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IMBEDDING THEOREMS FOR GENERAL SOBOLEV WEIGHT SPACES

ALOIS KUFNER

0. Introduction.

Let Ω be a bounded domain of the N-dimensional Euclidean space R^N; we assume that the boundary $\partial\Omega$ of Ω may be locally described by a function fulfilling the Lipschitz condition.

Let $h(x)$ be a function defined and positive almost everywhere on the domain Ω and called the weight function. We define the space $L_{p,h}(\Omega)$ for $p \geqq 1$ as the set of functions u defined almost everywhere on Ω and such that the norm

$$\| u \|_{0,p,h} = \left[\int_\Omega | u(x) |^p \, h(x) \, dx \right]^{1/p} \tag{0,1}$$

is finite.

Let $i = (i_1, i_2, \ldots, i_N)$ be a multi-index with $|i| = \sum_{s=1}^{N} i_s$, where i_s $(s = 1, 2, \ldots, N)$ are non-negative integers. We denote by $D^i u$ the (generalized) derivative of order $|i|$:

$$D^i u = \frac{\partial^{|i|} u}{\partial x_1^{i_1} \, \partial x_2^{i_2} \ldots \partial x_N^{i_N}} .$$

For each positive integer m we define the general Sobolev weight space $W_{p,h}^{(m)}(\Omega)$ as the Banach space of all functions u defined almost everywhere on Ω and such that generalized derivatives $D^i u$ belong to the space $L_{p,h}(\Omega)$ for all multi-indices i with $|i| \leqq m$. In the space $W_{p,h}^{(m)}(\Omega)$, we have the norm

$$\| u \|_{m,p,h} = \left[\sum_{|i|=0}^{m} \| D^i u \|_{0,p,h}^p \right]^{1/p} . \tag{0,2}$$

For $m = 0$ we define $W_{p,h}^{(0)}(\Omega) = L_{p,h}(\Omega)$.

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Let $\overset{\circ}{C}{}^{(\infty)}(\Omega)$ denote the set of all infinitely differentiable functions in R^N with a compact support in Ω. Then we define the space $\overset{\circ}{W}{}^{(m)}_{p,h}(\Omega)$ as the closure of $\overset{\circ}{C}{}^{(\infty)}(\Omega)$ in the norm (0.2).

The imbedding problem in these general Sobolev weight spaces is the problem of determining the (best) weight function $k(x)$ [depending on $h(x)$] such that the inclusion

$$W^{(1)}_{p,h}(\Omega) \subset L_{p,k}(\Omega) \tag{0.3}$$

holds and that this imbedding is continuous, i. e. that an estimate of the type

$$\| u \|_{0,p,k} \leqq c \| u \|_{1,p,h} \tag{0.4}$$

holds with a constant c not depending on the function u.

In this paper, this problem is solved for the case

$$h(x) = \sigma(\varrho(x)) \tag{0.5}$$

where $\sigma = \sigma(t)$ is an almost everywhere positive function of one variable $t \in (0, \infty)$ and $\varrho = \varrho(x)$ is the distance between the point $x \in \Omega$ and the boundary $\partial\Omega$ of Ω.

For some special functions σ there exist result in this direction: So for $\sigma(t) = t^\alpha$ with α a real number, this problem was solved by J. NEČAS [1]: Under some assumptions concerning the value of α [$\alpha > p - 1$ for the space $W^{(1)}_{p,h}(\Omega)$ and $\alpha < p - 1$ for the space $\overset{\circ}{W}{}^{(1)}_{p,h}(\Omega)$] he has shown that if $h(x) = \sigma(\varrho(x))$ with $\sigma(t) = t^\alpha$ then the weight function k is of the form $k(x) = \varkappa(\varrho(x))$ with

$$\varkappa = \varkappa(t) = t^{\alpha - p}. \tag{0.6}$$

His results were extended in the papers of J. KADLEC and the author [2] and [3]. E. g., it is shown, that for $\alpha = p - 1$ the imbedding

$$\overset{\circ}{W}{}^{(1)}_{p,h}(\Omega) \subset L_{p,k}(\Omega)$$

holds with $h(x) = \sigma(\varrho(x))$, $\sigma(t) = t^\alpha = t^{p-1}$ and

$$k(x) = \varkappa(\varrho(x)), \qquad \varkappa(t) = t^{\alpha - p}\, lg^{-p}(R/t) = \frac{1}{t}\, lg^{-p}\left(\frac{R}{t}\right) \tag{0.7}$$

with R a positive constant.

