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AN EXISTENCE THEOREM FOR BOUNDED VECTOR-VALUED FUNCTIONS

S. ZAIDMAN (*)

Introduction.

In professor's L. Amerio paper [1], supposing existence of bounded solutions for $t \geq 0$ ($t$-time), of non-linear almost-periodic differential equations, one proves existence of bounded solutions which are defined on the whole time axis, $-\infty < t < \infty$.

In our paper [2] we proved a very similar result for solutions of the heat equation, with almost-periodic known term. We shall see below that this situation can be extended to a certain class of Banach space valued functions admitting a certain representation through a given semi-group of class $C^0$.

§ 1. Let us consider first a reflexive Banach space $X$; then, a one-parameter semi-group of operators in $L(X, X)$: $T_t$, $t \geq 0$; such that $T_0 = I$, $T_{t+s} = T_t T_s$; $T_t \in L(X, X)$ $\forall t \geq 0$ and $T_t x$ is continuous from $0 \leq t < \infty$ to $X$.

Consider also a continuous function $-\infty < t < \infty$ to $X$, which is almost-periodic in Bochner's sense, that is:

*Each sequence $(f(t + a_n))_{n=1}^{\infty}$ contains a subsequence $(f(t + a_{n_p}))_{p=1}^{\infty}$ which is uniformly convergent on $-\infty < t < \infty$, in strong topology of $X$.*

Let now $u(t)$ be a continuous function: $0 \leq t < \infty$ to $X$, admitting representation

\begin{equation}
(1.1) \quad u(t) = T_t u(0) + \int_0^t T_{t-\zeta} f(\zeta) d\zeta, \quad \forall t \geq 0
\end{equation}

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and let us assume
\[ M = \sup_{t \geq 0} \| u(t) \|_X < \infty. \]

Then we have

**Theorem.** There exists a continuous function \( W(t), -\infty < t < \infty \) to \( X \), such that

\[ W(t) = T_{t-t_0} W(t_0) + \int_{t_0}^{t} T_{t-s} f(s) \, ds, \quad \forall t \geq t_0 \]

**Proof.** Let us consider the sequence of translates
\[ u_n(t) = u(t + n) \]

They are defined for \( t \geq -n \), and we have
\[ \sup_{t \geq -n} \| u_n(t) \| = M = \sup_{t \geq 0} \| u(t) \| \]

As well known, reflexivity implies weak sequential compactness of bounded sets in \( X \). Then, using almost-periodicity of \( f(t) \) and the diagonal procedure, we obtain a sequence of positive integers \( (n_k)_1^{\infty} \) with following properties:

\[ \lim_{k \to \infty} f(t + n_k) = g(t), \text{ uniformly on } -\infty < t < \infty \text{ (and consequently } g(t) \text{ is an almost-periodic function)} \]

For each \( N = 0, 1, 2 \), \( u_{n_k}(-N) \) is defined for \( k \geq N \), and

\[ (w) \lim_{k \to \infty} u_{n_k}(-N) = W_N \text{ exists and belongs to } X \text{ (here (w) means weak topology in } X \text{; remember that a reflexive space is weakly sequentially complete).} \]

Remark now, that for each real \( t \), \( u_{n_k}(t) \) is defined for \( k \geq k_t \). Then we shall see that, \( \forall t \in (-\infty, \infty) \)

\[ (w) \lim_{k \to \infty} u_{n_k}(t) = V(t) \text{ exists.} \]
In fact (1.10) is a consequence of (1.9) and (1.5). To prove (1.9) we use the following

**Lemma 1.** Let \( t \in (-\infty, \infty) \) be given, and \( N \) a positive integer such that \( t + N > 0 \). Then, \( \forall k > N \), we have

\[
(1.12) \quad u_{n_k}(t) = T_{t+N} u_{n_k}(-N) + \int_{-N}^{t} T_{t-\tau} f(\tau + n_k) \, d\tau.
\]

This Lemma is a Corollary of a slightly more general result

**Lemma 2.** Let \( u(t), t \geq 0 \rightarrow \mathcal{X} \) (arbitrary Banach space), be a continuous function; \( T_t; t \geq 0 \rightarrow L(\mathcal{X}, \mathcal{X}) \) be a strongly continuous one parameter semigroup of linear bounded operators in \( \mathcal{X} \); \( f(t), -\infty < t < \infty \rightarrow \mathcal{X} \) be a continuous function.

