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ON SPECTRAL PROJECTIONS OF ELLIPTIC OPERATORS

T. BURAK

Let A be an elliptic operator associated with a regular elliptic boundary value problem $(\mathcal{A}, \{B_j\}_{j=1}^{m}, G)$, cf. [1]. We investigate in this paper the existence of a direct sum decomposition: $H_0(G) = \overline{M^+} \oplus \overline{M^-}$ where $\overline{M^+}$ is the closed subspace of $H_0(G)$ spanned by the generalized eigenfunctions of A with eigenvalues in an infinite sector determined by two rays of minimal growth of the resolvent of A, and $\overline{M^-}$ is the closed subspace spanned by the other generalized eigenfunctions.

Assuming that there exist rays of minimal growth of the resolvent of A which divide the complex plane into angles less than $2m\pi/n$, so that by [1] the generalized eigenfunctions are complete in $H_0(G)$, and assuming in addition that the coefficients of $\mathcal{A}$, the boundary of G and the coefficients of the B_j's are C^∞ we prove that $H_0(G) = \overline{M^+} \oplus \overline{M^-}$ and that A is completely reduced by the pair of subspaces $\overline{M^+}$ and $\overline{M^-}$.

Let E^+ be the bounded projection of $H_0(G)$ on $\overline{M^+}$ determined by the above direct sum decomposition. If $D(A)$ is the domain of A, then $E^+ D(A) \subset D(A)$, moreover, for f in $D(A)$ we have $AE^+f = E^+Af$.

Furthermore let $\Gamma_R = \{\lambda;\ \theta_1 \leq \operatorname{Arg} \lambda \leq \theta_2,\ |\lambda| \geq R\}$ be the infinite sector under consideration and assume that the boundary γ_R of Γ_R is in the resolvent set of A. Let A^+ be the restriction of A to $\overline{M^+}$, and let $\sigma(A)$ and $\sigma(A^+)$ be the spectra of A and of A^+ respectively. Then $\sigma(A^+) = \sigma(A) \cap \Gamma_R$ and $(\lambda - A^+)^{-1} = 0\left(\frac{1}{|\lambda|}\right)$ as $\lambda \to \infty$ outside Γ_R.

Finally, let $f \in H_0(G)$. Then

$$\text{(I)} \qquad E^+f = -\lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma_R^n} (\lambda - A)^{-1} f\, d\lambda + \frac{\theta_2 - \theta_1}{2\pi} f$$

where γ_R^n is that part of γ_R which joins $ne^{i\theta_2}$ to $ne^{i\theta_1}$.

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Using results of R. T. Seeley [2], [3] we show first in Sections 2 and 3, that the operator B defined in $D(A)$ by

$$\text{(II)} \qquad Bf = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma_R^n} (\lambda - A)^{-1} f \, d\lambda$$

is bounded (it is easily seen that the limit on the right hand side of II exists for every f in $D(A)$). It is then shown in Section 4 that the above results follow from the boundedness of B.

1. Notations.

We will use the following notations:

R^n is the n dimensional Euclidean space and C^n is the n dimensional complex space ($R^1 = R$ and $C^1 = C$). Points in R^n are denoted by $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$ or by $x = (x', x_n)$ $x' \in R^{n-1}$ and $x_n \in R$. $x \cdot \xi$ denotes the scalar product in R^n. Also, let R_+^n be the closed half space given by $\{x; x_n \geq 0\}$, and let S_+^n be the closed half sphere $\{x \in R_+^n; |x| \leq r\}$. ∂S_+^n is the part of the boundary of S_+^n where $x_n = 0$.

We shall also employ the usual notation $D_i = i \dfrac{\partial}{\partial x_i}$ and $D = (D_1, \dots, D_n)$, α will stand for the multiindex $(\alpha_1, \dots, \alpha_n)$. In addition we shall write $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ and $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$.

Let G be a subset of R^n, and let $\mathscr{A}(x, D) = \sum_{|\alpha| \leq l} a_\alpha(x) D^\alpha$ be a linear differential operator with coefficients a_α defined in G. The principal part of $\mathscr{A}(x, D)$ given by $\sum_{|\alpha| = l} a_\alpha(x) D^\alpha$ is denoted by $\mathscr{A}'(x, D)$, the characteristic polynomial $\sum_{|\alpha| \leq l} a_\alpha(x) \xi^\alpha$ is denoted by $\mathscr{A}(x, \xi)$.

For a bounded domain G with closure $\bar{G}$ and boundary ∂G, $H_j(G)$ ($j = 0, 1, 2, \dots$) is the completion of the space of complex valued $C^\infty(\bar{G})$ functions under the norm $\left(\sum_{|\alpha| \leq j} \int |D^\alpha u|^2 \, dx \right)^{1/2}$. Given an elliptic boundary value problem $(\mathscr{A}, \{B_j\}_{j=1}^m, G)$, $H_{2m}(G, \{B_j\}_{j=1}^m)$ is defined as the completion in $H_{2m}(G)$ of the class of functions in $C^{2m}(\bar{G})$ satisfying the boundary conditions $B_j u = 0$ on ∂G. For any C^∞ function φ with support in G, M_φ is the operator mapping $C^\infty(G)$ into $C^\infty(G)$ defined by $M_\varphi f = \varphi f$.

Finally, given a C^∞ function f with compact support in R^n, $\tilde{f}$ will denote the Fourier transform of f: $\tilde{f}(\xi) = \int e^{-ix \cdot \xi} f(x) \, dx$.

2. Some Lemmas.

Consider the following boundary value problem in R^+:

$$\sum_{j=0}^{2m} \varphi_j D_t^j U = 0 \tag{2.1}$$

$$\sum_{j=1}^{2m} \psi_{ij} D_t^{j-1} U = g_i \quad \text{at} \quad t = 0, \; i = 1, \ldots, m. \tag{2.2}$$

$$U(t) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty. \tag{2.3}$$

Here $\varphi_j \in C$ for $j = 0, \ldots, 2m$; $\psi_{ij} \in C$ for $j = 1, \ldots, 2m$ $i = 1, \ldots, m$ and $g_i \in C$ for $i = 1, \ldots, m$. Assume also that the polynomial $\sum_{j=0}^{2m} \varphi_j \sigma^j$ has no real zeros.

Let $\beta_k \in C$ $k = 1, \ldots, 2m$. Then the β_k's are the initial values $D^{k-1} U(0)$ of a solution U of (2.1) (2.3) if and only if

$$\beta_k = \sum_{j=1}^{2m} P_{kj} \beta_j \tag{2.4}$$

where

$$P_{kj} = \int_{L^+} \sigma^{k-1} \left(\sum_{l=0}^{2m} \varphi_l \sigma^l \right)^{-1} \delta_j \, d\sigma \tag{2.5}$$

L^+ is a closed curve in the half plane $\operatorname{Im} \sigma > 0$ enclosing all the roots of $\sum_{l=0}^{2m} \varphi_j \sigma^j = 0$ which lie in this half plane; δ_j is a polynomial in $\varphi_1, \ldots, \varphi_{2m}, \sigma$, of degree less than $2m$ such that the relation

$$\frac{1}{2\pi} \int_0^\infty e^{-it\sigma} \sum_{j=0}^{2m} \varphi_j D_t^j U(t) \, dt = - \sum_1^{2m} \delta_j D^{j-1} U(0) + \sum_{j=0}^{2m} \varphi_j \sigma^j \frac{1}{2\pi} \int_0^\infty e^{-it\sigma} U(t) dt \tag{2.6}$$

holds for every C^∞ function U with compact support in R_+, cf. [2].

