B. Fishel

An abstract Lebesgue-Nikodym theorem


<http://www.numdam.org/item?id=ASNSP_1969_3_23_2_363_0>
AN ABSTRACT LEBESGUE-NIKODYM THEOREM

B. FISHEL

The development of the theory of Radon measures, continuous linear forms on the vector space $\mathcal{C}(T)$ of continuous real-valued functions with compact supports on the locally compact space $T$, endowed with a suitable inductive-limit topology $[B_3]$, leans heavily on the order structure of $\mathcal{C}(T)$. We have shown elsewhere $[F_1, 2]$ how some aspects of the theory of integration for Radon measures can equally be developed by availing oneself of the algebraic structure of $\mathcal{C}(T)$ rather than its order structure. We note, however, that such a treatment does not yield a theory of integration in that, for example, the objects which there correspond to the integrable functions of the more familiar development are elements of an abstract completion of $\mathcal{C}(T)$, i.e. are not functions. Our object in investigating the rôle of the algebraic structure was to explore the possibility of its application to

(i) vector measures and the integration of vector-valued functions, where rarely is there an order structure which arises in a natural way from the structure of the space which we are concerned, and

(ii) the theory of distributions, where utilisation of a naturally-occurring order imposes excessive restrictions — a positive distribution is a measure.

In $[F_3]$ we attempted to establish a Lebesgue-Nikodym theorem within our order-free structure, but were unable to do so without the aid of a supplementary hypothesis (loc cit Prop 2.3) which is not verified for Radon measures, but which holds in the theory of distribution $[F_3]$.

In the present paper we obtain a Lebesgue-Nikodym in a form applicable to the theory of integration. § 1.1 gives an account of some ideas from the theory of duality for topological vector spaces, in particular, of an extension of Grothendieck's completion theorem. § 1.2 introduces the pre-
hilbert structure defined by a positive linear form on a normed algebra. § 1.3 describes dualities defined by a linear form on an algebra. §§ 1.4-5 formulate a concept of absolute continuity of one linear form on a normed algebra with respect to another such form and apply the completion theorem of § 1.1 to establish an abstract Lebesgue Nikodym theorem. In § 1.6 this theorem is applied to give the Lebesgue-Nikodym theorem for Radon measures on a compact space. § 2.1 describes an extension of the results of §§ 1.1-5 to the inductive limit (as a topological vector space) of a family of normed algebras, and in § 2.2 we obtain the Lebesgue-Nikodym theorem on a locally compact space.

§ 1.1 We make use of the ideas of the theory of duality for topological vector spaces. We summarize here, briefly, the definitions and results which we shall need.

Let \( \langle E_1, E_2 \rangle \) be a dual system (pairing) of vector spaces \( E_1, E_2 \) over the complex field \( C \).

A family of subsets of \( E_i (i = 1, 2) \) is said to be saturated with respect to \( \langle E_1, E_2 \rangle \) if it contains

1. the subsets of each of its members,
2. the scalar multiples of its members,
3. the absolutely convex, weakly closed hulls of finite unions of its members.

\( \mathcal{T}_1 \) will denote the family obtained by saturating the family of finite subsets of \( E_1 \).

A saturated family \( \mathcal{T}_1 \) of weakly bounded subsets of \( E_1 \) defines a locally convex topology \( \sigma_{\mathcal{T}_1} \) on \( E_2 \) (the topology of uniform convergence on the sets of \( \mathcal{T}_1 \). If \( \langle E_1, E_2 \rangle \) is separated then \( \sigma_{\mathcal{T}_1} \) is Hausdorff (separated) if \( \mathcal{T}_1 \subset \mathcal{T}_1 \). (The corresponding statements obtained by interchanging the suffixes 1 and 2 also apply). With this notation we write \( \sigma_{\mathcal{T}_2} \) for the weak topology \( \sigma(E_1, E_2) \) on \( E_1 \).

