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# DIFFERENTIABLE DISTRIBUTION SEMI-GROUPS

by VIOREL BARBU

## Introduction.

Distribution semi-groups of operators in a Banach space were introduced and studied by Lions [1] (cf. also Foiaş [2], Yoshinaga [10], [11], Peetre [3]). J. L. Lions has obtained the characterization of the infinitesimal generator of an exponential distribution semi-group and recently his result has been generalized by Chazarain [4], [5] (cf. also Foiaş [2], Larsson [9]), for regular distribution and hyper-distribution semi-groups. In their works, Da Prato-Mosco [6], [7] and Fujiwara [8] have generalized the notion of holomorphic semi-group (cf. Yosida [14]) to that of holomorphic distribution semi-groups and have given a characterization of the infinitesimal generator of such a distribution semi-group.

In this paper we extend some of their results for differentiable distribution semi-groups.

## §. 1. General results on distribution semi-groups.

We use the notations and the terminologies of L. Schwartz [12], [13] for infinitely differentiable functions and for distributions. We set  $R = ]-\infty, \infty[$  and denote:  $\mathcal{D}$  the space of all infinitely differentiable functions with compact support in  $R$ ,  $\mathcal{C}$  the space of infinitely differentiable function on  $R$ ;  $\mathcal{D}^+$  the space of all  $\varphi \in \mathcal{D}$  such that  $\text{supp } \varphi \subset [0, \infty)$  topologized as in Schwartz [12];  $\mathcal{S}$  the space of rapidly decreasing  $\mathcal{C}$  functions and  $\mathcal{C}'$  the space of scalar distributions with compact support. We denote also by  $\mathcal{D}_-$  the strict inductive limit of the spaces  $\mathcal{C}_a = \{\varphi \in \mathcal{C}; \text{supp } \varphi \subset ]-\infty, a]\}$ . Let  $X$  be a Banach space and  $L(X, X)$  the space of all continuous linear

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operator on  $X$  topologized with the operator norm. We denote also by  $\mathcal{D}'(L(X, X))$ ,  $\mathcal{D}'_+(L(X, X))$  and  $\mathcal{S}'(L(X, X))$  the vector-valued distribution spaces:  $L(\mathcal{D}, L(X, X))$ ,  $L(\mathcal{D}_-, L(X, X))$  and  $L(\mathcal{S}, L(X, X))$  respectively.

A vector-valued distribution  $T \in \mathcal{D}'_+(L(X, X))$  is called a distribution semi-group (D. S. G. in short) if it satisfies the following conditions:

i)  $T(\varphi^* \psi) = T(\varphi) T(\psi)$  for any  $\varphi, \psi \in \mathcal{D}^+$ .

ii) The support of  $T$  is contained in  $[0, \infty)$ .

iii) The linear subspace  $[T(\mathcal{D}^+)X]$  generated by  $T(\mathcal{D}^+)X$  is dense in  $X$ .

iv) If  $x \in X$  and  $T(\varphi)x = 0$  for any  $\varphi \in \mathcal{D}^+$ , then  $x = 0$ .

Let  $R_+[t; t > 0]$  and  $\bar{R}_+[t; t \geq 0]$ . If  $\mu \in \mathcal{C}'(\bar{R}_+)$  then we define a closable and densely defined operator  $T(\mu)$  on  $[T(\mathcal{D}^+)X]$ , by the formula

$$(1.1) \quad T(\mu)x = \sum_{i=1}^n T(\varphi_i^* \mu)x_i, \quad \text{for} \quad x = \sum_{i=1}^n T(\varphi_i)x_i; \quad x_i \in X, \varphi_i \in \mathcal{D}^+.$$

Let us denote the closure of  $T(\mu)$  again by  $T(\mu)$ . The linear operator  $A = T(-D\delta_0)$  is called the infinitesimal generator of  $T$ . Here  $\delta_t$  is the Dirac measure concentrated at  $\mathcal{C} = t$  and  $D$  is the derivation symbol.

For any  $\varphi(t)$  defined on  $R$  we denote by  $\varphi_+(t)$  the function

$$\varphi_+(t) = \begin{cases} \varphi(t) & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases}$$

We say that a D. S. G.  $T$  is regular if  $T(\varphi_+) = T(\varphi)$  for any  $\varphi \in \mathcal{D}_-$ . A regular D. S. G.,  $T$  is called of exponential growth (E. D. S. G. in short) if there exists a number  $\alpha$  such that  $e^{-\alpha t} T \in \mathcal{S}'(L(X, X))$ .

**THEOREM 1 (Lions).** A closed linear operator  $A$  in  $X$  with domain  $D(A)$  dense in  $X$ , generates an E. D. S. G. if and only if there is a number  $\alpha \geq 0$  such that

i) for any  $\lambda$  with  $\text{Re } \lambda > \alpha$ ,  $\lambda I + A$  defines an isomorphism of  $D(A)$  onto  $X$ .

ii)  $\|(\lambda I - A)^{-1}\| \leq \text{pol}(|\lambda|)$  for  $\text{Re } \lambda > \alpha$  where  $\text{pol}(|\lambda|)$  denote a polynomial with non-negative coefficients.

For the proof see [1] and [10]. The following theorem is due to Chazarain [4] (cf. also Foias [2]).

**THEOREM 2.** Let  $A$  be a closed and dense operator on  $X$ . Then  $A$  is the infinitesimal generator of a regular S. G. D. if and only if the following conditions hold:

i) There exist the constants  $\alpha, \beta, \gamma; \alpha, \gamma \geq 0$  such that  $(\lambda I - A)^{-1} \in L(X, X)$  for any  $\lambda$  in the domain

$$(1.2) \quad A = \{\lambda \in C; \operatorname{Re} \lambda \geq \alpha \log |\operatorname{Im} \lambda| + \beta; \operatorname{Re} \lambda \geq \gamma\}$$

ii)  $\|(\lambda I - A)^{-1}\|_{L(X, X)} \leq \operatorname{pol}(|\lambda|)$ , for  $\lambda \in A$ .

We shall give a sketch of the proof for this theorem.

