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# SOME INEQUALITIES INVOLVING TRIGONOMETRICAL POLYNOMIALS

E. BOMBIERI and H. DAVENPORT

## 1. Introduction.

Let  $N$  be a positive integer and let  $a_{M+1}, \dots, a_{M+N}$  be any real or complex numbers. Define

$$(1) \quad S(x) = \sum_{n=M+1}^{M+N} a_n e^{2\pi i n x}.$$

Let  $x_1, \dots, x_R$  be any real numbers which satisfy

$$(2) \quad \|x_r - x_s\| \geq \delta \text{ when } r \neq s,$$

where  $\|\theta\|$  denotes the difference between  $\theta$  and the nearest integer, taken positively, and  $0 < \delta \leq \frac{1}{2}$ .

In a recent paper <sup>1)</sup> we proved that

$$(3) \quad \sum_{r=1}^R |S(x_r)|^2 \leq (N^{1/2} + \delta^{-1/2})^2 \sum_{n=M+1}^{M+N} |a_n|^2,$$

and that

$$\sum_{r=1}^R |S(x_r)|^2 \leq 2 \max(N, \delta^{-1}) \sum_{n=M+1}^{M+N} |a_n|^2;$$

the latter represents an improvement on the former if  $N$  and  $\delta^{-1}$  are of about the same size.

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(<sup>1</sup>) « On the large sieve method », *Abhandlungen aus Zahlentheorie und Analysis zur Erinnerung an Edmund Landau*, Deutscher Verlag der Wissenschaften, Berlin 1968.

1. *Annali della Scuola Norm. Sup. - Pisa.*

In the present paper we investigate more deeply the two cases in which  $N\delta$  is either large or small. The factor on the right of (3) is a little greater than  $N$  in the first case, and a little greater than  $\delta^{-1}$  in the second case. Our object is to determine the order of magnitude of the term that must be added to  $N$  or  $\delta^{-1}$ , as the case may be, to ensure the validity of the inequality.

For the case  $N\delta$  large, we prove:

**THEOREM 1.** *If  $N\delta \geq 1$  then*

$$(4) \quad \sum_{r=1}^R |S(x_r)|^2 < N(1 + 5(N\delta)^{-1}) \sum_{n=M+1}^{M+N} |a_n|^2.$$

*On the other hand, if  $c$  is a constant less than 1 there exist sums  $S(x)$  with  $\delta$  arbitrarily small and  $N\delta$  arbitrarily large for which*

$$(5) \quad \sum_{r=1}^R |S(x_r)|^2 > N(1 + c(N\delta)^{-1}) \sum_{n=M+1}^{M+N} |a_n|^2.$$

There are two new features in the proof of (4), as compared with the proof of (3) in our previous paper. The first of these is a maximization argument (Lemma 1), which has the effect of allowing us to limit ourselves to sums  $S(x)$  in which the numbers  $|a_n|$  have a measure of approximate equality. The second feature is the use of the function  $\Phi_1(t)$ , defined in (13), in place of the simpler function  $\Phi(t)$  of our previous paper.

For the case  $N\delta$  small, we prove:

**THEOREM 2.** *If  $N\delta \leq \frac{1}{4}$ , then*

$$(6) \quad \sum_{r=1}^R |S(x_r)|^2 < \delta^{-1}(1 + 270N^3\delta^3) \sum_{n=M+1}^{M+N} |a_n|^2.$$

*On the other hand there exist sums  $S(x)$  with  $N\delta$  arbitrarily small for which*

$$(7) \quad \sum_{r=1}^R |S(x_r)|^2 > \delta^{-1} \left(1 + \frac{1}{12}N^3\delta^3\right) \sum_{n=M+1}^{M+N} |a_n|^2.$$

This case represents a problem which is entirely different from that of the first case; it has much in common with the problem of approximating to a Riemann integral by a finite sum. The arguments used in the proof of (6) are quite delicate.

The present paper is self-contained, except for the general inequality (27), due to Davenport and Halberstam, which is needed for the proof of (4).