\( \mathcal{K}_2 \) denotes the family obtained by saturating the family of absolutely convex weakly compact sets in \( E_2 \). \( \mathcal{T}_2 \subset \mathcal{K}_2 \).

Let \( \mathcal{T}_1 \subset \mathcal{M}_1 \) be a saturated family in \( E_1 \), and let \( \mathcal{T}_2 \subset \mathcal{K}_2 (\subset \mathcal{K}_2) \) be a saturated family in \( E_2 \), both families being of weakly bounded sets.

We shall need the following form of Grothendieck's completion theorem:

if \( \langle E_1, E_2 \rangle \) is separated,

\[
G = \{ f \in E_1^*: f \text{ \( \sigma_{\mathcal{M}_1} \)-continuous on the sets of } \mathcal{M}_1 \} \]

is the (separated) completion \( (E_2, \sigma_{\mathcal{M}_1})^* \) of \( E_2 \) for the topology \( \sigma_{\mathcal{M}_1} \).

\( E_1^* \) denotes the algebraic dual of \( E_1 \).
This result may be established by showing

(i) (as in [K] p. 272) that $G$ is complete, and separated for $\mathcal{M}_4$ and $E^*$ is separated,

and

(ii) that $E$ is $\mathcal{M}_4$-dense in $G$.

The argument establishing the necessity of [S] Th. 6.2, p. 148 proves (ii) when we put $(E, \tau) = (E, \mathcal{M}_4)$ since $(E, \tau)' = (E_1, \mathcal{M}_4)' = E_2$ (where $(E, \tau)'$ denotes the topological dual of $E$ for the topology $\tau$) by the Mackey-Arens theorem ([S] Th. 3.2, p. 131), since $\mathcal{F}_2 \subset \mathcal{M}_2 \subset \mathcal{M}_2$.

(A "proof" of this theorem given in [F] is false. It relies on the false inequality (3) of [K] p. 272).

§ 1.2. We shall apply this theorem to a situation where we take (essentially) for $E_1, E_2$ two copies of a $\ast$-normed algebra $A$ (i.e. a normed algebra over $C$ on which there is defined an involution $\ast$ such that $\|\xi\ast\| = \|\xi\|$ for all $\xi \in A$, see [R] p. 180). Elements $\xi$ for which $\xi = \xi^\ast$ are said to be self-adjoint.

We consider a linear form $\mu$ on $A$ having the properties

\[ F_1: \quad \mu(\xi^\ast) = \overline{\mu(\xi)}, \]

(this is equivalent to: $\mu$ is real on self-adjoint elements of $A$, and implies that the sesqui-linear forms

\[ A \times A \rightarrow C \]

\[ (\xi, \eta) \mapsto \mu(\xi \eta^\ast) \]

and

\[ (\xi, \eta) \mapsto \mu(\eta^\ast \xi) \]

are hermitian, it is an easy matter to construct an example for which they do not coincide), and

\[ F_2: \quad \mu \text{ is positive,} \]

i.e.

\[ \mu(\xi^\ast) \geq 0 \quad \forall \xi \in A, \]

(and so \[ \mu(\xi^\ast \xi) = \mu(\xi^\ast \xi^\ast) \geq 0 \quad \forall \xi \in A), \]

so that the sesqui-linear forms are positive-definite.

We shall henceforth consider only the first of these two forms defined by $\mu$, and shall write it $\mu(\cdot)$. It defines a prehilbert structure on $A$ with corresponding semi-norm $p_0(\xi) = (\mu(\xi^\ast \xi))^\frac{1}{2}$. If $Z = \{\xi: \mu(\xi^\ast \xi) = 0\}$ =
\[ \xi : \mu (\xi \eta) = 0 \ \forall \eta \in A \] the positive hermitian form an \( A/Z \) associated with \( \mu (, ) \), (which we shall denote by \( \mu (, ) \)) defines a separated prehilbert structure. The norm \( p \) for this structure is that associated with the semi-norm \( p_0 : \mu (, ) \) extends uniquely to the separated completion \( (A/Z, p_0)^\ast \) (which we write \( (A/Z)^\ast \)) of \( A/Z \) for \( p_0 \), and makes it a Hilbert space. We can equally construct the separated completion \( (A, p_0)^\ast \) (or \( A^\ast \)) of \( A \) for \( p_0 \), and \( ([B_4] II \S 3.7) (A/Z)^\ast \) is isomorphic with \( A^\ast \), to which we may therefore transport the extended form \( \mu (, ) \), and if \( i \) is the map of \([B_4] II \S 3.7, Th. 3, i(A) \) is dense in \( A^\ast \) (loc. cit. \S 3.8), \( i \) is here a linear map, and is in fact the canonical map \( A \rightarrow A/Z \), so that \( i(A) \) is isomorphic with \( A/Z \).

