on G' whose zero-set is V'.

Suppose $(\varphi_0, \varphi_1) : (V', {}_{N'}\mathcal{O}/\mathcal{J}') \to (V, {}_N\mathcal{O}/\mathcal{J})$ is a holomorphic map. If $f_1, \dots, f_N \in \Gamma(V, {}_N\mathcal{O}/\mathcal{J})$ are induced by the coordinate-functions of $\mathbf{C}^N$ and $f_i' = \varphi_1(f_i)$, then we call $f_1', \dots, f_N'$ the *coordinates* of (φ_0, φ_1).

LEMMA 6.1. (φ_0, φ_1) is uniquely determined by its coordinates $f_1', \dots, f_N'$.

PROOF. Let ${}^0f_i'$ be the holomorphic function on the reduced space V which is induced by f_i'. (This notation is consistent with our earlier notations when $\mathcal{J}' = \mathcal{J}_{V'}(p)$). The map $({}^0f_1', \dots, {}^0f_N') : V' \to \mathbf{C}^N$ agrees with φ_0. Hence φ_0 is uniquely determined.

Take $x' \in V'$. Let $x = \varphi_0(x')$. We want to prove that $\varphi_1 : ({}_N\mathcal{O}/\mathcal{J})_x \to \to ({}_{N'}\mathcal{O}/\mathcal{J}')_{x'}$ is uniquely determined. Without loss of generality we can assume that $x = 0$.

Suppose $a \in ({}_N\mathcal{O}/\mathcal{J})_x$. Then a is induced by some $b \in \Gamma(U, {}_N\mathcal{O})$ for some polydisc neighborhood U of x in G. Let $b = \Sigma\, b_\nu z^\nu$ be the Taylor series expansion, where $\nu = (\nu_1, \dots, \nu_n) \in \mathbf{N}_*^N$ and $z^\nu = z_1^{\nu_1} \dots z_N^{\nu_N}$. a is also induced by $\Sigma\, b_\nu f_1^{\nu_1} \dots f_N^{\nu_N} \in \Gamma(U \cap V, {}_N\mathcal{O}/\mathcal{J})$. Since $\varphi_1(\Sigma_{|\nu| \leq m} b_\nu f_1^{\nu_1} \dots f_N^{\nu_N}) = = \Sigma_{|\nu| \leq m} b_\nu (f_1')^{\nu_1} \dots (f_N')^{\nu_N}$ for $m \in \mathbf{N}_*$, by letting $m \to \infty$, we conclude that $\varphi_1(a)$ is the germ of $\Sigma\, b_\nu (f_1')^{\nu_1} \dots (f_N')^{\nu_N}$. Hence φ_1 is uniquely determined. q. e. d.

Suppose $p \in \mathbf{N}_*$. We are going to describe holomorphic maps from $(V', {}_{N'}\mathcal{O}/\mathcal{J}_{V'}(p))$ to $(V, {}_N\mathcal{O}/\mathcal{J}_V(p))$.

The chain rule for differentiation gives us the following.

LEMMA 6.2. Suppose $\alpha \in \mathbf{N}_*^{N'} - \{0\}$. Then there exists a polynomial $P_\alpha(\{X_\beta\}, \{Y_{\gamma i}\}) = \Sigma_\beta\, Q_{\alpha\beta}(\{Y_{\gamma i}\}) X_\beta$ with non-negative integral coefficients, where $1 \leqq i \leqq N$, β runs through all elements of $\mathbf{N}_*^N - \{0\}$ with $|\beta| \leqq |\alpha|$, and γ runs through all elements of $\mathbf{N}_*^{N'} - \{0\}$ with $\gamma \leqq \alpha$, such that, if $g_i(z_1, \dots, z_N)$, $1 \leqq i \leqq N$, is a holomorphic function on an open neighborhood of $a \in \mathbf{C}^{N'}$ with $(g_1(a), \dots, g_N(a)) = b$ and $f(w_1, \dots, w_N)$ is a holomorphic function on an open neighborhood of b, then $(D_z^\alpha h)(a) = P_\alpha(\{(D_w^\beta f)(b)\}, \{(D_z^\gamma g_i)(a)\})$, where $h(z_1, \dots, z_{N'}) = f(g_1(z_1, \dots, z_{N'}), \dots, g_N(z_1, \dots, z_{N'}))$.

PROPOSITION 6.3. Suppose $(\psi_0, \psi_1) : (V', {}_{N'}\mathcal{O}/\mathcal{J}_{V'}(p)) \to (V, {}_N\mathcal{O}/\mathcal{J}_V(p))$ is a holomorphic map with coordinates $f_1', \dots, f_N'$. If $g \in \Gamma(U, {}_N\mathcal{O}/\mathcal{J}_V(p))$ for some open subset U of V, then ${}^\alpha(\psi_1(g)) = P_\alpha(\{({}^\beta g) \circ \psi_0\}, \{{}^\gamma f_i'\})$. Hence, if U is contained in a relatively compact open subset Q of V, then ${}^p\|\psi_1(g)\|_{\varphi_0^{-1}(U)} \leqq C({}_p\|g\|_U)$, where C is a constant depending only on Q and is independent of g and U.

PROOF. Take $x' \in \psi_0^{-1}(U)$. Let $x = \psi_0(x')$. For some open neighborhood D of x in G, g is induced by some $g^* \in \Gamma(D, {}_N\mathcal{O})$. For some open neighborhood H of x' in G', $f_i' \mid H \cap V'$ is induced by some $f_i^* \in \Gamma(H, {}_{N'}\mathcal{O})$ such that $f_i^*(H) \subset D$.

Let $h = g^*(f_1^*, \dots, f_N^*)$. Then h induces $\psi_1(g)$ at x'. Hence

$$^\alpha(\psi_1(g))(x') = (D^\alpha h)(x') = P_\alpha(\{(D^\beta g^*)(x)\}, \{(D^\gamma f_i^*)(x')\}) =$$

$$= P_\alpha(\{(^\beta g)(\psi_0(x'))\}, \{(^\gamma f_i')(x')\}).$$

Since x' is arbitrary, $^\alpha(\psi_1(g)) = P_\alpha(\{(^\beta g) \circ \psi_0\}, \{^\gamma f_i'\})$ on $\psi_0^{-1}(U)$.

Let $B = \sup_i({}_p\|f_i'\|_Q)$. Then ${}_p\|\psi_1(g)\|_{\psi_0^{-1}(U)} \leqq (\sup_{|\alpha| \leqq p} \Sigma_\beta Q_{\alpha\beta}(\{B\}))({}_p\|g\|_U)$.

q. e. d.

PROPOSITION 6.4. Suppose $\psi_0: V' \to V$ is a holomorphic map of reduced spaces. Suppose $f_i' \in \Gamma(V', {}_{N'}\mathcal{O}/\mathcal{J}_{V'}(p))$, $1 \leqq i \leqq N$, such that $(^0f_1', \dots, {}^0f_N'): V' \to \mathbf{C}^N$ agrees with ψ_0. Then there exists a unique holomorphic map $(\psi^0, \psi_1): (V', {}_{N'}\mathcal{O}/\mathcal{J}_{V'}(p)) \to (V, {}_N\mathcal{O}/\mathcal{J}_V(p))$ with $f_1', \dots, f_N'$ as coordinates.

PROOF. Uniqueness follows from Lemma 6.2.

Take $x' \in V'$. Let $x = \psi_0(x')$. We are going to define ψ_1: $({}_N\mathcal{O}/\mathcal{J}_V(p))_x \to$ $\to ({}_{N'}\mathcal{O}/\mathcal{J}_{V'}(p))_{x'}$. Without loss of generality we can assume that $x = 0$.

Take $a \in ({}_N\mathcal{O}/\mathcal{J}_V(p))_0$. a is induced by some $b \in \Gamma(D, {}_N\mathcal{O})$ for some polydisc neighborhood D of 0 in G. Let $b = \Sigma b_\nu z^\nu$ be the Taylor series expansion, wher $\nu = (\nu_1, \dots, \nu_N) \in \mathbf{N}_*^N$ and $z^\nu = z_1^{\nu_1} \dots z_N^{\nu_N}$.

Since $(^0f_1'(x'), \dots, {}^0f_N'(x')) = 0$, for some open neighborhood H of x in $G', f_i' \mid H \cap V'$ is induced by some $f_i^* \in \Gamma(H, {}_N\mathcal{O})$ such that $f_i^*(H) \subset D$. Since $\Sigma b_\nu (f_1^*)^{\nu_1} \dots (f_N^*)^{\nu_N}$ converges on H, $\Sigma b_\nu (f_1')^1 \dots (f_N')^{\nu_N}$ converges on $H \cap V'$. Define $\psi_1(a)$ to be the germ of $\Sigma b_\nu (f_1')^{\nu_1} \dots (f_N')^{\nu_N}$ at x'. To finish the proof we need only prove that $\psi_1(a)$ is independent of the choice of b.

