Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 23, nº 2 (1969), p. 363-372

<http://www.numdam.org/item?id=ASNSP_1969_3_23_2_363_0>

AN ABSTRACT LEBESGUE-NIKODYM THEOREM

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The developement of the theory of Radon measures, continuous linear forms on the vector space $\mathcal{K}(T)$ of continuous real-valued functions with compact supports on the locally compact space T, endowed with a suitable inductive-limit topology [B_3], leans heavily on the order structure of $\mathcal{K}(T)$. We have shown elsewhere [$F_{1,2}$] how some aspects of the theory of integration for Radon measures can equally be developed by availing oneself of the algebraic structure of $\mathcal{K}(T)$ rather than its order structure. We note, however, that such a treatment does not yield a theory of integration in that, for example, the objects which there correspond to the integrable functions of the more familiar development are elements of an abstract completion of $\mathcal{K}(T)$, i.e. are not functions. Our object in investigating the rôle of the algebraic structure was to explore the possibility of its application to

(i) vector measures and the integration of vector — valued functions, where rarely is there an order structure which arises in a natural way from the structure of the space which we are concerned, and

(ii) the theory of distributions, where utilisation of a naturally-occurring order imposes excessive restrictions — a positive distribution is a measure.

In [F_2] we attempted to establish a Lebesgue-Nikodym theorem within our order-free structure, but were unable to do so without the aid of a supplementary hypothesis (loc cit Prop 2.3) which is not verified for Radon measures, but which holds in the theory of distribution [F_3].

In the present paper we obtain a Lebesgue-Nikodym in a form applicable to the theory of integration. § 1.1 gives an account of some ideas from the theory of duality for topological vector spaces, in particular, of an extension of Grothendieck's completion theorem. § 1.2 introduces the pre-

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Pervenuto alla Redazione il 21 Ottobre 1968.

hilbert structure defined by a positive linear form on a normed algebra. § 1.3 describes dualities defined by a linear form on an algebra. §§ 1.4-5 formulate a concept of absolute continuity of one linear form on a normed algebra with respect to another such form and apply the completion theorem of § 1.1 to establish an abstract Lebesgue Nikodym theorem. In § 1.6 this theorem is applied to give the Lebesgue-Nikodym theorem for Radon measures on a compact space. § 2.1 describes an extension of the results of §§ 1.1-5 to the inductive limit (as a topological vector space) of a family of normed algebras, and in § 2.2 we obtain the Lebesgue-Nikodym theorem on a locally compact space.

§ 1.1 We make use of the ideas of the theory of duality for topological vector spaces. We summarize here, briefly, the definitions and results which we shall need.

Let $\langle E_1, E_2 \rangle$ be a dual system (pairing) of vector spaces E_1, E_2 over the complex field C.

A family of subsets of $E_i (i = 1,2)$ is said to be saturated with respect to $\langle E_1, E_2 \rangle$ if it contains

(1) the subsets of each of its members,
(2) the scalar multiples of its members,
(3) the absolutely convex, weakly closed hulls of finite unions of its members.

$\mathcal{F}_i$ will denote the family obtained by saturating the family of finite subsets of E_i).

A saturated family $\mathcal{T}_1$ of weakly bounded subsets of E_1 defines a locally convex topology $\sigma_{\mathcal{T}_1}$ on E_2 (the topology of uniform convergence on the sets of $\mathcal{T}_1$. If $\langle E_1, E_2 \rangle$ is separated then $\sigma_{\mathcal{T}_1}$ is Hausdorff (separated) if $\mathcal{F}_1 \subset \mathcal{T}_1$. (The corresponding statements obtained by interchanging the suffixes 1 and 2 also apply). With this notation we write $\sigma_{\mathcal{F}_2}$ for the weak topology $\sigma(E_1, E_2)$ on E_1.

$\mathcal{K}_2$ denotes the family obtained by saturating the family of absolutely convex weakly compact sets in E_2. $\mathcal{F}_2 \subset \mathcal{K}_2$.

Let $(\mathcal{F}_1 \subset) \mathcal{M}_1$ be a saturated family in E_1, and let $(\mathcal{F}_2 \subset) \mathcal{N}_2 (\subset \mathcal{K}_2)$ be a saturated family in E_2, both families being of weakly bounded sets.