We now impose upon \( \mu \) the further requirement

\( F_{3,} \mu \) \( \mu \) is continuous on \( A \).

(If \( A \) has a unit and is complete, i.e. is a Banach \( \ast \)-algebra, the continuity of \( \mu \) follows from the hypothesis that it is positive ([N], p. 200)). If

\[ U = \{ \xi : \| \xi \| \leq 1 \}, \quad V = \{ \xi : p_0 (\xi) \leq 1 \}, \]

since

\[ (p_0 (\xi))^2 = \mu (\xi \xi^\ast) \leq \| \mu \| \| \xi \xi^\ast \| \leq \| \mu \| \| \xi \| \| \xi^\ast \| = \| \mu \| \| \xi \| ^2, \]

and so

\[ U \subset mV \text{ for some } m > 0, \]

\[ i(U) \subset mi(V). \]

Now \( i(V) \) is contained in the unit ball of the Hilbert space \( A^\ast \), and the closed unit ball is compact for the weak topology \( \sigma (A^\ast, A^\ast) \) defined by the canonical duality \( (A^\ast, A^\ast) \) of \( A^\ast \) with its Hilbert-space dual \( A^\ast \), so that \( i(V) \) and therefore \( i(U) \) are relatively weakly compact in \( A^\ast \). The the weak closure of \( i(U) \) in \( A \) is compact for \( \sigma (A^\ast, A^\ast) \) and is therefore compact for the coarser topology \( \sigma (A^\ast, i(A)) \).

\[ \S 1.3. \text{ Our form } \mu \text{ defines a duality } (D_0) \text{ on } A \times A \] — which it is convenient to write \( A_1 \times A_2 \) — by

\[ A_1 \times A_2 \rightarrow C \]

\[ (\xi, \eta) \mapsto \mu (\xi \eta). \]

It is clear that this duality will not in general be separated.

Let \( \mathcal{U} \) be the family obtained by saturating the family \( \{ U \} \) in \( A_1 \) for the duality \( (D_0) \), and let \( \mathcal{U} \) be the same family considered as a family in \( A_2 \).
The sets of $\mathcal{H}$ and $\mathcal{N}$ are weakly bounded for $(D_0)$ since

$$\sup \{ |\mu(\xi,\eta)| : \|\xi\| \leq 1 \} \leq \|\mu\| \|\eta\|.$$ 

The extension of $\mu$ to $A_1 \times A_2$ defines a duality $(D)$ on $i(A_1) \times A_2$.

$$i(A_1) \times A_2 \rightarrow C$$

$$\langle \xi, \eta \rangle \rightarrow \mu(\xi,\eta).$$

Let $\mathcal{M}_1 = i(\mathcal{H})$. Since $i$ is continuous for $p_0$ and $p$, $i(U)$ is bounded for $p$ and so for the weak topology $\sigma(i(A_1), A_2)$, which is the weak topology defined by $(D)$. It follows that the sets of $\mathcal{M}_1$ are weakly bounded for $(D)$.