For weight functions of the type

$$h(x) = \sigma(r(x))$$

with $r(x)$ the distance between the point $x \in \Omega$ and a fixed point x_0 on the boundary $\partial\Omega$, similar results were obtained by the author [4]: for $\sigma(t) = t^\alpha$ there is $k(x) = \varkappa(r(x))$ with $\varkappa(t) = t^{\alpha-p}$.

There also similar results concerning, e. g., unbounded domains (L. D. KUDRJAVCEV [5]) or weight functions defined as a power of the distance from a n-dimensional manifold ($n < N$) (see G. N. JAKOVLEV [6]) etc.

1. A generalization of Hardy's inequality.

Important for the proof of an imbedding of the type (0.3) with weight functions of the form $h(x) = \varrho^\alpha(x)$ or $h(x) = r^\alpha(x)$ is the inequality of HARDY

$$\int_0^\infty |f(t)|^p t^{\alpha-p}\, dt \leqq \left(\frac{p}{|\alpha - p + 1|}\right)^p \int_0^\infty \left|\frac{df}{dt}\right|^p t^\alpha\, dt \tag{1.1}$$

which holds

i) for $\alpha > p - 1$ if $\lim\limits_{t \to \infty} f(t) = 0$,

ii) for $\alpha < p - 1$ if $\lim\limits_{t \to 0} f(t) = 0$

(see [7], Theorem 330); the value $\alpha = p - 1$ is a singular value.

For the proof of imbedding theorems with $h(x) = \sigma(\varrho(x))$ and $k(x) = \varkappa(\varrho(x))$, where σ and $\varkappa$ are more general, we need a generalization of Hardy's inequality [see (1.4)]. We will state it here as a Lemma:

LEMMA. Let be $p > 1$ and $\sigma = \sigma(t)$ defined and positive almost everywhere on (a, b) $[-\infty \leqq a < b \leqq \infty]$. Let us define a function $S(\tau)$ by the formula

$$S(\tau) = \sigma^{\frac{1}{1-p}}(\tau). \tag{1.2}$$

Further let $f = f(t)$ be a function differentiable in (a, b) and such that

$$\int_a^b \left|\frac{df}{dt}\right|^p \sigma(t)\, dt < \infty.$$

Let at least one of the following two conditions be fulfilled:

(i) $\lim\limits_{t \to b} f(t) = 0$ and the function $S^*(t) = \int_t^b S(\tau)\, d\tau$ is finite for every $t \in (a, b)$;

(ii) $\lim_{t \to a} f(t) = 0$ and the function $S^*(t) = \int_a^t S(\tau)\, d\tau$ is finite for every $t \in (a, b)$.

If we define the function $\varkappa$ by

$$\varkappa(t) = S(t)\,[S^*(t)]^{-p} \tag{1.3}$$

then the generalized inequality of Hardy holds:

$$\int_a^b |f(t)|^p \varkappa(t)\, dt \leqq \left(\frac{p}{p-1}\right)^p \int_a^b \left|\frac{df}{dt}\right|^p \sigma(t)\, dt. \tag{1.4}$$

The proof of the generalized Hardy's inequality is similar to the proof of inequality (1.1), e. g. in [8]. A proof of (1.4) has given, e. g., V. N. SEDOV in his (yet unpublished) Thesis. We will give here two simple examples:

EXAMPLE 1. For $(a, b) = (0, \infty)$ and $\sigma(t) = t^\alpha$ $(\alpha \neq p - 1)$ we have by condition (i) of the Lemma for $\alpha > p - 1$ and by condition (ii) for $\alpha < p - 1$ the following expression for $\varkappa$:

$$\varkappa(t) = \left(\frac{|\alpha + 1 - p|}{p - 1}\right)^p t^{\alpha - p}.$$

In this way, we obtained the Hardy's inequality (1.1).