Assume that for every $g = (g_1, \ldots, g_m)$ in C^m (2.1)-(2.3) has a unique solution. Then the matrix $\psi = (\psi_{ij})$ defines a one-to one linear transformation from the range of $P = (P_{ij})$ onto C^m. Let $\mathcal{E}$ be the inverse of this transformation.

We quote the following lemma which is proved in [2].

I. The set of $\varphi = (\varphi_0, \dots, \varphi_{2m})$ such that half the zeros of $\sum_{j=0}^{2m} \varphi_j \sigma^j$ lie above the real axis and half lie below is an open set and the projection P is an analytic function of φ on this set.

II. The set of values (φ, ψ) such that (2.1)-(2.3) has a unique solution for every choice of g_i, $i = 1, \dots, m$, is an open set and $\mathcal{E}$ is an analytic function of (φ, ψ) on this set.

III. The solution U of (2.1)(2.3) is given by:

$$U(t) = \frac{1}{2\pi} \int_{L^+} e^{-it\sigma} \left(\sum_{k=0}^{2m} \varphi_k \sigma^k \right)^{-1} \delta\, \mathcal{E}\, g\, d\sigma \tag{2.7}$$

where $\delta = (\delta_1, \dots, \delta_{2m})$ and δ_j is as in (2.6).

In particular, if $U_j(t)$ is the solution of (2.1)-(2.3) with $g_i = 0$ for $i \neq j$ and $g_j = 1$, and if $\mathcal{E}$ is given by the $2m \times m$ matrix (ε_{ij}) then

$$U_j(t) = \sum_{i=1}^{2m} \int_{L^+} e^{it\sigma} \left(\sum_{k=0}^{2m} \varphi_k \sigma^k \right)^{-1} \delta_i\, \varepsilon_{ij}\, d\sigma. \tag{2.8}$$

Let $\mathcal{A}(x, D)$ be an elliptic homogeneous differential operator, of even degree $2m$, with C^∞ coefficients in an R^n_+ neighborhood of S^n_+. Let $B_j(x, D)$, $j = 1, \dots, m$, be a system of m homogeneous differential operators with C^∞ coefficients in an R^n neighborhood of ∂S^n_+.

We make the following assumptions:

ASSUMPTION A. The B_j's form a normal system of boundary operators; i. e. the boundary ∂S^n_+ is noncharacteristic to B_j at each point and the orders of the different operators B_j are distinct.

In the case $n = 2$ we assum in addition that $\mathcal{A}$ satisfies the roots condition.

ASSUMPTION B. For every $x \in \partial S^n_+$ and $\xi' \neq 0$ the polynomials in τ, $B_j(x, \xi', \tau)$, $j = 1, \dots, m$, are linearly independent modulo the polynomial $\prod_{k=1}^{m} (\tau - \tau_k^+(x, \xi'))$, where $\tau_k^+(x, \xi')$, $k = 1, \dots, m$, are the m roots of $\mathcal{A}(x, \xi'\tau) = 0$ with positive imaginary parts.

Finally we assume that for $\theta = \theta_i$, $i = 1, 2$, the following conditions hold, cf. [1].

CONDITION I. $(-1)^m \mathscr{A}(x, \xi) \neq \lambda$ for every λ with $\operatorname{Arg} \lambda = \theta$, $x \in S_+^n$, and $\xi \neq 0$.

Let $\theta_1 < \theta_2$ and write $\Gamma = \{\lambda, \theta_1 \leq \operatorname{Arg} \lambda \leq \theta_2\}$. From the assumptions on $\mathscr{A}$ it readily follows that $(-1)^m \mathscr{A}(x, \xi) \neq \lambda$ for either all $\lambda \in \Gamma$ or all $\lambda \notin \Gamma$. We may thus assume without loss of generality that $(-1)^m \mathscr{A}(x, \xi) \neq \lambda$ for all $\lambda \in \Gamma$.

CONDITION II. The polynomials in $\tau\ B_j(x, \xi', \tau)$, $j = 1, \ldots, m$, are linearly independent modulo the polynomial $\prod_{k=1}^{m} (\tau - \tau_k^+(x, \xi', \lambda))$, where for every $x \in \partial S_+^n$ $\xi' \neq 0$ and λ with $\operatorname{Arg} \lambda = \theta$, $\tau_k^+(x, \xi', \lambda)$ $k = 1, \ldots, m$ are the m roots of $(-1)^m \mathscr{A}(x, \xi', \tau) - \lambda = 0$ with positive imaginary parts.

Let $\Delta = \{q\,;\ \theta_1/2m \leq \operatorname{Arg} q \leq \theta_2/2m\}$ and let ∂ be the boundary of Δ orineted from $\infty e^{i\theta_2}$ to $\infty e^{j\theta_1}$. For $|x'| \leq r$ and $q \in \partial$ let

$$(2.9) \qquad \mathscr{A}(x', 0, \xi', D_n) - q^{2m} = \sum_{j=0}^{2m} \varphi_j(x', \xi', q)\, D_n^j$$

and

$$(2.10) \qquad B_i(x', 0, \xi', D_n) = \sum_{j=1}^{2m} \psi_{ij}(x', \xi')\, D_n^{j-1}.$$

The assumptions on $\mathscr{A}$ and on the B_j's guarantee that, provided that $|\xi'|^2 + |q|^2 \neq 0$ the system (2.1)-(2.3) with $\varphi_j = \varphi_j(x', \xi' q)$ and $\psi_{ij} = \psi_{ij}(x', \xi')$ has a unique solution $u(t, x', \xi' q)$, for every g in C^m.

$u(t, x', \xi', q)$ is given by (2.7) where $L^+ = L^+(\xi', q)$ represents the boundary of $\left\{\frac{1}{2} r(|\xi'|^2 + |q|^2)^{1/2} < \operatorname{Im} \tau,\ |\tau| < 2 r_2 (|\xi'|^2 + |q|^2)^{1/2}\right\}$ and r_1, r_2 are chosen so that the roots of $\mathscr{A}(x', 0, \xi', \tau) - q^{2m} = 0$ in the upper half plane satisfy $\{r_1(|\xi'|^2 + |q|^2)^{1/2} < \operatorname{Im} \tau,\ |\tau| < r_2(|\xi|^2 + |q|^2)^{1/2}\}$. As observed in [2], $\delta_j = \delta_j(x', \xi', \sigma, q)$ is homogeneous of degree $2m - j$ in (σ, ξ', q) while $\varepsilon_{ij} = \varepsilon_{ij}(x', \xi', q)$ is homogeneous of degree $i - m_j - 1$ in (ξ', q).

DEFINITION 2.1. For $|x'| \leq r$, $q \in \partial$ and $|q|^2 + |\xi'|^2 \neq 0$ let $\Omega_j(x', \xi', q, t)$ be the solution of the system (2.1)-(2.3) with $g_i = \delta_{ij}$ (here δ_{ij} is Kronecker's delta).