Suppose another $\widetilde{b} \in \Gamma(\widetilde{D}, {}_N\mathcal{O})$ is chosen. We can obviously assume that $D = \widetilde{D}$. $D^\beta \widetilde{b} = D^\beta b$ on $D \cap V$ for $|\beta| \leqq p$. Let $\widetilde{b} = \Sigma \widetilde{b}_\nu z^\nu$ be the Taylor series expansion. $\Sigma b_\nu (f_1')^{\nu_1} \dots (f_N')^{\nu_N}$ and $\Sigma \widetilde{b}(f_1')^{\nu_1} \dots (f_N')^{\nu_N}$ are respectively induced by $h = b(f_1^*, \dots, f_N^*)$ and $\widetilde{h} = \widetilde{b}(f_1^*, \dots, f_N^*)$. For $|\alpha| \leqq p$ and $y' \in H \cap V'$,

$$(D^\alpha \widetilde{h})(y') = P_\alpha(\{(D^\beta \widetilde{b})(y)\}, \{(D^\gamma f_i^*)(y')\}) =$$

$$= P_\alpha(\{(D^\beta b)(y)\}, \{(D^\gamma f_i^*)(y')\}) = (D^\alpha h)(y'),$$

where $y = \psi_0(y') \in D \cap V$. Hence $D^\alpha \tilde{h} = D^\alpha h$ on $A \cap H'$. The germs of $\Sigma b_\nu (f_1')^{\nu_1} \dots (f_N')^{\nu_N}$ and $\Sigma \tilde{b}_\nu (f_1')^{\nu_1} \dots (f_N')^{\nu_N}$ at x' agree. q.e.d.

PROPOSITION 6.5. Suppose V' is Stein, $\mathcal{I}_V(p) \subset \mathcal{I}$, and $\mathcal{I}_{V'}(p) \subset \mathcal{I}'$. Then for every holomorphic map $(\varphi_0, \varphi_1): (V', {}_{N'}\mathcal{O}\mathcal{I}') \to (V, {}_N\mathcal{O}/\mathcal{I})$ there exists a holomorphic map $(\psi_0, \psi_1): (V', {}_{N'}\mathcal{O}/\mathcal{I}_{V'}(p)) \to (V, {}_N\mathcal{O}/\mathcal{I}_V(p))$ such that the following diagram is commutative

$$\begin{array}{ccc} (V', {}_{N'}\mathcal{O}/\mathcal{I}') & \xrightarrow{(\varphi_0, \varphi_1)} & (V, {}_N\mathcal{O}/\mathcal{I}) \\ {\scriptstyle (id_{V'}, \eta')}\big\downarrow & & \big\downarrow{\scriptstyle (id_V, \eta)} \\ (V', {}_{N'}\mathcal{O}/\mathcal{I}_{V'}(p)) & \xrightarrow{(\psi_0, \psi_1)} & (V, {}_N\mathcal{O}/\mathcal{I}_V(p)), \end{array}$$

where $\eta: {}_N\mathcal{O}/\mathcal{I}_V(p) \to {}_N\mathcal{O}/\mathcal{I}$ and: $\eta': {}_{N'}\mathcal{O}/\mathcal{I}_{V'}(p) \to {}_{N'}\mathcal{O}/\mathcal{I}'$ are the natural sheaf-homomorphisms.

PROOF. Set $\psi_0 = \varphi_0$. Suppose $f_1', \dots, f_N' \in \Gamma(V', {}_{N'}\mathcal{O}/\mathcal{I}')$ are the coordinates of (φ_0, φ_1). Since V' is Stein, $f_i' = \eta'(g_i')$ for some $g_i' \in \Gamma(V', {}_{N'}\mathcal{O}/\mathcal{I}_{V'}(p))$. The map $({}^0g_1', \dots, {}^0g_N'): V' \to \mathbf{C}^N$ agrees with ψ_0, because g_i' and f_i' induce the same holomorphic function on the reduced space V'. By Proposition 6.4 there exists a unique holomorpic map $(\psi_0, \psi_1): (V', {}_{N'}\mathcal{O}/\mathcal{I}_{N'}(p)) \to (V, {}_N\mathcal{O}/\mathcal{I}_V(p))$ with coordinates $g_1', \dots, g_N'$. The commuativity of the diagram follows from Lemma 6.1, because the holomorphic maps $(\psi_0, \psi_1) \circ (id_{V'}, \eta')$ and $(id_V, \eta) \circ (\varphi_0, \varphi_1)$ from $(V', {}_{N'}\mathcal{O}/\mathcal{I}')$ to $(V, {}_N\mathcal{O}/\mathcal{I}_V(p))$ have both $f_1', \dots, f_N'$ as coordinates. q.e.d.

C. Suppose V is a subvariety of an open subset G of $\mathbf{C}^N$ and $\mathcal{I}$ is an ideal-sheaf on G whose zero-set is V. For $x \in V$, let $\mathfrak{n}(x)$ be the maximal ideal of the local ring $({}_N\mathcal{O}/\mathcal{I})_x$. Consider the following statement.

$$(6.1)_p \qquad \begin{cases} \text{If } f \in \Gamma(U, {}_N\mathcal{O}/\mathcal{I}) \text{ for some open subset } U \text{ of } V \\ \text{and } f_x \in \mathfrak{n}(x)^{p+1} \text{ for } x \in U, \text{ then } f = 0. \end{cases}$$

The following Lemma is obvious.

LEMMA 6.3. $(6.1)_p$ implies $\mathcal{I}_V(p) \subset \mathcal{I}$.
The following Proposition is proved as Theorem 2 of [8].

PROPOSITION 6.6. Suppose $\mathcal{F}$ is a coherent analytic sheaf on a complex space $(X, \mathcal{O})$. For $x \in X$, let $\mathfrak{m}(x)$ be the maximal ideal of $\mathcal{O}_x$. The for every relatively compact open subset Q of X there exists $p = p(Q) \in \mathbf{N}$ such that, if $f \in \Gamma(U, \mathcal{F})$ for some subset U of Q and $f_x \in \mathfrak{m}(x)^p \mathcal{F}_x$ for every $x \in U$, then $f = 0$ on U.

COROLLARY. For every relatively compact open subset Q of G there exists $p \in \mathbf{N}_*$ such that $\mathcal{I}_V(p) \subset \mathcal{I}$ on Q.

PROOF. Apply Proposition 6.6 to the coherent analytic sheaf ${}_N\mathcal{O}/\mathcal{I}$ on $(V, {}_N\mathcal{O}/\mathcal{I})$ and the relatively compact open subset $Q \cap V$ of V and make use of Lemma 6.3. q. e. d.

DEFINITION. A non-negative integer p is called the *reduction order* of a complex space $(X, \mathcal{O})$ if p is the smallest non-negative integer such that the following holds. If $f \in \Gamma(U, \mathcal{O})$ for some open subset U of X and $f_x \in \mathfrak{m}(x)^{p+1}$ for $x \in U$, where $\mathfrak{m}(x)$ is the maximal ideal of the local ring $\mathcal{O}_x$, then $f = 0$ on U. When no such p exists, the reduction order of $(X, \mathcal{O})$ is defined to be ∞.

From the definition it is clear that a complex space is reduced if and only if its reduction order is 0.

Proposition 6.6 implies that a complex space which can be realized as a relatively compact open subset of another complex space has finite reduction order.

Suppose $(X, \mathcal{O})$ is a complex space of reduction order $\leqq p < \infty$ and U is a relatively compact open subset of Stein open subset $\widetilde{U}$ of X. Take of X. Take $f \in \Gamma(U, \mathcal{O})$. We are going to define a norm ${}_p\|f\|_U$. This norm cannot be defined intrinsically. It will depend on some embedding we choose at random. However, any two different norms obtained this way will be equivalent.

By shrinking $\widetilde{U}$, we can assume that $\widetilde{U}$ is relatively compact in X. There exists a biholomorphic map Φ from $\widetilde{U}$ onto a complex subspace V of an open subset G of $\mathbf{C}^N$. (In fact, we can choose $G = \mathbf{C}^N$. However, this is not important. G is not even required to be Stein). Let $\mathcal{I}$ be the ideal sheaf on G defining the complex subspace V. By Lemma 6.3, $\mathcal{I}_V(p) \subset \mathcal{I}$. The biholomorphic map Φ carries f uniquely to an element $f_* \in \Gamma(\Phi(U), {}_N\mathcal{O}/\mathcal{I})$. Define ${}_p\|f\|_U$ to be

$$\inf\{{}_p\|f_\#\|_{\Phi(U)} \mid f_\# \in I(\Phi(U), {}_N\mathcal{O}/\mathcal{I}_V(p)),\ f_\# \text{ induces } f_*\}.$$

Suppose we have another set of $U', \Phi', V', G', \mathbf{C}^{N'}, \mathcal{I}'$, and f'_* and obtain another norm ${}_p\|f\|'_U$. By replacing both $\widetilde{U}$ and $\widetilde{U}'$ by $\widetilde{U} \cap \widetilde{U}'$, we can assume that $\widetilde{U}=\widetilde{U}'$. We have a biholomorphic map (φ_0, φ_1): $(V', {}_{N'}\mathcal{O}/\mathcal{I}') \to (V, {}_N\mathcal{O}/\mathcal{I})$ such that $(\varphi_0, \varphi_1) \circ \Phi' = \Phi$.