Since $i$ is linear, to prove that $\mathcal{M}_1$ is saturated for $(D)$ it suffices to prove that the $i$-images of absolutely convex weakly closed (for $(D_0)$) sets of $\mathcal{M}$ are again weakly closed. Such a set $M$ is closed for $p_0$, and since $i$ is the canonical map $A \rightarrow A/Z$ $i$ is closed, so that $i(M)$ is closed for $p$. Since it is absolutely convex and $A_1^\circ = (i(A_1), p)^\circ$, $i(M)$ is also $\sigma(i(A_1), A_2^\circ)$ closed, i.e. is weakly closed for $(D)$.

Finally, we define $\mathcal{M}_2$ to be the saturate, for $\langle i(A_1), A_2 \rangle$ of $i(\mathcal{N})$. It is clear that the sets of $\mathcal{M}_2$ are weakly bounded for $(D)$.

§ 1.4. If $\lambda$ is another linear form on $A$ we now define:

$$\lambda$$ absolutely continuous with respect to $\mu$

to mean:

$$\lambda$$ is $\sigma_{\mathcal{H}}$-continuous on the sets of $\mathcal{H}$,

(while $\sigma_{\mathcal{H}}$ is defined by the duality $(D_0)$).

It follows that $\lambda(Z) = 0$:

$$\zeta \in Z \quad \mu(\zeta,\eta) = 0 \quad \forall \eta \in A$$

$$\mu(\zeta,N) = 0 \quad \forall N \in \mathcal{N}.$$ 

Since $\zeta \in M$ for some $M \in \mathcal{M} (F \subset \mathcal{M})$, the $\sigma_{\mathcal{H}}$-continuity of $\lambda$ on $M$ shows that $\lambda(\zeta) = 0$.

$\lambda$ is thus well-defined on $A/Z$. We shall denote by $\hat{\lambda}$ the functional so defined, and shall prove that it is $\sigma_{\mathcal{H}_2}$-continuous on the sets of $\mathcal{M}_2$. For this it suffices to establish continuity for the coarser topology $\sigma_{i(\mathcal{H})}$. 

10 Annali della Scuola Norm. Sup. - Pisa.
Since \( \lambda = \dot{\lambda} \circ i \) and \( i \) is the canonical map of \( M \) onto \( M/Z \), it suffices ([B1] § 3.4) to prove that \( \sigma_{\mathcal{N}} \) is the quotient by \( Z \) of \( \sigma_{\mathcal{M}} \). This is immediate from consideration of the neighbourhoods for the two topologies since a \( \sigma_{\mathcal{N}} \)-neighbourhood \( \{ \xi : \sup_{\eta \in \mathcal{N}} |\mu(\xi\eta)| < \varepsilon \} \) is saturated for the equivalence relation defined by \( Z ([B_1] \ II \ § 3.4) \).

§ 1.5 We now apply Grothendieck's completion theorem to establish a Lebesgue-Nikodym theorem.

**Lemma.** \( \sigma_{\mathcal{M}} \) is coarser than the \( p_0 \)-topology,

\[
\sigma_{\mathcal{M}} \text{ is coarser then the } p \text{-topology.}
\]

**Proof.** Since \( \mu \) is positive

\[
|\mu(\xi\eta)|^2 \leq \mu(\xi\xi^*) \mu(\eta^* \eta^{**}) \leq (p_0(\xi))^2 \| \mu \| \| \eta \|^2.
\]

Therefore

\[
p(\xi) < 1 \implies \sup \{ |\mu(\xi\eta)| : |\eta| \leq 1 \} \leq \| \mu \|^2
\]

so that a \( \sigma_{\mathcal{M}} \)-neighbourhood defined by \( M = U \) contains a \( p_0 \) neighbourhood, i.e. \( \sigma_{\mathcal{M}} \subset p \)-topology. The second assertion is now immediate.

**Theorem 1.** If \( \zeta \in (A, \sigma_{\mathcal{M}})^* \)

\[
A \to C
\]

\[
\xi \to \mu(\xi \zeta)
\]

defines a linear form \( A \) which is absolutely continuous with respect to \( \mu \).