The $\Omega_j(x', \xi', q, t), j = 1, \ldots, m$, form a basis for the space of the solutions of (2.1) which tend to zero at infinity. Also by (2.8)

$$(2.11) \quad \Omega_j(x', \xi', q, t) = \sum_{i=1}^{2m} \int_{L^+(\xi', q)} e^{it\sigma} (\mathscr{A}(x', 0, \xi', \sigma) - q^{2m})^{-1} \delta_i(x', \xi', \sigma, q)\, \varepsilon_{ij}(x', \xi', q)\, d\sigma.$$

From (2.11) recalling the definition of $L^+(\xi', q)$ and keeping in mind the respective homogeneities of $(\mathscr{A}(x', 0, \xi', \sigma) - q^{2m})^{-1}$, $\delta_j(x', \xi', \sigma, q)$ and $\varepsilon_{ij}(x', \xi', q)$ in (ξ', σ, q), it follows that there exist constants c_1 and c_2 such that

$$|\Omega_j(x', \xi', q, t)| \leq c_1(|\xi'|^2 + |q|^2)^{-m_j/2} e^{-c_2 t(|\xi'|^2 + |q|^2)^{1/2}} \tag{2.12}$$

DEFINITION 2.2 For $|x'| \leq r, q \in \partial, |\xi'|^2 + |q|^2 \neq 0$ let $d(x', \xi, q, t)$ be the (unique) solution of

$$(\mathscr{A}(x', 0, \xi', D_n) - q^{2m})\, d(x', \xi, q, t) = 0 \tag{2.13}$$

$$B_i(x', 0, \xi' D_n)\, d(x', \xi, q, t) = B_i(x', 0, \xi)(\mathscr{A}(x', 0, \xi) - q^{2m})^{-1} \text{ at } t = 0 \tag{2.14}$$

$$d(x', \xi, q, t) \longrightarrow 0 \text{ as } t \longrightarrow \infty \text{ cf. [2].} \tag{2.15}$$

It is easily seen that

$$d(x', \xi, q, t) = \sum_{i=1}^{m} B_i(x', 0, \xi)(\mathscr{A}(x', 0, \xi) - q^{2m})^{-1}\, \Omega_i(x', \xi', q, t). \tag{2.16}$$

DEFINITION 2.3. Let $\theta_1(\xi', q)$ be an infinitely differentiable function in ξ', Re q and Im q such that $\theta_1(\xi', q) = 0$ whenever $|\xi'|^2 + |q|^2 \leq 1/2$ and $\theta_1(\xi', q) = 1$ whenever $|\xi'|^2 + |q|^2 \geq 1$. For $|x'| \leq r$ and $t > 0$ let $k(x', \xi, t)$ be defined by

$$k(x', \xi, t) = \frac{2m}{2\pi i} \int_{\partial} q^{2m-1} \theta_1(\xi', q)\, d(x', \xi, q, t)\, dq. \tag{2.17}$$

The convergence of the right hand side of (2.17) follows from (2.12) and (2.16). Note also that as can easily be verified $k(x', \xi, t)$ is infinitely differentiable.

DEFINITION 2.4. Let K be the operator mapping the C^∞ functions with compact support in R^n_+ into the C^∞ functions in $\{x; |x'| < r, x_n > 0\}$ defined by

$$Kf(x', x_n) = (2\pi)^{-n} \int e^{ix'\xi'} k(x', \xi, x_n) \tilde{f}(\xi)\, d\xi. \tag{2.18}$$

It follows from (2.16), (2.17) and (2.18) that for every $\delta > 0$ there exists a constant C_δ such that if $x_n \geq \delta$ and $|x'| < r$ then

$$|Kf(x', x_n)| \leq C_\delta \, \|f\| \tag{2.19}$$

where $\|f\|$ denotes the $L_2(R^n)$ norm of f.

The following lemma constitutes the main result of this section.

LEMMA 2.1. Let φ' be a C^∞ function in R^{n-1} with support in $\{x'; |x'| < r\}$. Let θ be a function in $C^\infty(R)$ with compact support in R_+. Let φ be the function in $C^\infty(R^n)$ given by $\varphi(x) = \varphi'(x')\,\theta(x_n)$. There exists a constant C such that if $f \in C^\infty(R^n)$ and has compact support in R^n_+, then

$$\| M_\varphi Kf \|_+ \leq C \, \|f\| . \tag{2.20}$$

Note that if $g \in C^\infty(R^n_+)$ then $\|g\|_+$ denotes the norm of g in $L_2\{x; x_n > 0\}$.

To prove this result, we shall first establish a number of lemmas.

Consider the following boundary value broblem in R_+.

$$(\mathscr{A}(x', 0, \xi', D_n) - q^{2n})\, U = 0 \tag{2.21}$$

$$D_t^{i-1}\, U(t) = g_i \text{ at } t = 0, i = 1, \dots, m, \tag{2.22}$$

$$U(t) \to 0 \text{ as } t \to \infty . \tag{2.23}$$

Since condition I holds for all $\lambda \in \Gamma$, the above system has a unique solution for every $g = (g_1, \dots, g_m)$ in C^m, provided that $|x'| \leq r$, $q \in \Delta$ and $|q| + |\xi'|^2 \neq 0$.

DEFINITION 2.5. For $|x'| \leq r$, $q \in \Delta$ and $|q|^2 + |\xi'|^2 \neq 0$ let $W_j(x', \xi', q, t)$ be the solution of (2.21)-(2.23) pertaining to the value $g_i = \delta_{ij}$. The functions $W_j(x', \xi', q, t), j = 1, \dots, m$, span the space of those solutions of (2.21) which tend to zero at infinity. Furthermore $W_j(x', \xi', q, t)$ is given by (2.7) where the $\varphi_j = \varphi_j(x, \xi', q)$ are determined by (2.9) and $\psi_{ij} = \delta_{ij}$. As a result the corresponding $\delta_i(x', \xi', \sigma, q)$ is homogeneous of degree $2m - i$ in (σ, ξ', q) and the corresponding $\varepsilon_{ij}(x', \xi', q)$ is homogeneous of degree $i - j$ in (ξ', q). Also the $W_j(x', \xi', q, t)$'s, for $|x'| \leq r$, $q \in \Delta$, $|\xi'|^2 + |q|^2 \neq 0$ and $t \geq 0$, are infinitely differentiable in x', ξ', Re q, Im q and t and are analytic functions of q in the interior of Δ.

Now, as the $W_j(x' \xi', q, t)$, $j = 1, \dots, m$ form a basis for the solutions of (2.21) which tend to zero at infinity the system

$$(\mathscr{A}(x', 0, \xi', D_n) - q^{2m})\, U = 0 \tag{2.24}$$

(2.25) $$B_i(x', 0, \xi', D_n)\, U(t) = g_i \quad \text{at } t = 0, i = 1, \dots, m$$

(2.26) $$U(t) \to 0 \text{ as } t \to \infty$$

with $|x'| \leq r$, and $|q|^2 + |\xi'|^2 \neq 0$ has a unique solution for every choice of g_i if and only if

(2.27) $$\det(B_i(x', 0, \xi', D_n)\, W_j(x', \xi', q, t)) \neq 0 \text{ at } t = 0.$$

We shall denote the value at $t = 0$ of the left hand side of (2.27) by $D(x', \xi', q)$.

It follows from the definition of D from 2.27 and the above properties of the W_j's that $D(x', \xi', q)$ is infinitely differentiable in x', ξ' Re q and Im q and is analytic in q throughout the interior of Δ. Also, as is easily seen $D(x', \xi', q)$ is homogeneous of degree $\sum_{j=1}^{m} m_j - \frac{m(m-1)}{2}$ in (ξ', q).

We now have:

Lemma 2.2. There exists a $\varrho > 0$ such that for all $|x'| \leq r, q \in \Delta$ and provided $D(x', \xi', q) = 0$ then $|q| < \varrho |\xi'|$.

