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1. Introduction.

Let $P(D)$ be a partial differential operator on $E^n$ with complex constant coefficients. If $C_0^\infty$ denotes the set of infinitely differentiable complex valued functions with compact supports on $E^n$, then $P(D)$ on $C_0^\infty$ is a closable operator in $L^p = L^p(E^n)$ for $1 \leq p \leq \infty$. Let $P_0 = P_{op}$ denote its closure in $L^p$. It is the purpose of this note to describe the spectrum of $P_0$ under certain assumption on $P(D)$.

In describing our results, we shall need to define the polynomial $P(\xi)$ associated with $P(D)$. If we set

\begin{equation}
D_k = \partial_i \partial x_k, \quad 1 \leq k \leq n,
\end{equation}

we can consider $P(D)$ as a « polynomial » in $D_1, \ldots, D_n$. If we replace $D_1, \ldots, D_n$ by real variables $\xi_1, \ldots, \xi_n$, we obtain a polynomial $P(\xi)$ called the polynomial associated with $P(D)$.

Our first result is

**Theorem 1.1.** Let $P(D)$ be an operator of order $m$ such that the associated polynomial $P(\xi)$ satisfies

\begin{equation}
1/P(\xi) = 0 \text{ (1/} |\xi|^b) \text{ as } |\xi| \to \infty,
\end{equation}

where $b > 0$ and $|\xi|^2 = \xi_1^2 + \ldots + \xi_n^2$. Let $p$ satisfy $1 < p < \infty$ and

\begin{equation}
|1/p - 1/2| < b/n (m - b).
\end{equation}
Then
\[
\sigma(P_0) = \{P(\xi), \xi \in \mathbb{R}^n\},
\]
where \(\xi = (\xi_1, \ldots, \xi_n)\). Thus (1.4) holds for all \(p\) satisfying \(1 < p < \infty\) when
\[
b \geq mn/(n + 2).
\]

**Corollary 1.2.** If \(P(D)\) is such that
\[
|P(\xi)| \to \infty \text{ as } |\xi| \to \infty,
\]
then there is an \(\eta > 0\) such that (1.4) holds when
\[
\frac{1}{p} - 1/2 < \eta.
\]

In describing our next result we let \(\mu = (\mu_1, \ldots, \mu_n)\) denote a multi-index of non-negative integers and put
\[
P^{(\mu)}(\xi) = \frac{1}{|\xi|^{n+\mu}} P(\xi) / \partial_{\xi_1}^{\mu_1} \cdots \partial_{\xi_n}^{\mu_n},
\]
where \(|\mu| = \mu_1 + \cdots + \mu_n\).

**Theorem 1.3.** Assume that \(1 < p < \infty\) and that \(P(\xi)\) satisfies (1.2) and
\[
P^{(\mu)}(\xi)/P(\xi) = o(1/|\xi|^{n+|\mu|}) \quad \text{as} \quad |\xi| \to \infty, \quad |\mu| \leq l,
\]
for real \(a \leq 1\) and
\[
b > (1 - a)n \left| \frac{1}{p} - 1/2 \right|,
\]
where \(l\) is the smallest integer greater than \(n \left| \frac{1}{p} - 1/2 \right|\). Then (1.4) holds.

Balslev [6] proved (1.4) for elliptic operators. Iha and Schubert [4, 5] proved that for \(1 < p < \infty\) and any operator satisfying (1.6) either \(\sigma(P_0)\) is the whole complex plane or (1.4) holds. They also proved a theorem slightly weaker than Theorem 1.1. They give an example of a fourth order operator for which (1.4) does not hold when
\[
\left| \frac{1}{p} - 1/2 \right| > 3/8.
\]
Our Theorem 1.3 applied to this operator shows that (1.4) does hold for
\[
\left| \frac{1}{p} - 1/2 \right| < 2/9.
It would be of interest to know the exact point of demarcation. I would like to thank F.T. Iha and C.F. Schubert for informing me about their work.

2. A. Multiplier Theorem.

Let $S$ denote the set of all complex valued functions $v \in C^\infty (E^n)$ such that

$$|x|^k |D^\mu v|$$

is bounded on $E^n$ for each integer $k \geq 0$ and multi-index $\mu$, where

$$D^\mu = D_1^{\mu_1} \ldots D_n^{\mu_n}.$$

If $F$ denotes the Fourier transform, then both $F$ and $F^{-1}$ map $S$ into itself. A function $m(\xi)$ on $E^n$ is called a multiplier in $L^p$ if there exists a constant $C$ such that

$$(2.1) \quad \| F^{-1} [m(\xi) Fu] \|_p \leq C \| u \|_p, \ u \in S,$$

where $\xi$ is the argument of the Fourier transform and the norm is that of $L^p$.

The following is a special case of a theorem due to Littman [1].