Since V' is Stein, by Proposition 6.5 there exists a holomorphic map (ψ_0, ψ_1): $(V', {}_{N'}\mathcal{O}/\mathcal{I}_{V'}(p)) \to (V, {}_N\mathcal{O}/\mathcal{I}_V(p))$ such that the following diagram is commutative:

$$\begin{array}{ccc}
(V', {}_{N'}\mathcal{O}/\mathcal{I}') & \xrightarrow{(\varphi_0, \varphi_1)} & (V, {}_N\mathcal{O}/\mathcal{I}) \\
{\scriptstyle (id_{V'}, \eta')}\downarrow & & \downarrow{\scriptstyle (id_V, \eta)} \\
(V', {}_{N'}\mathcal{O}/\mathcal{I}_{V'}(p)) & \xrightarrow{(\psi_0, \psi_1)} & (V, {}_N\mathcal{O}/\mathcal{I}_V(p))
\end{array}$$

where η: ${}_N\mathcal{O}/\mathcal{I}_V(p) \to {}_N\mathcal{O}/\mathcal{I}$ and η' : ${}_{N'}\mathcal{O}/\mathcal{I}_{V'}(p) \to {}_{N'}\mathcal{O}/\mathcal{I}'$ are the natural sheaf-homomorphisms.

Since $\Phi(U)$ is a relatively compact in V, by Proposition 6.3, for $f_\# \in \Gamma(\Phi(U), {}_N\mathcal{O}/\mathcal{I}_V(p))$ we have ${}_p\|\psi_1(f_\#)\|_{\Phi'(U)} \leqq C({}_p\|f_\#\|_{\Phi(U)})$, where C is a constant independent of $f_\#$. Since $\eta(f_\#) = f_*$ implies $\eta'(\psi_1(f_\#)) = f'_*$ for $f_\# \in \Gamma(\Phi(U), {}_N\mathcal{O}/\mathcal{I}_V(p))$, we have ${}_p\|f\|'_U \leqq C({}_p\|f\|_U)$. By reversing the roles of V and V', we obtain another C' independent of f such that ${}_p\|f\|_U \leqq$ $\leqq C'({}_p\|f\|'_U)$. Hence the two norms ${}_p\|f\|_U$ and ${}_p\|f\|'_U$ are equivalent.

For notational semplicity, in what follows, whenever such a norm ${}_p\|f\|_U$ arises, we assume that we choose a fixed norm from the class of all those equivalent norms. Whenever possible, we choose always the one which is obviously the most convenient for the purpose.

If no confusion can arise, we simply write $\|f\|_U$ for ${}_p\|f\|_U$.

Again, for $g \in \Gamma(U, \mathcal{O})$, we do not have $\|fg\|_U \leqq \|f\|_U \|g\|_U$, but we have $\|fg\|_U \leqq C''\|f\|_U \|g\|_U$, where C'' is a constant depending only on U and is independent of f and g.

For $g = (g_1, \dots, g_q) \in \Gamma(U, \mathcal{O}^q)$, we define $\|g\|_U = \sup_i \|g_i\|_U$.

Suppose $\mathcal{F}$ is a coherent analytic sheaf on X and $h \in \Gamma(U, \mathcal{F})$. We are going to define a norm for h. By shrinking $\widetilde{U}$, we can assume that there is a sheaf-epimorphism φ: $\mathcal{O}^q \to \mathcal{F}$ on $\widetilde{U}$. Define

$$\|h\|^\varphi_U = \inf\{\|h'\|_U \mid h' \in \Gamma(U, \mathcal{O}^q), \varphi(h') = h\}.$$

When no such h' exists, $\|h\|^\varphi_U$ is defined to be $+\infty$.

Suppose ψ: $\mathcal{O}^r \to \mathcal{F}$ is another sheaf-epimorphism. Since $\widetilde{U}$ is Stein, there exists a sheaf-homomorphism σ: $\mathcal{O}^q \to \mathcal{O}^r$ on $\widetilde{U}$ such that $\psi\sigma = \varphi$. There exists $C_1 \in \mathbf{R}_+$ such that $\|\sigma(h')\|_U \leqq C_1\|h'\|_U$ for $h' \in \Gamma(U, \mathcal{O}^q)$. Hence $\|h\|^\psi_U \leqq C_1\|h\|^\varphi_U$.

This together with the result obtained by interchanging the roles of φ and ψ shows that the two norms $\| h \|_U^{\varphi}$ and $\| h \|_U^{\psi}$ are equivalent. For the sake of notational simplicity, in what follows, whenever such a norm arises, we assume that we choose a fixed norm from the class of all the equivalent norms and denote it simply by $\| h \|_U$ if no confusion can arise. Whenever possible, we choose always the one which is obviously the most convenient.

Suppose $\mathfrak{U} = \{U_i\}$ is a finite collection of open subsets of X and each U_i is relatively compact in some Stein open subset of X. If $\xi \in C^r(\mathfrak{U}, \mathcal{F})$, then $\| \xi \|_{\mathfrak{U}}$ denotes $\sup_{i_0, \dots, i_r} \| \xi_{i_0 \dots i_r} \|_{U_{i_0 \dots i_r}}$.

§ 7. Stein Open Subsets of a Subvariety.

In this section we approximate Stein open subsets of a subvariety embedded in a Stein domain in a complex number space by Stein open subsets of the complex number space.

The following Proposition is proved as Satz 3.2 of [6].

PROPOSITION 7.1. Suppose X is a subvariety of an open subset W of $\mathbf{C}^N$ and A is a compact subset of W. If φ is a C^∞ strictly plurisubharmonic function on X, then there exists an open neighborhood G of A in W and C^∞ strictly plurisubharmonic function $\tilde{\varphi}$ on G which agrees with φ on $G \cap X$.

PROPOSITION 7.2. Suppose X is a subvariety of a Stein open subset G of $\mathbf{C}^N$. Suppose L is a Stein open subset of X and A is a compact subset of L. Then there exists a Stein open subset H of G such that $A \subset X \cap H \subset L$.

PROOF. Since L is Stein, there exist holomorphic functions $f_1, \dots, f_m$ on L such that $(f_1, \dots, f_m): L \to \mathbf{C}^m$ imbeds L as a subvariety of $\mathbf{C}^m$. Let $\varphi = \Sigma_{i=1}^N |z_i|^2 + \Sigma_{j=1}^m |f_j|^2$ on L, where $z_1, \dots, z_N$ are coordinates of $\mathbf{C}^N$. φ is *a* C^∞ strictly plurisubharmonic function on L. Let $c_1 = \sup_{x \in A} \varphi(x)$. Take $c_2 > c_1$. Let $M = \{x \in L \mid \varphi(x) \leq c_2\}$. M is a compact subset of L.

Since L is an open subset of $X, L = G' \cap X$ for some open subset G' of G. L is a subvariety of G'. By Proposition 7.1 there exist an open neighborhood G^* of M in G' and a C^∞ strictly plurisubharmonic function $\tilde{\varphi}$ on G which agrees with φ on $G^* \cap L$. Let D and E be open subsets of G^* such that $M \subset D \subset\subset E \subset\subset G^*$.

Since G is Stein, there exist holomorphic functions $g_1, \dots, g_l$ on G defining X. For $\varepsilon \in \mathbf{R}_+$ let $H_\varepsilon = \{x \in E \mid \tilde{\varphi}(x) < c_2, |g_j(x)| < \varepsilon, 1 \leqq j \leqq l\}$. We clain that $H_\varepsilon \subset D$ for some $\varepsilon \in \mathbf{R}_+$. Suppose the contrary. Then for every $r \in \mathrm{N}$ there exists $x_r \in H_{1/r} - D$. Since $E^- - D$ is compact, there exists a subsequence $\{x_{r_\nu}\}$ of $\{x_r\}$ converting to some $x^* \in E^- - D$. Hence $\tilde{\varphi}(x^*) \leqq c_2$ and $g_j(x^*) = 0$ for $1 \leqq j \leqq l$. $g_1(x^*) = \dots = g_l(x^*) = 0$ implies that $x^* \in X \cap G^* \subset L$. $\varphi(x^*) = \tilde{\varphi}(x^*) \leqq c_2$ implies that $x^* \in M \subset D$, contradicting $x^* \in E^- - D$. Therefore we can find $\varepsilon \in \mathbf{R}_+$ such that $H_\varepsilon \subset D$.

Let $H = H_\varepsilon$. $A \subset H \cap X \subset L$. To finish the proof, we need only show that H is Stein. We are going to show that H is p-convex (Definition IX. C. 11, [4]). Take arbitrarily a compact subset A of H. Let A_p be the p-convex hull of A in H, i. e. $A_p = \{x \in H \mid \psi(x) \leqq \sup_{y \in A} \psi(y)$ for all continuous plurisubharmonic function on $H\}$.

Since A is compact, $\sup_{x \in A} \tilde{\varphi}(x) = c_3 < c_2$ and $\sup\ \{ | g_j(x) | \mid x \in A, 1 \leqq j \leqq l\} = \varepsilon' < \varepsilon$. Hence $A_p \subset \{x \in E \mid \tilde{\varphi}(x) \leqq c_3, |g_j(x)| \leqq \varepsilon'$ for $1 \leqq j \leqq l\} \subset D$. Since D is relatively compact in E and A_p is a closed subset of E, A_p is compact. Therefore H is p-convex. H is Stein (Theorem IX. D. 14, [4]).

q. e. d.