EXAMPLE 2. For $(a, b) = (0, 1)$ and $\sigma(t) = t^{p-1}\, lg^{\beta+p}\, \frac{1}{t}$ with $\beta \neq -1$ we obtain by condition (i) for $\beta < -1$ and by condition (ii) for $\beta > -1$ the following expression for the function $\varkappa$:

$$\varkappa(t) = \left(\frac{|\beta + 1|}{p - 1}\right)^p \frac{1}{t}\, lg^\beta \frac{1}{t}\,.$$

This is exactly the inequality (4.4) from [3] (see also formula (0.7) which is a special case of our function $\varkappa(t)$ for $\beta = -p$).

2. Imbedding theorems in a special case.

For the points $x \in R^N$ we use the notation $x = (x', x_N)$ where $x' = (x_1, x_2, \dots, x_{N-1})$. In this section, special cylindrical domains will be considered: $\Omega = K \times (0, 1)$ with K the unit ball in R^{N-1}:

$$K = \left\{ x' \in R^{N-1} : \sum_{j=1}^{N-1} x_j^2 < 1 \right\}.$$

Further, it will be supposed in this Section, that all functions $u = u(x', x_N)$ vanish in the neighbourhood of the part of the boundary $\partial\Omega$ described by

$$|x'| = 1 \quad \text{or} \quad x_N = 1 \tag{2.1}$$

(i. e. the functions vanish in the neighbourhood of the sides and of the upper base of the cylindre Ω).

For such functions u we define the space $L_{p,\sigma}(\Omega)$ as the space of all functions u, for which the norm

$$\| u \|_{0,p,\sigma} = \left[\int_K dx' \int_0^1 | u(x', x_N) |^p \, \sigma(x_N) \, dx_N \right]^{1/p} \tag{2.2}$$

is finite. Here, the weight function depends on x_N only. The Sobolev weight space $W_{p,\sigma}^{(1)}(\Omega)$ is the space of all functions u for which

$$u \in L_{p,\sigma}(\Omega) \quad \text{and} \quad \frac{\partial u}{\partial x_j} \in L_{p,\sigma}(\Omega) \quad \text{for } j = 1, 2, \ldots, N$$

with the norm

$$\| u \|_{1,p,\sigma} = \left[\| u \|_{0,p,\sigma}^p + \sum_{j=1}^N \left\| \frac{\partial u}{\partial x_j} \right\|_{0,p,\sigma}^p \right]^{1/p}. \tag{2.3}$$

Further, we set (similarly as in Section 1)

$$S(t) = \sigma^{\frac{1}{1-p}}(t) \tag{2.4}$$

with $p > 1$ and suppose that

$$\int_t^1 S(\tau) \, d\tau < \infty \quad \text{for } t > 0 \tag{2.5}$$

and define a new weight function $\varkappa$ by the formula

$$\varkappa(t) = S(t) \left[\int_t^1 S(\tau) \, d\tau \right]^{-p}. \tag{2.6}$$

Finally, we suppose that the space $C^{(\infty)}(\bar{\Omega})$ [i. e. the space of functions infinitely differentiable in Ω and continuous with all derivatives in the closure $\bar{\Omega}$] forms a dense subset of the spaces $W^{(1)}_{p,\sigma}(\Omega)$ and $L_{p,\varkappa}(\Omega)$. Let us note that in the paper of O. V. BESOV and the author [9] conditions are given which guarantee the density of smooth functions in weight spaces.

Under all these assumptions, we have the following

THEOREM 1. For all $u \in W^{(1)}_{p,\sigma}(\Omega)$ the inequality

$$\| u \|_{0,p,\varkappa} \leqq \frac{p}{p-1} \| u \|_{1,p,\sigma} \tag{2.7}$$

holds with $\varkappa$ given by (2.6).