**2. The maximization argument.**

Let  $\delta$  and  $x_1, \dots, x_R$  be fixed. For each positive integer  $N$ , we define  $G(N)$  by

$$(8) \quad N + G(N) = \max \sum_{r=1}^R |S(x_r)|^2,$$

where

$$(9) \quad S(x) = \sum_{n=1}^N a_n e^{2\pi i n x}$$

and the maximum is taken over all complex numbers  $a_1, \dots, a_N$  satisfying

$$(10) \quad \sum_{n=1}^N |a_n|^2 = 1.$$

The maximum is obviously attained, and we call a set of coefficients  $a_n$  for which it is attained a *maximal set*.

LEMMA 1. *Suppose  $N$  is such that  $G(N) > 0$  and*

$$(11) \quad G(N') \leq G(N) \text{ for } N' \leq N.$$

*Then, for a maximal set of coefficients, we have*

$$(12) \quad \sum_{M+1}^{M+H} |a_n|^2 = \frac{H + \theta G}{N + G} \sum_1^N |a_n|^2,$$

where  $|\theta| \leq 1$  and  $G = G(N)$ ; and this holds for all  $M, H$  satisfying

$$0 \leq M < M + H \leq N.$$

*Proof.* Write  $e(\theta) = e^{2\pi i \theta}$ , and

$$a_n = A_n e(\alpha_n),$$

where  $A_n \geq 0$  and  $\alpha_n$  is real. For a maximal set of coefficients we have a stationary value of  $\sum |S(x_r)|^2$  subject to  $\sum A_n^2 = 1$ . Hence we must have

$$\frac{\partial}{\partial A_m} \sum_{r=1}^R |S(x_r)|^2 - 2\lambda A_m = 0 \quad (m = 1, \dots, N)$$

for some  $\lambda$ . Writing  $|S|^2 = S\bar{S}$ , and noting that

$$\frac{\partial S(x)}{\partial A_m} = e(mx + \alpha_m),$$

we see that the conditions are

$$\sum_{r=1}^R e(mx_r + \alpha_m) \overline{S(x_r)} + \sum_{r=1}^R e(-mx_r - \alpha_m) S(x_r) = 2\lambda A_m.$$

If this is multiplied by  $A_m$  and summed for  $m = 1, \dots, N$ , it gives

$$2 \sum_{r=1}^R |S(x_r)|^2 = 2\lambda \sum_{m=1}^N A_m^2 = 2\lambda,$$

whence

$$\lambda = N + G.$$

If however the sum is restricted to  $m = M + 1, \dots, M + H$ , we get

$$\sum_{r=1}^R S_H(x_r) \overline{S(x_r)} + \sum_{r=1}^H \overline{S_H(x_r)} S(x_r) = 2\lambda \sum_{m=M+1}^{M+H} A_m^2,$$

where

$$S_H(x) = \sum_{m=M+1}^{M+H} a_m e(mx).$$

Hence

$$\lambda \sum_{m=M+1}^{M+H} |a_m|^2 \leq \sum_{r=1}^R |S_H(x_r)| |S(x_r)| \leq \left( \sum_{r=1}^R |S_H(x_r)|^2 \right)^{1/2} \left( \sum_{r=1}^R |S(x_r)|^2 \right)^{1/2}.$$

Now

$$\sum_{r=1}^R |S(x_r)|^2 = \lambda = N + G.$$

Also, by the definition of  $G(H)$  and the hypothesis (11), we have

$$\sum_{r=1}^R |S_H(x_r)|^2 \leq (H + G(H)) \sum_{m=M+1}^{M+H} |a_m|^2 \leq (H + G) \sum_{m=M+1}^{M+H} |a_m|^2.$$

Hence

$$(N + G)^{1/2} \sum_{m=M+1}^{M+H} |a_m|^2 \leq \left( (H + G) \sum_{m=M+1}^{M+H} |a_m|^2 \right)^{1/2},$$

that is,

$$\sum_{m=M+1}^{M+H} |a_m|^2 \leq \frac{H+G}{N+G}.$$

This is on the hypothesis that  $\sum_1^N |a_n|^2 = 1$ . Plainly that hypothesis can be omitted if we modify the inequality so that it reads

$$\sum_{m=M+1}^{M+H} |a_m|^2 \leq \frac{H+G}{N+G} \sum_{n=1}^N |a_n|^2.$$

We can obtain a complementary inequality by applying this to the sums over

$$0 < m \leq M \quad \text{and} \quad M+H < m \leq N,$$

and subtracting from the complete sum. We obtain

$$\sum_{m=M+1}^{M+H} |a_m|^2 \geq \left(1 - \frac{M+G}{N+G} - \frac{N-M-H+G}{N+G}\right) \sum_{n=1}^N |a_n|^2 = \frac{H-G}{N+G} \sum_{n=1}^N |a_n|^2.$$

The two inequalities together prove (12).