PROPOSITION 7.3. Suppose X is a subvariety of a Stein open subset G of $\mathbf{C}^N$ and $\mathfrak{U}_1 << \mathfrak{U}_2$ are finite collections of open subsets of X such that every member of $\mathfrak{U}_2$ is Stein. Then there exists a finite collection $\mathfrak{D}$ of Stein open subsets of G such that $\mathfrak{U}_1 << \mathfrak{D} \cap X << \mathfrak{U}_2$. Moreover, if H is a relatively compact Stein open subset of G with $|\mathfrak{U}_1| \subset\subset X \cap H$ and $X \cap H^- \subset |\mathfrak{U}_2|$, then we can choose $\mathfrak{D}$ to satisfy in addition that $H = |\mathfrak{D}|$.

PROOF. Suppose $\mathfrak{U}_i = \{U_j^{(i)}\}_{j \in I_i}$ and $\tau : I_1 \to I_2$ is the index map for $\mathfrak{U}_1 << \mathfrak{U}_2$. For $j \in I_2$, $\bigcup_{k \in \tau^{-1}(j)} U_k^{(1)} \subset\subset U_j^{(2)}$. Since $U_j^{(2)}$ is Stein, we can chose open subsets W_j and W_j' of X such that W_j' is Stein and $\bigcup_{k \in \tau^{-1}(j)} U_k^{(1)} \subset\subset W_j \subset\subset W_j' \subset\subset U_j^{(2)}$.

For $j \in I_2$, choose by Proposition 7.2 a Stein open subset D_j of G such that $W_j \subset D_j \cap X \subset W_j'$. $\mathfrak{D} = \{D_j\}_{j \in I_2}$ satisfies the requirement.

Suppose H is a relatively compact stein open subset of G with $|\mathfrak{U}_1| \subset\subset X \cap H$ and $X \cap H^- \subset |\mathfrak{U}_2|$. Since $X \cap H^-$ is a compact subset of $|\mathfrak{U}_2|$, we can find a finite collection $\mathfrak{U}_3$ of open subsets of X such that $\mathfrak{U}_1 < \mathfrak{U}_3 << \mathfrak{U}_2$ and $X \cap H^- \subset |\mathfrak{U}_3|$. By the preceding argument, we can find a finite collection $\mathfrak{D}_1$ of Stein open subsets of G such that $\mathfrak{U}_3 << \mathfrak{D}_1 \cap X << \mathfrak{U}_2$. $H^- - |\mathfrak{D}_1|$ is a compact subset of $G - X$. We can find a finite collection $\mathfrak{D}_2$ of Stein open subsets of $G - X$ such that $H^- -$

$-|\mathfrak{D}_1| \subset |\mathfrak{D}_2|$. Let $\mathfrak{D}' = (\mathfrak{D}_1 \cup \mathfrak{D}_2) \cap H$. Then $\mathfrak{D}'$ is a finite collection of Stein open subsets of G, $\mathfrak{U}_1 << \mathfrak{D}' \cap X << \mathfrak{U}_2$, and $H = |\mathfrak{D}'|$. q. e. d.

§ 8. Grauert Norms.

In this section we define Grauert norms for sheaf sections defined on a complex space equipped with a projection. Some elementary properties of these norms are then derived.

A. Suppose $(X, \mathcal{O})$ is a complex space of reduction order $\leqq p < \infty$ and $\pi: (X, \mathcal{O}) \longrightarrow K(\varrho^0)$ is a holomorphic map, where $\varrho^0 \in \mathbf{R}^n_+$. Suppose U is a relatively compact open subset of a Stein open subset $\widetilde{U}$ of X. $U(\varrho)$ will denote $U \cap \pi^{-1}(K(\varrho))$ for $\varrho \leqq \varrho^0$ in $\mathbf{R}^n_+$. Suppose $f \in \Gamma(U(\varrho), \mathcal{O})$ for some $\varrho \leqq \varrho^0$. We denote by ${}^{\pi}_{p}\|f\|_{U,\varrho}$ or simply by $\|f\|_{U,\varrho}$ the following *Grauert norm :*

$$\|f\|_{U,\varrho} = \inf \{\sup_\nu \|f_\nu\|_U \mid f_\nu \in \Gamma(U, \mathcal{O}),$$

$$\Sigma_{\nu \in \mathbf{N}^n_*} f_\nu \left(\frac{t_1 \circ \pi}{\varrho_1}\right)^{\nu_1} \cdots \left(\frac{t_n \circ \pi}{\varrho_n}\right)^{\nu_n}$$

converges on $U(\varrho)$ and is equal to $f\}$.

At first sight this definition of Grauert norms seems very unnatural. To shed some light on the motive behind this definition we are going to give a second description of Grauert norms. This second description will not be used in the rest of this paper. The only purpose of this second description is to help clarify the preceding definition of Grauert norms.

From the complex structure of $(X, \mathcal{O})$ we obtain a natural product complex structure for $K(\varrho^0) \times X$ and we denote the structure sheaf by $\widetilde{\mathcal{O}}$. ($(X, \mathcal{O})$ is reduced if and only if $(K(\varrho^0) \times X, \widetilde{\mathcal{O}})$ is reduced). Denote the projections $(K(\varrho^0) \times X, \widetilde{\mathcal{O}}) \longrightarrow K(\varrho^0)$ and $(K(\varrho^0) \times X, \widetilde{\mathcal{O}}) \longrightarrow (X, \mathcal{O})$ by Π_1 and Π_2 respectively. For every $g \in \Gamma(K(\varrho) \times U, \widetilde{\mathcal{O}})$, we have a unique « Taylor series expansion »:

$$g = \Sigma (g_{\nu_1 \ldots \nu_n} \circ \Pi_2) \left(\frac{t_1 \circ \Pi_1}{\varrho_1}\right)^{\nu_1} \cdots \left(\frac{t_n \circ \Pi_1}{\varrho_n}\right)^{\nu_n},$$

where $g_{\nu_1 \ldots \nu_n} \in \Gamma(U, \mathcal{O})$). Define $\|g\|^*_{U,\varrho} = \sup_{\nu_1, \ldots, \nu_n} \|g_{\nu_1 \ldots \nu_n}\|_U$. $\pi: (X, \mathcal{O}) \longrightarrow K(\varrho^0)$ and the identity holomorphic map $id_X: (X, \mathcal{O}) \longrightarrow (X, \mathcal{O})$ give rise to a holomorphic embedding $\tau: (X, \mathcal{O}) \longrightarrow (K(\varrho^0) \times X, \widetilde{\mathcal{O}})$ satisfying $\Pi_1 \circ \tau = \pi$

and $\Pi_2 \circ \tau = id_X$. Let $\mathcal{O}^* = \tau_0(\mathcal{O})$. Since $\mathcal{O}^* \mid \tau(X)$ is the structure shea· of the complex subspace $\tau(X)$ of $(K(\varrho^0) \times X, \widetilde{\mathcal{O}})$, we have a natural sheaff epimorphism $\eta : \widetilde{\mathcal{O}} \longrightarrow \mathcal{O}^*$. f corresponds uniquely to $f_* \in \Gamma(\tau(U(\varrho)), \mathcal{O}^*)$. Since $\operatorname{Supp} \mathcal{O}^* = \tau(X)$ and $(K(\varrho) \times U) \cap \tau(X) = \tau(U(\varrho))$, we can regard f_* naturally as an element of $\Gamma(K(\varrho) \times U, \mathcal{O}^*)$. It is easily seen that the Grauert norm $\|f\|_{U,\varrho}$ is equal to

$$\inf\{\, \|g\|^*_{U,\varrho} \mid g \in \Gamma(K(\varrho) \times U, \widetilde{\mathcal{O}}), \eta(g) = f^*\}.$$

The second description of Grauert norms is complete.

For $h = (h_1, \dots, h_r) \in \Gamma(U(\varrho), \mathcal{O}^r)$, define $\|h\|_{U,\varrho} = \sup_i \|h_i\|_{U,\varrho}$.

LEMMA 8.1. Suppose $\sigma : \mathcal{O}^q \longrightarrow \mathcal{O}^r$ is a sheaf-homomorphism on $\widetilde{U}$. Then there exists $C \in \mathbf{R}_+$ such that $\|\sigma(g)\|_{U,\varrho} \leqq C \|g\|_{U,\varrho}$ for $g \in \Gamma(U(\varrho), \mathcal{O}^q)$ and $\varrho \leqq \varrho^0$.

PROOF. σ is represented by an $r \times q$ matrix (σ_{ij}) of holomorphic functions on $\widetilde{U}$. Let $C_1 = \sup_{i,j} \|\sigma_{ij}\|_U$. C_1 is finite, because σ_{ij} is defined on $\widetilde{U}$ which contains U as a relatively compact subset.