PROOF. At first, let us prove (2.7) for a function $u \in C^{(\infty)}(\bar{\Omega})$ which vanishes in the neighbourhood of points $x = (x', x_N)$ with $|x'| = 1$ or $x_N = 1$. We want to estimate the integral

$$I = \| u \|^p_{0,p,\varkappa} = \int_K dx' \int_0^1 | u(x', x_N) |^p \varkappa(x_N)\, dx_N .$$

Let us denote by J the inner integral and by $f(x_N)$ the function $u(x', x_N)$ for a fixed $x' \in K$. The function f vanishes for values x_N near to 1. So, we can use the Lemma, condition (i) [with $(a, b) = (0, 1)$] and have from inequality (1.4)

$$J = \int_0^1 |f(x_N)|^p \varkappa(x_N)\, dx_N \leqq \left(\frac{p}{p-1}\right)^p \int_0^1 \left| \frac{df}{dx_N} \right|^p \sigma(x_N)\, dx_N .$$

Because $f(x_N) = u(x', x_N)$ and $\dfrac{df}{dx_N}(x_N) = \dfrac{\partial u}{\partial x_N}(x', x_N)$, we can write the last inequality in the form

$$J = \int_0^1 | u(x', x_N) |^p \varkappa(x_N)\, dx_N \leqq \left(\frac{p}{p-1}\right)^p \int_0^1 \left| \frac{\partial u}{\partial x_N}(x', x_N) \right|^p \sigma(x_N)\, dx_N .$$

Integrating this inequality by x' over K we obtain

$$\| u \|^p_{0,p,\varkappa} \leqq \left(\frac{p}{p-1}\right)^p \left\| \frac{\partial u}{\partial x_N} \right\|^p_{0,p,\sigma} .$$

Using the obvious estimation

$$\left\| \frac{\partial u}{\partial x_N} \right\|^p_{0,p,\sigma} \leqq \| u \|^p_{1,p,\sigma}$$

we have (2.7) for a smooth function u.

Now, we suppose $u \in W_{p,\sigma}^{(1)}(\Omega)$. Because $C^{(\infty)}(\bar{\Omega})$ forms a dense subset in this space, a sequence of functions $u_n \in C^{(\infty)}(\bar{\Omega})$ exists such that $u_n \longrightarrow u$ for $n \longrightarrow \infty$ in the norm (2.3). In the first part of the proof we have shown that (2.7) holds for u_n, $n = 1, 2, \ldots$. The functions u_n form a Cauchy sequence in $W_{p,\sigma}^{(1)}(\Omega)$ and by (2.7) also in $L_{p,\varkappa}(\Omega)$. So, a limit of u_n exists in $L_{p,\varkappa}(\Omega)$, and we obtain (2.7) for u by a passage to the limit for $n \longrightarrow \infty$ in (2.7) for u_n.

Theorem 1 shows that for weight functions σ fulfilling the condition (2.5) the imbedding

$$W_{p,\sigma}^{(1)}(\Omega) \subset L_{p,\varkappa}(\Omega) \tag{2.8}$$

holds with $\varkappa$ given by (2.6). If we define the space $\overset{\circ}{W}{}_{p,\sigma}^{(1)}(\Omega)$ as the closure of $\overset{\circ}{C}{}^{(\infty)}(\Omega)$ in the norm (2.3) (see also Section 0), then obviously $\overset{\circ}{W}{}_{p,\sigma}^{(1)}(\Omega) \subset W_{p,\sigma}^{(1)}(\Omega)$ and so, the imbedding (2.8) holds for $\overset{\circ}{W}{}_{p,\sigma}^{(1)}(\Omega)$ too:

$$\overset{\circ}{W}{}_{p,\sigma}^{(1)}(\Omega) \subset L_{p,\varkappa}(\Omega). \tag{2.9}$$

An imbedding of the form (2.9) holds also if the weight function σ fulfils the condition

$$\int_0^t S(\tau)\, d\tau < \infty \qquad \text{for } t < 1 \tag{2.10}$$

instead of (2.5): then the weight function $\varkappa$ is given by

$$\varkappa(t) = S(t) \left[\int_0^t S(\tau)\, d\tau \right]^{-p} : \tag{2.11}$$

THEOREM 2. For all functions $u \in \overset{\circ}{W}{}_{p,\sigma}^{(1)}(\Omega)$ the inequality (2.7) holds with $\varkappa$ given by (2.11).

PROOF. We can use the same method as in the proof of Theorem 1.