*Necessity.* Since  $T \in \mathcal{D}'_+(L(X, X))$  is regular it follows (cf. Yoshinaga [10]) that  $T \in \mathcal{D}'_+(L(X, D_A))$  and

$$(1.3) \quad \left(\frac{d}{dt} - A\right) * T = \delta_0 \otimes I_X; \quad T * \left(\frac{d}{dt} - A\right) = \delta_0 \otimes I_{D_A}$$

where  $D_A$  is the domain of  $A$  topologized by the norm  $\|x\| = \|x\| + \|Ax\|$  and  $I_X$  (resp.  $I_{D_A}$ ) is the identical application on  $X$  (resp.  $D_A$ ). Let  $\varrho(t)$  be a  $\mathcal{D}$ -function such that  $\varrho(t) = 1$  on  $\{t; |t| < 1\}$  and  $\varrho(t) = 0$  for  $|t| > 2$ . We denote by  $E$  (resp.  $\Phi$ ) the distribution  $\varrho T$  (resp.  $\varrho' T$ ) and set  $\widehat{E}(\varrho) = E(e^{-\lambda t}); \widehat{\Phi}(\lambda) = \Phi(e^{-\lambda t})$  for any complex  $\lambda$ . From (1.3) we have

$$(1.4) \quad (\lambda I - A) \widehat{E}(\lambda) = I_X - \widehat{\Phi}(\lambda); \quad \widehat{E}(\lambda)(\lambda I - A) = I_{D_A} - \widehat{\Phi}(\lambda).$$

Since  $\Phi \in \mathcal{C}'(L(X, X))$  and  $\operatorname{supp} \Phi \geq 1$ , by a well known argument it follows

$$(1.5) \quad \|\widehat{\Phi}(\lambda)\|_{L(X, X)} \leq C(1 + |\lambda|)^N \exp(-\operatorname{Re} \lambda), \quad \text{for any } \lambda \in C.$$

From (1.4) this implies that  $(\lambda I - A)^{-1} \in L(X, X)$  for  $\lambda \in A = \{\lambda; \operatorname{Re} \lambda \geq \alpha \log |\operatorname{Im} \lambda| + \beta; \operatorname{Re} \lambda \geq \gamma\}$ , with  $\alpha, \gamma > 0$  convenient chosen. Moreover we get

$$(1.6) \quad \|(\lambda I - A)^{-1}\|_{L(X, X)} \leq c \|\widehat{E}(\lambda)\|_{L(X, X)}, \quad \text{for } \lambda \in A.$$

But  $\operatorname{supp} E \subset [0, 1]$  and by a Paley-Wiener theorem argument it follows

$$\|\widehat{E}(\lambda)\|_{L(X, X)} \leq \operatorname{pol}(|\lambda|) \quad \text{for any } \lambda \in A.$$

This inequality together (1.6) proves (ii).

*Sufficiency.* Define  $T \in \mathcal{D}'(L(X, X))$  by the formula

$$(1.7) \quad T(\varphi) = (2\pi i)^{-1} \int_{\Gamma} (\lambda I - A)^{-1} \widehat{\varphi}(-\lambda) d\lambda,$$

where  $\Gamma$  is the frontier of  $A$  and

$$\widehat{\varphi}(\lambda) = \int e^{-\lambda t} \varphi(t) dt.$$

From (i), (ii) it follows that  $T$  is a regular D. S. G.

## §. 2. Differentiable distribution semi-groups.

**DEFINITION.** A regular D. S. G. T. is called differentiable if for every  $t > 0$ ,  $T(\delta_t) \in L(X, X)$  and the application  $t \rightarrow T(\delta_t)$  from  $R^+$  in  $L(X, X)$  is differentiable.

**REMARKS.** 1° If  $T$  is differentiable, then the distribution  $T \in \mathcal{D}'(L(X, X))$  is given on  $R^+$  by a differentiable  $L(X, X)$ -valued function. In fact for any  $x \in [T(\mathcal{D}^+)X]$ , we have

$$(2.1) \quad T(\varphi)x = \int_0^{\infty} T(\delta_t)x \varphi(t) dt \quad \forall \varphi \in \mathcal{D}(R^+).$$

Since the space  $[T(\mathcal{D}^+)X]$  is dense in  $X$ , this implies that  $T = T(\delta_t)$  on  $R^+$ .

2° Let  $T$  be a differentiable D. S. G. and  $A = T(-\delta'_0)$  its infinitesimal generator. Then for every  $t > 0$ ,  $T(\delta_t)X \subset D_A$  and

$$(2.2) \quad \frac{d}{dt} T(\delta_t)x = AT(\delta_t)x, \quad \text{for any } x \in X \text{ and } t > 0.$$

To prove this, we consider  $x$  an arbitrary element of  $X$  and set  $y(t) = T(\delta_t)x$  for  $t > 0$ . Let  $x_n$  be a sequence of  $[T(\mathcal{D}^+)X]$  such that  $x_n \rightarrow x$ . It is obvious that  $AT(\delta_t)x_n = d/dt T(\delta_t)x_n \rightarrow d/dt T(\delta_t)x$  for  $n \rightarrow \infty$ . Since  $A$  is closed, this implies that  $y(t) \in D(A)$  and  $y'(t) = Ay(t)$  for any  $t > 0$ .

The following theorem gives a characterization for the generator of a differentiable D. S. G.

**THEOREM 3.** Let  $A$  be a closed operator on  $X$  with domain  $D_A$  dense in  $X$ . A necessary and sufficient condition for  $A$  generate a differentiable regular D. S. G. is: for every  $\delta > 0$  there exist positive constants  $C_\delta$  and  $M_\delta$  such that  $(\lambda I - A)^{-1} \in L(X, X)$  for any complex  $\lambda$  in the domain

$$A_\delta = \{\lambda; \operatorname{Re} \lambda \geq -\delta \log |\operatorname{Im} \lambda| + C_\delta\} \cup \{\lambda; \operatorname{Re} \lambda \geq \gamma\}$$

and for  $\lambda \in A_\delta$ ,

$$(2.3) \quad \|(\lambda I - A)^{-1}\|_{L(X, X)} \leq M_\delta \text{pol}(|\lambda|),$$

where  $\gamma$  is a non-negative constant independent of  $\delta$ .