### 3. A particular Fourier series.

LEMMA 2. *Let*

$$(13) \quad \Phi_1(t) = \frac{\sin \pi \lambda t}{t} \cdot \frac{\sin \pi t}{\pi t} \quad \text{for } 0 < t < 1.$$

*Suppose that  $\lambda > |\alpha| + 1$ . Then*

$$(14) \quad \int_0^1 \{\Phi_1(t)\}^2 dt < \frac{1}{2} \pi^2 \lambda,$$

$$(15) \quad \int_0^1 \Phi_1(t) \cos \pi \alpha t dt > \frac{\pi}{2} - \frac{1}{2\pi^2} \left( \frac{1}{(\lambda + \alpha)^2 - 1} + \frac{1}{(\lambda - \alpha)^2 - 1} \right).$$

*Proof.* (14) is almost immediate, for since  $\sin \pi t < \pi t$  we have

$$\int_0^1 \{\Phi_1(t)\}^2 dt < \int_0^\infty \left(\frac{\sin \pi \lambda t}{t}\right)^2 dt = \frac{1}{2} \pi^2 \lambda.$$

To prove (15) we start from the relations

$$\begin{aligned} \int_0^1 \Phi_1(t) \cos \pi \alpha t dt &= \frac{1}{2} \int_0^1 \frac{\sin \pi(\lambda + \alpha)t + \sin \pi(\lambda - \alpha)t}{t} \cdot \frac{\sin \pi t}{\pi t} dt \\ &= \frac{1}{4} \int_0^1 \frac{\cos \pi(\lambda + \alpha - 1)t - \cos \pi(\lambda + \alpha + 1)t}{\pi t^2} dt \\ &\quad + \frac{1}{4} \int_0^1 \frac{\cos \pi(\lambda - \alpha - 1)t - \cos \pi(\lambda - \alpha + 1)t}{\pi t^2} dt. \end{aligned}$$

For  $B > A > 0$  we have

$$\begin{aligned} \int_0^\infty \frac{\cos At - \cos Bt}{\pi t^2} dt &= \int_0^\infty \frac{2 \sin^2 \frac{1}{2} Bt - 2 \sin^2 \frac{1}{2} At}{\pi t^2} dt \\ &= \frac{B - A}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{1}{2} (B - A) = \pi \end{aligned}$$

if  $A = \pi(\lambda + \alpha - 1)$  and  $B = \pi(\lambda + \alpha + 1)$ . Hence

$$(16) \quad \int_0^1 \Phi_1(t) \cos \pi \alpha t dt = \frac{\pi}{2} - \frac{1}{4\pi} \mathcal{R}(J(\alpha) + J(-\alpha)),$$

where

$$J(\alpha) = \int_1^\infty (e^{i\pi(\lambda+\alpha-1)t} - e^{i\pi(\lambda+\alpha+1)t}) t^{-2} dt.$$

To estimate  $J(\alpha)$  we rotate the line of integration through an angle  $\frac{\pi}{2}$  in the complex plane, so that it becomes the line  $1 + iu$ ,  $u > 0$ . The contribution of the quadrant at infinity vanishes.

We get

$$\begin{aligned}
 J(\alpha) &= \int_0^{\infty} (e^{i\pi(\lambda+\alpha-1)(1+iu)} - e^{i\pi(\lambda+\alpha+1)(1+iu)}) (1+iu)^{-2} i \, du \\
 &= -ie^{i\pi(\lambda+\alpha)} \int_0^{\infty} (e^{-\pi(\lambda+\alpha-1)u} - e^{-\pi(\lambda+\alpha+1)u}) (1+iu)^{-2} \, du.
 \end{aligned}$$

Hence

$$\begin{aligned}
 |J(\alpha)| &\leq \int_0^{\infty} (e^{-\pi(\lambda+\alpha-1)u} - e^{-\pi(\lambda+\alpha+1)u}) \, du \\
 &= \frac{1}{\pi} \left( \frac{1}{\lambda+\alpha-1} - \frac{1}{\lambda+\alpha+1} \right) = \frac{2}{\pi((\lambda+\alpha)^2-1)}.
 \end{aligned}$$

Putting this, for  $\alpha$  and  $-\alpha$ , in (16), we obtain (15).