**Proof.** \( \mu(\xi \cdot) \) is \( \sigma_{\mathcal{M}} \)-continuous on \( A^*_1 \) for all \( \xi \in A_1 \),

\[
(\{\xi \in \mathcal{M}_1\}), \text{ and so extends (uniquely) to } (A_2, \sigma_{\mathcal{M}})^* \supset (A_2, \sigma_{\mathcal{N}})^*.
\]

This extension defines \( \hat{\mu}(\xi \zeta) \).

\( \zeta_{\mu} : \xi \to \hat{\mu}(\xi \zeta) \) is clearly linear on \( A \). To prove that it is absolute continuous with respect to \( \mu \), let \( M \in \mathcal{M}, \varepsilon > 0 \). If \( \zeta = \{P\} \) (a \( \sigma_{\mathcal{M}} \) minimal
Cauchy filter on $A_2 ([B_1] II \S 3.7))$, there exists $P$ such that
\[
\sup \{ |\mu (\xi P) : \xi \in M) | < \varepsilon /2.
\]
Choose $\pi \in P, |\pi | \in H$, and so $| \mu (\xi \pi) | < \varepsilon /2$ implies that
\[
| \xi \mu (\xi) | = | \mu (\xi \xi) | < \varepsilon \text{ for } \xi \in M,
\]
i.e., $\xi \mu$ is $\sigma_{\mathcal{H}}$-continuous on $M$.

**Theorem 2.** If $\lambda$ is absolutely continuous with respect to $\mu$ then there exists $\xi \in (A, \sigma_{\mathcal{H}})^*$ such that $\lambda = \xi \mu$.

**Proof.** $\lambda$ is $\sigma_{\mathcal{H}}$-continuous on the sets of $\mathcal{H}$, by definition, therefore $\lambda$ is $\sigma_{\mathcal{H}}$-continuous on the sets of $\mathcal{H}$, and so $\lambda$ is $\sigma_{\mathcal{H}}$-continuous on the sets of $\mathcal{H}$ since $\sigma_{\mathcal{H}} \subseteq \sigma_{\mathcal{H}}$. Thus $\lambda(\mathcal{A}_2, \sigma_{\mathcal{H}})$ by Grothendieck's theorem.

Now $\sigma_{\mathcal{H}} \subset \mathcal{P}$-topology and so, if $Z_{\mathcal{H}}$ denotes the adherence of 0 in $(A_2, \sigma_{\mathcal{H}})^*$,
\[
(A_2, \sigma_{\mathcal{H}})^* / Z_{\mathcal{H}} = A_2^* / Z_{\mathcal{H}} \subseteq (A_2, \sigma_{\mathcal{H}})^*
\]
by [F,] Prop. 2.3. It follows that
\[
\text{(1)} \quad (A_2^* / Z_{\mathcal{H}}, \sigma_{\mathcal{H}})^* \subset (A_2, \sigma_{\mathcal{H}})^*.
\]
Since $\sigma_{\mathcal{H}}$ is clearly the associated separated topology on $(A_2, \sigma_{\mathcal{H}})^*$, we have
\[
\text{(2)} \quad ((A_2, \sigma_{\mathcal{H}})^*, \sigma_{\mathcal{H}})^* = (A_2, \sigma_{\mathcal{H}})^*.
\]
Finally, since, denoting by $Z_0$ the adherence of 0 in $(A_2, \mathcal{P})^*$, it is easy to verify that $Z_0 = Z_{\mathcal{H}}$, we have
\[
A_2^* = A_2^* / Z_0 \quad [B_1] II \S 3.8,
\]
\[
= A_2^* / Z_{\mathcal{H}},
\]
there follows, from (1) and (2),
\[
(A_2^*, \sigma_{\mathcal{H}})^* \subset (A_2, \sigma_{\mathcal{H}})^*.
\]
This shows that \( \hat{\lambda} \), as a linear form an \( A_2^\wedge \), is represented via the duality \( (D) \) by \( \xi \in (A_2, \sigma_{\mathcal{M}})^\wedge \). Thus

\[
\lambda(\xi) = \hat{\lambda}(\xi) = \lim_{P} \mu(\xi P),
\]

where \( \xi = |P| \) is a \( \sigma_{\mathcal{M}_i} \) Cauchy filter on \( A_2 \).