**Theorem 2.1.** Let $m_1, m_2$ be non-negative integers and suppose $\theta, \rho$ satisfy $0 \leq \theta \leq 1$, $1 < p < \infty$ and

$$(2.2) \quad (1 - \theta) m_1 + \theta m_2 > n \left| \frac{1}{p} - \frac{1}{2} \right|.$$

Suppose $w(\xi)$ is a function in $C^\infty (E^n)$ satisfying

$$(2.3) \quad \max_{R < |\xi| < 2R} |D^\mu w|^{1-\theta} \max_{R < |\xi| < 2R} |D^\nu w|^\theta \leq C/R^{(1-\theta)\mu + \theta \nu}$$

for all $\mu, \nu$ such that $|\mu| \leq m_1$ and $|\nu| \leq m_2$, where $m = \max (m_1, m_2)$. Then $w$ is a multiplier in $L^p$.

We shall use this theorem to prove

**Theorem 2.2.** Suppose $1 < p < \infty$, and let $l$ be the smallest integer greater than $n \left| \frac{1}{p} - \frac{1}{2} \right|$. Assume that $w(\xi)$ is a function in $C^1 (E^n)$ such that

$$(2.4) \quad |D^\mu w(\xi)| \leq C/|\xi|^{|\mu|+b}, \quad |\mu| \leq l,$$
for real \( a \leq 1 \) and
\[
(2.5) \quad b > (1 - a) n \left| \frac{1}{p} - 1 \right|.
\]

Then \( w \) is a multiplier in \( L^p \).

**Proof.** First assume that \( b \geq (1 - a) l \). In this case we take \( m_1 = m_2 = l \) and \( \theta = 0 \). Inequality (2.4) implies
\[
(2.6) \quad \max_{R < \| \xi \| < 2R} |D^n w(\xi)| \leq C/R^n |\mu|^{1+b}.
\]
Since \( b \geq (1-a)l \), \( b \geq (1-a)|\mu| \) for \( |\mu| \leq l \), and consequently \( a |\mu| + b \geq |\mu| \). Thus (2.6) implies (2.3). Since all of the hypotheses are satisfied, we obtain the desired conclusion from Theorem 2.1.

Next assume \( b \leq (1 - a) l \). In this case we must have \( a < 1 \). By (2.5) and the definition of \( l \) we have
\[
0 < l - \frac{b}{1-a} < l - n \left| \frac{1}{p} - \frac{1}{2} \right| \leq 1,
\]
and consequently there is a \( \theta \) satisfying \( 0 < \theta < 1 \) such that
\[
0 < l - \frac{b}{1-a} < \theta < l - n \left| \frac{1}{p} - \frac{1}{2} \right| \leq 1.
\]
In particular we have
\[
(2.7) \quad n \left| \frac{1}{p} - \frac{1}{2} \right| < l - \theta < \frac{b}{1-a}.
\]
We now take \( m_1 = l, ~ m_2 = l - 1 \) and \( \theta \) satisfying (2.7). Note that (2.6) implies
\[
(2.8) \quad \max_{R < \| \xi \| < 2R} |D^n w|^{-\theta} \max_{R < \| \xi \| < 2R} |D^r w|^{\theta} \leq C/R^{n(1-\theta)|\mu| + \theta |\nu| + \theta |\nu|}.
\]
If \( |\mu| \leq m_1 = l \) and \( |\nu| \leq m_2 = l - 1 \), then
\[
(1 - \theta) |\mu| + \theta |\nu| \leq l - \theta < b/(1 - a)
\]
by (2.7). This is equivalent to
\[
a [(1 - \theta) |\mu| + \theta |\nu|] + b \geq (1 - \theta) |\mu| + \theta |\nu|
\]
for such \( \mu \) and \( \nu \). Thus (2.8) implies (2.3) and all of the hypotheses of Theorem 2.1 are satisfied. This completes the proof.
The connection between multipliers and the spectrum of a partial differential operator is given by

**THEOREM 2.3.** For \( 1 \leq p < \infty \), a point \( \lambda \) is in \( \mathcal{Q}(P_{op}) \) if and only if

\[
\frac{1}{|P(\xi) - \lambda|} \text{ is a multiplier in } L^p.
\]

**Proof.** We may take \( \lambda = 0 \). If \( 0 \in \mathcal{Q}(P_{op}) \), then there is a constant \( C \) such that

\[
\| u \|_p \leq C \| P_{op} u \|_p , \quad u \in D(P_{op}).
\]

In particular we have

\[
\| v \|_p \leq C \| P(D) v \|_p , \quad v \in S.
\]

Now (2.10) implies that

\[
|P(\xi)| \geq 1/C , \quad \xi \in E^n.
\]