LEMMA 3. Suppose that  $K$  satisfies  $K\delta \geq 2$ . Let  $N_0$  be any positive integer. There exists a Fourier series

$$(17) \quad \psi(x) = \sum_{-\infty}^{\infty} b_n e(nx),$$

with real coefficients  $b_n$  satisfying  $b_{-n} = b_n$ , such that

$$(18) \quad \psi(x) = 0 \quad \text{if} \quad \|x\| > \frac{1}{2} \delta,$$

$$(19) \quad \sum_{-\infty}^{\infty} b_n^2 < \frac{1}{2} \pi^2 (N_0 + K) \delta^2,$$

and, for  $|n| \leq N_0$

$$(20) \quad b_n^{-2} < \frac{4}{\pi^2 \delta^2} \left( 1 + \frac{1}{7 \{ (N_0 + K - |n|)^2 \delta^2 - 1 \}} \right).$$

*Proof.* We define  $\lambda = (N_0 + K) \delta$ , and we define  $\psi(x)$  for  $|x| \leq \frac{1}{2} \delta$  by

$$\psi(x) = \begin{cases} \Phi_1(2\delta^{-1}|x|) & \text{if } |x| < \frac{1}{2} \delta, \\ 0 & \text{if } \frac{1}{2} \delta \leq |x| \leq \frac{1}{2} \delta. \end{cases}$$



We define  $\psi(x)$  for other real  $x$  by periodicity with period 1. Then  $\psi(x)$  is an even function of  $x$  and satisfies (18). The Fourier coefficients  $b_n$  of  $\psi(x)$  are given by

$$b_n = \int_{-\delta/2}^{\delta/2} \psi(x) e(-nx) dx = \delta \int_0^1 \Phi_1(t) \cos \pi \alpha t dt,$$

where  $\alpha = n\delta$ .

By Parseval's formula,

$$\sum_{-\infty}^{\infty} b_n^2 = \int_{-\delta/2}^{\delta/2} \psi^2(x) dx = \delta \int_0^1 \{\Phi_1(t)\}^2 dt.$$

Hence, by (14) of Lemma 2,

$$\sum_{-\infty}^{\infty} b_n^2 < \frac{1}{2} \pi^2 \lambda \delta = \frac{1}{2} \pi^2 (N_0 + K) \delta^2.$$

It remains only to prove (20). For  $|n| \leq N_0$ , we have

$$\lambda - |\alpha| = (N_0 + K - |n|) \delta \geq K \delta \geq 2.$$

By (15) of Lemma 2,

$$b_n > \frac{1}{2} \pi \delta \left( 1 - \frac{2}{\pi^3 \{(\lambda - |\alpha|)^2 - 1\}} \right).$$

Now

$$\frac{2}{\pi^3 \{(\lambda - |\alpha|)^2 - 1\}} < \frac{2}{3\pi^3} < \frac{1}{46},$$

and for  $0 < \gamma < \frac{1}{46}$  we have

$$(1 - \gamma)^{-2} < 1 + 2 \cdot 1 \gamma.$$

Hence

$$b_n^{-2} < \frac{4}{\pi^2 \delta^2} \left( 1 + \frac{4 \cdot 2}{\pi^3 \{(\lambda - |\alpha|)^2 - 1\}} \right).$$

Since  $4 \cdot 2 \pi^{-3} < \frac{1}{7}$  this gives (20), on recalling that

$$\lambda - |\alpha| = (N_0 + K - |n|) \delta.$$

4. Proof of the first part of Theorem 1.

We observe first that there is no loss of generality in taking  $M = 0$  in the definition of  $S(x)$  in (1), for we can reduce the general case to this by putting  $n = M + n'$ . Thus we can take  $S(x)$  to be defined by (9).