We shall need the following form of *Grothendieck's completion theorem*:

if $\langle E_1, E_2 \rangle$ is separated,

$G = \{f \in E_1^* : f\ \sigma_{\mathcal{N}_2}\text{-continuous on the sets of } \mathcal{M}_1\}$ (1) is the (separated) completion $(E_2, \sigma_{\mathcal{M}_1})^\wedge$ of E_2 for the topology $\sigma_{\mathcal{M}_1}$.

(1) E_1^* denotes the algebraic dual of E_1.

This result may be established by showing

(i) (as in [K] p. 272) that G is complete, and separated for $\sigma_{\mathcal{M}_1}$ ($\mathcal{F}_1 \subset \mathcal{M}_1$ and $\langle E_1, E_1^* \rangle$ is separated),

and

(ii) that E is $\sigma_{\mathcal{M}_1}$-dense in G.

The argument establishing the necessity of [S] Th. 6.2, p. 148 proves (ii) when we put $(E, \tau) = (E, \sigma_{\mathcal{N}_2})$ since $(E, \tau)' = (E_1, \sigma_{\mathcal{N}_2})' = E_2$ (where $(E, \tau)'$ denotes the topological dual of E for the topology τ) by the Mackey-Arens theorem ([S] Th. 3.2, p. 131), since $\mathcal{F}_2 \subset \mathcal{N}_2 \subset \mathcal{K}_2$.

(A «proof» of this theorem given in $[F_2]$ is false. It relies on the false inequality (3) of [K] p. 272).

§ 1.2. We shall apply this theorem to a situation where we take (essentially) for E_1, E_2 two copies of a $*$-normed algebra A (i.e. a normed algebra over C on which there is defined an involution $*$ such that $\| \xi^* \| = \| \xi \|$ for all $\xi \in A$, (see [R] p. 180). Elements ξ for which $\xi = \xi^*$ are said to be self-adjoint.

We consider a linear form μ on A having the properties

$$\text{F}_1) \qquad \mu(\xi^*) = \overline{\mu(\xi)},$$

(this is equivalent to: μ is real on self-adjoint elements of A, and implies that the sesqui-linear forms

$$A \times A \rightarrow C$$

$$(\xi, \eta) \longrightarrow \mu(\xi \eta^*)$$

and

$$(\xi, \eta) \longrightarrow \mu(\eta^* \xi)$$

are hermitian, it is an easy matter to construct an example for which they do not coincide), and

F$_2$) μ is positive,

i. e.
$$\mu(\xi\xi^*) \geq 0 \qquad \forall\, \xi \in A,$$

(and so
$$\mu(\xi^* \xi) = \mu(\xi^* \xi^{**}) \geq 0 \qquad \forall\, \xi \in A),$$

so that the sesqui-linear forms are positive-definite.

We shall henceforth consider only the first of these two forms defined by μ, and shall write it $\mu(,)$. It defines a prehilbert structure on A with corresponding semi-norm $p_0(\xi) = (\mu(\xi\xi^*))^{\frac{1}{2}}$. If $Z = \{\xi : \mu(\xi\xi^*) = 0\} =$

$\{\xi : \mu(\xi\eta) = 0 \ \forall \eta \in A\}$ the positive hermitian form an A/Z associated with $\mu(\ ,\ )$, (which we shall denote by $\dot{\mu}(\ ,\ )$ defines a separated prehilbert structure. The norm p for this structure is that associated with the semi-norm $p_0 \cdot \dot{\mu}(\ ,\ )$ extends uniquely to the separated completion $(A/Z, p_0)^\wedge$ (which we write $(A/Z)^\wedge$) of A/Z for p_0, and makes it a Hilbert space. We can equally construct the separated completion $(A, p_0)^\wedge$ (or $A^\wedge$) of A for p, and ([B$_1$] II § 3.7) $(A/Z)^\wedge$ is isomorphic with $A^\wedge$, to which we may therefore transport the extended form $\dot{\mu}(\ ,\ )$, and if i is the map of [B$_1$] II § 3.7, Th. 3, $i(A)$ is dense in $A^\wedge$ (loc. cit. § 3.8). i is here a linear map, and is in fact the canonical map $A \to A/Z$, so that $i(A)$ is isomorphic with A/Z.

We now impose upon μ the further requirement

F$_3$) μ is continuous on A.