Using the fact that, for $u \in \overset{\circ}{C}{}^{(\infty)}(\Omega)$, it is $f(x_N) = u(x', x_N) = 0$ for x_N near to 0, we obtain from inequality (1.4) [condition (ii) of the Lemma] the estimate (2.7) for functions $u_n \in \overset{\circ}{C}{}^{(\infty)}(\Omega)$ and by a limit procedure with $n \longrightarrow \infty$ also for $u \in \overset{\circ}{W}{}_{p,\sigma}^{(1)}(\Omega)$.

Example 3. From Example 2 it follows that the estimation

$$\| u \|_{0, p, x_N^{-1} \lg^{\alpha} (x_N^{-1})} \leqq \frac{p}{|\alpha + 1|} \| u \|_{1, p, x_N^{p-1} \lg^{\alpha+p} (x_N^{-1})}$$

holds (i) for $u \in W_{p, \sigma}^{(1)}(\Omega)$ with $\sigma(x_N) = x_N^{p-1} \lg^{\alpha+p}(x_N^{-1})$ if $\alpha < -1$ (by using Theorem 1);

(ii) for $u \in \overset{\circ}{W}{}_{p, \sigma}^{(1)}(\Omega)$ with the same σ if $\alpha \neq -1$ (by using Theorem 1 for $\alpha < -1$ and Theorem 2 for $\alpha > -1$).

The just mentioned results may be carried over to the spaces $W_{p, \sigma}^{(m)}(\Omega)$ with $m > 1$, defined as the spaces of functions u with a support of the described type [i. e. vanishing in the neighbourhood of the part of the boundary $\partial\Omega$ given by (2.1)] and such that the norm

$$\| u \|_{m, p, \sigma} = \left[\sum_{|i|=0}^{m} \int_K dx' \int_0^1 | D^i u(x', x_N) |^p \sigma(x_N)\, dx_N \right]^{1/p}$$

is finite. But this is a technical question only, how may be seen from the procedure for $m = 2$:

If $u \in W_{p, \sigma}^{(2)}(\Omega)$ with σ fulfilling the condition (2.5), then $u \in W_{p, \sigma}^{(1)}(\Omega)$ and $v_j = \dfrac{\partial u}{\partial x_j} \in W_{p, \sigma}^{(1)}(\Omega)$ for $j = 1, 2, \dots, N$. Using now Theorem 1 for the functions u and v_j, we obtain

$$u \in L_{p, \varkappa}(\Omega) \text{ and } v_j = \frac{\partial u}{\partial x_j} \in L_{p, \varkappa}(\Omega) \qquad (j = 1, 2, \dots, N).$$

with $\varkappa$ given by (2.6). It means that $u \in W_{p, \varkappa}^{(1)}(\Omega)$, and so, we have the imbedding

$$W_{p, \sigma}^{(2)}(\Omega) \subset W_{p, \varkappa}^{(1)}(\Omega). \tag{2.12}$$

Let us denote

$$K(t) = \varkappa^{\frac{1}{1-p}}(t)$$

and suppose

$$\int_t^1 K(\tau)\, d\tau < \infty \qquad \text{for } t > 0.$$

Setting now

$$\lambda(t) = K(t)\left[\int_t^1 K(\tau)\,d\tau\right]^{-p}, \tag{2.13}$$

Theorem 1 gives the imbedding

$$W_{p,\varkappa}^{(1)}(\Omega) \subset L_{p,\lambda}(\Omega).$$

From this inclusion and from (2.12) now immediately follows the main result:

$$W_{p,\sigma}^{(2)}(\Omega) \subset L_{p,\lambda}(\Omega)$$

with λ from (2.13).

* * *

Let now $a = a(x')$ be a function defined and lipschitzian for $x' \in \bar{K}$ with K the unit ball in R^{N-1}. We can repeat all considerations made in Theorems 1 and 2 for $\Omega = K \times (0, 1)$ and $\sigma = \sigma(x_N)$, $\varkappa = \varkappa(x_N)$, also for Ω defined by

$$\Omega = \{x = (x', x_N) : x' \in K,\ a(x') < x_N < a(x') + 1\}$$

and for weight functions

$$h(x) = h(x', x_N) = \sigma(a(x') + x_N),$$

$$k(x) = k(x', x_N) = \varkappa(a(x') + x_N),$$

because here

$$\int_\Omega v(x)\,dx = \int_K dx' \int_{a(x')}^{a(x')+1} v(x', x_N)\,dx_N = \int_K dx' \int_0^1 v(x', a(x') + t)\,dt.$$

Also for such domains Ω Theorems 1 and 2 hold (with function u vanishing in the neighbourhood of points with $|x'| = 1$ or with $x_N = a(x') + 1$).