**PROOF. Necessity.** Let  $\varphi(t)$  be a  $\mathcal{D}$ -function so that  $\text{supp } \varphi \subset \{t; |t| \leq 1\}$  and  $\varphi(t) = 1$  in  $|t| < 2^{-1}$ . Denote by  $\varphi_\varepsilon(t)$ ,  $\varepsilon > 0$ , the function  $\varphi(t/\varepsilon)$  and by  $E_\varepsilon, \Phi_\varepsilon$  the vector-valued distribution  $\varphi_\varepsilon T$  and  $\varphi'_\varepsilon T$  respectively. It is obvious that  $\Phi_\varepsilon$  is differentiable and  $\text{supp } \Phi_\varepsilon \subset \{t; 2^{-1}\varepsilon \leq t \leq \varepsilon\}$ . Put

$$M_\varepsilon = \sup_{2^{-1}\varepsilon < t < \varepsilon} \|D^1(\varphi'_\varepsilon(t) T(t))\|.$$

It is easy to see that

$$\|\widehat{\Phi}_\varepsilon(\lambda)\| \leq 2^{-1}\varepsilon M_\varepsilon |\text{Im } \lambda|^{-1} \sup_{\varepsilon/2 \leq t \leq \varepsilon} \exp(-t \text{Re } \lambda), \quad \lambda \in C.$$

Hence  $\|\widehat{\Phi}_\varepsilon(\lambda)\| \leq 2^{-1}$  for any complex  $\lambda$  in the domain

$$\Sigma_\varepsilon = \{\lambda; \log |\text{Im } \lambda| \geq -\varepsilon \text{Re } \lambda + \log M_\varepsilon; \text{Re } \lambda \leq 0\} \cup \\ \cup \{\lambda; \log |\text{Im } \lambda| \geq -\varepsilon/2 \text{Re } \lambda + \varepsilon/2 \log M_\varepsilon; \text{Re } \lambda \geq 0\}.$$

Remembering (1.4) this implies that

$$(2.4) \quad \|(\lambda I - A)^{-1}\|_{L(X, X)} \leq 2^{-1} \|\widehat{E}_\varepsilon(\lambda)\|_{L(X, X)}, \quad \text{for } \lambda \in \Sigma_\varepsilon.$$

On the other hand, since  $\text{supp } E \subset [0, \varepsilon]$  we have (see L. Schwartz 12, Th.)

$$\|\widehat{E}_\varepsilon(\lambda)\|_{L(X, X)} \leq \sup_{t \in [0, \varepsilon]} \sum_{j=0}^m |D^j(e^{-\lambda t} \varphi_\varepsilon(t))|, \quad \lambda \in C.$$

Hence

$$(2.5) \quad \|\widehat{E}_\varepsilon(\lambda)\|_{L(X, X)} \leq M_\varepsilon \text{pol}(|\lambda|) |\text{Im } \lambda|, \quad \text{for } \lambda \in \Sigma_\varepsilon$$

where the degree of the polynomial  $\text{pol}(|\lambda|)$  is equal to the order of the distribution  $T$  in a neighbourhood of the origin. Therefore  $(\lambda I - A)^{-1} \in L(X, X)$  and satisfies (2.3) for any  $\lambda \in \Sigma_\varepsilon$ . From theorem 2 it follows then, that there exists a non-negative constant  $\gamma$  such that  $\|(\lambda I - A)^{-1}\| \leq \text{pol}(|\lambda|)$  for  $\text{Re } \lambda \geq \gamma$ . If we choose  $N_\varepsilon$  so that  $\log N_\varepsilon/M_\varepsilon \geq \gamma$ , we deduce that the

estimate (2.3) is verified for any  $\lambda$  in the domain

$$\{\lambda; \operatorname{Re} \lambda \geq -\varepsilon^{-1} \log |\operatorname{Im} \lambda| + \varepsilon^{-1} \log N_\varepsilon\} \cup \{\lambda; \operatorname{Re} \lambda \geq \gamma\}.$$

Choosing  $\delta = \varepsilon^{-1}$  this implies that  $(\lambda I - A)^{-1}$  satisfies (2.3) in any domain  $A_\delta$ .

*Sufficiency.* It is obvious that  $T$  is an E. D. S. G. Hence we may write

$$(2.6) \quad T(\varphi) = (2\pi i)^{-1} \int_{\operatorname{Re} \lambda = \gamma} (\lambda I - A)^{-1} \widehat{\varphi}(-\lambda) d\lambda, \quad \varphi \in \mathcal{D}.$$

As  $(\lambda I - A)^{-1}$  is holomorphic in every  $A_\delta$  and  $\|(\lambda I - A)^{-1} \widehat{\varphi}(-\lambda)\|$  rapidly tends to zero at infinity, we can change the path of integration and obtain

$$T(\varphi) = (2\pi i)^{-1} \int_{\Gamma_\delta} (\lambda I - A)^{-1} \widehat{\varphi}(-\lambda) d\lambda,$$

where  $\Gamma_\delta$  is the boundary of the domain  $\{\lambda; \operatorname{Re} \lambda \geq -\delta \log |\operatorname{Im} \lambda| + C_\delta; \operatorname{Re} \lambda \leq \gamma\}$ . Let  $\{\varrho_k\}_{k=0}^\infty \subset \mathcal{D}^+$  be a sequence of regularization for Dirac distribution, i. e.  $\varrho_n(t) \geq 0$ ,  $\int \varrho_n(t) dt = 1$  and  $\operatorname{supp} \varrho_n \rightarrow 0$ . We have

$$(2.7) \quad D_t^k T(\delta_t * \varrho_n) = (2\pi i)^{-1} \int_{\Gamma_\delta} e^{\lambda t} (\lambda I - A)^{-1} \lambda^k \widehat{\varrho}_n(-\lambda) d\lambda$$

for any non-negative integer  $k$ . We set  $\Gamma_\delta = \Gamma_\delta^1 \cup \Gamma_\delta^2$ , where  $\Gamma_\delta^1$  is given by  $\{\operatorname{Re} \lambda = -\delta \log |\operatorname{Im} \lambda| + C_\delta; |\operatorname{Im} \lambda| \geq A_\delta = \exp(\delta^{-1}(C_\delta - \gamma))\}$  and  $\Gamma_\delta^2$  by  $\{\operatorname{Re} \lambda = \gamma; |\operatorname{Im} \lambda| \leq \exp(\delta^{-1}(C_\delta - \gamma))\}$ . We write

$$T_j^{(k)}(t) = (2\pi i)^{-1} \int_{\Gamma_\delta^j} e^{\lambda t} (\lambda I - A)^{-1} \lambda^k \widehat{\varrho}_n(-\lambda) d\lambda; \quad j = 1, 2, \dots$$

It is obvious that  $T_2^{(k)}(t)$  is defined for every  $t \geq 0$  and

$$(2.8) \quad \|T_2^{(k)}(t)\|_{L(X, X)} \leq M_\delta^{m+k+1} \exp(\gamma t) \quad \text{for } t \geq 0.$$

where  $M_\delta$  is another non-negative constant. Let  $\lambda = \sigma + i\eta$ ; then since on  $\Gamma_\delta^1$ ,  $\sigma = -\delta \log |\eta| + C_\delta$  we have

$$\|T_1^{(k)}(t)\|_{L(X, X)} \leq M_\delta \exp(C_\delta t) \int |\eta|^{m+k-t\delta} d\eta.$$