(If A has a unit and is complete, i.e. is a Banach $*$-algebra, the continuity of μ follows from the hypothesis that it is positive ([N], p. 200)).
If

$$U = \{\xi : \|\xi\| \leq 1\}, \qquad V = \{\xi : p_0(\xi) \leq 1\},$$

since

$$(p_0(\xi))^2 = \mu(\xi\xi^*) \leq \|\mu\| \, \|\xi\xi^*\| \leq \|\mu\| \, \|\xi\| \, \|\xi^*\| = \|\mu\| \, \|\xi\|^2,$$

$$U \subset mV \quad \text{for some } m > 0,$$

and so

$$i(U) \subset mi(V).$$

Now $i(V)$ is contained in the unit ball of the Hilbert space $A^\wedge$, and the closed unit ball is compact for the weak topology $\sigma(A^\wedge, A^\wedge)$ defined by the canonical duality $\langle A^\wedge, A^\wedge \rangle$ of $A^\wedge$ with its Hilbert-space dual $A^\wedge$, so that $i(V)$ and therefore $i(U)$ are relatively weakly compact in $A^\wedge$. The the weak closure of $i(U)$ in A is compact for $\sigma(A^\wedge, A^\wedge)$ and is therefore compact for the coarser topology $\sigma(A^\wedge, i(A))$.

§ 1.3. Our form μ defines a duality (D_0) on $A \times A$ — which it is convenient to write $A_1 \times A_2$ — by

$$A_1 \times A_2 \to C$$

$$(\xi, \eta) \to \mu(\xi\eta).$$

It is clear that this duality will not in general be separated.

Let $\mathcal{M}$ be the family obtained by saturating the family $\{U\}$ in A_1 for the duality (D_0), and let $\mathcal{N}$ be the same family considered as a family in A_2.

The sets of $\mathcal{M}$ and $\mathcal{N}$ are weakly bounded for (D_0) since

$$\sup \{ | \mu (\xi \eta) | : \| \xi \| \leq 1 \} \leq \| \mu \| \| \eta \| .$$

The extension of $\dot{\mu}$ to $A_1 \times A_2$ defines a duality (D) on $i(A_1) \times A_2$.

$$i(A_1) \times A_2 \rightarrow C$$

$$(\dot{\xi}, \eta) \rightarrow \dot{\mu} (\dot{\xi} \eta).$$

Let $\mathcal{M}_1 = i(\mathcal{M})$. Since i is continuous for p_0 and p, $i(U)$ is bounded for p and so for the weak topology $\sigma(i(A_1), A_2)$, which is the weak topology defined by (D). It follows that the sets of $\mathcal{M}_1$ are weakly bounded for (D).

Since i is linear, to prove that $\mathcal{M}_1$ is saturated for (D) it suffices to prove that the i-images of absolutely convex weakly closed (for (D_0)) sets of $\mathcal{M}$ are again weakly closed. Such a set M is closed for p_0, and since i is the canonical map $A \rightarrow A/Z$ i is closed, so that $i(M)$ is closed for p. Since it is absolutely convex and $A_2^{\wedge} = (i(A_1), p)^{\wedge}$, $i(M)$ is also $\sigma(i(A_1), A_2^{\wedge})$ - closed, i. e. is weakly closed for (D).

Finally, we define $\mathcal{N}_2$ to be the saturate, for $\langle i(A_1), A_2 \rangle$ of $i(\mathcal{N})$. It is clear that the sets of $\mathcal{N}_2$ are weakly bounded for (D).

§ 1.4. If λ is another linear form on A we now define:

λ absolutely continuous with respect to μ

to mean:

λ is $\sigma_{\mathcal{N}}$-continuous on the sets of $\mathcal{M}$,

(where $\sigma_{\mathcal{N}}$ is defined by the duality (D_0)).

It follows that $\lambda(Z) = 0$:

$$\zeta \in Z \qquad \mu(\zeta\eta) = 0 \qquad \forall\, \eta \in A$$

$$\mu(\zeta N) = 0 \qquad \forall\, N \in \mathcal{N}.$$

Since $\zeta \in M$ for some $M \in \mathcal{M}$ $(\mathcal{F} \subset \mathcal{M})$, the $\sigma_{\mathcal{N}}$-continuity of λ on M shows that $\lambda(\zeta) = 0$.

λ is thus well-defined on A/Z. We shall denote by $\dot{\lambda}$ the functional so defined, and shall prove that it is $\sigma_{\mathcal{N}_2}$-continuous on the sets of $\mathcal{M}_1$. For this it suffices to establish continuity for the coarser topology $\sigma_{i(\mathcal{N})}$.