Let $g = (g_1, \dots, g_q)$ and $\sigma(g) = (h_1, \dots, h_r)$, where $g_j, h_i \in \Gamma(U(\varrho), \mathcal{O})$. $h_i = \Sigma_{j=1}^q \sigma_{ij} g_j$ on $U(\varrho)$. We can assume that $\|g\|_{U,\varrho} < \infty$. Choose arbitrarily $e > \|g\|_{U,\varrho}$. There exists $g_{j\nu} \in \Gamma(U, \mathcal{O})$, $\nu \in \mathbf{N}_*^n$, such that $\|g_{j\nu}\|_U < e$ and $\Sigma_\nu g_{j\nu} \left(\frac{t_1 \circ \pi}{\varrho_1}\right)^{\nu_1} \cdots \left(\frac{t_n \circ \pi}{\varrho_n}\right)^{\nu_n}$ converges on $U(\varrho)$ to g_j.

Let $h_{i\nu} = \Sigma_{j=1}^q g_{j\nu} (\sigma_{ij} \mid U)$. Then $h_{j\nu} \in \Gamma(U, \mathcal{O})$ and $\|h_{i\nu}\|_U < q\, C_2\, C_1\, e$, where C_2 is a constant depending only on U. $\Sigma_\nu h_{i\nu} \left(\frac{t_1 \circ \pi}{\varrho_1}\right)^{\nu_1} \cdots \left(\frac{t_n \circ \pi}{\varrho_n}\right)^{\nu_n}$ converges on $U(\varrho)$ to h_i. Hence $\|h\|_{U,\varrho} < q\, C_2\, C_1\, e$. Since e is arbitrary, $\|\sigma(g)\|_{U,\varrho} \leqq q\, C_2\, C_1 \|g\|_{U,\varrho}$. q. e. d.

LEMMA 8.2. Suppose $\varrho \leqq \varrho^* \leqq \varrho^0$ in $\mathbf{R}_+^n$ such that $\varrho_i = \varrho_i^*$ for $1 \leqq i \leqq n-1$ and $\varrho_n < \varrho_n^*$. Suppose $a_k \in \Gamma(U(\varrho^*), \mathcal{O}^q)$ for $k \in \mathbf{N}_*$ such that $\|a_k\|_{U,\varrho^*} < C$ for some $C \in \mathbf{R}_+$. Then $\left\| \Sigma_{k=0}^\infty a_k \left(\frac{t_n \circ \pi}{\varrho_n}\right)^k \right\|_{U,\varrho} < \left(1 - \frac{\varrho_n}{\varrho_n^*}\right)^{-1} C$.

PROOF. Since $\|a_k\|_{U,\varrho^*} < C$, there exists $a_{k\mu} \in \Gamma(U, \mathcal{O}^q)$ for $\mu \in \mathbf{N}_*^n$ such that $\|a_{k\mu}\|_U < C$ and $\Sigma_\mu a_{k\mu} \left(\frac{t_1 \circ \pi}{\varrho_1^*}\right)^{\mu_1} \cdots \left(\frac{t_n \circ \pi}{\varrho_n^*}\right)^{\mu_n}$ converges to a_k con $U(\varrho^*)$.

Let $b_\lambda = \Sigma_{k+i=\lambda_n} a_{k,(\lambda_1, \dots, \lambda_{n-1}, i)} \left(\frac{\varrho_n}{\varrho_n^*}\right)^i$ for $\lambda \in \mathbf{N}_*^n$. Then $b_\lambda \in \Gamma(U, \mathcal{O}^q)$ and

$\| b_\lambda \|_U < \left(1 - \frac{\varrho_n}{\varrho_n^*}\right)^{-1} C$. $\Sigma_\lambda b_\lambda \left(\frac{t_1 \circ \pi}{\varrho_1}\right)^{\lambda_1} \cdots \left(\frac{t_n \circ \pi}{\varrho_n}\right)^{\lambda_n}$ converges to $\Sigma_{k=0}^\infty a_k \left(\frac{t_n \circ \pi}{\varrho_n}\right)^k$ on $U(\varrho)$. Hence $\left\| \Sigma_{k=0}^\infty a_k \left(\frac{t_n \circ \pi}{\varrho_n}\right)^k \right\|_{U, \varrho} < \left(1 - \frac{\varrho_n}{\varrho_n^*}\right)^{-1} C$. q. e. d.

Suppose $\mathcal{F}$ is a coherent analytic sheaf on $(X, \mathcal{O})$. Suppose $h \in \Gamma(U(\varrho), \mathcal{F})$ for some $\varrho \leqq \varrho^0$. We are going to define a norm for h. By shrinking $\widetilde{U}$, we can assume that there is a sheaf-epimorphism $\varphi : \mathcal{O}^q \longrightarrow \mathcal{F}$ on $\widetilde{U}$. We denote by ${}^{\pi}_{p}\| h \|^{\varphi}_{U, \varrho}$ or simply by $\| h \|_{U, \varrho}$ the following *Grauert norm:*

$$\| h \|_{U, \varrho} = \inf \{ \| g \|_{U, \varrho} \mid g \in \Gamma(U(\varrho), \mathcal{O}^q),\ \varphi(g) = h \}.$$

When no such g exists, $\| h \|_{U, \varrho}$ is defined to be $+ \infty$.

Suppose $\psi : \mathcal{O}^r \longrightarrow \mathcal{F}$ is another sheaf-epimorphism. Since $\widetilde{U}$ is Stein, there exists a sheaf-homomorphism $\sigma : \mathcal{O}^q \longrightarrow \mathcal{O}^r$ on $\widetilde{U}$ such that $\psi\sigma = \varphi$. By Lemma 8.1, $\| \sigma(g) \|_{U, \varrho} \leqq C \| g \|_{U, \varrho}$ for $g \in \Gamma(U(\varrho), \mathcal{O}^q)$, where C is a constant independent of g and ϱ. Since $\sigma(g) = g'$ implies $\psi(g') = \varphi(g)$ for $g \in \Gamma(U(\varrho), \mathcal{O}^q)$, we have ${}^{\pi}_{p}\| h \|^{\psi}_{U, \varrho} \leqq C ({}^{\pi}_{p}\| h \|^{\varphi}_{U, \varrho})$. This together with the result obtained by interchanging the roles of φ and ψ shows that the two norms are equivalent. For the sake of notational simplicity, in what follows, whenever such a norm arises, we assume that a fixed norm from the class of all the equivalent norms is chosen. Whenever possible, the obviously most convenient one is always chosen.

Suppose $\mathfrak{U} = \{U_i\}$ is a finite collection of open subsets of X and each U_i is relatively compact in some Stein open subset of X. For $\varrho \leqq \varrho^0$ in $\mathbf{R}^n_+$, we denote $\{U_i(\varrho)\}$ by $\mathfrak{U}(\varrho)$. Suppose $\xi \in C^r(\mathfrak{U}(\varrho), \mathcal{F})$ for some $\varrho \leqq \varrho^0$, then $\| \xi \|_{\mathfrak{U}, \varrho}$ denotes $\sup_{i_0, \ldots, i_r} \| \xi_{i_0 \ldots i_r} \|_{U_{i_0 \ldots i_r}, \varrho}$.

It is easily seen that the Grauert norm $\| \xi \|_{\mathfrak{U}, \varrho}$ can also be defined as follows:

$$\| \xi \|_{\mathfrak{U}, \varrho} = \sup \{ \| \xi_\nu \|_{\mathfrak{U}} \mid \xi_\nu \in C^r(\mathfrak{U}, \mathcal{F}),$$

$$\Sigma_{\nu \in \mathbf{N}^n_*} \xi_\nu \left(\frac{t_1 \circ \pi}{\varrho_1}\right)^{\nu_1} \cdots \left(\frac{t_n \circ \pi}{\varrho_n}\right)^{\nu_n} \text{converges to } \xi \text{ on } \mathfrak{U}(\varrho)\}.$$

This alternative definition will be used in § 14.

The following Lemma follows trivially from the definition of Grauert norms.

LEMMA 8.3 Suppose $\mathcal{G}$ is a coherent analytic sheaf on X and $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a sheaf-epimorphism. If $g \in C^r(\mathfrak{U}(\varrho), \mathcal{G})$ for some $\varrho \leqq \varrho^0$ and $\| g \|_{\mathfrak{U}, \varrho} < e$, then there exists $f \in C^r(\mathfrak{U}(\varrho), \mathcal{F})$ such that $\varphi(f) = g$ and $\| f \|_{\mathfrak{U}, \varrho} < e$ (when the two norms are suitably chosen).

The following is a consequence of Lemma 8.2.

LEMMA 8.4. Suppose $\varrho \leqq \varrho^* \leqq \varrho^0$ in $\mathbf{R}_+^n$ such that $\varrho_i = \varrho_i^*$ for $1 \leqq i \leqq \leqq n-1$ and $\varrho_n < \varrho_n^*$. Suppose $\eta_k \in \Gamma(\mathfrak{U}(\varrho^*), \mathcal{F})$ for $k \in \mathbf{N}_*$ such that $\| \eta_k \|_{\mathfrak{U}, \varrho^*} < C$ for some $C \in \mathbf{R}_+$. Then

$$\left\| \Sigma_{k=0}^{\infty} \eta_k \left(\frac{t_n \circ \pi}{\varrho_n} \right)^k \right\|_{\mathfrak{U}, \varrho} < \left(1 - \frac{\varrho_n}{\varrho_n^*}\right)^{-1} C.$$

B. Suppose $(X, \mathcal{O})$ is a complex subspace of an open subset G of $\mathbf{C}^N$ and the reduction order of $(X, \mathcal{O})$ is $\leqq p < \infty$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on $(X, \mathcal{O})$. Suppose $\varrho^0 \in \mathbf{R}_+^n$ and $\pi : (X, \mathcal{O}) \longrightarrow K(\varrho^0)$ is a holomorphic map.

