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AN EXISTENCE THEOREM FOR BOUNDED VECTOR-VALUED FUNCTIONS

S. ZAIDMAN (*)

Introduction.

In professor's L. Amerio paper [1], supposing existence of bounded solutions for $t \geq 0$ (t-time), of non-linear almost-periodic differential equations, one proves existence of bounded solutions which are defined on the whole time axis, $-\infty < t < \infty$.

In our paper [2] we proved a very similar result for solutions of the heat equation, with almost-periodic known term. We shall see below that this situation can be extended to a certain class of Banach-space valued functions admitting a certain representation through a given semi-group of class C^0.

§ 1. Let us consider first a reflexive Banach space X; then, a one-parameter semi-group of operators in $L(X, X)$: T_t, $t \geq 0$; such that $T_0 = I$, $T_{t+\zeta} = T_t T_\zeta$; $T_t \in L(X, X)$ $\forall\, t \geq 0$ and $T_t x$ is continuous from $0 \leq t < \infty$ to X.

Consider also a continuous function $-\infty < t < \infty$ to X, which is almost-periodic in Bochner's sense, that is:
Each sequence $(f(t + a_n))_{n=1}^{\infty}$ *contains a subsequence* $(f(t + a_{n_p}))_{p=1}^{\infty}$ *which is uniformly convergent on* $-\infty < t < \infty$, *in strong topology of* X.

Let now $u(t)$ be a continuous function: $0 \leq t < \infty$ to X, admitting representation

$$u(t) = T_t u(0) + \int_0^t T_{t-\zeta} f(\zeta)\, d\zeta, \qquad \forall\, t \geq 0 \tag{1.1}$$

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and let us assume

$$M = \sup_{t \geq 0} \| u(t) \|_X < \infty . \tag{1.2}$$

Then we have

THEOREM. *There exists a continuous function* $W(t)$, $-\infty < t < \infty$ *to* X, *such that*

$$W(t) = T_{t-t_0} W(t_0) + \int_{t_0}^{t} T_{t-\zeta} f(\zeta) \, d\zeta, \qquad \forall t \geq t_0 \tag{1.3}$$

$$\sup_{-\infty < t < \infty} \| W(t) \| < \infty. \tag{1.4}$$

PROOF. Let us consider the sequence of translates

$$u_n(t) = u(t + n)$$

They are defined for $t \geq -n$, and we have

$$\sup_{t \geq -n} \| u_n(t) \| = M = \sup_{t \geq 0} \| u(t) \| \tag{1.5}$$

As well known, reflexivity implies weak sequential compactness of bounded sets in X. Then, using almost-periodicity of $f(t)$ and the diagonal procedure, we obtain a sequence of positive integers $(n_k)_1^\infty$ with following properties:

(1.6) $\lim_{k \to \infty} f(t + n_k) = g(t)$, uniformly on $-\infty < t < \infty$ (and consequently $g(t)$ is an almost-periodic function)

(1.7) For each $N = 0, 1, 2,$ $u_{n_k}(-N)$ is defined for $k > N$,

and

(1.8) $(\omega) \lim_{k \to \infty} u_{n_k}(-N) = W_N$ exists and belongs to X (here (w) — means weak topology in X; remember that a reflexive space is weakly sequentially complete).

Remark now, that for each real t, $u_{n_k}(t)$ is defined for $k \geq k_t$. Then we shall see that, $\forall t \in (-\infty, \infty)$

$$(w) \lim_{k \to \infty} u_{n_k}(t) = V(t) \text{ exists.} \tag{1.9}$$

$$\sup_{-\infty<t<\infty} \| V(t) \| < \infty \tag{1.10}$$

$$V(t) = T_{t-t_0} V(t_0) + \int_{t_0}^{t} T_{t-\sigma} g(\sigma) \, d\sigma, \qquad \forall t \geq t_0, \quad \forall t_0 \in R^1. \tag{1.11}$$

In fact (1.10) is a consequence of (1.9) and (1.5). To prove (1.9) we use following

LEMMA 1. *Let* $t \in (-\infty, \infty)$ *be given, and* N *a positive integer such that* $t + N > 0$. *Then,* $\forall k > N$, *we have*

$$u_{n_k}(t) = T_{t+N} u_{n_k}(-N) + \int_{-N}^{t} T_{t-\tau} f(\tau + n_k) \, d\tau. \tag{1.12}$$

This Lemma is a Corollary of a slightly more general result

LEMMA 2. *Let* $u(t)$, $t \geq 0 \rightarrow \mathcal{X}$ *(arbitrary Banach space), be a continuous function ;* T_t; $t \geq 0 \rightarrow L(\mathcal{X}, \mathcal{X})$ *be a strongly continuous one parameter semi-group of linear bounded operators in* $\mathcal{X}$; $f(t)$, $-\infty < t < \infty \rightarrow \mathcal{X}$ *be a continuous function.*

Suppose

$$u(t) = T_t u(0) + \int_0^t T_{t-\tau} f(\tau) \, d\tau, \qquad \forall t \geq 0.$$

Then, if $t \in (-\infty, \infty)$, *is given and* $b > a > 0$, $a + t > 0$, *we have*

$$u(t+b) = T_{t+a} u(b-a) + \int_{-a}^{t} T_{t-\zeta} f(\zeta + b) \, d\zeta \tag{1.13}$$

REMARK. Lemma 1 follows from Lemma 2 if we take $b = n_k$, $a = N$.