This fact will be used in the following Section.

3. Imbedding theorems in the general case.

The domains Ω considered in Section 2 were rather special and also the condition concerning the support of functions u was very restrictive.

But results analoguous to Theorems 1 and 2 hold also for general Sobolev weight spaces defined in Section 0 with weight functions of the type

$$h(x) = \sigma(\varrho(x)).$$

For such spaces, we must make some additional assumptions concerning the weight function σ:

(I) $\sigma = \sigma(t)$ is a non-negative function defined for $t \in (0, \infty)$ and such that for every interval $[c, d]$ with $0 < c < d < \infty$ a number η exists ($\eta > 1$ and depending on the interval $[c, d]$) such that

$$\frac{1}{\eta} \leqq \sigma(t) \leqq \eta \qquad \text{for } t \in [c, d]; \tag{3.1}$$

(II) if c_1 and c_2 are positive constants such that $c_1 \leqq t/s \leqq c_2$, then there exist positive constants C_1 and C_2 such that for this s and t the inequalities

$$C_1 \leqq \frac{\sigma(t)}{\sigma(s)} \leqq C_2 \tag{3.2}$$

hold.

Some conditions for σ under which (3.2) holds are given in [9].

Now, we again define $S(t)$ by (2.4) and suppose that (2.5) holds. We define a new weight function $\varkappa$ by (2.6) and suppose that conditions (I) and (II) hold for both functions σ and $\varkappa$ and that functions from $C^{(\infty)}(\bar{\Omega})$ are dense in $W^{(1)}_{p,h}(\Omega)$ and in $L_{p,k}(\Omega)$ where

$$h(x) = \sigma(\varrho(x)); \quad k(x) = \varkappa(\varrho(x)); \quad \varrho(x) = \operatorname{dist}(x, \partial\Omega). \tag{3.3}$$

Then we have

THEOREM 3. Let $p > 1$. Let Ω be a bounded domain with locally lipschitzian boundary $\partial\Omega$. Then a constant $c > 0$ exists such that for all $u \in W^{(1)}_{p,h}(\Omega)$ the inequality

$$\| u \|_{0,p,k} \leqq c \, \| u \|_{1,p,h} \tag{3.4}$$

holds.

PROOF. For $u \in W^{(1)}_{p,h}(\Omega)$ a sequence of functions $u_n \in C^{(\infty)}(\bar{\Omega})$ (without assumptions about the support of u_n!!) exists such that $u_n \to u$ in $W^{(1)}_{p,h}(\Omega)$ for $n \to \infty$. Thus, it suffices to prove (3.4) for functions from $C^{(\infty)}(\bar{\Omega})$.

The boundary $\partial\Omega$ may be described locally by functions fulfilling the Lipschitz condition; more precisely (see [1]):

(i) Coordinate systems (x'_r, x_{rN}) $(r = 1, 2, \dots, R)$ and functions $a_r = a_r(x'_r)$ defined and lipschitzian on $(N-1)$-dimensional balls $\bar{K}_r = \{x'_r \in R^{N-1} : |x'_r| \leqq 1\}$ exist such that for every point $x \in \partial\Omega$ at least one index r exists such that the point x has the coordinates $(x'_r, a_r(x'_r))$ [i. e. that the boundary $\partial\Omega$ is in the neighbourhood of x described by the function $a_r : x_{rN} = a_r(x'_r)$].

(ii) A constant β $(0 < \beta < 1)$ exists such that the « cylindrical » domain

$$B_r = \{x = (x'_r, x_{rN}) : x'_r \in K_r,\ a_r(x'_r) - \beta < x_{rN} < a_r(x'_r)\}$$

is contained in Ω for $r = 1, 2, \dots, R$ and that the domains

$$D_r = \{x = (x'_r, x_{rN}) : x'_r \in K_r,\ a_r(x'_r) - \beta < x_{rN} < a_r(x'_r) + \beta\}$$

cover the boundary $\partial\Omega$ and $D_r \cap \Omega = B_r$ $(r = 1, 2, \dots, R)$.