Since $\lambda = \dot{\lambda} \circ i$ and i is the canonical map of M onto M/Z, it suffices ([B_1] §. 3.4) to prove that $\sigma_{i(\mathcal{N})}$ is the quotient by Z of $\sigma_{\mathcal{N}}$. This is immediate from consideration of the neighbourhoods for the two topologies since a $\sigma_{\mathcal{N}}$-neighbourhood $\{\xi : \sup_{\eta \in N} |\mu(\xi\eta)| < \varepsilon\}$ is saturated for the equivalence relation defined by Z ([B_1] II § 3.4).

§ 1.5 We now apply Grothendieck's completion theorem to establish a Lebesgue-Nikodym theorem.

LEMMA. $\sigma_{\mathcal{M}}$ is coarser than the p_0-topology,

$\sigma_{\mathcal{M}_1}$ is coarser then the p-topology.

PROOF. Since μ is positive

$$|\mu(\xi\eta)|^2 \leq \mu(\xi\xi^*)\,\mu(\eta^*\eta^{**}) \leq (p_0(\xi))^2 \, \|\mu\| \, \|\eta\|^2 .$$

Therefore

$$p(\xi) < 1 \Longrightarrow \sup\{|\mu(\xi\eta)| : \|\eta\| \leq 1\} \leq \|\mu\|^{\frac{1}{2}}$$

so that a $\sigma_{\mathcal{M}}$ neighbourhood defined by $M = U$ contains a p_0 neighbourhood, i. e. $\sigma_{\mathcal{M}} \subset p$-topology. The second assertion is now immediate.

THEOREM 1. If $\zeta \in (A, \sigma_{\mathcal{M}})^\wedge$

$$A \to C$$

$$\xi \to \dot{\mu}(\dot{\xi}\zeta)$$

defines a linear form A which is absolutely continuous with respect to μ.

PROOF. $\dot{\mu}(\dot{\xi}\,\cdot)$ is $\sigma_{\mathcal{M}_1}$-continuous on $A_2^\wedge$ for all $\xi \in A_1$,

$(\{\dot{\xi}\} \in \mathcal{M}_1)$, and so extends (uniquely) to $(A_2, \sigma_{\mathcal{M}_1})^\wedge \supset (A_2, \sigma_{\mathcal{M}})^\wedge$.

This extension defines $\dot{\mu}(\dot{\xi}\zeta)$.

$\zeta\mu : \xi \to \dot{\mu}(\dot{\xi}\zeta)$ is clearly linear on A. To prove that it is absolute continuous with respect to μ, let $M \in \mathcal{M}$, $\varepsilon > 0$. If $\zeta = \{P\}$ (a $\sigma_{\mathcal{M}}$ minimal

Cauchy filter on A_2 ([B_1] II § 3.7)), there exists P such that

$$\sup \{|\dot{\mu}(\dot{\xi}(\zeta - P))| : \xi \in M\} < \varepsilon/2.$$

Choose $\pi \in P$. $\{\pi\} \in \mathcal{N}$, and so $|\mu(\xi\pi)| < \varepsilon/2$ implies that

$$|\zeta\mu(\xi)| = |\dot{\mu}(\dot{\xi}\zeta)| < \varepsilon \quad \text{for } \xi \in M,$$

i.e., $\zeta\mu$ is $\sigma_{\mathcal{N}}$-continuous on M.

THEOREM 2. If λ is absolutely continuous with respect to μ there exists $\zeta \in (A, \sigma_{\mathcal{M}})^\wedge$ such that $\lambda = \zeta\mu$.

PROOF. λ is $\sigma_{\mathcal{N}}$-continuous on the sets of $\mathcal{M}$, by definition, therefore $\dot{\lambda}$ is $\sigma_{i(\mathcal{N})}$-continuous on the sets of $i(\mathcal{M}) = \mathcal{M}_1$, and so λ is $\sigma_{\mathcal{N}_2}$-continuous on the sets of $\mathcal{M}_1$ since $\sigma_{i(\mathcal{N})} \subset \sigma_{\mathcal{N}_2}$. Thus $\dot{\lambda} \in (A_2^\wedge, \sigma_{\mathcal{M}_1})^\wedge$ by Grothendieck's theorem.