To see this, let \( \xi \) be any vector in \( E^n \) and let \( \psi \) be any function in \( C_0^\infty \) such that \( \| \psi \|_p = 1 \). Set

\[
\varphi_k(x) = \exp \left( i \left( x_1 \xi_1 + \cdots + x_n \xi_n \right) \right) \psi \left( x/k \right) \left/ k^{n/p} \right. , \quad k = 1, 2, \ldots.
\]

Then \( \varphi_k \in C_0^\infty \) for each \( k \) and \( \| \varphi_k \|_p = 1 \). Furthermore by Leibnitz's formula

\[
P(D) \varphi_k = \sum_{\mu} P^{(\xi)}(\xi) \psi_{\mu}(x/k) \exp \left( i \left( x_1 \xi_1 + \cdots + x_n \xi_n \right) / k^{\mu_1 \cdots \mu_n} \right),
\]

where \( \psi_{\mu}(x) = D^{\mu} \psi(x) \) and \( \mu_1 \cdots \mu_n = \mu ! \). Since

\[
\| \psi_{\mu}(x/k) \left/ k^{n/p} \right. \|_p = \| \psi_{\mu} \|_p,
\]

We see that

\[
\| P(D) \varphi_k \|_p \to |P(\xi)| \quad \text{as} \quad k \to \infty.
\]

But by (2.10)

\[
1 = \| \varphi_k \|_p \leq C \| P(D) \varphi_k \|_p , \quad k = 1, 2, \ldots.
\]

This gives (2.11). Let \( f \) be any function in \( S \), and set

\[
u = F^{-1} \left[ \frac{1}{P} f \right].
\]
By (2.11) we know that \( u \in S \). Furthermore \( P(D)u = f \). Thus (2.10) implies
\[
\left\| F^{-1} \left[ \frac{1}{P} Ff \right] \right\|_p \leq C \| f \|_p.
\]
This shows that \( 1/P(\xi) \) is a multiplier in \( L^p \).

Conversely, suppose \( 1/P(\xi) \) is a multiplier. Thus (2.14) holds. Since all multipliers are bounded, (2.11) holds for some constant \( C \). Thus for each \( f \in S \) there is a \( u \in S \) such that \( P(D)u = f \) and
\[
\| u \|_p \leq C \| f \|_p
\]
by (2.14). Since \( S \) is dense in \( L^p \), this shows that \( 0 \in \mathcal{G}(P_0) \). This completes the proof.

3. Proofs of the Theorems.

We first give the

**Proof of Theorem 1.3.** It is well known that
\[
\sigma(P_0) \supset \{ P(\xi), \xi \in \mathbb{R}^n \}
\]
(see [3, 4]). Thus it suffices to show that each \( \lambda \) satisfying \( P(\xi) = \lambda \) for \( \xi \in \mathbb{R}^n \) is in \( \mathcal{G}(P_0) \). We may take \( \lambda = 0 \). By Theorem 2.3 it suffices to show that \( 1/P(\xi) \) is a multiplier in \( L^p \). Since \( a \leq 1 \), we must have \( b > 0 \) and consequently (1.6) holds. This means that there is a constant \( C \) such that (2.11) is satisfied. Now for each \( \mu \) the derivative \( D^\mu (1/P) \) consists of a sum of terms of the form
\[
\text{constant \ } P^{(\mu(1))}(\xi) \ldots P^{(\mu(k))}(\xi)/P(\xi)^{k+1},
\]
where \( \mu(1) + \ldots + \mu(k) = \mu \) (this is easily verified by a simple induction argument). Thus
\[
| D^\mu (1/P) | \leq C/| \xi |^{\mu(1)+\ldots+\mu(k)}, \quad | \mu(1)+\ldots+\mu(k) | \leq l.
\]
We may apply Theorem 2.2 to conclude that \( 1/P(\xi) \) is an \( L^p \) multiplier. This completes the proof.

Next we give the
Proof of Theorem 1.1. Since $P(\xi)$ is of degree $m$, $P^{(\mu)}(\xi)$ is of degree at most $m - |\mu|$. Hence

$(3.4) \quad P^{(\mu)}(\xi)/P(\xi) = 0 (|\xi|^{m - |\mu| - 1}) as |\xi| \to \infty.$

For $|\mu| \geq 1$ we have $m - |\mu| - b \leq (m - b - 1) |\mu|$. Hence (3.4) gives

$(3.5) \quad P^{(\mu)}(\xi)/P(\xi) = 0 (|\xi|^{m - b - 1} |\mu|) as |\xi| \to \infty.$

This shows that (1.8) holds with $a = b + 1 - m$. If we now apply Theorem 1.3 we see that (1.4) holds provided (1.3) is satisfied. If (1.5) holds, then the right hand side of (1.3) is $\geq 1/2$. This allows $p$ to take on all values between 1 and $\infty$.

Proof of Corollary 1.2. We merely note that (1.6) implies (1.2) for some constant $b > 0$. Thus by Theorem 1.1 we may take $\eta = b/n (m - b)$.

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