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The spectrum of operators on $L^p(E^n)$

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THE SPECTRUM OF OPERATORS ON $L^p(E^n)$ [1]

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1. Introduction.

Let $P(D)$ be a partial differential operator on E^n with complex constant coefficients. If C_0^∞ denotes the set of infinitely differentiable complex valued functions with compact supports on E^n, then $P(D)$ on C_0^∞ is a closable operator in $L^p = L^p(E^n)$ for $1 \leq p \leq \infty$. Let $P_0 = P_{0p}$ denote its closure in L^p. It is the purpose of this note to describe the spectrum of P_0 under certain assumption on $P(D)$.

In describing our results, we shall need to define the polynomial $P(\xi)$ associated with $P(D)$. If we set

$$D_k = \partial / i \partial x_k, \quad 1 \leq k \leq n, \tag{1.1}$$

we can consider $P(D)$ as a « polynomial » in $D_1, \dots, D_n$. If we replace $D_1, \dots, D_n$ by real variables $\xi_1, \dots, \xi_n$, we obtain a polynomial $P(\xi)$ called the polynomial associated with $P(D)$.

Our first result is

THEOREM 1.1. *Let $P(D)$ be an operator of order m such that the associated polynomial $P(\xi)$ satisfies*

$$1/P(\xi) = 0\,(1/|\xi|^b) \text{ as } |\xi| \to \infty, \tag{1.2}$$

where $b > 0$ and $|\xi|^2 = \xi_1^2 + \dots + \xi_n^2$. Let p satisfy $1 < p < \infty$ and

$$|1/p - 1/2| < b/n\,(m - b). \tag{1.3}$$

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Then

$$\sigma(P_0) = \{P(\xi), \xi \in E^n\}, \tag{1.4}$$

where $\xi = (\xi_1, \dots, \xi_n)$. *Thus* (1.4) *holds for all* p *satisfying* $1 < p < \infty$ *when*

$$b \geq mn/(n+2). \tag{1.5}$$

COROLLARY 1.2. *If* $P(D)$ *is such that*

$$|P(\xi)| \to \infty \text{ as } |\xi| \to \infty, \tag{1.6}$$

then there is an $\eta > 0$ *such that* (1.4) *holds when*

$$|1/p - 1/2| < \eta. \tag{1.7}$$

In describing our next result we let $\mu = (\mu_1, \dots, \mu_n)$ denote a multi-index of non-negative integers and put

$$P^{(\mu)}(\xi) = \partial^{|\mu|} P(\xi)/\partial\xi_1^{\mu_1} \dots \partial\xi_n^{\mu_n},$$

where $|\mu| = \mu_1 + \dots + \mu_n$.

THEOREM 1.3. *Assume that* $1 < p < \infty$ *and that* $P(\xi)$ *satisfies* (1.2) *and*

$$P^{(\mu)}(\xi)/P(\xi) = 0\,(1/|\xi|^{a|\mu|}) \text{ as } |\xi| \to \infty, |\mu| \leq l, \tag{1.8}$$

for real $a \leq 1$ *and*

$$b > (1-a)\, n\, |1/p - 1/2|, \tag{1.9}$$

where l *is the smallest integer greater than* $n\,|1/p - 1/2|$. *Then* (1.4) *holds.*

Balslev [6] proved (1.4) for elliptic operators. Iha and Schubert [4, 5] proved that for $1 < p < \infty$ and any operator satisfying (1.6) either $\sigma(P_0)$ is the whole complex plane or (1.4) holds. They also proved a theorem slightly weaker than Theorem 1.1. They give an example of a fourth order operator for which (1.4) does not hold when

$$|1/p - 1/2| > 3/8.$$

Our Theorem 1.3 applied to this operator shows that (1.4) does hold for

$$|1/p - 1/2| < 2/9.$$

It would be of interest to know the exact point of demarcation. I would like to thank F.T. Iha and C.F. Schubert for informing me about their work.

2. A. Multiplier Theorem.

Let S denote the set of all complex valued functions $v \in C^\infty(E^n)$ such that

$$|x|^k \, |D^\mu v|$$

is bounded on E^n for each integer $k \geq 0$ and multi-index μ, where

$$D^\mu = D_1^{\mu_1} \dots D_n^{\mu_n}.$$

If F denotes the Fourier transform, then both F and F^{-1} map S into itself. A function $m(\xi)$ on E^n is called a multiplier in L^p if there exists a constant C such that

$$\| F^{-1}[m(\xi)\, Fu] \|_p \leq C \| u \|_p, \; u \in S, \tag{2.1}$$

where ξ is the argument of the Fourier transform and the norm is that of L^p.