Let $P_1 : K(\varrho^0) \times G \longrightarrow K(\varrho^0)$ and $P_2 : K(\varrho^0) \times G \longrightarrow G$ be the projections. There exists a unique holomorphic embedding $\theta : (X, \mathcal{O}) \longrightarrow (K(\varrho^0) \times G, {}_{n+N}\mathcal{O})$, such that $P_2 \circ \theta$ is the inclusion map $(X, \mathcal{O}) \hookrightarrow (G, {}_N\mathcal{O})$ and $P_1 \circ \theta = \pi$. Let $\mathcal{F}^* = \theta_0(\mathcal{F})$.

If U is an open subset of X, D is an open subset of G with $D \cap X = U$, and $\varrho \leqq \varrho^0$ in $\mathbf{R}_+^n$, then there is a natural isomorphism θ_D: $\Gamma(U(\varrho), \mathcal{F}) \longrightarrow \Gamma(K(\varrho) \times D, \mathcal{F}^*)$, because $\theta(U(\varrho)) = (K(\varrho) \times D) \cap \theta(X)$.

If $\mathfrak{U}$ is a finite collection of open subsets of X, $\mathfrak{D}$ is a finite collection of open subsets of G with $\mathfrak{D} \cap X = \mathfrak{U}$, and $\varrho \leqq \varrho^0$ in $\mathbf{R}_+^n$, then for $r \in \mathbf{N}_*$ we have a natural isomorphism $\theta_{\mathfrak{D}} : C^r(\mathfrak{U}(\varrho), \mathcal{F}) \longrightarrow C^r(K(\varrho) \times \mathfrak{D}, \mathcal{F}^*)$. Since $\theta_{\mathfrak{D}}$ commutes with the coboundary operator δ, $\theta_{\mathfrak{D}}$ maps $Z^r(\mathfrak{U}(\varrho), \mathcal{F})$ onto $Z^r(K(\varrho) \times \mathfrak{D}, \mathcal{F}^*)$ and maps $B^r(\mathfrak{U}(\varrho), \mathcal{F})$ onto $B^r(K(\varrho) \times \mathfrak{D}, \mathcal{F}^*)$.

We are going to investigate the relations between $\|f\|_{U, \varrho}$ and $| \theta_{\mathfrak{D}}(f) |_{\mathfrak{D}, \varrho}$ for $f \in \Gamma(U(\varrho), \mathcal{F})$ and the relations between $\| g \|_{\mathfrak{U}, \varrho}$ and $| \theta_{\mathfrak{D}}(g) |_{\mathfrak{D}, \varrho}$ for $g \in C^r(\mathfrak{U}(\varrho), \mathcal{F})$. In § 9 we will transplant the results of § 5 to general complex spaces equipped with projections by means of these relations.

PROPOSITION 8.1. Suppose H is a relatively compact open subset of a Stein open subset $\widetilde{H}$ of G and $U \subset\subset U^*$ are open subsets of X such that $H \cap X = U^*$ Then there exist $C \in \mathbf{R}_+$ such that, if $\varrho \leqq \varrho^0$ and $f \in \Gamma(U^*(\varrho), \mathcal{F})$ then, $\|f\|_{U, \varrho} \leqq C \, | \theta_H(f) |_{H, \varrho}$.

PROOF. By replacing G by $\widetilde{H}$ and by shrinking $\widetilde{H}$ we can assume that there is a scheaf-epimorphism $\sigma : \mathcal{O}^q \longrightarrow \mathcal{F}$ on X. σ is defined by q elements $s_1, \ldots, s_q \in \Gamma(X, \mathcal{F})$. The q elements $\theta_G(s_1), \ldots, \theta_G(s_q)$ of $\Gamma(K(\varrho^0) \times G, \mathcal{F}^*)$ defines a sheaf-epimorphism $\sigma^* : {}_{n+N}\mathcal{O}^q \longrightarrow \mathcal{F}^*$.

Let $f^* = \theta_H(f)$. We can assume that $|f^*|_{H, \varrho} < \infty$. Take arbitrarily $e > |f^*|_{H, \varrho}$. By definition of the norm there exists $g^* \in \Gamma(K(\varrho) \times H, {}_{n+N}\mathcal{O}^q)$

such that $\sigma^*(g^*) = f^*$ and $|g^*|_{H,\varrho} < e$. Let $g^* = \Sigma_{\nu \in \mathbf{N}_*^n} (g_\nu^* \circ P_2) \left(\frac{t_1 \circ P_1}{\varrho_1}\right)^{\nu_1} \dots$ $\dots \left(\frac{t_n \circ P_1}{\varrho_1}\right)^{\nu_n}$ be the Taylor series expansion, where $g_\nu^* \in \Gamma(H, {}_N\mathcal{O}^q)$. Then $|g_\nu^*|_H < e$.

Let $g_\nu \in \Gamma(U^*, \mathcal{O}^q)$ be induced by g_ν^*. By Cauchy's inequality $\| g_\nu \|_U = = \sup \{ | D^\alpha g_\nu^*(x) | \mid \alpha \in \mathbf{N}_*^N, |\alpha| \leqq p, x \in U \} < Ce$, where C is a constant independent of g_ν^*.

Let $g = \Sigma g_\nu \left(\frac{t_1 \circ \pi}{\varrho_1}\right)^{\nu_1} \dots \left(\frac{t_n \circ \pi}{\varrho_n}\right)^{\nu_n} \in \Gamma(U(\varrho), \mathcal{O}^q)$. Then $\| g \|_{U,\varrho} < Ce$. Since $\sigma(g) = f$ on $U(\varrho)$, $\| f \|_{U,\varrho} < Ce$. q. e. d.

The following is a consequence of Proposition 8.1.

PROPOSITION 8.2. Suppose $\mathfrak{D}$ is a finite collection of open subsets of G such that every member of $\mathfrak{D}$ is relatively compact is some Stein open subset of G. Suppose $\mathfrak{U} << \mathfrak{U}^*$ are finite collections of open subsets of X such that $\mathfrak{D} \cap X = \mathfrak{U}^*$. Then there exists $C \in \mathbf{R}_+$ such that, for $\varrho \leqq \varrho^0$ and $f \in C^r(\mathfrak{U}'(\varrho), \mathcal{F})$, $\| f \|_{\mathfrak{U},\varrho} \leqq C \,|\, \theta_{\mathfrak{D}}(f) |_{\mathfrak{D},\varrho}$.

PROPOSITION 8.3. Suppose U is a relatively compact open subset of a Stein open subset $\widetilde{U}$ of X. Suppose $H \subset\subset H^* \subset D$ are open subsets of G such that H^* is Stein and $D \cap X = U$. Then there exists $C \in \mathbf{R}_+$ such that, for $\varrho \leqq \varrho^0$ and $f \in \Gamma(U(\varrho), \mathcal{F})$, $| \theta_D(f) |_{H,\varrho} \leqq C \| f \|_{U,\varrho}$.

PROOF. By replacing X by $\widetilde{U}$ and shrinking $\widetilde{U}$, we can assume that there is a sheaf-epimorphism $\sigma : \mathcal{O}^q \to X$. σ is defined by q elements $s_1, \dots, s_q$ of $\Gamma(X, \mathcal{F})$. The q elements $\theta_G(s_1), \dots, \theta_G(s_q)$ of $\Gamma(K(\varrho^0) \times G, \mathcal{F}_*)$ define a sheaf-epimorphism $\sigma^* : {}_{n+N}\mathcal{O}^q \to \mathcal{F}^*$.

We can assume that $\| f \|_{U,\varrho} < \infty$. Take arbitrarily $e > \| f \|_{U,\varrho}$. There exists $g \in \Gamma(U(\varrho), \mathcal{O}^q)$ such that $\sigma(g) = f$ and $\| g \|_{U,\varrho} < e$. There exist $g_\nu \in \in \Gamma(U, \mathcal{O}^q)$, $\nu \in \mathbf{N}_*^n$, such that $\| g_\nu \|_U < e$ and $\Sigma g_\nu \left(\frac{t_1 \circ \pi}{\varrho_1}\right)^{\nu_1} \dots \left(\frac{t_n \circ \pi}{\varrho_n}\right)^{\nu_n} = g$ on $U(\varrho)$.

Since the quotient map $\eta : \Gamma(H^*, {}_N\mathcal{O}) \to \Gamma(H^* \cap X, \mathcal{O})$ is a continuous linear surjection of Fréchet spaces, by the open mapping theorem there exists $g_\nu^* \in \Gamma(H^*, {}_N\mathcal{O})$ such that $\eta(g_\nu^*) = g_\nu$ on $H^* \cap X$ and $| g_\nu^* |_H < Ce$, where C is a constant independent of g_ν.