For fixed  $\delta$  and fixed  $x_1, \dots, x_R$ , let  $N$  be the least positive integer for which  $G(N)$ , defined in § 2, satisfies

$$(21) \quad G(N) > 5\delta^{-1};$$

if there is no such integer the desired conclusion holds. For this  $N$ , the hypothesis (11) of Lemma 1 is satisfied. For a maximal set of coefficients, (12) holds; and of course we also have

$$(22) \quad \sum_{r=1}^R |S(x_r)|^2 = (N + G) \sum_1^N |a_n|^2.$$

Suppose first that  $N$  is odd, say  $N = 2N_0 + 1$ . We define

$$(23) \quad a'_n = a_{n+N_0+1} \quad \text{for} \quad -N_0 \leq n \leq N_0,$$

and have

$$(24) \quad |S(x)| = |S_0(x)|,$$

where

$$(25) \quad S_0(x) = \sum_{n=-N_0}^{N_0} a'_n e(nx).$$

By (12),

$$(26) \quad \sum_{-m}^m |a'_n|^2 = \sum_{-m+N_0+1}^{m+N_0+1} |a_n|^2 \geq \frac{2m+1-G}{N+G} \sum_1^N |a_n|^2$$

if  $0 \leq m \leq N_0$ .

Suppose next that  $N$  is even, say  $N = 2N_0$ .

We define  $a'_n$  as above in (23) for  $-N_0 \leq n < N_0$ , and put  $a'_{N_0} = 0$ . Then (24) is still valid, with the same definition (25). Also (26) is still valid, provided  $0 \leq m < N_0$ .

Let  $\psi(x)$  be any even function of period 1 satisfying the condition (18) of Lemma 3. It was proved by Davenport and Halberstam<sup>(2)</sup> that

$$(27) \quad \sum_{r=1}^R |S(x_r)|^2 \leq \left( \sum_{-\infty}^{\infty} b_n^2 \right) \left( \sum_{-N_0}^{N_0} b_n^{-2} |a_n|^2 \right).$$

<sup>(2)</sup> « The values of a trigonometrical polynomial at well spaced points », *Mathematika* 13 (1966), 91-96, and 14 (1967), 229-232.

With the particular function  $\psi(x)$  of Lemma 3, this gives

$$\sum_{r=1}^R |S(x_r)|^2 \leq 2(N_0 + K) \left( \sum_{-N_0}^{N_0} |a'_n|^2 \left( 1 + \frac{1}{7} c_n \right) \right),$$

where

$$c_n = \frac{1}{(N_0 + K - |n|)^2 \delta^2 - 1}.$$

In view of (22) we can write the result as

$$(28) \quad (N + G) \sum_1^N |a_n|^2 \leq 2(N_0 + K) \left( \sum_1^N |a_n|^2 + \frac{1}{7} S \right),$$

where

$$S = \sum_{-N_0}^{N_0} c_n |a'_n|^2.$$

Applying partial summation, and using the fact that  $c_{-n} = c_n$ , we obtain

$$\sum_{-N_0}^{N_0} c_n |a'_n|^2 = c_{N_0} \sum_{-N_0}^{N_0} |a'_n|^2 - \sum_{m=0}^{N_0-1} (c_{m+1} - c_m) \sum_{n=-m}^m |a'_n|^2.$$

Since  $c_{m+1} - c_m > 0$  for  $m \geq 0$ , we can apply the inequality (26) in the inner sum on the right. This gives

$$\begin{aligned} N &\leq \left\{ c_{N_0} - \sum_{m=0}^{N_0-1} (c_{m+1} - c_m) \frac{2m+1-G}{N+G} \right\} \sum_1^N |a_n|^2 \\ &= \frac{1}{N+G} \left\{ \sum_{-N_0}^{N_0} c_m + (2c_{N_0} - c_0) G + (N - 2N_0 - 1) c_{N_0} \right\} \sum_1^N |a_n|^2. \end{aligned}$$

Since  $2N_0 + 1 \geq N$ , the last term in the bracket can be omitted.