(iii) An open set D_{R+1} exists such that $\bar{D}_{R+1} \subset \Omega$, $\bigcup_{r=1}^{R+1} D_r \supset \bar{\Omega}$ and $\Omega = \bigcup_{r=1}^{R} B_r + D_{R+1}$.

Then functions $\varphi_r(x)$, $r = 1, 2, \dots, R+1$, exist such that : $0 \leqq \varphi_r(x) \leqq 1$, $\varphi_r \in \overset{\circ}{C}{}^{\infty)}(D_r)$ and for $x \in \bar{\Omega}$ it is $\sum_{r=1}^{R+1} \varphi_r(x) = 1$ (the functions φ_r form a partition of the unity in $\bar{\Omega}$).

Let now be $u \in C^{(\infty)}(\bar{\Omega})$ and let us denote $u_r = u\,\varphi_r$. Let us fix the index r between 1 and R. We want to estimate the integral

$$(3.5) \qquad I_r = \| u_r \|^p_{0,\,p,\,k} = \int_{B_r} | u_r(x'_r, x_{rN}) |^p\, k(x'_r, x_{rN})\, dx =$$

$$= \int_{K_r} dx'_r \int_{a_r(x'_r)-\beta}^{a_r(x'_r)} | u_r(x'_r, x_{rN}) |^p\, \varkappa(\varrho(x'_r, x_{rN}))\, dx_{rN}\,.$$

[We can integrate over B_r instead of the whole Ω, because u_r vanishes in $\Omega - B_r$ for $\varphi_r \in \overset{\circ}{C}{}^{(\infty)}(D_r)$].

Because the function $a_r(x'_r)$ which describes the boundary $\partial\Omega$ for $x'_r \in K_r$ is lipschitzian, the distance of the point $x = (x'_r, x_{rN}) \in B_r$ from $\partial\Omega$ *in the direction* x_{rN} [this distance is given by $a_r(x'_r) - x_{rN}$] is equivalent with the usual distance from $\partial\Omega$, given by $\varrho(x_r', x_{rN})$, i.e. constants c_1 and c_2 exist such that

$$0 < c_1 \leqq \frac{\varrho(x_r', x_{rN})}{a_r(x_r') - x_{rN}} \leqq c_2 < \infty$$

for all $x = (x_r', x_{rN}) \in B_r$. Using now condition (II) for the weight function $\varkappa$ [see (3.2)] we have

$$k(x_r', x_{rN}) = \varkappa(\varrho(x_r', x_{rN})) \leqq C_2\, \varkappa(a_r(x_r') - x_{rN})$$

and from (3.5) it follows

$$I_r \leqq C_2 \int\limits_{K_r} dx_r' \int\limits_{a_r(x_r')-\beta}^{a_r(x_r')} |u_r(x_r', x_{rN})|^p\, \varkappa(a_r(x_r') - x_{rN})\, dx_{rN} =$$

$$= C_2 \int\limits_{K_r} dx_r' \int\limits_0^\beta |u_r(x_r', a_r(x_r') - t)|^p\, \varkappa(t)\, dt.$$

But for a fixed $x_r' \in K_r$ the function $u_r(x_r', x_{rN})$ vanishes for x_{rN} near to $a_r(x_r') - \beta$, and so, $u(x_r', a_r(x_r') - t) = 0$ for t near to $\beta < 1$. Thus, we have the situation considered in Section 2 and Theorem 1 gives immediately

$$I_r \leqq C_2 \left(\frac{p}{p-1}\right)^p \int\limits_{K_r} dx_r' \int\limits_0^\beta \left| \frac{\partial u_r}{\partial x_{rN}}(x_r', a_r(x_r') - t) \right|^p \sigma(t)\, dt =$$

$$= C_2 \left(\frac{p}{p-1}\right)^p \int\limits_{B_r} \left| \frac{\partial u_r}{\partial x_{rN}}(x_r', x_{rN}) \right|^p \sigma(a_r(x_r') - x_{rN})\, dx.$$