Proof. As is shown in [1] it follows from Assumption A and Condition I that for all $q \in \Delta$, $q \neq 0$ and for all $|x'| \leq r$ the boundary value problem (2.24)-(2.26) with $\xi' = 0$ has a unique solution for every choice of g. Consequently one has

(2.28) $$D(x', 0, q) \neq 0 \text{ for all } q \in \Delta, q \neq 0, |x'| \leq r.$$

The existence of ϱ now follows from 2.28 and from the continuity of $D(x', \xi', q)$ in the region $|x'| \leq r$, $q \in \Delta$ and $|q|^2 + |\xi'|^2 \neq 0$ combined with the homogeneity of $D(x', \xi', q)$ in (ξ', q).

In view of the above we can extend definitions (2.1) and (2.2) as follows:

Definition 2.6. Let $|x'| \leq r, q \in \Delta, |q|^2 + |\xi'|^2 \neq 0$ and $D(x', \xi', q) \neq 0$ $\Omega_j(x', \xi', q, t)$ is defined as the solution (2.24)-(2.26) with $g_i = \delta_{ij}$, $i = 1, \dots, m$, and $d(x', \xi, q, t)$ as the solution of that system with

$$g_i = B_i(x', 0, \xi)\,(\mathcal{A}(x', 0, \xi) - q^{2m})^{-1}.$$

We have,

(2.29) $$\Omega_j(x', \xi', q, t) = \sum_{k=1}^{m} \frac{\alpha_{jk}(x', \xi', q)}{D(x', \xi', q)}\, W_k(x', \xi', q, t)$$

The $\alpha_{jk}(x', \xi', q)'s$ are infinitely differentiable in x', ξ', $\operatorname{Re} q$ and $\operatorname{Im} q$ and homogeneous in (ξ', q) of degree $\sum_{i=1}^{m} m_i - \frac{m(m-1)}{2} - m_j + k - 1$ and the $\alpha_{jk}(x', \xi', q)$'s are analytic in q in the interior of Δ. Also (2.16) continues to hold.

DEFINITION 2.7. Let $\psi(\xi')$ be a C^∞ function such that $\psi(\xi') = 0$ when $|\xi'| \leq 1$ and $\psi(\xi') = 1$ when $|\xi'| \geq 2$. Let $\chi = 1 - \psi$. For f in $C^\infty(R^n)$ with compact support in R^n_+, $|x'| \leq r$ and $x_n > 0$ we define:

$$K_1 f(x', x_n) = (2\pi)^{-n} \int e^{ix'\xi'} \psi(\xi') k(x', \xi, x_n) \tilde{f}(\xi)\, d\xi. \tag{2.30}$$

$$K_2 f(x', x_n) = (2\pi)^{-n} \int e^{ix'\xi'} \chi(\xi') k(x', \xi, x_n) \tilde{f}(\xi)\, d\xi \tag{2.31}$$

where $k(x', \xi, x_n)$ is given by 2.17. Note that the operator K, defined by Definition 2.4, which appears in Lemma 2.1 equals $K_1 + K_2$. For K_2 we have

LEMMA 2.3. Let f be a function in $C^\infty(R^n)$ with compact support in R^n_+. There exists a constant C such that if $|x| < r$ and $x_n > 0$, then

$$|K_2 f(x', x_n)| \leq C \|f\|. \tag{2.32}$$

PROOF. Recalling the definitions of K_2 and χ one can see that it suffices to show that there exists a constant C such that

$$|\chi(\xi') k(x', \xi, x_n)| \leq C(1 + |\xi_n|)^{-1}. \tag{2.33}$$

The homogeneity of $D(x', \xi', q)$ and Lemma 2.2 imply that if $|x'| \leq r$, $q \in \Delta$ and $|q| > 2\varrho |\xi'|$, then

$$|D(x', \xi', q)| \leq C(|\xi'|^2 + |q|^2)^{-\sum_{j=1}^{m} m_j + \frac{m(m-1)}{2}}. \tag{2.34}$$

Also

$$|W_j(x', \xi', q, x_n)| \leq C(|\xi'|^2 + |q|^2)^{-j-1/2} e^{-c_1 x_n (|\xi'|^2 + |q|^2)^{1/2}} \tag{2.35}$$

as can be seen from (2.11) by substituting $B_j = D_n^{j-1}$ in the system (2.1)-(2.3). From (2.34), (2.35) and the properties of the α_{jk}'s one can deduce that for $|x'| < r$ and $x_n > 0$

(2.36) $$\lim_{R\to\infty} \int_{\Delta \cap \{q\,;\,|q|=R\}} q^{2m-1}\, d(x', \xi, q, x_n)\, dq = 0.$$

Therefore if $|\xi'| \leq 2$

(2.37) $$K(x', \xi, x_n) = \frac{2m}{2\pi i} \int_{\gamma} q^{2m-1}\, \theta_1(\xi', q)\, d(x', \xi, q, x_n)\, dq$$

where γ is the boundary of $\{q\,;\, q \in \Delta, |q| \leq M\}$ and $M = \max(1, 4\varrho)$. The conclusion (2.33) can now be easily verified by considering (2.37) together with (2.16), the estimates (2.34) and (2.35), and the properties of the α_{jk}'s.

Lemma 2.4. Let φ' be a function in $C^\infty(R^{n-1})$ with support in $|x'| < r$. There exists a constant C such that if $f \in C^\infty(R^n)$ and has compact support in R^n_+ then

(2.38) $$\| M_{\varphi'} K_1 f \| \leq C \| f \|.$$

As in the proof of Lemma 2.3 we notice that if $|\xi'| \geq 2$

(2.39) $$k(x', \xi, x_n) = \frac{2m}{2\pi i} \int_{\gamma(\xi')} q^{2m-1}\, d(x', \xi, q, x_n)\, dq$$

where $\gamma(\xi')$ is the boundary of $\{q\,;\, q \in \Delta,\ |q| \leq 2\varrho\,|\xi'|\}$. For $q \in \gamma(\xi')$ we have $D(x', \xi', q) \neq 0$, hence (2.16) and (2.29) are valid and consequently

(2.40) $$k(x', \xi, x_n) = \Sigma_{ij}\, k_{ij}(x', \xi, x_n)$$

with

(2.41) $$k_{ij}(x', \xi, x_n) =$$

$$B_i(x', 0, \xi) \cdot \frac{2m}{2\pi i} \int_{\gamma(\xi)} q^{2m-1} (\mathcal{A}(x', 0, \xi) - q^{2m})^{-1}\, \alpha_{ij}(x', \xi', q)\, D(x', \xi', q)^{-1}\, W_j(x', \xi', q, x_n)\, dq.$$

From equation (2.11) by inserting $B_j = D_n^{j-1}$ in the system (2.1)-(2.3).

(2.42) $$W_j(x', \xi', q, x_n) = \int_{L^+(\xi', q)} e^{i\mu x_n} (\mathcal{A}(x', 0, \xi', \mu) - q^{2m})^{-1} M_j(x', \xi', \mu, q)\, d\mu$$

where $M_j(x', \xi', \mu, q)$ is homogeneous of degree $2m - j$ in (ξ', μ, q).

Thus

(2.43) $k_{ij}(x', \xi, x_n) =$

$$B_i(x',0,\xi)\frac{2m}{2\pi i}\int\limits_{L^+(\xi')} e^{i\mu x_n}\left(\int\limits_{\gamma(\xi')} q^{2m-1}\frac{\alpha_{ij}(x',\xi',q)\,(M_j(x',\xi',\mu,q)\,dq}{(\mathscr{A}(x',0,\xi)-q^{2m})\,D(x',\xi',q)(\mathscr{A}(x',0,\xi',\mu)-q^{2m})}\right)d\mu$$

where $L^+(\xi')$ is a finite contour in the half plane $\operatorname{Im}\mu > 0$ which encloses all the μ roots with positive imaginary part of $\mathscr{A}(x', 0, \xi'\,\mu) - q^{2m} = 0$ for x' and q restricted by $|x'| \leq r$, $q \in \Delta$ and $|q| \leq 2\varrho\,|\xi'|$. Also $L^+(\xi')$ is so chosen that for all $\mu \in L^+(\xi')$

$$\operatorname{Im}\mu > c\,|\xi'|\,. \tag{2.44}$$

It follows from (2.43) that the singularities of $k_{ij}(x', \xi, x_n)$ in the half plane $\operatorname{Im}\xi_n < 0$ are contained inside a closed contour $L^-(\xi')$, which is the boundary of $\{\xi_n\,;\ \operatorname{Im}\xi_n < -C_1|\xi'|,\ |\xi_n| < C_2|\xi'|\}$.