Hence

$$(2.9) \quad \|T_1^{(k)}(t)\| \leq M_\delta \exp(C_\delta t), \quad \text{for } t > (m+k+1)\delta^{-1}.$$

Therefore we find a constant  $M_{k, \delta}$  such that

$$(2.10) \quad \|D_t^k T(\delta_t * \varrho_n)\|_{L(X, X)} \leq M_{k, \delta} \exp(C_\delta t)$$

for  $t > (m+k+1)\delta^{-1}$ . But for any  $x \in [T(\mathcal{D}^+)X]$  we have

$$D_t^k T(\delta_t * \varrho_n)x \rightarrow D_k^k T(\delta_t)x, \quad t > 0, \quad k = 0, 1, \dots$$

uniformly on every compact. Since the space  $[T(\mathcal{D}')X]$  is dense in  $X$  this implies that  $D_t^k T(\delta_t) \in L(X, X)$  for  $t > (m+k+1)\delta^{-1}$ . Since  $\delta$  is arbitrary this proves the differentiability of  $T$ . Moreover we have proved that  $D_t^k T(\delta_t) \in L(X, X)$  for any  $t > 0$  and  $k = 0, 1, \dots$ . Combining with the first part of the proof it follows that if a regular D. S. G.,  $T$  is differentiable then the application  $t \rightarrow T(\delta_t)$  from  $R^+$  in  $L(X, X)$  is infinitely differentiable.

**COROLLARY.** Let  $A$  be a closed and densely defined operator on the Banach space  $X$ . If the conditions of Theorem 3 are satisfied, then the abstract Cauchy problem  $(ACP)_0$ :

$$(2.11) \quad \begin{aligned} \frac{du(t)}{dt} - Au(t) &= 0, & \text{for } t > 0, \\ u(0) &= 0, \end{aligned}$$

has a solution  $u \in C^\infty(R^+, X)$  for every  $x \in X$ .

**PROOF.** Let  $T$  be the D. S. G. generated by  $A$ . Then from remark 2 it follows that  $T(\delta_t)x$  solves  $(ACP)_0$  for any  $x \in X$ .

**REMARKS 1<sup>0</sup>** If  $T$  is a differentiable regular D. S. G., then

$$(2.1') \quad \|D^k T(\delta_t)\|_{L(X, X)} = 0(\exp(\gamma_0 t)), \quad \text{for } t \rightarrow \infty$$

and  $k = 0, 1, \dots$ , where  $\gamma_0$  is a non-negative constant.



2° In particular if  $T$  is a strongly continuous semi-group of bounded linear operators on  $X$ , then according to formula (2.5) it follows that  $T$  is differentiable if and only if for  $\lambda \in A_\delta$

$$(2.12) \quad \|(\lambda I - A)^{-1}\|_{L(X, X)} \leq M_\delta |\operatorname{Im} \lambda|.$$

Thus we find a result proved by Pazy [16].

### § 3. Analytic and non-quasianalytic D. S. G.

Let  $Y$  be a Banach space and  $L = \{L_k\}_{k=0}^\infty$  and increasing sequence of non-negative numbers such that

$$(3.1) \quad L_k^{2k} \leq L_{k-1}^{k-1} L_{k+1}^{k+1}; \quad L_{m+k+n} \leq q(m) L_k, \quad k = 0, 1, \dots$$

where  $m$  and  $n$  are non-negative integers and  $r \rightarrow q(r)$  is a positive and monotone increasing function. If  $\Omega$  is an open set of  $\mathbb{R}$  we denote by  $C^L(\Omega, Y)$  the space of infinitely differentiable  $Y$ -valued functions  $u(t)$  in  $\Omega$ , such that for any compact subset  $K$  there exists  $M > 0$  such that

$$(3.2) \quad \sup_{t \in K} \|D^j u(t)\| \leq M^{j+1} L_j^j; \quad j = 0, 1, \dots$$

The space  $C^L(\Omega, Y)$  is topologized as projective limit of all  $\{C^L(K, Y); K \subset \Omega\}$ . The function class  $C^L(\Omega, Y)$  is called non-quasi-analytic if it contains a non-trivial regular function with compact support contained in  $\Omega$ . The Carleman-Denjoy criterion states that  $C^L$  is non-quasianalytic if and only if

$$\sum L_j^{-1} < \infty.$$

If  $Y = \mathbb{R}$  we often omit  $\mathbb{R}$  and write  $C^L(Y)$ . In particular, if  $L_j = (j!)^{\rho^j}$ ,  $C^L$  is the classical Gevrey class  $G$  which is non-quasianalytic for  $1 < \rho < \infty$ . For  $\rho = 1$  we obtain the class of real analytic functions. If  $L$  is a non-quasianalytic sequence we denote by  $C_0^L(\Omega, Y)$  the space  $C^L(\Omega, Y) \cap C_0^\infty(\Omega, Y)$ .

DEFINITION. A D. S. G.,  $T \in \mathcal{D}'_+(L(X, X))$  is said to be of class  $C^L$  if the mapping  $t \rightarrow T(\delta_t)$  is of class  $C^L$  on  $\mathbb{R}^+$ .

In particular, for  $L_j = (j!)^{\rho^j}$  the semi-group  $T$  is called  $\rho$ -hypoanalytic;  $1 \leq \rho < \infty$ . As above we remark that if the semi-group  $T$  is of class  $C^L$  then the distribution  $T \in \mathcal{D}'(L(X, X))$  is defined on  $\mathbb{R}^+ = \{t; t > 0\}$  by a  $L(X, X)$ -valued function of class  $C^L$ .

Let  $L = \{L_j\}_{j=0}^\infty$  be a non-quasianalytic sequence and  $\omega_L(t)$  be a scalar function defined by

$$\omega_L(t) = \sum_{j=0}^\infty t^j / L_j^j; \quad t \geq 0.$$

Then we have

**THEOREM 4.** Let  $T$  be a regular D.S.G. and  $A = T(-\delta'_0)$  be its infinitesimal generator.  $T$  is of class  $C^L$  if and only if for every  $0 < \varepsilon < 1$  there exist  $C_\varepsilon$  and  $M_\varepsilon > 0$  such that

i)  $(\lambda I - A)^{-1} \in L(X, X)$  for any  $\lambda$  in the domain

$$(3.3) \quad \Sigma_\varepsilon = \{\lambda; \operatorname{Re} \lambda \geq -\log \omega_L(C_\varepsilon |\operatorname{Im} \lambda|) + \gamma\}$$

and

ii)  $\|(\lambda I - A)^{-1}\|_{L(X, X)} \leq M_\varepsilon \operatorname{pol}(|\lambda|) \exp(\varepsilon |\operatorname{Re} \lambda|)$ , for  $\lambda \in \Sigma_\varepsilon$  where  $\gamma$  is a positive constant independent of  $\varepsilon$ .