Now, we again use condition (II) — in this case for the function σ — and have

$$\sigma(a_r(x_r') - x_{rN}) \leqq \frac{1}{C_1}\, \sigma(\varrho(x_r', x_{rN})) = \frac{1}{C_1}\, h(x_r', x_{rN}).$$

So we have

$$\int\limits_{B_r} |u_r(x)|^p\, k(x)\, dx \leqq \frac{C_2}{C_1} \left(\frac{p}{p-1}\right)^p \int\limits_{B_r} \left| \frac{\partial u_r}{\partial x_{rN}}(x) \right|^p h(x)\, dx$$

and

$$(3.6) \qquad \| u_r \|_{0,p,k} \leqq c^* \| u_r \|_{1,p,h} \leqq c^{**} \| u \|_{1,p,h}$$

because from the properties of φ_r it follows immediately that

$$\| u_r \|_{1,p,h} = \| u \varphi_r \| \leqq \tilde{c} \| u \|_{1,p,h} .$$

Inequality (3.6) holds for $r = 1, 2, \dots, R$; but it holds also for $r = R + 1$, i. e for the function $u_{R+1} = u \varphi_{R+1}$ with compact support in D_{R+1}:

Setting $\delta_1 = \operatorname{dist}(D_{R+1}, \partial\Omega) > 0$, $\delta_2 = \operatorname{diam} \Omega < \infty$, we have

$$\delta_1 \leqq \varrho(x) \leqq \delta_2 \qquad \text{for } x \in D_{R+1}$$

and from property (I) of the weight functions σ and $\varkappa$ it follows, that then $\eta > 1$ and $\eta^* > 1$ exist such that

$$\frac{1}{\eta} \leqq h(x) = \sigma(\varrho(x)) \leqq \eta \; ; \; \frac{1}{\eta^*} \leqq k(x) = \varkappa(\varrho(x)) \leqq \eta^*$$

for $x \in D_{R+1}$. So, for these x we have $(x)\, k \leqq \eta^* \eta\, h(x)$ and

$$\| u_{R+1} \|_{0,p,k}^p = \int_{D_{R+1}} | u_{R+1}(x) |^p k(x)\, dx \leqq \eta^* \eta \int_{D_{R+1}} | u_{R+1}(x) |^p h(x)\, dx \leqq$$

$$\leqq \eta^* \eta \int_{D_{R+1}} \left[| u_{R+1}(x) |^p + \sum_{j=1}^{N} \left| \frac{\partial u_{R+1}}{\partial x_j}(x) \right|^p \right] h(x)\, dx =$$

$$= \eta^* \eta \| u_{R+1} \|_{1,p,h}^p \leqq \eta^* \eta \tilde{c} \| u \|_{1,p,h}^p .$$

Thus (3.6) holds for $r = 1, 2, \dots, R + 1$. Because $u(x) = u(x) \sum_{r=1}^{R+1} \varphi_r(x) = \sum_{r=1}^{R+1} u_r(x)$, we have from (3.6)

$$\| u \|_{0,p,k} = \left\| \sum_{r=1}^{R+1} u_r \right\|_{0,p,k} \leqq \sum_{r=1}^{R+1} \| u_r \|_{0,p,k} \leqq (R+1)\, c^{**} \| u \|_{1,p,h}$$

what is (3.4) for $u \in C^{(\infty)}(\bar{\Omega})$. So, Theorem 3 is proved.

Theorem 3 shows that the imbedding

$$W_{p,h}^{(1)}(\Omega) \subset L_{p,k}(\Omega)$$

holds for the general Sobolev weight space with $h(x) = \sigma(\varrho(x))$ and $k(x) = \varkappa(\varrho(x))$. If we suppose that (2.10) holds instead of (2.5) and then define $\varkappa = \varkappa(t)$ by (2.11), we obtain by the same way as in Theorem 2 the imbedding

$$\overset{\circ}{W}{}^{(1)}_{p,h}(\Omega) \subset L_{p,k}(\Omega).$$

Analoguously as in Section 2 we can now derive imbedding theorems for the spaces $W^{(m)}_{p,h}(\Omega)$ with $m \geqq 2$.

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