\[
\lambda(\xi) = \mu(\xi \lim P)
\]

since \( \mu(\xi) \) is \( \sigma_{\mathcal{M}_i} \)-continuous \( A_2 \), \( (\xi \in \mathcal{M}_i) \), and so

\[
\lambda(\xi) = \mu(\xi \xi) = (\xi \mu)(\xi).
\]

§ 1.6. We apply our abstract Lebesgue-Nikodym theorem to Radon measures on a compact space.

Let \( T \) be a compact Hausdorff space. We take \( A \) to be \( \mathcal{K}(T) \), the algebra of continuous real-valued functions with compact supports, (which here coincides with \( C(T) \)—continuous functions). Our involution is the identity map, and the norm on \( A \) is the uniform = \( \sup_{t \in T} |\xi(t)| \). \( A \) has a unit and is complete for this norm. We take \( \mu \) to be a positive Radon measure, and so \( \mu \) is continuous for the norm of \( A \).

If \( \lambda \) is a Radon measure on \( T \) which is absolutely continuous with respect to \( \mu \) in the sense of \([B_2]\) V § 5.5 we have shown \([F_3]\) Prop. 3.1 that it is \( \sigma_{\mathcal{K}} \)-continuous on \( \mathcal{K} \) (in the present context \( \mathcal{M} = \mathcal{C}_Y \mathcal{N} = \mathcal{S}_Y \) and \( \mathcal{K} \mathcal{S} = \mathcal{S} \), in the notation of \([F_3]\)). In order to apply our abstract Lebesgue-Nikodym theorem we must show that \( \lambda \) is \( \sigma_{\mathcal{M}_i} \)-continuous on \( \mathcal{M} \), the family obtained by saturating \( \{U\} \). It clearly suffices to establish \( \sigma_{\mathcal{M}_i} \)-continuity on the weak closure of \( U \) for the duality \( (D_0) \). However, \( U \) is already weakly closed (in \( \mathcal{K}_i \)), for if this were not so there would exist \( f \in \mathcal{K}_i \cap U \) \((|f(t_0)| > 1 \text{ for some } t_0 \in T) \) which is weakly adherent to \( U \), i.e. given \( \varepsilon > 0 \) and \( \varphi \in \mathcal{K}_2 \) there would exist \( h \in U \) such that \( |\mu((f - h) \varphi)| < \varepsilon \), which is clearly false.

Finally, we observe that the space \( (A, \sigma_{\mathcal{M}})^\wedge \) of Theorems 1 and 2 is here \( L(\mu) \), by \([F_3]\) Prop. 3.4, since for compact \( T \) \( L_{\text{loc}}(\mu) = L(\mu) \) and \( \mathcal{K} \mathcal{S} = \mathcal{S} \). Now \([F_3]\) Prop. 2.7 and Prop. 3.4 ensure that \( \xi \mu \) as here defined for \( \xi \in L(\mu) \) is none other than the product \( \xi \mu \) of integration theory. Our Lebesgue-Nikodym theorem now asserts that \( \lambda \) is absolutely continuous with respect to \( \mu \) if and only if \( \lambda = \xi \mu \).
§ 2.1 We now consider an extension of Theorems 1 and 2 which will enable us to establish the Lebesgue-Nikodym theorem for Radon measures on a locally-compact space. The extension has very little in the way of novel features now that the more restricted Theorems 1 and 2 have been established, but in the interests of ease of presentation it seemed worthwhile to establish the simpler results first.