Substitution in (28) gives

$$(N + G)^2 \leq 2(N_0 + K) \left( N + G + \frac{1}{7} A \right),$$

where

$$(29) \quad A = \sum_{-N_0}^{N_0} c_m + (2c_{N_0} - c_0) G.$$

By the inequality of the arithmetic and geometric means, we have

$$N + G \leq N_0 + K + \frac{1}{2} N + \frac{1}{2} G + \frac{1}{14} A,$$

and since  $N_0 \leq \frac{1}{2} N$  this implies that

$$(30) \quad G \leq 2 \cdot K + \frac{1}{7} A.$$

Since the function  $c_n$ , for a continuous variable  $n$ , has a positive second derivative, we have

$$\begin{aligned} \sum_{-N_0}^{N_0} c_n &< \int_{-N_0 - \frac{1}{2}}^{N_0 + \frac{1}{2}} c_x dx = 2 \int_0^{N_0 + \frac{1}{2}} \frac{dx}{(N_0 + K - x)^2 \delta^2 - 1} \\ &< 2 \delta^{-1} \int_{(K - \frac{1}{2})\delta}^{\infty} \frac{du}{u^2 - 1} = \delta^{-1} \log \frac{(K - \frac{1}{2})\delta + 1}{(K - \frac{1}{2})\delta - 1}. \end{aligned}$$

We now take  $K = 2\delta^{-1}$ , and have

$$\sum_{-N_0}^{N_0} c_n < \delta^{-1} \log \frac{3 - \frac{1}{2} \delta}{1 - \frac{1}{2} \delta} < \delta^{-1} \log \frac{11}{3} < 1.3 \delta^{-1}.$$

Also

$$2c_{N_0} - c_0 < 2c_{N_0} = \frac{2}{K^2 \delta^2 - 1} = \frac{2}{3}.$$

Hence

$$A < 1.3 \delta^{-1} + \frac{2}{3} G.$$

From (30) we now obtain

$$G < 4\delta^{-1} + \frac{1}{7} \left( 1.3 \delta^{-1} + \frac{2}{3} G \right) < 4.2 \delta^{-1} + \frac{1}{10} G,$$

which gives a contradiction to (21). This contradiction proves the first part of Theorem 1.

We have not used the hypothesis that  $N\delta \geq 1$ , but the result (4) becomes of little value if  $N\delta < 1$ .

### 5. Proof of the second part of Theorem 1.

We give a simple example, with  $\delta$  arbitrarily small and  $N\delta$  arbitrarily large, for which

$$(31) \quad \sum_{r=1}^R |S(x_r)|^2 = (N + \delta^{-1} - 1) \sum |a_n|^2.$$

This suffices for (5), since  $\delta^{-1} - 1 > c\delta^{-1}$  if  $\delta$  is sufficiently small.

Let  $h$  and  $L$  be arbitrarily large positive integers, and take

$$S(x) = \sum_{\nu=-L}^L e^{2\pi i(2h+1)\nu x}.$$

For this sum, regarded as a case of (1), we have

$$N = 2(2h+1)L + 1, \quad \sum |a_n|^2 = 2L + 1.$$

Take  $\delta = 1/(2h+1)$ , and take the points  $x_r$  to be

$$0, \pm \frac{1}{2h+1}, \pm \frac{2}{2h+1}, \dots, \pm \frac{h}{2h+1}.$$

At each of these points we have  $S(x) = 2L + 1$ , and therefore

$$\sum_r |S(x_r)|^2 = (2h+1)(2L+1)^2.$$

Also

$$\begin{aligned} (N + \delta^{-1} - 1) \sum |a_n|^2 &= (2(2h+1)L + 1 + 2h + 1 - 1)(2L + 1) \\ &= (2h+1)(2L+1)^2. \end{aligned}$$

This proves (31).

### 6. Lemmas for Theorem 2.

As observed at the beginning of § 4, we can take  $M = 0$  in (1), so that  $S(x)$  is defined by (9). We suppose that

$$(32) \quad N\delta \leq \frac{1}{4}.$$

Define  $\Phi(z)$  by

$$(33) \quad \Phi(z) = \sum_{m=1}^N \sum_{n=1}^N a_m \bar{a}_n \left( 1 - \frac{\pi\delta(m-n)}{\sin \pi\delta(m-n)} \right) e((m-n)z).$$

We note that in every term  $|\delta(m-n)| \leq \delta N \leq \frac{1}{4}$ .