Let $g^* = \Sigma (g_\nu^* \circ P_2) \left(\frac{t_1 \circ P_1}{\varrho_1}\right)^{\nu_1} \dots \left(\frac{t_n \circ P_1}{\varrho_n}\right)^{\nu_n} \in \Gamma(K(\varrho) \times H, {}_{n+N}\mathcal{O})$. $|g^*|_{H,\varrho} < < Ce$. Since $\sigma^*(g^*) = \theta_D(f)$, $| \theta_D(f) |_{H,\varrho} < Ce$. q. e. d.

The following is a consequence of Proposition 8.3.

PROPOSITION 8.4. Suppose $\mathfrak{U}$ is a finite collection of open subsets of X such that every member of $\mathfrak{U}$ is relatively compact in some Stein open subset of X. Suppose $\mathfrak{W} << \mathfrak{W}^* < \mathfrak{D}$ are finite collections of open subsets of G such that $\mathfrak{D} \cap X = \mathfrak{U}$ and every member of $\mathfrak{W}^*$ is Stein. Then there exists $C \in \mathbf{R}_+$ such that, for $\varrho \leqq \varrho^0$ and $f \in C^r(\mathfrak{U}(\varrho), \mathcal{F})$, $|\theta_{\mathfrak{D}}(f)|_{\mathfrak{W}, \varrho} \leqq$ $\leqq C \|f\|_{\mathfrak{U}, \varrho}$.

C. Suppose $(X, \mathcal{O})$ is a complex space of reduction $\leqq p < \infty$ and $\mathcal{F}$ is a coherent analytic sheaf on $(X, \mathcal{O})$. Suppose $\varrho^0 \in \mathbf{R}_+^n$ and $\pi : (X, \mathcal{O}) \to K(\varrho^0)$ is a holomorphic map.

If U and U' are open subsets of X and $U(\varrho) = U'(\varrho)$, from the definition of Grauert norms we cannot immediately say anything about the relationship between $\|\cdot\|_{U, \varrho}$ and $\|\cdot\|_{U', \varrho}$. We are going to investigate this situation. It turns out that we can draw some conclusions if we shrink U or U' a little.

LEMMA 8.5. Suppose W, $\widetilde{W}$, U are relatively compact open subsets of a Stein open subset $\widetilde{U}$ of X and $\varrho'' < \varrho' \leqq \varrho^0$ in $\mathbf{R}_+^n$ such that $W \subset\subset \widetilde{W}$, $\widetilde{W}(\varrho') \subset U$, and $\widetilde{W}$ is Stein. Then there exists $C \in \mathbf{R}_+$ such that, if $\varrho \leqq \varrho''$ and $f \in \Gamma(U(\varrho), \mathcal{F})$, then $\|f \mid W(\varrho)\|_{W, \varrho} \leqq C \|f\|_{U, \varrho}$.

PROOF. Obviously we need only prove the special case $\mathcal{F} = \mathcal{O}$. By replacing X by $\widetilde{U}$ and by shrinking $\widetilde{U}$, we can assume that X is a complex subspace of a Stein open subset G of $\mathbf{C}^N$. We use the notations of § 8B.

We can suppose $\|f\|_{U, \varrho} < \infty$. Take arbitrarily $e > \|f\|_{U, \varrho}$. There exist $f_\nu \in \Gamma(U, \mathcal{O})$, $\nu \in \mathbf{N}_*^n$, such that $\|f_\nu\|_U < e$ and $\Sigma f_\nu \left(\frac{t_1 \circ \pi}{\varrho_1}\right)^{\nu_1} \dots \left(\frac{t_n \circ \pi}{\varrho_n}\right)^{\nu_n} = f$ on $U(\varrho)$.

Since $\widetilde{W}$ is Stein, by Proposition 7.2 we can choose Stein open subsets $D_1 \subset\subset D_2$ of G such that $W \subset\subset D_1$ and $D_2 \cap X \subset \widetilde{W}$. Let $W_i = D_i \cap X$. Choose $\varrho'' < \varrho^* < \varrho'$ in $\mathbf{R}_+^n$.

From the assumption we have $W_2(\varrho') \subset U$. Let $f_\nu' = f_\nu \mid W_2(\varrho')$. Then $\|f_\nu'\|_{W_2(\varrho')} < e$.

Let $\alpha : \Gamma(K(\varrho') \times D_2, {}_{n+N}\mathcal{O}) \to \Gamma(K(\varrho') \times D_2, \mathcal{O}^*)$ be induced by the quotient map ${}_{n+N}\mathcal{O} \to \mathcal{O}^*$ (recall that $\mathcal{O}^* = \theta_0(\mathcal{O})$). Let $\beta : \Gamma(K(\varrho') \times$ $\times D_2, {}_{n+N}\mathcal{O}) \to \Gamma(W_2(\varrho'), \mathcal{O})$ be defined by $\beta = (\theta_{D_2})^{-1} \alpha$.

Since β is a continuous linear surjection on Fréchet spaces, by the open mapping theorem there exists $g_\nu \in \Gamma(K(\varrho') \times D_2, {}_{n+N}\mathcal{O})$ such that $\beta(g_\nu) = f_\nu'$ and $|g_\nu|_{K(\varrho^*) \times D_1} < C_1$ e, where C_1 is a constant independent of f_ν'.

Let $g_\nu = \Sigma_{\mu \in \mathbf{N}_*^n} (g_{\nu\mu} \circ P_2) \left(\frac{t_1 \circ P_1}{\varrho_1^*}\right)^{\mu_1} \dots \left(\frac{t_n \circ P_1}{\varrho_n^*}\right)^{\mu_n}$ be the Taylor series expansion. By Cauchy's inequality $|g_{\nu\mu}|_{D_1} < C_1 e$.

Let $f_{\nu\mu} \in \Gamma(W_1, \mathcal{O})$ be induced by $g_{\nu\mu}$. By Cauchy's inequality $\|f_{\nu\mu}\|_W = \sup\{|D^\alpha g_{\nu\mu}(x)| \mid \alpha \in \mathbf{N}_*^N, |\alpha| \leqq p, x \in W\} < C_2 C_1 e$, where C_2 is a constant independent of $f_{\nu\mu}$.

Let $f_\lambda^* = \Sigma_{\mu+\nu=\lambda} f_{\nu\mu}\left(\frac{\varrho_1}{\varrho_1^*}\right)^{\mu_1} \cdots \left(\frac{\varrho_n}{\varrho_n^*}\right)^{\mu_n}$ for $\lambda \in \mathbf{N}_*^n$. Then we have $f = \Sigma_\lambda f_\lambda^* \left(\frac{t_1 \circ \pi}{\varrho_1}\right)^{\lambda_1} \cdots \left(\frac{t_n \circ \pi}{\varrho_n}\right)^{\lambda_n}$ on $W(\varrho)$ and $\|f_\lambda^*\|_W < C_3 C_2 C_1 e$, where $C_3 = \left(1 - \frac{\varrho_n''}{\varrho_1^*}\right)^{-1} \cdots \left(1 - \frac{\varrho_n''}{\varrho_n^*}\right)^{-1}$. Hence $\|f\|_{W,\varrho} < C_3 C_2 C_1 e$. q. e. d.

LEMMA 8.6. Suppose $U_1 \subset\subset U_2 \subset\subset \widetilde{U}$ are open subsets of X and U_2, $\widetilde{U}$ are Stein. If $\varrho^1 < \varrho^2 \leqq \varrho^0$ in $\mathbf{R}_+^n$ and $f \in \Gamma(U_2(\varrho^2), \mathcal{F})$, then $\|f\|_{U_1, \varrho^1} < \infty$.

PROOF. Choose a Stein open subset U_3 of X such that $U_1 \subset\subset U_3 \subset\subset U_2$. Choose $\varrho^1 < \varrho^3 < \varrho^2$ in $\mathbf{R}_+^n$. Since $U_3(\varrho^3) \subset\subset U_2(\varrho^2)$, $\|f\|_{U_3(\varrho^3)} < \infty$. Obviously $\|f\|_{U_3(\varrho^3), \varrho^1} \leqq \|f\|_{U_3(\varrho^3)}$. The Lemma follows from Lemma 8.5 by setting $W = U_1$, $\widetilde{W} = U_3$, $U = U_3(\varrho^3)$, $\varrho'' = \varrho = \varrho^1$, and $\varrho' = \varrho^3$. q.e.d.

The following two Propositions are consequences of Lemmas 8.5 and 8.6.

PROPOSITION 8.5. Suppose $\mathfrak{W}$, $\widetilde{\mathfrak{W}}$, $\mathfrak{U}$ are finite collections of Stein open subsets of X such that each member of $\mathfrak{W}$, $\widetilde{\mathfrak{W}}$, $\mathfrak{U}$ is relatively compact in some Stein open subset of X. Suppose $\varrho'' < \varrho' \leqq \varrho^0$ in $\mathbf{R}_+^n$, $\mathfrak{W} << \widetilde{\mathfrak{W}}$, and $\widetilde{\mathfrak{W}}(\varrho') < \mathfrak{U}$. Then there exists $C \in \mathbf{R}_+$ such that, if $\varrho \leqq \varrho''$ and $f \in C^r(\mathfrak{U}(\varrho), \mathcal{F})$, then $\|f \mid \mathfrak{W}(\varrho)\|_{\mathfrak{W}, \varrho} \leqq C \|f\|_{\mathfrak{U}, \varrho}$.