Now $\sigma_{\mathcal{M}} \subset p_0$-topology and so, if $Z_{\mathcal{M}}$ denotes the adherence of 0 in $(A_2, \sigma_{\mathcal{M}})^\wedge$,

$$(A_2, p_0)^\wedge / Z_{\mathcal{M}} = A_2^\wedge / Z_{\mathcal{M}} \subset (A_2, \sigma_{\mathcal{M}})^\wedge$$

by [F_1] Prop. 2.3. It follows that

$$(1) \qquad (A_2^\wedge / Z_{\mathcal{M}}, \sigma_{\mathcal{M}_1})^\wedge \subset ((A_2, \sigma_{\mathcal{M}})^\wedge, \sigma_{\mathcal{M}_1})^\wedge.$$

Since $\sigma_{\mathcal{M}_1}$ is clearly the associated separated topology on $(A_2, \sigma_{\mathcal{M}})^\wedge$, we have

$$(2) \qquad ((A_2, \sigma_{\mathcal{M}})^\wedge, \sigma_{\mathcal{M}_1})^\wedge = (A_2, \sigma_{\mathcal{M}})^\wedge.$$

Finally, since, denoting by Z_0 the adherence of 0 in $(A_2, p_0)^\wedge$, it is easy to verify that $Z_0 = Z_{\mathcal{M}}$, we have

$$A_2^\wedge = A_2^\wedge / Z_0 \quad \text{by } [B_1] \text{ II § 3.8},$$

$$= A_2^\wedge / Z_{\mathcal{M}},$$

there follows, from (1) and (2),

$$(A_2^\wedge, \sigma_{\mathcal{M}_1})^\wedge \subset (A_2, \sigma_{\mathcal{M}})^\wedge.$$

This shows that $\dot{\lambda}$, as a linear form an $A_2^\wedge$, is represented via the duality (D) by $\zeta \in (A_2, \sigma_{\mathcal{M}})^\wedge$. Thus

$$\lambda(\xi) = \dot{\lambda}(\dot{\xi}) = \lim \dot{\mu}(\dot{\xi} P),$$

where $\xi = \{P\}$ is a $\sigma_{\mathcal{M}_1}$ Cauchy filter on A_2.

$$\lambda(\xi) = \dot{\mu}(\dot{\xi} \lim P)$$

since $\dot{\mu}(\dot{\xi})$ is $\sigma_{\mathcal{M}_1}$-continuous A_2, $(\{\dot{\xi}\} \in \mathcal{M}_1)$, and so

$$\lambda(\xi) = \dot{\mu}(\dot{\xi}\zeta) = (\zeta\mu)(\xi).$$

§ 1.6. We apply our abstract Lebesgue-Nikodym theorem to Radon measures on a compact space.

Let T be a compact Hausdorff space. We take A to be $\mathcal{K}(T)$, the algebra of continuous real-valued functions with compact supports, (which here coincides with $\mathcal{C}(T)$—continuous functions). Our involution is the identity map, and the norm on A is the uniform norm, $\|\xi\| = \sup_{t \in T} |\xi(t)|$. ($A$ has a unit and is complete for this norm). We take μ to be a positive Radon measure, and so μ is continuous for the norm of A.

If λ is a Radon measure on T which is absolutely continuous with respect to μ in the sense of [B$_3$] V § 5.5 we have shown [F$_2$] Prop. 3.1 that it is $\sigma_{\mathcal{N}}$-continuous on K (in the present context $\mathcal{M} = \mathcal{S}_p$ $\mathcal{N} = \mathcal{S}_2$ and $\mathcal{K}\mathcal{S} = \mathcal{S}$, in the notation of [F$_2$]). In order to apply our abstract Lebesgue-Nikodym theorem we must show that λ is $\sigma_{\mathcal{S}_2}$-continuous on $\mathcal{M}$, the family obtained by saturating $\{U\}$. It clearly suffices to establish $\sigma_{\mathcal{S}_2}$-continuity on the weak closure of U for the duality (D_0). However, U is already weakly closed (in $\mathcal{K}_1$), for if this were not so there would exist $f \in \mathcal{K}_1 \cap U^c$ ($|f(t_0)| > 1$ for some $t_0 \in T$) which is weakly adherent to U, i.e. given $\varepsilon > 0$ and $\varphi \in \mathcal{K}_2$ there would exist $h \in U$ such that $|\mu((f - h)\varphi)| < \varepsilon$, which is clearly false.

Finally, we observe that the space $(A, \sigma_{\mathcal{M}})^\wedge$ of Theorems 1 and 2 is here $L(\mu)$, by [F$_1$] Prop. 3.4, since for compact T $L_{\text{loc}}(\mu) = L(\mu)$ and $\mathcal{K}\mathcal{S} = \mathcal{S}$. Now [F$_1$] Prop. 2.7 and Prop. 3.4 ensure that $\zeta\mu$ as here defined for $\zeta \in L(\mu)$ is none other than the product $\xi\mu$ of integration theory. Our Lebesgue-Nikodym theorem now asserts that λ is absolutely continuous with respect to μ if and only if $\lambda = \zeta\mu$.