By a method similar to the one employed in [2] we put:

$$\widehat{k}(x', \xi', s, x_n) = \int\limits_{L^-(\xi')} e^{-is\xi_n}\,k(x', \xi, x_n)\,d\xi_n\,, \tag{2.45}$$

and for $f \in C^\infty(R^n)$ with compact support in R^n_+ we set

$$\widehat{f}(\xi', s) = \int e^{-i\xi' y'} f(y', s)\,dy'\,. \tag{2.46}$$

Then

$$M_{\varphi'}\,Kf(x', x_n) = (2\pi)^{-n}\iint e^{i\xi' x'}\psi(\xi')\,\varphi'(x')\widehat{k}(x',\xi',s',x_n)\widehat{f}(\xi'(s)\,d\xi'\,ds. \tag{2.47}$$

Let

$$\widehat{k}_{ij}(x', \xi, s, x_n) = -\int\limits_{L^-(\xi')} e^{-is\xi_n}\,k_{ij}(x', \xi, x_n)\,d\xi_n\,. \tag{2.48}$$

It follows then from 2.43, from the definition of $L^+(\xi')$ and $L^-(\xi')$ and from the properties of B_i, α_{ij}, M_j and D that for every $\alpha' = (\alpha_1, \dots, \alpha_{n-1})$ there exist C_1 and C_2 such that

$$|D^{\alpha'}_{x'}\,\psi(\xi')\,\varphi'(x')\,\widehat{k}_{ij}(x', \xi, s, x_n)| \leq C_1\,|\xi'|\,e^{-C_2|\xi'|(x_n+s)}\,. \tag{2.49}$$

Let

$$L(\eta', \xi', s, x_n) = \int e^{ix'(\xi'-\eta')} \varphi'(x') \psi(\xi') \widehat{k}(x', \xi', s, x_n)\, dx'. \tag{2.50}$$

Then

$$\int e^{-i\eta' x'} M_{\varphi'} K_1 f(x', x_n)\, dx' = \int L(\eta', \xi', x_n, s) \widehat{f}(\xi', s)\, d\xi'\, ds\,. \tag{2.51}$$

From (2.50), (2.45), (2.40), (2.48), and (2.49) one concludes that for every p there exist C_1 and C_2 such that

$$|L(\eta', \xi' s, x_n)| \leq C_1 (1 + |\xi' - \eta'|^2)^{-p} e^{-C_2 |\xi'|(s+x_n)} |\xi'|. \tag{2.52}$$

Therefore for every p there exists a constant C such that

$$\int_0^\infty |L(\eta', \xi' s, x_n)|\, dx_n \leq C(1 + |\xi' - \eta'|^2)^{-p} \tag{2.53}$$

and

$$\int_0^\infty |L(\eta' \xi', s, x_n)|\, ds \leq C(1 + |\xi' - \eta'|^2)^{-p}\,. \tag{2.54}$$

Now choosing $2p > n + 1$, we obtain

$$\iint |L(\eta', \xi', s, x_n)|\, d\eta'\, dx_n \leq C \tag{2.55}$$

and

$$\iint |L(\eta', \xi', x_n, s)|\, d\xi'\, ds \leq C. \tag{2.56}$$

Finally, using the above inequalities together with (2.51) and Plancheral's formula, proposition (2.38) follows.

Lemma 2.1, the result that we set to prove in the beginning of this sections, is now obtained as an immediate consequence of Lemma 2.3 and Lemma 2.4.

3. The boundedness of B.

We shall assume throughout this section and the following one that $(\mathscr{A}, \{B_j\}_{j=1}^m G)$ is a regular elliptic boundary value problem. We assume in addition that the coefficients of $\mathscr{A}$ are in $C^\infty(\overline{G})$ that the boundary ∂G is C^∞, and that the B_j's are C^∞ in a neighborhood of ∂G.

Let us suppuse further that for every λ such that $\operatorname{Arg}\lambda = \theta_i$, $i = 1, 2$, $-\pi < \theta_1 \leq 0$ $0 < \theta_2 \leq \pi$, Agmon's Conditions I' and II' below are satisfied

Condition I'. $\mathscr{A}'(x, \xi) \neq \lambda$

Condition II'. At any point x of ∂G, let v be the normal vector and let $\xi \neq 0$ be any real vector parallel to ∂G. Let $t_k^+(\xi, \lambda)$ $k = 1, \ldots, m$ be the m roots with positive imaginary parts of the polynomial in t $(-1)^m \mathscr{A}'(x, \xi + tv) - \lambda$ where λ is on the ray $\operatorname{Arg}\lambda = \theta_i$. Then the polynomials in t $B_j'(x, \xi + tv)$ are linearly independent modulo the polynomial $\pi_{k=1}^m (t - t_k^+(\xi, \lambda))$.

As before we may assume without loss of generality that Condition I' holds for $\theta_1 \leq \operatorname{Arg}\lambda \leq \theta_2$.

Let A be the unbounded linear operator in $H_0(G)$ defined as follows: The domain, $D(A)$, of A is $H_{2m}(\{B_j\}_{j=1}^m\, G)$. For every $f \in D(A)$, $Af = \mathscr{A}(xD)f$.

It follows from [1] that the spectrum, $\sigma(A)$, of A is discrete and that $\operatorname{Arg}\lambda = \theta_i$ is a ray of minimal growth of the resolvent of A; i.e., there exist constants C and M such that for $\operatorname{Arg}\lambda = \theta_i$ and $|\lambda| > M$, λ is in the resolvent set of A and

$$\|(\lambda - A)^{-1}\| \leq \frac{C}{|\lambda|}. \tag{3.1}$$

DEFINITION 3.1. Let B be the linear operator from $D(A)$ to $H_0(G)$ given by

$$Bf = \frac{1}{2\pi i} \lim_{n\to\infty} \int_{\gamma_R^n} (\lambda - A)^{-1} f\, d\lambda \tag{3.2}$$

where R and γ_R^n are defined as in the introduction.

The existence of the limit in the right hand side of (3.2) follows from (3.1) and from the relation

$$(\lambda - A)^{-1} f = \frac{(\lambda - A)^{-1} Af}{\lambda} + \frac{f}{\lambda} \tag{3.3}$$

which holds for every $f \in D(A)$.

The main result of this section is

THEOREM 3.1. There exists a constant C such that for every $f \in D(A)$,

$$\|Bf\|_{H_0(G)} \leq C\, \|f\|_{H_0(G)}. \tag{3.4}$$

In the proof of Theorem 3.1 we will use the prarametrix $P_0(\lambda)$ of $(\lambda - A)^{-1}$ which was constructed by R. T. Seeley [2] as follows:

Let us assume first that $\mathscr{A}(x, D)$ is an elliptic differential operator of even degree $2m$ with C^∞ coefficients in an R^n_+ neighborhood of S^n_+. Also suppose that $B_j(x, D)$, $j = 1, \dots, m$, is a system of m linear differential operators of respective degrees m_j with C^∞ coefficients in an R^n neighborhood of ∂S^n_+, and that $\mathscr{A}'(x, D)$ and the $B_j'(x, D)$ satisfy assumptions A and B and Conditions I and II of Section 2.