**PROOF. Necessity.** Assume that for every  $0 < \varepsilon < 1$ .

$$(3.4) \quad \|D^k T(t)\|_{L(X, X)} \leq M_\varepsilon^{k+1} L_k^k, \quad t \in [\varepsilon/2, \varepsilon], \quad k = 0, 1, \dots$$

We choose  $\varphi \in C_0^L$  such that  $\operatorname{supp} \varphi \subset \{t; |t| \in 1\}$ ,  $\varphi(t) = 1$  in  $|t| \leq 2$ , and denote:  $\Phi_\varepsilon = \varphi'_\varepsilon T$ ;  $E_\varepsilon = \varphi_\varepsilon T$  where  $\varphi_\varepsilon(t) = \varphi(t/\varepsilon)$ . From (3.4) we obtain

$$(3.5) \quad \|\Phi_\varepsilon^{(k)}(t)\| \leq M M_\varepsilon \varepsilon^{-1} (2 N_\varepsilon)^{-k} L_k^k, \quad k = 0, 1, \dots$$

where  $M > 0$  and  $N_\varepsilon^{-1} = 2 \max(M\varepsilon^{-1}, M\varepsilon)$ . Or,

$$\|\widehat{\Phi}_\varepsilon(\lambda)\| = M (L_k / 2N_\varepsilon |\operatorname{Im} \lambda|)^k \int_{\varepsilon/2}^\varepsilon \exp(-t \operatorname{Re} \lambda) dt, \quad k = 0, 1, \dots$$

Thus for any  $\lambda$  complex in the domain

$$A_\varepsilon = \{\lambda; \operatorname{Re} \lambda \geq \varepsilon^{-1} \log \omega_L(N_\varepsilon |\operatorname{Im} \lambda|) + M_\varepsilon^{-1}; \operatorname{Re} \lambda \leq \gamma_\varepsilon\}$$

we have  $\|\widehat{\Phi}_\varepsilon(\lambda)\| \leq 2^{-1}$ . Here  $M_\varepsilon^{-1}$  and  $\gamma_\varepsilon$  are another non-negative constants. Since the semi group  $T$  is regular we may assume that  $\gamma_\varepsilon = \infty$ . As in the proof of theorem 3, this implies that  $(\lambda I - A)^{-1} \in L(X, X)$ , and

$$(3.6) \quad \|(\lambda I - A)^{-1}\| \leq p_\varepsilon(|\lambda|) \exp(\varepsilon |\operatorname{Re} \lambda|), \quad \text{for } \lambda \in A_\varepsilon.$$

where  $p_\varepsilon$  is a polynomial with non-negative coefficients. Since the sequence  $\{L_k\}$  satisfies (3.1), the function  $r \rightarrow \log \omega_L(r)$  is sub-additive. Hence we may find another constant  $C_\varepsilon > 0$  such that (3.6) to be satisfied for any  $\lambda$  in the domain

$$\{\lambda; \operatorname{Re} \lambda \geq -2 \log \omega_L(C_\varepsilon |\operatorname{Im} \lambda|) + M_\varepsilon^1\}.$$

Without loss of the generality we may assume that  $\varepsilon \rightarrow C_\varepsilon$  is bounded and  $\lim_{\varepsilon \rightarrow 0} M_\varepsilon^1 = \infty$ . Let  $a$  be a non-negative constant such that  $C_\varepsilon \leq a$  for  $0 < \varepsilon < 1$ . Using the above argument it follows that there exist  $b > 0$  such that

$$(3.8) \quad \|(\lambda I - A)^{-1}\| \leq \operatorname{pol}(|\lambda|) \exp(N |\operatorname{Re} \lambda|)$$

for  $\operatorname{Re} \lambda \geq -\log \omega_L(a |\operatorname{Im} \lambda|) + b$ . For  $\varepsilon$  enough small we may suppose  $b < M_\varepsilon^1$ . Hence

$$(3.9) \quad \|(\lambda I - A)^{-1}\| \leq p_\varepsilon(|\lambda|) \exp(\varepsilon |\operatorname{Re} \lambda|)$$

for  $\operatorname{Re} \lambda \geq -\log \omega_L(C_\varepsilon |\operatorname{Im} \lambda|) + b$ ; and,  $|\operatorname{Im} \lambda| \geq C_\varepsilon^{-1} \omega_L^{-1}(\exp(M_\varepsilon^{-1} - b))$ . Using (3.8) we get that the estimate (3.3) satisfied in the whole domain  $\Sigma_\varepsilon$  with  $\gamma = b$ .

*Sufficiency.* From (3.3) it follows that  $\|(\lambda I - A)^{-1}\| = 0(\operatorname{pol}(|\lambda|))$  for  $\operatorname{Re} \lambda > \gamma$ . Hence, as in the proof of theorem 3, we get

$$(3.10) \quad D_t^k T(\delta_t * \varrho_n) = (2\pi i)^{-1} \int_{\Gamma_\varepsilon} e^{it} (\lambda I - A)^{-1} \lambda^k \widehat{\varrho}_n(-\lambda) d\lambda$$

where  $\{\varrho_n\}$  is a sequence of regularization of  $\mathcal{D}^+$  and  $\Gamma_\varepsilon$  is the frontier of the domain

$$\{\operatorname{Re} \lambda \geq -\log \omega_L(C_\varepsilon |\operatorname{Im} \lambda|) + \gamma; \operatorname{Re} \lambda \leq \gamma\}.$$

It is easily verified that

$$(3.11) \quad \|D_t^k T(\delta_t * \varrho_n)\|_{L(\mathbb{R}, \mathbb{R})} \leq M_\varepsilon^{m+k+1} \exp(\gamma t) \int_{\bar{K}_+} \eta^{m+k} \omega_L^{\varepsilon-t}(\eta) d\eta$$

where  $m$  is the degree of the polynomial  $\operatorname{pol}(|\lambda|)$ .