§ 2.1 We now consider an extension of Theorems 1 and 2 which will enable us to establish the Lebesgue-Nikodym theorem for Radon measures on a locally-compact space. The extension has very little in the way of novel features now that the more restricted Theorems 1 and 2 have been established, but in the interests of ease of presentation it seemed worthwhile to establish the simpler results first.

We consider a $*$-algebra A which is the inductive limit, as a topological vector space ([B_2] II § 4), of a directed family $\{A_j\}_J$ of normed $*$ sub-algebras A_j with norms $\| \ \|_j$. Let μ be a linear form on A having the properties $F_{1,2,3}$) of § 1.2. As in § 1.2 μ defines a prehilbert structure on A, with semi-norm p_0 and associated norm p, and $A^\wedge = (A, p_0)^\wedge$ with norm p is a Hilbert space. If $U_j = \{\xi \in A_j : \| \xi \|_j \leq 1\}$ and $V = \{\xi \in A : p_0(\xi) \leq 1\}$, since

$$(p_0(\xi))^2 = \mu(\xi\xi^*) \leq \| \mu \|_j \| \xi\xi^* \|_j \leq \| \mu \|_j \| \xi \|_j^2$$

if $\xi \in A_j$ (where $\| \mu \|_j$ is the norm of the restriction of μ to A_j), we have

$$U_j \subset m_j V \quad \text{for some} \quad m_j > 0.$$

It follows that if, as before, i is the canonical map of A into the separated completion $A^\wedge$, then the weak closures in $A^\wedge$ of the sets $i(U_j)$ are compact for $\sigma(A^\wedge, i(A))$.

We define, as before, a duality (D_0) on $A_1 \times A_2$ and now take as $\mathcal{M}$ the saturate for (D_0) of $\{U_j\}_J$ considered as a family of subsets of A_1. $\mathcal{N}$ will be the same family considered as a family of subsets of A_2. $\mathcal{M}_1$ and $\mathcal{N}_2$, families in $i(A_1)$ and A_2, respectively, are defined as before. We define absolute continuity of another linear form λ with respect to μ as in § 1, and Theorems 1 and 2 can then be established in this new context, the only change on the argument that is needed is an obvious modification of the proof of the lemma preceeding Theorem 1.

§ 2.2 In applying the theorems to Radon measures on a locally compact space T we take the A_j to be the $*$-normed algebras $\mathcal{K}(K_j)$, where $\{K_j\}_J$ are the compact sets of T. A Radon measure μ on T has the properties $F_{1,2,3}$). If λ is absolutely continuous with respect to μ (in the sense of [B_3] V § 5.5) we have seen ([F_2] Prop. 3.1) that it is $\sigma_{\mathcal{K}_2\mathcal{S}_2}$-continuous on the sets of $\mathcal{K}_1\mathcal{S}_1$, in the notation of [$F_2$], so that to show that λ is absolutely continuous with respect to μ in the present sense it suffices to observe i) that each U_j is contained in a set of $\mathcal{K}_1\mathcal{S}_1$ of the form fS_0 where $f \in \mathcal{K}$ has the value 1 on K_j and $S_0 = \{\varphi \in \mathcal{K} : |\varphi| \leq 1\}$, and ii) that $\mathcal{K}_2\mathcal{S}_2 \subset \mathcal{N}$ since each gS is contained in a set gcS_0, where c is a constant, and $gcS_0 \subset dU_j$ for some constant d and support $(g) \subset K_j$.

Finally, in interpreting the conclusions of our abstract Lebesgue-Nikodym theorem in the present context we note that $\sigma_{\mathcal{M}} = \sigma_{\mathcal{K}_1 \mathcal{S}_1}$: i) above, shows that $\sigma_{\mathcal{M}} \subset \sigma_{\mathcal{K}_1 \mathcal{S}_1}$, and ii) shows that $\sigma_{\mathcal{K}_1 \mathcal{S}_1} \subset \sigma_{\mathcal{M}}$. Thus $(\mathcal{K}, \sigma_{\mathcal{M}})$ is isomorphic, as a topological vector space, with $L_{\mathrm{loc}}(\mu)$.

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