LEMMA 4. We have

$$(34) \quad \delta \int_{x-\delta/2}^{x+\delta/2} (|S(z)|^2 - \Phi(z)) dz.$$

*Proof.* Since

$$|S(z)|^2 = \sum_{m=1}^N \sum_{n=1}^N a_m \bar{a}_n e((m-n)z),$$

we have

$$|S(z)|^2 - \Phi(z) = \sum_{m=1}^N \sum_{n=1}^N a_m \bar{a}_n \frac{\pi\delta(m-n)}{\sin \pi\delta(m-n)} e((m-n)z).$$

The result now follows from

$$\int_{-\delta/2}^{\delta/2} e((m-n)z) dz = \frac{\sin \pi\delta(m-n)}{\pi(m-n)}.$$

LEMMA 5. Let  $F(z) = |S(z)|^2$ . Then

$$(35) \quad \Phi(z) = \sum_{k=1}^{\infty} (-1)^{k-1} c_k \delta^{2k} F^{(2k)}(z),$$

where the  $c_k$  are positive constants, and  $c_1 = \frac{1}{24}$ , and, for  $0 < \theta < 2\pi$ ,

$$(36) \quad \sum_1^{\infty} c_k \theta^{2k} = \frac{\frac{1}{2} \theta}{\sin \frac{1}{2} \theta} - 1.$$

*Proof.* For any  $\theta$  with  $|\theta| < 2\pi$ , we have

$$\frac{\frac{1}{2} \theta}{\sin \frac{1}{2} \theta} = \prod_{n=1}^{\infty} \left\{ 1 - \left( \frac{\theta}{2n\pi} \right)^2 \right\}^{-1} = 1 + \sum_{k=1}^{\infty} c_k \theta^{2k},$$

where the  $c_k$  are positive constants. Also  $c_1 = \frac{1}{24}$  from  $\sum_1^\infty n^{-2} = \frac{\pi^2}{6}$ . Putting  $\theta = 2\pi \delta(m - n)$  and substituting in (33), we obtain

$$\Phi(z) = - \sum_{m=1}^N \sum_{n=1}^N a_m \bar{a}_n \sum_{k=1}^\infty c_k (2\pi \delta(m - n))^{2k} e((m - n)z).$$

Now

$$F^{(2k)}(z) = (2\pi i)^{2k} \sum_{m=1}^N \sum_{n=1}^N a_m \bar{a}_n (m - n)^{2k} e((m - n)z).$$

These two equations yield (35).

LEMMA 6. *For any positive integer  $k$ ,*

$$(37) \quad \int_0^1 |F^{(k)}(z)| dz \leq (4\pi N)^k \sum_{n=1}^N |a_n|^2.$$

*Proof.* Let  $T(z) = \overline{S(z)}$ , so that  $F(z) = S(z)T(z)$ . By Leibniz's formula,

$$F^{(k)}(z) = \sum_{l=0}^k \binom{k}{l} S^{(l)}(z) T^{(k-l)}(z).$$

By Cauchy's inequality,

$$\int_0^1 |S^{(l)}(z) T^{(k-l)}(z)| dz \leq \left\{ \int_0^1 |S^{(l)}(z)|^2 dz \right\}^{1/2} \left\{ \int_0^1 |T^{(k-l)}(z)|^2 dz \right\}^{1/2}.$$

Now

$$S^{(l)}(z) = (2\pi i)^l \sum_{n=1}^N a_n n^l e(nz),$$

whence

$$\int_0^1 |S^{(l)}(z)|^2 dz = (2\pi)^{2l} \sum_{n=1}^N |a_n|^2 n^{2l}.$$

There is a similar result for the integral containing  $T$ . Finally

$$\int_0^1 |F^{(k)}(z)| dz \leq \sum_{l=0}^k \binom{k}{l} \left\{ (2\pi)^{2l} N^{2k} \left( \sum_{n=1}^N |a_n|^2 \right)^2 \right\}^{1/2} = (4\pi N)^k \sum_{n=1}^N |a_n|^2.$$

LEMMA 7. Let  $E$  denote the set of real numbers, considered modulo 1, for which

$$(38) \quad \Phi(z) - F(z) > 0.$$

Then

$$(39) \quad \delta \sum_r |S(x_r)|^2 \leq \sum_1^N |a_n|^2 + \int_E (\Phi(z) - F(z)) dz.$$