Let $\theta_1(\xi', q)$ be a C^∞ function in ξ' Re q and Im q such that $\theta_1(\xi', q) = 0$ when $|\xi'|^2 + |q|^2 \leq 1/2$ and $\theta_1(\xi', q) = 1$ when $|\xi'|^2 + |q|^2 \geq 1$.

Let $\theta_2(\xi, q)$ be a C^∞ function in ξ, Re q and Im q such that $\theta_2(\xi, q) = 0$ when $|\xi|^2 + |q|^2 \leq 1/2$ and $\theta_2(\xi, q) = 1$ when $|\xi|^2 + |q|^2 \geq 1$.

Let f be a C^∞ function with compact support in the interior of R^n_+. For $q \in \varDelta$, $q \in \partial$, we introduce respectively the one parameter families of operators $C(q)$, $D(q)$:

$$C(q) f(x) = (2\pi)^{-n} \int e^{ix\xi}\, \theta_2(\xi, q)\, (\mathscr{A}'(x, \xi) - q^{2m})^{-1} \tilde{f}(\xi)\, d\xi \tag{3.5}$$

$$D(q) f(x) = (2\pi)^{-n} \int e^{ix'\xi'}\, \theta_1(\xi', q)\, d(x', \xi, q, x_n)\, \tilde{f}(\xi)\, d\xi \tag{3.6}$$

where $d(x', \xi, q, x_n)$ is given by substituting $\mathscr{A}'(x, D)$ and $B_j'(x, D)$ for $\mathscr{A}(x, D)$ and $B_j(x, D)$ respectively in Definition 2.2.

Let $Q = \{x; x_n > 0, |x'| < r\}$. As in [2] there exists a constant C such that for $q \in \varDelta$

$$\| C(q) f \|_{L_2(Q)} \leq C (1 + |q|)^{-2m} \|f\| \tag{3.7}$$

and for $q \in \delta$

$$\| D(q) f \|_{L_2(Q)} \leq C (1 + |q|^{-2m} \|f\|_+ . \tag{3.8}$$

In the general case, let $\varphi_j, j = 1, \dots, N$, be a resolution of the identity subordinate to a covering of $\bar{G}$ by coordinate patches. Suppose that the coordinates in a coordinate patch that intersects the boundary assume values in S^n_+ and $x_n = 0$ on the boundary. Let $\psi_j, j = 1, \dots, N$, have support in the same patches and let $\psi_j = 1$ in a neighborhood of the support of φ_j.

When the support of φ_j does not intersect the boundary of G and for $q \in \varDelta$, define

$$P_0(q, \varphi_j, \psi_j) = M_\psi\; C(q)\, M_{\varphi_j} \tag{3.9}$$

with $C(q)$ as defined in (3.5).

When the support of φ_j intersects ∂G and for $q \in \partial$, define

$$P_0(q, \varphi_j, \psi_j) = M_{\psi_j}(C(q) - D(q)) M_{\varphi_j}. \tag{3.10}$$

Then for $q \in \partial$, $P_0(q)$ of [2] is given by

$$P_0(q) = \sum_{j=1}^{N} P_0(q, \varphi_j, \psi_j) \tag{3.11}$$

and there exists a constant C such that if f is C^∞ with support in G and $q \in \partial$,

$$\| P_0(q) f \|_{H_0(G)} \leq C(1 + |q|)^{-2m} \| f \|_{H_0(G)}. \tag{3.12}$$

It is proved in [2] that there exists a constant C such that if $q \in \partial$ $|q|^{2m} \geq R$, and f is C^∞ with support in G then

$$\| ((q^{2m} - A)^{-1} - P_0(q)) f \|_{H_0(G)} \leq C(1 + |q|)^{-(2m+1)} \| f \|_{H_0(G)}. \tag{3.13}$$

It follows from (3.13) and from the existence of the limit in the right hand side of (3.2), that for every $f \in D(A)$, the limit $\lim_{n\to\infty} \frac{2m}{2\pi i} \int_{\partial^n} q^{2m-1} P_0(q) f dq$ exists. Here $P_0(q)$ denotes the extension of $P_0(q)$ of (3.11) to a bounded operator from $H_0(G)$ to $H_0(G)$.

DEFINITION 3.2. Let B_0 be the linear operator from $D(A)$ to $H_0(G)$ given by

$$B_0 f = \lim_{n\to\infty} \frac{2m}{2\pi i} \int_{\partial^n} q^{2m-1} P_0(q) f dq. \tag{3.14}$$

Using (3.13) it becomes evident that the proof of Theorem 3.1 is reduced to a verification of the following lemma:

LEMMA 3.1. There exists a constant C such that if $f \in D(A)$ then

$$\| B_0 f \|_{H_0(G)} \leq C \| f \|_{H_0(G)}. \tag{3.15}$$

The proof of Lemma 3.1 is based on the following two lemmas:

LEMMA 3.2. Let ψ_ε be C^∞ with compact support in $\{x; x_n \geq \varepsilon\}$. The limit $\lim_{n\to\infty} \frac{2m}{2\pi i} \int_{\partial^n} M_{\psi_\varepsilon} D(q) q^{2m-1} f dq$ exist in $H_0(S_+^n)$, for every $f \in H_0(S_+^n)$ and is equal to $M_{\psi_\varepsilon} Kf$, where K is the extension of the operator defined by (2.18) to a bounded operator from $H_0(S^n_+)$ to $H_0(S^n_+)$.

Proof. For every C^∞ function f with support in S^n_+ we have

$$\lim_{n\to\infty} \frac{2m}{2\pi i} \int_{\partial n} M_{\psi_\varepsilon} q^{2m-1} D(q) f dq = M_{\psi_\varepsilon} Kf, \tag{3.16}$$

as follows from a consideration of relations (3.6), (2.16) and (2.12). Thus, to conclude the proof we need only to check that for every $f \in H_0(S^n_+)$, the limit appearing on the left-hand side of (3.16) exist (in $H_0(S^n_+)$). This, however, follows readily from the estimate

$$\| M_{\psi_\varepsilon} D(q) f \|_{H_0(S^n_+)} \leq C_1(\varepsilon)\, e^{-C_2(\varepsilon)|q|} \| f \|_{H_0(S^n_+)} \tag{3.17}$$

which also is a consequence of (3.6), (2.16) and (2.12).

Let ∂_R be the boundary of $\{q\,;\, q \in \Delta, |q| \geq R\}$ oriented from $\infty e^{i\theta_2/2m}$ to $\infty e^{i\theta/2m}$ and let ∂^n_R be that part of ∂_R which joins $ne^{i\theta_2/2m}$ to $ne^{i\theta_1/2m}$.

Lemma 3.3. The limit: $\lim\limits_{n\to\infty} \frac{2m}{2\pi i} \int_{\partial^n} q^{2m-1} C(q) f dq$ exists in $H_0(S^n_+)$ for every $f \in H_0(S^n_+)$.