Since, for any non-negative integer  $k$ ,  $\omega_L(\eta) \geq \eta^j L_j^{-j}$ , it follows that the right side of (3.11) is bounded by  $M^{m+k+1} \exp(\gamma t) L_{(m+k+1)P+1}^{m+k+2}$

where  $p$  is the largest integer smaller than  $(t - \varepsilon)^{-1}$ . Because of the properties of the sequence  $\{L_k\}$  we find another constant  $M_\varepsilon$  we such that

$$\| D_t^k T(\delta_t * \varrho_n) \|_{L(X, X)} \leq M_\varepsilon^{k+1} L_k^k (\exp(\gamma t))$$

for  $t > \varepsilon$  and  $k = 0, 1, \dots$ . According to an argument used in the proof of theorem 3, this implies that

$$(3.13) \quad \| D^k T(\delta_t) \|_{L(X, X)} \leq M_\varepsilon^{k+1} + L_k^k \exp(\gamma t), \quad k = 0, 1, \dots; \quad t > \varepsilon > 0.$$

Since  $\varepsilon$  is arbitrary, the proof is complete.

**COROLLARY.** Let  $T$  be a regular D.S.G. and  $A$  its infinitesimal generator. The semi-group  $T$  is  $\varrho$ -hypoanalytic;  $1 \leq \varrho < \infty$ , if and only if for every  $\varepsilon > 0$  there exist constants  $C_\varepsilon$  and  $M_\varepsilon$  such that  $(\lambda I - A)^{-1} \in L(X, X)$  and satisfies

$$(3.14) \quad \| (\lambda I - A)^{-1} \| \leq M_\varepsilon \text{pol}(|\lambda|) \exp(\varepsilon |\text{Re } \lambda|)$$

for

$$(3.15) \quad \text{Re } \lambda \geq - C_\varepsilon |\text{Im } \lambda|^{1/\varepsilon} + \gamma$$

where  $\gamma$  is a non-negative constant independent of  $\varepsilon$ .

**PROOF.** The non-quasianalytic case  $\varrho > 1$  is a consequence of theorem 4. We assume that  $\varrho = 1$ . It is easily proved that there exists a sequence  $\varphi_k \in \mathcal{D}$ ,  $k = 0, 1, \dots$  such that

$$\text{supp } \varphi_k \subset \{t; |t| \leq 1\}; \quad \varphi_k(t) \equiv 1 \quad \text{for } |t| \leq 2^{-1}$$

and

$$(3.16) \quad |\varphi_k^{(j)}(t)| \leq M^{j+1} k^j, \quad \text{for } j \leq k.$$

Put

$$\varphi_{\varepsilon, k}(t) = \varphi_k(t/\varepsilon); \quad E_{\varepsilon, k} = \varphi_{\varepsilon, k} T; \quad \Phi_{\varepsilon, k} = \varphi'_{\varepsilon, k} T.$$

If  $T(\delta_t) \in G^1(\mathbb{R}^+, L(X, X))$ , then as in the proof of theorem 4 we find a constant  $M_\varepsilon > 0$  such that

$$(3.17) \quad \| \widehat{\Phi}_{\varepsilon, k}(\lambda) \|_{L(X, X)} \leq M_\varepsilon (k/M_\varepsilon |\text{Im } \lambda|)^k \int_{\varepsilon/2}^\varepsilon \exp(-t \text{Re } \lambda) dt$$

for any non-negative integer  $k$ . Take  $k$  equal to the largest integer smaller than  $M_\varepsilon |\operatorname{Im} \lambda| e^{-1}$ . Thus from (3.17) we obtain

$$\|\widehat{\Phi}_{\varepsilon, k}(\lambda)\|_{L(X, X)} \leq M_\varepsilon^{-1} \exp(-M_\varepsilon e^{-1} |\operatorname{Im} \lambda|) \int_{\varepsilon/2}^{\varepsilon} \exp(-\operatorname{Re} \lambda t) dt.$$

As above this implies that,  $(\lambda I - A)^{-1} \in L(X, X)$  and satisfies

$$\|(\lambda I - A)^{-1}\| \leq \operatorname{pol}(|\lambda|) \exp(\varepsilon |\operatorname{Re} \lambda|)$$

for

$$\operatorname{Re} \lambda \geq 2\varepsilon^{-1} \log(2M_\varepsilon) - M_\varepsilon (e\varepsilon)^{-1} |\operatorname{Im} \lambda|; \quad \text{and} \quad |\operatorname{Im} \lambda| < M_\varepsilon^{-1} e(k+1).$$

Since  $k$  is arbitrary, this implies that  $(\lambda I - A)^{-1}$  satisfies the estimate (3.14) in a domain of the form

$$\operatorname{Re} \lambda \geq -C_\varepsilon |\operatorname{Im} \lambda| + D_\varepsilon; \quad \operatorname{Re} \lambda \geq \gamma$$

Sufficiency of (3.14) follows just in the proof of theorem 4.

#### § 4. Distribution semi-groups of class $A^e$ .

If a D.S.G.  $T$  is differentiable, then for any integer  $k \geq 0$ ,  $\|D_t^k T(\delta_t)\|_{L(X, X)}$  is of exponential growth for  $t \rightarrow \infty$ . In this section we also impose a restriction of the origin for  $D_t^k T(\delta_t)$ .

**DEFINITION.** Let  $1 \leq \varrho < \infty$ . A regular D.S.G.,  $T$  is said to be of class  $A^e$ , if for  $t > 0$ ,

$$(4.1) \quad \|D_t^k T(\delta_t)\|_{L(X, X)} \leq p(t^{-\varrho}) (Mt)^{-ek} (k!)^e \exp(\gamma t); \quad k = 0, 1, \dots$$

where  $M, \gamma$  are non-negative constants and  $p(r)$  is a polynomial with non negative coefficients.

The semi-groups of class  $A^e$  can be characterized in the following way (see theorem 4).