The following is a special case of a theorem due to Littman [1].

THEOREM 2.1. *Let* m_1, m_2 *be non-negative integers and suppose* θ, p *satisfy* $0 \leq \theta \leq 1$, $1 < p < \infty$ *and*

$$(1-\theta)\, m_1 + \theta m_2 > n \,|\, 1/p - 1/2 \,|. \tag{2.2}$$

Suppose $w(\xi)$ *is a function in* $C^m(E^n)$ *satisfying*

$$\max_{R<|\xi|<2R} |D^\mu w|^{1-\theta} \max_{R<|\xi|<2R} |D^\nu w|^\theta \leq C/R^{(1-\theta)|\mu|+\theta|\nu|} \tag{2.3}$$

for all μ, ν *such that* $|\mu| \leq m_1$ *and* $|\nu| \leq m_2$, *where* $m = \max(m_1, m_2)$. *Then* w *is a multiplier in* L^p.

We shall use this theorem to prove

THEOREM 2.2. *Suppose* $1 < p < \infty$, *and let* l *be the smallest integer greater than* $n\,|\,1/p - 1/2\,|$. *Assume that* $w(\xi)$ *is a function in* $C^l(E^n)$ *such that*

$$|D^\mu w(\xi)| \leq C / |\xi|^{a|\mu|+b}, \qquad |\mu| \leq l, \tag{2.4}$$

for real $a \leq 1$ *and*

$$b > (1-a)\, n \,|\, 1/p - 1/2 \,|. \tag{2.5}$$

Then w *is a multiplier in* L^p.

PROOF. First assume that $b \geq (1-a)\, l$. In this case we take $m_1 = m_2 = l$ and $\theta = 0$. Inequality (2.4) implies

$$\max_{R < |\xi| < 2R} | D^\mu w(\xi) | \leq C/R^{a|\mu|+b}. \tag{2.6}$$

Since $b \geq (1-a)\, l$, $b \geq (1-a)|\mu|$ for $|\mu| \leq l$, and consequently $a|\mu| + b \geq |\mu|$. Thus (2.6) implies (2.3). Since all of the hypotheses are satisfied, we obtain the desired conclusion from Theorem 2.1.

Next assume $b < (1-a)\, l$. In this case we must have $a < 1$. By (2.5) and the definition of l we have

$$0 < l - \frac{b}{1-a} < l - n \left| \frac{1}{p} - \frac{1}{2} \right| \leq 1,$$

and consequently there is a θ satisfying $0 < \theta < 1$ such that

$$0 < l - \frac{b}{1-a} < \theta < l - n \left| \frac{1}{p} - \frac{1}{2} \right| \leq 1.$$

In particular we have

$$n \left| \frac{1}{p} - \frac{1}{2} \right| < l - \theta < \frac{b}{1-a}. \tag{2.7}$$

We now take $m_1 = l$, $m_2 = l - 1$ and θ satisfying (2.7). Note that (2.6) implies

$$\max_{R < |\xi| < 2R} | D^\mu w |^{1-\theta} \max_{R < |\xi| < 2R} | D^\nu w |^{\theta} \leq C/R^{a[(1-\theta)|\mu| + \theta|\nu|] + b}. \tag{2.8}$$

If $|\mu| \leq m_1 = l$ and $|\nu| \leq m_2 = l - 1$, then

$$(1-\theta)|\mu| + \theta|\nu| \leq l - \theta < b/(1-a)$$

by (2.7). This is equivalent to

$$a[(1-\theta)|\mu| + \theta|\nu|] + b \geq (1-\theta)|\mu| + \theta|\nu|$$

for such μ and ν. Thus (2.8) implies (2.3) and all of the hypotheses of Theorem 2.1 are satisfied. This completes the proof.

The connection between multipliers and the spectrum of a partial differential operator is given by

THEOREM 2.3. *For* $1 \leq p < \infty$, *a point* λ *is in* $\varrho(P_{0p})$ *if and only if* $1/[P(\xi) - \lambda]$ *is a multiplier in* L^p.