We consider a *-algebra $A$ which is the inductive limit, as a topological vector space $([B_2] II § 4)$, of a directed family $\{A_j\}_{0}$ of normed *-subalgebras $A_j$ with norms $\| \|_j$. Let $\mu$ be a linear form on $A$ having the properties $F_{1,2,3}$ of § 1.2. As in § 1.2 $\mu$ defines a prehilbert structure on $A_j$, with semi-norm $p_0$ and associated norm $p$, and $A^\sim = (A, p_0)^\sim$ with norm $p$ is a Hilbert space. If $U_j = \{ \xi \in A_j : \| \xi \|_j \leq 1 \}$ and $V = \{ \xi \in A : p_0(\xi) \leq 1 \}$, since

\[
(p_0(\xi))^2 = \mu(\xi^* \xi) \leq \| \mu \|_j \| \xi^* \xi \|_j \leq \| \mu \|_j \| \xi \|^2,
\]

if $\xi \in A_j$ (where $\| \mu \|_j$ is the norm of the restriction of $\mu$ to $A_j$), we have $U_j \subset m_j V$ for some $m_j > 0$.

It follows that if, as before, $i$ is the canonical map of $A$ into the separated completion $A^\sim$, then the weak closures in $A^\sim$ of the sets $i(U_j)$ are compact for $\sigma(A^\sim, i(A))$.

We define, as before, a duality $(P_0)$ on $A_1 \times A_2$ and now take as $\mathcal{N}$ the saturate for $(P_0)$ of $\{ U_j \}_{j}$ considered as a family of subsets of $A_1$. $\mathcal{N}$ will be the same family considered as a family of subsets of $A_2$. $\mathcal{N}_1$ and $\mathcal{N}_2$, families in $i(A_1)$ and $A_2$, respectively, are defined as before. We define absolute continuity of another linear form $\lambda$ with respect to $\mu$ as in § 1, and Theorems 1 and 2 can then be established in this new context, the only change on the argument that is needed is an obvious modification of the proof of the lemma preceding Theorem 1.

§ 2.2 In applying the theorems to Radon measures on a locally compact space $T$ we take the $A_j$ to be the *-normed algebras $\mathcal{K}(K_j)$, where $\{K_j\}_{j}$ are the compact sets of $T$. A Radon measure $\mu$ on $T$ has the properties $F_{1,2,3}$. If $\lambda$ is absolutely continuous with respect to $\mu$ (in the sense of $[B_2] V$ § 5.5) we have seen ([F_2] Prop. 3.1) that it is $\sigma_{\mathcal{K}_2 \mathcal{C}_2}$-continuous on the sets of $\mathcal{K}_1 \mathcal{C}_1$, in the notation of $[F_2]$, so that to show that $\lambda$ is absolutely continuous with respect to $\mu$ in the present sense it suffices to observe i) that each $U_j$ is contained in a set of $\mathcal{K}_1 \mathcal{C}_1$ of the form $fS_0$ where $f \in \mathcal{K}$ has the value 1 on $K_j$ and $S_0 = \{ \varphi \in \mathcal{K} : \| \varphi \| \leq 1 \}$, and ii) that $\mathcal{K}_2 \mathcal{C}_2 \subset \mathcal{N}$ since
each $gS$ is contained in a set $gS_0$, where $c$ is a constant, and $gS_0 \subseteq dU$, for some constant $d$ and support $(g) \subseteq K_j$.

Finally, in interpreting the conclusions of our abstract Lebesgue-Nikodym theorem in the present context we note that $\sigma_{\mathcal{M}} = \sigma_{\mathcal{K}_1 \mathcal{D}_1}$: i) above, shows that $\sigma_{\mathcal{M}} \subseteq \sigma_{\mathcal{X}_1 \mathcal{D}_1}$, and ii) shows that $\sigma_{\mathcal{K}_1 \mathcal{D}_1} \subseteq \sigma_{\mathcal{M}}$. Thus $(\mathcal{X}, \sigma_{\mathcal{M}})$ is isomorphic, as a topological vector space, with $L_{\text{loc}}(\mu)$.

REFERENCES


Istituto Matematico «Leoniud Tonelli»
Via Dener, Pisa.

Westfield College,
(University of London),
Hampstead, N. W. 3.