**THEOREM 5.** Let  $A$  be a closed operator on  $X$  with the domain  $D_A$  dense in  $X$ . Then  $A$  is the infinitesimal generator for a D.S.G. of class  $A^e$  if and only if there exist positive constants  $\alpha$  and  $\beta$  such that

$$(i) \quad (\lambda I - A)^{-1} \in L(X, X) \text{ for}$$

$$(4.2) \quad \lambda \in A = \{\lambda \mid \operatorname{Re} \lambda > -\alpha \mid \operatorname{Im} \lambda \mid^{1/e} + \beta\}.$$

$$(ii) \quad \|(\lambda I - A)^{-1}\| \leq \operatorname{pol}(|\lambda|) \text{ for any } \lambda \in A.$$

PROOF. *Necessity.* From (4.1) it is obvious that  $e^{-\gamma t} T \in \mathcal{S}'(L(X, X))$  where  $\gamma$  is a non-negative constant. Therefore the semigroup  $T$  is of exponential growth and from Lions's theorem it follows that  $(\lambda I - A)^{-1}$  exists and satisfies the estimate (ii) for  $\text{Re } \lambda > \gamma$ . Moreover by a well known result (cf. Schwartz [12], Yoshinaga [11]) there exists a function  $f \in \mathcal{C}^0(L(X, X))$  and an integer  $m \geq 0$  such that

$$(4.3) \quad \|f(t)\|_{L(X, X)} = o((1 + t^2)^m) \quad \text{for } t \rightarrow \infty$$

and  $e^{-\gamma t} T$  may be expressed as

$$(4.4) \quad e^{-\gamma t} T = D^m f.$$

The function  $f(t)$  is regular on  $R^+$  and from (4.1) we have

$$(4.5) \quad \|D^k f(t)\|_{L(X, X)} \leq M p (t^{-\rho}) (M t^{-\rho})^k (k!)^e, \quad k \geq m, t > 0$$

and

$$(4.6) \quad \|D^k f(t)\|_{L(X, X)} \leq M_1 p (t^{-\rho}) t^{m-k} \quad \text{for } 0 \leq k \leq m$$

Let  $j$  and  $k$  be two non-negative integers such that  $\rho(k+p) < j \leq \rho(k+p) + 1$ , where  $p$  is the degree of  $p(r)$ . Since  $\text{supp } f \subset [0, \infty)$ , for  $j$  and  $k$  taken as above we have

$$\lambda^k D^j \widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} D^k (t^j f(t)) dt, \quad \text{Re } \lambda > \varepsilon$$

where  $\varepsilon$  is an arbitrary positive number. Then (4.5) and (4.6) imply that

$$\|D^j \widehat{f}(\lambda)\|_{L(X, X)} \leq M^{j+1} j! |\lambda|^{-k}$$

for  $\text{Re } \lambda > \varepsilon$  and  $\rho(k+p) < j \leq \rho(k+p) + 1$ . Hence

$$(4.7) \quad \|D^j \widehat{f}(\lambda)\|_{L(X, X)} \leq M^{j+1} j! |\lambda|^{-(j-1)e^{-1} + p}, \quad \text{for } \text{Re } \lambda > \varepsilon > 0.$$

Then the analyticity of  $\widehat{f}(\lambda)$  in the domain  $\{\lambda : \text{Re } \lambda > 0\}$  and the estimate (4.7) imply that  $\widehat{f}(\lambda)$  can be extended holomorphically in a domain of the form

$$\Sigma = \{\lambda \in \mathbb{C}; |\text{Re } \lambda - \varepsilon| < M^{-1} |\text{Im } \lambda|^{1/e}\}$$

and  $\|\widehat{f}(\lambda)\|_{L(X, X)} \leq M |\lambda|^{p+e^{-1}}$  for  $\lambda \in \Sigma$ . We observe that

$$T(e^{-\lambda t}) = \widehat{T}(\lambda) = (\lambda - \gamma)^m \widehat{f}(\lambda - \gamma), \quad \text{for } \operatorname{Re} \lambda > \gamma.$$

Hence we have proved that  $\widehat{T}(\lambda)$  exists and satisfies the estimate

$$(4.8) \quad \|\widehat{T}(\lambda)\|_{L(X, X)} \leq \operatorname{pol}(|\lambda|)$$

for  $|\operatorname{Re} \lambda - \gamma - \varepsilon| < M^{-1} |\operatorname{Im} \lambda|^{1/e}$ . Because  $\widehat{T}(\lambda) = (\lambda I - A)^{-1} \in L(X, X)$  for  $\operatorname{Re} \lambda > \gamma$ , the analyticity of  $(\lambda I - A)^{-1}$  implies that it satisfies the estimate (ii) for

$$\operatorname{Re} \lambda > -M^{-1} |\operatorname{Im} \lambda|^{1/e} + \gamma.$$

*Sufficiency.* From (i) and (ii) it follows that the operator  $A$  generates an E. D. S. G. and

$$T(\varphi) = (2\pi i)^{-1} \int_{\operatorname{Re} \lambda = \beta + \varepsilon} (\lambda I - A)^{-1} \widehat{\varphi}(-\lambda) d\lambda, \quad \varphi \in \mathcal{D}$$

where  $\varepsilon$  is an arbitrary positive number. As in the proof of theorem 5 we have

$$(4.9) \quad D_t^k T(\delta_t * \varrho_n) = (2\pi i)^{-1} \int_{\Gamma_\varepsilon} e^{\lambda t} (\lambda I - A)^{-1} \lambda^k \widehat{\varrho}_n(-\lambda) d\lambda$$

where  $\Gamma_\varepsilon$  is the curve given by

$$\Gamma_\varepsilon = \{\lambda = \sigma + i\eta; \sigma = -\alpha |\eta|^{1/e} + \beta + \varepsilon; -\infty < \sigma \leq \beta + \varepsilon\}.$$

Then our estimates of  $(\lambda I - A)^{-1}$  and  $\widehat{\varrho}_n(-\lambda)$  imply that

$$\|D_t^k T(\delta_t * \varrho_n)\|_{L(X, X)} \leq C^{k+1} \exp(\beta + \varepsilon) t \int_0^\infty \eta^{p+k} \exp(-t \alpha \eta^{1/e}) d\eta$$

for any  $t > 0$  and  $k = 0, 1, \dots$ . Hence

$$(4.10) \quad \|D_t^k T(\delta_t * \varrho_n)\|_{L(X, X)} \leq \operatorname{pol}(t^{-e}) (C t^{-e})^k \Gamma(\rho k) \exp(\beta + \varepsilon) t.$$

Here  $\Gamma(r)$  is Euler's function and  $C$  is a positive constant independent of  $\varepsilon$ . Consequently the semi-group  $T$  is of class  $A^e$  and the proof is complete.

For  $\varrho \geq 1$  and  $\gamma \geq 0$  we denote by  $A_\gamma^e$  the class of regular D. S. G.,  $T$  such that for any  $\varepsilon > 0$

$$(4.11) \quad \| D_t^k T(\delta_t) \|_{L(X, X)} \leq \underset{\varepsilon}{\text{pol}}(t^{-e})(Mt)^{-ek} (k!)^e \exp(\gamma + \varepsilon)t.$$

As a consequence of theorem 5 and its proof we obtain (see also Da Prato-Mosco [7]).