Proof. To prove this lemma it is sufficient to check that $\lim\limits_{n\to\infty} \frac{2m}{2\pi i} \cdot \int_{\partial^n_2} q^{2m-1} C(q) f dq$ exists, in $H_0(S^n_+)$ for $f \in H_0(S^n_+)$. It follows from the definition (3.5) that $C(q)$, considered as a family of operators from $H_0(S^n_+)$ to $H_0(S^n_+)$, is analytic in q throughout $\Delta \cap \{q' \mid |q| \geq 2\}$. As a result

$$\int_{\partial^n_2} q^{2m-1} C(q) f dq = \int_{\Delta \cap \{q\,;\,|q|=n\}} q^{2m-1} C(q) f dq. \tag{3.18}$$

Therefore in view of (3.7) there exists a constant C such that for $n = 3, 4, \ldots$,

$$\left\| \frac{2m}{2\pi i} \int_{\partial^n_2} q^{2m-1} C(q) f dq \right\|_{H_0(S^n_+)} \leq C \| f \|_{H_0(S^n_+)}. \tag{3.19}$$

Also if f is a C^∞ function with support in S_+^n, then $\frac{2m}{2\pi i}\int_{\partial_2^n} q^{2m-1}\, C(q) f dq \in C^\infty(S_+^n)$. It is now easily seen that

$$\frac{2m}{2\pi i}\int_{\partial_2^n} q^{2m-1}\, C(q) f dq\,(x) = \frac{2m}{2\pi i}(2\pi)^{-n}\int e^{ix\xi}\left(\int_{\partial_2^n}\frac{q^{2m-1}}{\mathscr{A}(x\xi)-q^{2m}}\,dq\right)\tilde{f}(\xi)\,d\xi. \tag{3.20}$$

Hence

$$\lim_{n\to\infty}\frac{2m}{2\pi i}\int_{\partial_2^n} q^{2m-1}\, C(q) f dq\,(x) = \frac{\theta_2-\theta_1}{2\pi} f(x) \tag{3.21}$$

and

$$\left|\frac{2m}{2\pi i}\int_{\partial_2^n} q^{2m-1}\, C(q) f dq\,(x)\right| \le C\int |\tilde{f}(\xi)|\, d\xi \tag{3.22}$$

so that $\lim \frac{2m}{2\pi i}\int_{\partial_2^n} q^{2m-1}\, C(q) f dq$ exists in $H_0(S_+^n)$.

The rest of the proof follows from (3.19) when combined with the Banach-Steinhaus theorem.

We turn now to the proof of Lemma 3.1.

THE PROOF OF LEMMA 3.1. As in (3.11) let $P_0(q) = \sum_{j=1}^{N} P_0(q, \varphi_j, \psi_j)$.

Assume that the supports of $\varphi_j, j = 1, \dots, K$, do not intersect ∂G and that the supports of $\varphi_j, j = K+1,, \dots, N$, intersect ∂G. Then:

$$P_0(q) = P_1(q) - P_2(q) \tag{3.23}$$

where

$$P_1(q) = \sum_{j=1}^{N} M_{\psi_j}\, C(q)\, M_{\psi_j} \tag{3.24}$$

and

$$P_2(q) = \sum_{j=K+1}^{N} M_{\psi_j}\, D(q)\, M_{\varphi_j}. \tag{3.25}$$

By Lemma 3.3 the limit $\lim_{n\to\infty}\frac{2m}{2\pi i}\int_{\partial^n} q^{2m-1}\, P_1(q) f dq$ exists in $H_0(G)$ for every $f \in H_0(G)$ and therefore

$$\lim_{n \to \infty} \int_{\partial^n} \frac{2m}{2\pi i} q^{2m-1} P_1(q) f dq = K_1 f \tag{3.26}$$

where K_1 is a bounded operator from $H_0(G)$ to $H_0(G)$.

Let $f \in H_0(G)$ and let ψ be C^∞ with support in G. From Lemma 3.2 one sees that $\lim_{n \to \infty} \frac{2m}{2\pi i} \int_{\partial^n} M_\psi q^{2m-1} P_2(q) f dq$ exists in $H_0(G)$ and that there exists a bounded operator K_2 from $H_0(G)$ to $H_0(G)$ such that

$$\lim_{n \to \infty} \frac{2m}{2\pi i} \int_{\partial^n} M_\psi q^{2m-1} P_2(q) f dq = M_\psi K_2 f. \tag{3.27}$$

It follows from (3.23), (3.26) and (3.27) that if $f \in D(A)$ then for every C^∞ function ψ with support in G

$$M_\psi B_0 f = M_\psi (K_1 + K_2) f. \tag{3.28}$$

Hence $B_0 f = (K_1 + K_2) f$. Therefore B_0 is bounded.

4. The Reducibility of A.

Let $(\mathscr{A}, \{B_j\}_{j=1}^m, G)$ be a regular elliptic boundary value problem satisfying the assumptions mentioned at the beginning of Section 2.

We assume in addition that there exist rays $\operatorname{Arg} \lambda = \beta_i$, $i = 1, \ldots, N$ which are rays of minimal growth of the resolvent of A and are such that the angles into which they divide the complex plane are all less than $2m\pi/n$.

As in the introduction let $\operatorname{Arg} \lambda = \theta_i$ $i = 1, 2$ be rays of minimal growth of the resolvent of A and assume that γ_R belongs to the resolvent set of A. Let M be the algebraic subspace of $H_0(G)$ spanned by the generalized eigenfunctions of A and let $M^+ (M^-)$ be the algebraic subspace spanned by the generalized eigenfunctions of A with eigenvalues inside (outside) Γ_R.

Let P^+ and P^- be the projections of M on M^+ and M^- respectively, corresponding to the direct sum decomposition $M = M^+ \oplus M^-$.

Let us first verify:

LEMMA 4.1. For every $f \in M$

$$P^+ f = -Bf + \frac{\theta_2 - \theta_1}{2\pi} f \tag{4.1}$$

$$P^- f = Bf + \frac{2\pi - (\theta_2 - \theta_1)}{2\pi} f, \tag{4.2}$$

where Bf is given by (3.2).

PROOF. We first show that for $f \in M$

$$Bf = \frac{\theta_2 - \theta_1}{2\pi} P^- f - \left(\frac{2\pi - (\theta_2 - \theta_1)}{2\pi}\right) P^+ f. \tag{4.3}$$

Using (3.2)

$$Bf = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma_R^n} (\lambda - A)^{-1} P^+ f \, d\lambda + \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma_R^n} (\lambda - A)^{-1} P^- f \, d\lambda. \tag{4.4}$$

Let $(\lambda - A)^{-1} P^+ = R_\lambda^+$ and let $(\lambda - A)^{-1} P^- = R_\lambda^-$. It follows from the definitions of P^+ and P^- that for $f \in M$, $R_\lambda^+ f$ is holomorphic for $\lambda \notin \Gamma_R$ and $R_\lambda^- f$ is holomorphic for $\lambda \in \Gamma_R$. Consequently

$$Bf = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{C_n^-} R_\lambda^+ f \, d\lambda + \lim_{n \to \infty} \frac{1}{2\pi i} \int_{C_n} R_\lambda^- f \, d\lambda. \tag{4.5}$$

Here $R_\lambda^+ f$ denotes the analytic extension of $(\lambda - A)^{-1} P^+ f$, $R_\lambda^- f$ denotes the analytic extension of $(\lambda - A)^{-1} P^- f$, C_n^+ is $\Gamma \cap \{\lambda ; |\lambda| = n\}$ oriented from $ne^{i\theta_2}$ to $ne^{i\theta_1}$ and C_n^- is the complementary arc of C_n^+ oriented from $ne^{i\theta_2}$ to $ne^{i\theta_1}$.