PROOF OF LEMMA 2.

As $t + b > t + a > 0$, we have using (1.1)

$$u(t+b) = T_{t+b} u(0) + \int_0^{t+b} T_{t+b-\zeta} f(\zeta) \, d\zeta = T_{t+a} T_{b-a} u(0) + \tag{1.14}$$

$$+ \int_0^{t+b} T_{t+b-\sigma} f(\sigma) \, d\sigma \, .$$

Next remark, again by (1.1), the representation

$$u(b-a)=T_{b-a}\,u(0)+\int_{0}^{b-a}T_{b-a-\sigma}f(\sigma)\,d\sigma.$$

Introducing in (1.14) the value of $T_{b-a}\,u(0)$ we get

$$u(t+b)=T_{t+a}\left(u(b-a)-\int_{0}^{b-a}T_{b-a-\sigma}f(\sigma)\,d\sigma\right)+\int_{0}^{t+b}T_{t+b-\sigma}f(\sigma)\,d\sigma= \tag{1.15}$$

$$T_{t+a}\,u(b-a)+\int_{b-a}^{t+b}T_{t+b-\sigma}f(\sigma)\,d\sigma.$$

Now, set $\sigma=\zeta+b$; it follows $\int_{b-a}^{t+b}T_{t+b-\sigma}f(\sigma)\,d\sigma=\int_{-a}^{t}T_{t-\zeta}f(\zeta+b)\,d\zeta$ which proves our Lemma, and consequently Lemma 1 too.

Actually we see that (1.9) is true in the following way: Fix an arbitrary real t; then take N a positive integer, such that $t+N>0$, and take $k>N$. We use then (1.12); as $f(t+n_k)\rightarrow g(t)$ uniformly on $(-\infty,\infty)$ and in X strong, we have obviously

$$\lim_{k\to\infty}\int_{-N}^{t}T_{t-\zeta}f(\zeta+n_k)\,d\zeta=\int_{-N}^{t}T_{t-\zeta}\,g(\zeta)\,d\zeta.$$

Then we have also

$$(w)\lim_{k\to\infty}T_{t+N}\,u_{n_k}(-N)=T_{t+N}\,W_N.$$

because a linear continuous operator in a B-space is continuous also in respect to the weak convergence.

Now, we shall see that for function $V(t)$, $-\infty<t<\infty\rightarrow X$ defined by (1.9), the representation formula (1.11) holds for each semi axis $t\geq t_0$.

Take in fact two reals $t\geq t_0$, and choose an integer N such that $-N<t_0$. Apply Lemma 1 to t, t_0, N. We have

$$u_{n_k}(t)=T_{t+N}\,u_{n_k}(-N)+\int_{-N}^{t}T_{t-\zeta}f(\zeta+n_k)\,d\zeta$$

$$u_{n_k}(t_0)=T_{t_0+N}\,u_{n_k}(-N)+\int_{-N}^{t_0}T_{t_0-\zeta}f(\zeta+n_k)\,d\zeta.$$

Then, reasoning as above, we obtain

$$V(t) = T_{t+N} W_N + \int_{-N}^{t} T_{t-\zeta}\, g(\zeta)\, d\zeta$$

$$V(t_0) = T_{t_0+N} W_N + \int_{-N}^{t_0} T_{t_0-\zeta}\, g(\zeta)\, d\zeta$$

Trying now to get (1.11) we write

$$T_{t-t_0} V(t_0) = T_{t-t_0}\left(T_{t_0+N} W_N + \int_{-N}^{t_0} T_{t_0-\zeta}\, g(\zeta)\, d\zeta\right) = T_{t+N} W_N + \int_{-N}^{t_0} T_{t-\zeta}\, g(\zeta)\, d\zeta.$$

Hence

$$T_{t-t_0} V(t_0) + \int_{t_0}^{t} T_{t-\zeta}\, g(\zeta)\, d\zeta = T_{t+N} W_N + \int_{-N}^{t} T_{t-\zeta}\, g(\zeta)\, d\zeta = V(t)$$

that is (1.11).

Now we give the idea of the final step in the proof. Using uniform (on real axis) convergence of sequence $f(t+n_k)$ to $g(t)$, we obtain uniform convergence of sequence $g(t-n_k)$ to $f(t)$. Starting now with $V(t)$ and repeating above procedure, we obtain function $W(t)$, $-\infty < t < \infty$ which is continuous, bounded, and admits representation (1.3) $\forall\, t \geq t_0$.

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