Suppose

\[
(1.13) \quad u(t) = T_t u(0) + \int_{0}^{t} T_{t-\tau} f(\tau) \, d\tau, \quad \forall t \geq 0.
\]

Then, if \( t \in (-\infty, \infty) \), is given and \( b > a > 0, a + t > 0 \), we have

\[
(1.14) \quad u(t+b) = T_{t+a} u(b-a) + \int_{a}^{t} T_{t-\zeta} f(\zeta + b) \, d\zeta.
\]

**Remark.** Lemma 1 follows from Lemma 2 if we take \( b = n_k, a = N \).

**Proof of Lemma 2.**

As \( t + b > t + a > 0 \), we have using (1.1)
Next remark, again by (1.1), the representation

\[ u(b - a) = T_{b-a} u(0) + \int_0^{b-a} T_{b-a-s} f(s) \, ds. \]

Introducing in (1.14) the value of \( T_{b-a} u(0) \) we get

\[ (1.15) \quad u(t + b) = T_{t+a} \left( u(b - a) - \int_0^{b-a} T_{b-a-s} f(s) \, ds \right) + \int_0^{t+b} T_{t+b-s} f(s) \, ds = \]

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\[ T_{t+a} u(b - a) + \int_{b-a}^{t+b} T_{t+b-s} f(s) \, ds. \]

Now, set \( a = \zeta + b \); it follows \( \int_{b-a}^{t+b} T_{t+b-s} f(s) \, ds = \int_{-a}^{t} T_{t-\zeta} f(\zeta + b) \, d\zeta \) which proves our Lemma, and consequently Lemma 1 too.

Actually we see that (1.9) is true in the following way: Fix an arbitrary real \( t \); then take \( N \) a positive integer, such that \( t + N > 0 \), and take \( k \geq N \). We use then (1.12); as \( f(t + n_k) \to g(t) \) uniformly on \(( -\infty, \infty)\) and in \( X \) strong, we have obviously

\[ \lim_{k \to \infty} \int_{-N}^{t} T_{t-\zeta} f(\zeta + n_k) \, d\zeta = \int_{-N}^{t} T_{t-\zeta} g(\zeta) \, d\zeta. \]

Then we have also

\[ (w) \lim_{k \to \infty} T_{t+N} u_{n_k}(-N) = T_{t+N} W_N. \]

because a linear continuous operator in a \( B \)-space is continuous also in respect to the weak convergence.

Now, we shall see that for function \( V(t) \), \( -\infty < t < \infty \to X \) defined by (1.9), the representation formula (1.11) holds for each semi axis \( t \geq t_0 \).

Take in fact two reals \( t \geq t_0 \), and choose an integer \( N \) such that \( -N < t_0 \). Apply Lemma 1 to \( t, t_0, N \). We have

\[ u_{n_k}(t) = T_{t+N} u_{n_k}(-N) + \int_{-N}^{t} T_{t-\zeta} f(\zeta + n_k) \, d\zeta \]

\[ u_{n_k}(t_0) = T_{t_0+N} u_{n_k}(-N) + \int_{-N}^{t_0} T_{t_0-\zeta} f(\zeta + n_k) \, d\zeta. \]
Then, reasoning as above, we obtain

\[ V(t) = T_{t+N} W_N + \int_{-N}^{t} T_{t-\zeta} g(\zeta) \, d\zeta \]

\[ V(t_0) = T_{t_0+N} W_N + \int_{-N}^{t_0} T_{t_0-\zeta} g(\zeta) \, d\zeta \]

Trying now to get (1.11) we write

\[ T_{t-t_0} V(t_0) = T_{t-t_0} \left( T_{t_0+N} W_N + \int_{-N}^{t_0} T_{t_0-\zeta} g(\zeta) \, d\zeta \right) = T_{t+N} W_N + \int_{-N}^{t} T_{t-\zeta} g(\zeta) \, d\zeta. \]

Hence

\[ T_{t-t_0} V(t_0) + \int_{t_0}^{t} T_{t-\zeta} g(\zeta) \, d\zeta = T_{t+N} W_N + \int_{-N}^{t} T_{t-\zeta} g(\zeta) \, d\zeta = V(t) \]

that is (1.11).

Now we give the idea of the final step in the proof. Using uniform (on real axis) convergence of sequence \( f(t + n_k) \) to \( g(t) \), we obtain uniform convergence of sequence \( g(t - n_k) \) to \( f(t) \). Starting now with \( V(t) \) and repeating above procedure, we obtain function \( W(t), -\infty < t < \infty \) which is continuous, bounded, and admits representation (1.3) \( \forall t \geq t_0 \).

REFERENCES