PROPOSITION 8.6. Suppose $\mathfrak{U}_1 << \mathfrak{U}_2$ are finite collections of Stein open subsets of X such that each member of $\mathfrak{U}_2$ is relatively compact in some Stein open subset of X. If $\varrho^1 < \varrho^2 \leqq \varrho^0$ in $\mathbf{R}_+^n$ and $f \in C^r(\mathfrak{U}_2(\varrho^2), \mathcal{F})$, then $\|f\|_{\mathfrak{U}_1, \varrho^1} < \infty$.

D. Suppose X is a complex space of reduction order $\leqq p < \infty$, $\varrho^0 \in \mathbf{R}_+^n$, and $\pi : X \rightarrow K(\varrho^0)$ is a holomorphic map. Suppose $(X', \mathcal{O}')$ is a complex subspace of X and the reduction order of X' is $\leqq p' < \infty$. Let $\pi' = \pi \mid X'$. Suppose $\mathcal{F}'$ is a coherent analytic sheaf on X' and $\mathcal{F}$ is the trivial extension of $\mathcal{F}'$ on X.

If U is an open subset of X, $U \cap X'$ will be denoted by U'. If $\mathfrak{U}$ is a collection of open subsets of X, $\mathfrak{U} \cap X'$ will be denoted by $\mathfrak{U}'$.

For $\varrho \leqq \varrho^0$ and any open subset U of X there is a natural isomorphism $\sigma_U : \Gamma(U(\varrho), \mathcal{F}) \rightarrow \Gamma(U'(\varrho), \mathcal{F}')$. For $\varrho \leqq \varrho^0$, $r \in \mathbf{N}_*$, and any collection $\mathfrak{U}$ of open subsets of X there is a natural isomorphism $\sigma_{\mathfrak{U}}: C^r(\mathfrak{U}(\varrho), \mathcal{F}) \to$ $\to C^r(\mathfrak{U}'(\varrho), \mathcal{F}')$.

LEMMA 8.7. Suppose U_1, U_2, U are relatively compact Stein open subsets of a Stein open subset of X and $\varrho'' < \varrho' \leqq \varrho^0$ in $\mathbf{R}^n_+$ such that $U_1 \subset\subset U_2$ and $U_2(\varrho') \subset\subset U$. Then there exists $C \in \mathbf{R}_+$ such that, if $\varrho \leqq \varrho''$, then

(i) $$\| \sigma_U(f) \|_{U'_1, \varrho} \leqq C \| f \|_{U, \varrho} \text{ for } f \in \Gamma(U(\varrho), \mathcal{F}), \text{ and}$$

(ii) $$\| \sigma_U^{-1}(f') \|_{U_1, \varrho} \leqq C \| f' \|_{U', \varrho} \text{ for } f' \in \Gamma(U'(\varrho), \mathcal{F}').$$

PROOF. We can assume without loss of generality that X is a complex subspace of a Stein open subset G of $\mathbf{C}^N$. We use the notations of § 8 B. For corresponding notations for X' we add a prime. Note that $(\mathcal{F}')^* = \mathcal{F}^*$.

Let $U_3 = U_2(\varrho')$. By Proposition 7.2 we can choose relatively compact Stein open subsets $D_1 \subset\subset D_2$ of G such that $U_3 \subset\subset D_1$ and $D_2 \cap X \subset U$. Choose a relatively compact open subset D of G such that $D_2 \subset D$ and $D \cap X = U$.

(i) Take $\varrho \leqq \varrho''$ and $f \in \Gamma(U(\varrho), \mathcal{F})$. By Proposition 8.3, $| \theta_D(f) |_{D_1, \varrho} \leqq$ $\leqq C_1 \| f \|_{U, \varrho}$, where C_1 is a constant. By Proposition 8.1, $\| (\theta'_D)^{-1}(\theta_D(f)) \|_{U'_3, \varrho} \leqq$ $\leqq C_2 | \theta_D(f) |_{D_1, \varrho}$, where C_2 is constant. Since $\sigma_U(f) = (\theta'_D)^{-1}(\theta_D(f))$, $\| \sigma_U(f) \|_{U'_3, \varrho} \leqq C_2 C_1 \| f \|_{U, \varrho}$. By Lemma 8.6, $\| \sigma_U(f) \|_{U'_1, \varrho} \leqq C_3 \| \sigma_U(f) \|_{U'_3, \varrho}$, where C_3 is a constant.

(ii) Take $\varrho \leqq \varrho''$ and $f' \in \Gamma(U'(\varrho), \mathcal{F})$. By Proposition 8.3, $| \theta'_D(f') |_{D_1, \varrho} \leqq$ $\leqq C'_1 \| f' \|_{U', \varrho}$, where C'_1 is a constant. By Proposition 8.1, $\| \theta_D^{-1}(\theta'_D(f)) \|_{U_3, \varrho} \leqq$ $\leqq C_2 | \theta'_D(f) |_{D_1, \varrho}$, where C'_2 is a constant. Since $\sigma_U^{-1}(f') = \theta_D^{-1}(\theta'_D(f'))$, $\| \sigma_U^{-1}(f') \|_{U_3, \varrho} \leqq C'_2 C'_1 \| f' \|_{U', \varrho}$. By Lemma 8.6, $\| \sigma_U^{-1}(f') \|_{U_1, \varrho} \leqq C'_3 \| \sigma_U^{-1}(f') \|_{U_3, \varrho}$, where C'_3 is a constant. q.e.d.

The following Proposition follows from Lemma 8.7.

PROPOSITION 8.7. Suppose $\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}$ are finite collections of Stein open subsets of X such that each member of $\mathfrak{U}_2, \mathfrak{U}$ is relatively compact in some Stein open subset of X. Suppose $\varrho'' < \varrho' \leqq \varrho^0$ in $\mathbf{R}^n_+$, $\mathfrak{U}_1 << \mathfrak{U}_2$, and $\mathfrak{U}_2(\varrho') < \mathfrak{U}$. Then for $r \in \mathbf{N}_*$ there exists $C \in \mathbf{R}_+$ such that, if $\varrho \leqq \varrho''$, then

(i) $$\| \sigma_{\mathfrak{U}}(f) \|_{\mathfrak{U}'_1, \varrho} \leqq C \| f \|_{\mathfrak{U}, \varrho} \text{ for } f \in C^r(\mathfrak{U}(\varrho), \mathcal{F}), \text{ and}$$

(ii) $$\| \sigma_{\mathfrak{U}}^{-1}(f) \|_{\mathfrak{U}_1, \varrho} \leqq C \| f' \|_{\mathfrak{U}', \varrho} \text{ for } f' \in C^r(\mathfrak{U}'(\varrho), \mathcal{F}').$$

REMARK. In the special case $X = X'$, Lemma 8.7 and Proposition 8.7 spell out the effect on Grauert norms when p is changed.

Suppose, in addition, $\pi'(X') \subset \{t_n = 0\}$ and $(t_n \circ \pi)\,\mathcal{O}' = 0$. For $\varrho = (\varrho_1, \dots, \varrho_n) \in \mathbf{R}_+^n$, we denote $(\varrho_1, \dots, \varrho_{n-1}) \in \mathbf{R}_+^{n-1}$ by $\bar{\varrho}$. We can regard π' as a holomorphic map from X' to $K^{n-1}(\bar{\varrho}^0)$. If $\varrho \leqq \varrho^0$ and $\mathfrak{V}' << \mathfrak{V}_1'$ are finite collections of Stein open subsets of X', then we have two Grauert norms: One is $\|\cdot\|_{\mathfrak{V}',\varrho}$ when we regard π' as a holomorphic map from X' to $K(\varrho^0)$; another is $\|\cdot\|_{\mathfrak{V}',\bar{\varrho}}$ when we regard π' as a holomorphic map from X' to $K^{n-1}(\bar{\varrho}^0)$. From the definition of Grauert norms, we can easily see that $(t_n \circ \pi)\,\mathcal{O}' = 0$ implies $\|\cdot\|_{\mathfrak{V}',\varrho} = \|\cdot\|_{\mathfrak{V},\bar{\varrho}}$.

Hence under these additional assumptions we have the following.

PROPOSITION 8.8. Suppose $\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}$, and ϱ'', ϱ' are the same as in Proposition 8.7. Then there exists $C \in \mathbf{R}_+$ such that, if $\varrho \leqq \varrho''$, then

(i) $$\|\sigma_{\mathfrak{U}}(f)\|_{\mathfrak{U}_1',\bar{\varrho}} \leqq C\,\|f\|_{\mathfrak{U},\varrho} \text{ for } f \in C^r(\mathfrak{U}(\varrho), \mathcal{F}), \text{ and}$$

(ii) $$\|\sigma_{\mathfrak{U}}^{-1}(f')\|_{\mathfrak{U}_1,\varrho} \leqq C\,\|f'\|_{\mathfrak{U}',\bar{\varrho}} \text{ for } f' \in C^r(\mathfrak{U}'(\varrho), \mathcal{F}').$$