PROOF. We may take $\lambda = 0$. If $0 \in \varrho(P_{0p})$, then there is a constant C such that

$$\| u \|_p \leq C \| P_{0p} u \|_p , \qquad u \in D(P_{0p}). \tag{2.9}$$

In particular we have

$$\| v \|_p \leq C \| P(D) v \|_p , \qquad v \in S. \tag{2.10}$$

Now (2.10) implies that

$$| P(\xi) | \geq 1/C, \qquad \xi \in E^n. \tag{2.11}$$

To see this, let ξ be any vector in E^n and let ψ be any function in C_0^∞ such that $\| \psi \|_p = 1$. Set

$$\varphi_k(x) = \exp\{i(x_1\xi_1 + \dots + x_n\xi_n)\} \, \psi(x/k)/k^{n/p} , \quad k = 1, 2, \dots . \tag{2.12}$$

Then $\varphi_k \in C_0^\infty$ for each k and $\| \varphi_k \|_p = 1$. Furthermore by Leibnitz's formula

$$P(D) \varphi_k = \sum_\mu P^{(\mu)}(\xi) \, \psi_\mu(x/k) \exp\{i(x_1\xi_1 + \dots + x_n\xi_n)\}/\mu ! \, k^{|\mu| + (n/p)} ,$$

where $\psi_\mu(x) = D^\mu \psi(x)$ and $\mu ! = \mu_1 ! \dots \mu_n !$. Since

$$\| \psi_\mu(x/k)/k^{n/p} \|_p = \| \psi_\mu \|_p ,$$

We see that

$$\| P(D) \varphi_k \|_p \to | P(\xi) | \quad \text{as} \quad k \to \infty.$$

But by (2.10)

$$1 = \| \varphi_k \|_p \leq C \| P(D) \varphi_k \|_p , \quad k = 1, 2, \dots .$$

This gives (2.11). Let f be any function in S, and set

$$u = F^{-1}\left[\frac{1}{P} Ff\right]. \tag{2.13}$$

By (2.11) we know that $u \in S$. Furthermore $P(D)u = f$. Thus (2.10) implies

$$\left\| F^{-1}\left[\frac{1}{P} Ff\right]\right\|_p \leq C\|f\|_p. \tag{2.14}$$

This shows that $1/P(\xi)$ is a multiplier in L^p.

Conversely, suppose $1/P(\xi)$ is a multiplier. Thus (2.14) holds. Since all multipliers are bounded, (2.11) holds for some constant C. Thus for each $f \in S$ there is a $u \in S$ such that $P(D)u = f$ and

$$\|u\|_p \leq C\|f\|_p$$

by (2.14). Since S is dense in L^p, this shows that $0 \in \varrho(P_0)$. This completes the proof.

3. Proofs of the Theorems.

We first give the

PROOF OF THEOREM 1.3. It is well known that

$$\sigma(P_0) \supset \{P(\xi),\ \xi \in E^n\} \tag{3.1}$$

(see [3, 4]). Thus it suffices to show that each λ satisfying $P(\xi) \neq \lambda$ for $\xi \in E^n$ is in $\varrho(P_0)$. We may take $\lambda = 0$. By Theorem 2.3 it suffices to show that $1/P(\xi)$ is a multiplier in L^p. Since $a \leq 1$, we must have $b > 0$ and consequently (1.6) holds. This means that there is a constant C such that (2.11) is satisfied. Now for each μ the derivative $D^\mu(1/P)$ consists of a sum of terms of the form

$$\text{constant } P^{(\mu^{(1)})}(\xi) \dots P^{(\mu^{(k)})}(\xi)/P(\xi)^{k+1}, \tag{3.2}$$

where $\mu^{(1)} + \dots + \mu^{(k)} = \mu$ (this is easily verified by a simple induction argument). Thus

$$|D^\mu(1/P)| \leq C/|\xi|^{a|\mu|+b}, \qquad |\mu| \leq l. \tag{3.3}$$

We may apply Theorem 2.2 to conclude that $1/P(\xi)$ is an L^p multiplier. This completes the proof.

Next we give the

PROOF OF THEOREM 1.1. Since $P(\xi)$ is of degree m, $P^{(\mu)}(\xi)$ is of degree at most $m - |\mu|$. Hence

(3.4) $$P^{(\mu)}(\xi)/P(\xi) = 0\,(|\xi|^{m-|\mu|-b}) \text{ as } |\xi| \to \infty.$$

For $|\mu| \geq 1$ we have $m - |\mu| - b \leq (m - b - 1)\,|\mu|$. Hence (3.4) gives

(3.5) $$P^{(\mu)}(\xi)/P(\xi) = 0\,(|\xi|^{(m-b-1)|\mu|}) \text{ as } |\xi| \to \infty.$$

This shows that (1.8) holds with $a = b + 1 - m$. If we now apply Theorem 1.3 we see that (1.4) holds provided (1.3) is satisfied. If (1.5) holds, then the right hand side of (1.3) is $\geq 1/2$. This allows p to take on all values between 1 and ∞.

PROOF OF COROLLARY 1.2. We merely note that (1.6) implies (1.2) for some constant $b > 0$. Thus by Theorem 1.1 we may take $\eta = b/n\,(m - b)$.

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