**COROLLARY.** A closed and densely defined operator  $A$  on  $X$  generates a D. S. G. of class  $A_\gamma^e$  if and only if there exists  $\alpha > 0$  such that  $(\lambda I - A)^{-1} \in L(X, X)$  and satisfies

$$\| (\lambda I - A)^{-1} \| \leq \underset{\varepsilon}{\text{pol}}(|\lambda|)$$

for  $\text{Re } \lambda > -\alpha |\text{Im } \lambda|^{1/e} + \gamma + \varepsilon$ , where  $\varepsilon$  is an arbitrary non-negative number.

Let  $T \in \mathcal{D}_+^1(L(X, X))$  be a regular D. S. G.  $T$  is said holomorphic (cf. Fujiwara [8], Da Prato-Mosco [6]) in the sector  $\Sigma = \{\mu; |\arg \mu| < \alpha; 0 < \alpha < \pi/2\}$  if  $t \rightarrow T(\delta_t)$  can be extended at an holomorphic function  $T_\mu$  in this sector. It is obvious that a D. S. G. of class  $A^e$  with  $\varrho = 1$  is holomorphic in a sector of the complex plane. Conversely from Cauchy's formula it follows that any holomorphic D. S. G. in a sector  $\Sigma$  is of class  $A^1$ . We can now formulate the following result (cf [8]).

**THEOREM 6.** A closed and dense linear operator  $A$  generates a D. S. G. which is holomorphic in the sector  $\Sigma = \{\mu; |\arg \mu| < \alpha < \pi/2\}$  if and only if there exists a real  $\gamma$  such that for any  $\varepsilon > 0$  and any  $\lambda$  in the sector

$$A = \{\lambda \mid |\arg(\lambda - \gamma)| < \pi/2 + \alpha - \varepsilon\}$$

we have  $(\lambda I - A)^{-1} \in L(X, X)$  with the estimate

$$(4.12) \quad \| (\lambda I - A)^{-1} \|_{L(X, X)} \leq \text{pol}(|\lambda|).$$

**PROOF.** The sufficiency of condition (4.12) is a consequence of theorem 5. Also the necessity can be obtained by an adaptation of the proof of theorem 5, but we shall give a direct proof. If the semi-group  $T$  is holomorphic in the sector  $\Sigma = \{\mu; |\arg \mu| < \alpha < \pi/2\}$ , then according to theorem 3, there exists a real  $\gamma$  such that  $e^{-\gamma t} T = D^m f$  where  $f(t)$  is a  $\mathcal{C}^0$



$(L(X, X))$ -function satisfying (4.3). Put

$$\widehat{f}(\lambda) = \int_0^{\infty} e^{-\lambda t} f(t) dt, \quad \text{for } \operatorname{Re} \lambda > 0.$$

Because  $f(t)$  is analytic in  $\Sigma$  and  $\|e^{-\lambda t} f(t)\|$  rapidly tends to zero at infinity, we may write

$$(4.13) \quad \widehat{f}(\lambda) = \int_{\Gamma} e^{-\lambda \mu} f(\mu) d\mu, \quad \text{for } \operatorname{Re} \lambda > 0.$$

where  $\Gamma = \{\mu; \mu = te^{-i(\alpha-\varepsilon)}; t > 0\}$  for  $\operatorname{Im} \lambda \geq 0$

and  $\Gamma = \{\mu; \mu = te^{i(\alpha-\varepsilon)}; t > 0\}$  for  $\operatorname{Im} \lambda < 0$ .

This implies that  $\widehat{f}(\lambda)$  can be extended at an holomorphic function  $\widehat{f}(\lambda)$  in the domain

$$\{\lambda; \operatorname{Re} \lambda > -(\alpha - \varepsilon) |\operatorname{Im} \lambda|\}.$$

Again following the proof of theorem 5 we obtain that  $(\lambda I - A)^{-1} \in (L(X, X))$  and satisfies (4.12) for

$$\operatorname{Re} \lambda > -(\alpha - \varepsilon) |\operatorname{Im} \lambda| + \gamma.$$

Thus theorem 6 is proved.

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## BIBLIOGRAPHY

- [1] J. L. LIONS, *Les semi-groupes distributions*. Portugaliae Math., 19 (1960), 141-164.
- [2] C. FOIAS, *Remarques sur le semi-groupes distributions d'opérateurs*. Portugaliae Math. 19 (1960), 227-243.
- [3] J. PEETRE, *Sur la théorie des semi-groupes distributions. Sémin. sur les équations aux dérivées partielles*. Coll. France, 1963-1964, 76-98.
- [4] J. CHAZARAIN, *Problèmes de Cauchy au sens des distributions vectorielles et applications*. C. R. Acad. Sc. Paris, t. 226 (1968), 10-13.
- [5] J. CHAZARAIN, *Problèmes dans les espaces d'ultra-distributions*. C. R. Acad. Sc. Paris, t. 266 (1968), 564-566.
- [6] G. DA PRATO - U. MOSCO, *Semigrupperi Distribuzioni Analitici*. Annali Scuola Normale Superiore di Pisa, XIX (1965), 367-396.
- [7] G. DA PRATO - U. MOSCO, *Regolarizzazione dei semigrupperi Distribuzioni Analitici*. Annali Scuola Normale Superiore di Pisa, XIX (1965), 563-570.
- [8] D. FUJIWARA, *A characterization of exponential distribution semi-groups*. J. Math. Soc. Japan, 18, 3 (1966), 265-274.
- [9] E. LARSSON, *Generalized distributions semi-groups of bounded linear operators*. Ann. Scuola Norm. Sup. Pisa, 19 (1967), 137-140.
- [10] K. YOSHINAGA, *Ultradistributions and semi-groups distributions*. Bull. Kyushu Inst. Techn. 10 (1963), 1-24.
- [11] K. YOSHINAGA, *Values of vector-valued distributions and smoothness of semi-groups distributions*. Bul. Kyushu Inst. Techn. 12 (1965), 1-27.
- [12] L. SCHWARTZ, *Théorie des distributions*. Hermann, Paris, 1966.
- [13] L. SCHWARTZ, *Théorie des distributions à valeurs vectorielles*. Ann. Inst. Fourier, 7 (1957), 1-141.
- [14] K. YOSIDA, *Functional Analysis*. Springer, Verlag (1965).
- [15] G. DA PRATO, *Semigrupperi di crescita n*. Ann. Scuola Norm. Sup. Pisa, XX (1966), 753-782.
- [16] A. PAZZY, *On the differentiability and compactness of semigroups of linear operators*. J. of Math. and Mech., 17, 12 (1968), 1131-1141.
- [17] V. BARBU, *Les semi-groupes distributions différentiables*. C. R. Acad. Sc. Paris, t. 268 (1968).