Observe the following relations:

$$R_\lambda^+ f = \frac{R_\lambda^+ Af}{\lambda} + \frac{P^+ f}{\lambda} \tag{4.6}$$

$$R_\lambda^- f = \frac{R_\lambda^- Af}{\lambda} + \frac{P^- f}{\lambda}. \tag{4.7}$$

Also note that for every $f \in M$ there exists a constant C_f such that for $\lambda \notin \Gamma_R$

$$\| R_\lambda^+ f \| \leq \frac{C_f}{|\lambda|} \| f \| . \tag{4.8}$$

We point out here that $\|f\|$ denotes (throughout this section) the norm of f in $H_0(G)$.

Similarly for $\lambda \in \Gamma_R$

$$\| R_\lambda^- f \| \leq \frac{C_f}{|\lambda|} \| f \| \tag{4.9}$$

(4.7)-(4.10) now imply that:

$$Bf = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\sigma_n^-} \frac{P^+ f}{\lambda} d\lambda + \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\sigma_n^+} \frac{P^- f}{\lambda} d\lambda \tag{4.10}$$

which proves (4.3). Now (4.4) together with the identity $P^+ + P^- = I$ (in M) lead to the conclusions (4.1), (4.2).

The following theorem is a consequence of Lemma 4.1 and Theorem 3.1.

Theorem 4.1. There exists a constant C such that for every $f \in M$

$$\| P^+ f \| \leq C \| f \| \tag{4.11}$$

$$\| P^- f \| \leq C \| f \| . \tag{4.12}$$

Let E^+ and E^- be the extensions of P^+ and P^-, respectively, to bounded projections defined in $H_0(G)$. Then the range of E^+ is $\overline{M^+}$, the range of E^- is $\overline{M^-}$ ($\overline{M^+}$, $\overline{M^-}$ being the closures of M^+, M^- respectively) and, as the generalized eigenfunctions are complete in $H_0(G)$, the following holds:

Theorem 4.2. $H_0(G) = \overline{M^+} \oplus \overline{M^-}$.

Theorem 4.3. For every $f \in D(A)$ we have $E^+ f \in D(A)$ and $AE^+ f = E^+ A f$. Let A^+ be the operator, in $\overline{M^+}$ which is the restriction of A to $D(A) \cap \overline{M^+}$. Let $\sigma(A)$ ($\sigma(A^+)$) be the spectrum of A (A^+).

Then $\sigma(A^+) = \sigma(A) \cap \Gamma_R$ and

$$(\lambda - A^+)^{-1} = 0\left(\frac{1}{|\lambda|}\right) \quad \text{as} \quad \lambda \to \infty \quad \text{outside} \quad \Gamma_R . \tag{4.13}$$

PROOF. It follows from (4.2) and (3.2) that for λ_0 in the resolvent set of A $(\lambda_0 - A)^{-1} E^+ = E^+ (\lambda_0 - A)^{-1}$. Therefore if $f \varepsilon D(A)$ then $E^+ f \in D(A)$ and $AE^+ f = E^+ Af$. $(\lambda - A^+)^{-1}$ is a meromorphic function of λ and $(\lambda - A^+)^{-1} f$ is holomorphic for $\lambda \in \Gamma_R$ and for $f \in M^+$. Since M^+ is dense in $\overline{M^+}$, $(\lambda_0 - A^+)^{-1}$ is holomorphic for $\lambda \in \Gamma_R$. The relation $\sigma(A^+) = \sigma(A) \cap \Gamma_R$ then follows from the obvious inclusion relations $\sigma(A) \cap \Gamma_R \subset \sigma(A^+) \subset \sigma(A)$.

It is a consequence of the assumptions on the α_i's that if $\operatorname{Arg} \lambda = \alpha_i$ and $|\lambda|$ is large enough, then λ is in the resolvent set of A^+ and

$$\| (\lambda - A^+)^{-1} \| = 0 \left(\frac{1}{|\lambda|} \right) \quad \text{as} \quad |\lambda| \longrightarrow \infty \quad \text{and} \quad \operatorname{Arg} \lambda = \alpha_i . \tag{4.14}$$

It is shown in [1] that if λ_0 is in the resolvent set of A, then for every ε there exists a sequence of positive numbers μ_k with $\mu_k \longrightarrow \infty$ such that

$$\| (\lambda - A)^{-1} \| \leq \exp |\lambda|^{n/2m+\varepsilon} \tag{4.15}$$

if $|\lambda - \lambda_0| = \mu_k$ $k = 1, 2, \ldots$. Hence if $|\lambda - \lambda_0| = \mu_k$ $k = 1, 2, \ldots$ then

$$\| (\lambda - A^+)^{-1} \| \leq \exp |\lambda|^{n/2m+\varepsilon} . \tag{4.16}$$

As in [1], (4.13) follows from (4.14) and (4.16) applying the Phragman-Lindelof principle to the intersection of the complement of Γ_R with each of the angles determined by the rays $\operatorname{Arg} \lambda = \alpha_j$, $j = 1, \ldots, N$.

THEOREM 4.4. The limit $\lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma_R^n} (\lambda - A)^{-1} f \, d\lambda$ exists, in $H_0(G)$, for every $f \in H_0(G)$.

PROOF. Since $\lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma_R^n} (\lambda - A)^{-1} f \, d\lambda$ exists for every $f \in D(A)$, to prove Theorem 4.4 it is sufficient to demonstrate the existence of a constant C such that for every $f \in H_0(G)$ and $n = [R] + 1, [R] + 2 \ldots$

$$\left\| \frac{1}{2\pi i} \int_{\gamma_R^n} (\lambda - A)^{-1} f \, d\lambda \right\| \leq C \| f \| . \tag{4.17}$$

Let A^- be the operator, in $\overline{M^-}$, defined as the restriction of A to $D(A) \cap \overline{M^-}$. By a reasoning similar to that of Theorem 4.3, if $\lambda \in \Gamma_R$ then

λ is in the resolvent set of A^- and furthermore

$$\text{(4.18)} \qquad \| (\lambda - A^-)^{-1} \| = 0 \left(\frac{1}{|\lambda|} \right) \quad \text{as} \quad \lambda \to \infty \quad \text{and} \quad \lambda \in \Gamma_R .$$

Let λ be in the resolvent set of A; then

$$\text{(4.19)} \qquad (\lambda - A)^{-1} f = (\lambda - A^+)^{-1} E^+ f + (\lambda - A^-)^{-1} E^- f$$

and therefore

$$\text{(4.20)} \quad \frac{1}{2\pi i} \int_{\gamma_R^n} (\lambda - A)^{-1} f \, d\lambda = \frac{1}{2\pi i} \int_{\gamma_R^n} (\lambda - A^+)^{-1} E^+ f \, d\lambda + \int_{\gamma_R^n} (\lambda - A^-)^{-1} E^- f \, d\lambda.$$

It follows from (4.20) and from the properties of $(\lambda - A^+)^{-1}$ and $(\lambda - A^-)^{-1}$ mentioned above that

$$\text{(4.21)} \quad \frac{1}{2\pi i} \int_{\gamma_R^n} (\lambda - A)^{-1} f \, d\lambda = \frac{1}{2\pi i} \int_{C_n^-} (\lambda - A^+)^{-1} E^+ f \, d\lambda + \frac{1}{2\pi i} \int_{C_n^+} (\lambda - A^-)^{-1} E^- f \, d\lambda.$$

Here C_n^+ is $\Gamma \cap \{\lambda ; |\lambda| = n\}$ oriented from $ne^{i\theta_2}$ to $ne^{i\theta_1}$ and C_n^- is the complementary arc oriented from $ne^{i\theta_2}$ to $ne^{i\theta_1}$ and (4.13).

As a result (4.17) follows from (4.21), (4.18) and (4.13).

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