*Proof.* By Lemma 4 the left hand side is

$$\sum_r \int_{x_r - \delta/2}^{x_r + \delta/2} \left\{ |S(z)|^2 - \Phi(z) \right\} dz,$$

and by (2) the intervals of integration are disjoint. We have

$$\int_0^1 |S(z)|^2 dz = \sum_1^N |a_n|^2, \quad \int_0^1 \Phi(z) dz = 0,$$

the latter being a consequence of the definition of  $\Phi(z)$  in (33), since the terms with  $m = n$  in the double sum have coefficients 0. It follows that

$$\delta \sum_r |S(x_r)|^2 = \sum_1^N |a_n|^2 + \int_{E'} \{ \Phi(z) - F(z) \} dz,$$

where  $F(z) = |S(z)|^2$  as before, and where  $E'$  denotes the complement of the set of intervals  $(x_r - \delta/2, x_r + \delta/2)$ . The integral in the last expression can only be increased if we replace  $E'$  by  $E$ , since  $E$  comprises all  $z$  for which the integrand is positive.

### 7. Proof of the first part of Theorem 2.

If we represent numbers  $z$  by points on the circumference of a circle of perimeter 1, the set  $E$  of Lemma 7 consists of a finite number of open intervals. Each interval has length less than  $\delta$ , for by Lemma 4 it is impossible for (38) to hold throughout any interval of length  $\delta$ .

We divide the intervals of  $E$  into two types. The first type are those for which  $F'(z)$  does not vanish in the interval, and we denote the union of these by  $E_1$ . The second type are those for which  $F'(z)$  vanishes at some point of the (open) interval, and we denote the union of these by  $E_2$ .



Let  $I$  be one of the intervals of  $E_1$ . Since  $\Phi(z) - F(z) = 0$  at the end points of  $I$ , we have

$$\int_I \{\Phi(z) - F(z)\} dz = - \int_I (z - \gamma) \{\Phi'(z) - F'(z)\} dz$$

for any real number  $\gamma$ . Since  $F'(z)$  is of constant sign, we can choose  $\gamma$  in  $I$  so that

$$\int_I (z - \gamma) F'(z) dz = 0.$$

We now have

$$\left| \int_I \{\Phi(z) - F(z)\} dz \right| = \left| \int_I (z - \gamma) \Phi'(z) dz \right| \leq \delta \int_I |\Phi'(z)| dz.$$

Hence

$$\int_{E_1} \{\Phi(z) - F(z)\} dz \leq \delta \int_0^1 |\Phi'(z)| dz.$$

By lemmas 5 and 6, the right hand side is

$$\begin{aligned} &\leq \delta \sum_{k=1}^{\infty} c_k \delta^{2k} \int_0^1 |F^{(2k+1)}(z)| dz \\ (40) \quad &\leq \left\{ \sum_{k=1}^{\infty} c_k (4\pi N\delta)^{2k+1} \right\} \left\{ \sum_1^N |a_n|^2 \right\}. \end{aligned}$$

We now turn to the set  $E_2$ . Since  $F(z) \geq 0$  always, we have

$$\int_{E_2} \{\Phi(z) - F(z)\} dz \leq \int_{E_2} \Phi(z) dz.$$

By Lemmas 5 and 6, this is

$$\begin{aligned} &\leq \frac{\delta^2}{24} \int_{E_2} F''(z) dz + \sum_{k=2}^{\infty} c_k \delta^{2k} \int_{E_2} |F^{(2k)}(z)| dz \\ (41) \quad &\leq \frac{\delta^2}{24} \int_{E_2} F''(z) dz + \left\{ \sum_{k=2}^{\infty} c_k (4\pi N\delta)^{2k} \right\} \sum_1^N |a_n|^2. \end{aligned}$$

It remains to estimate the integral over  $E_2$  of  $F''(z)$ . Suppose first that  $E_2$  consists of only one or two intervals; as noted earlier, each has length less than  $\delta$ . Thus in this case

$$\int_{E_2} F''(z) dz \leq 2\delta \max |F''(z)|.$$

Now

$$F''(z) = - (2\pi)^2 \sum_{m=1}^N \sum_{n=1}^N a_m \bar{a}_n (m-n)^2 e((m-n)z),$$

and therefore

$$|F''(z)| \leq (2\pi)^2 N^2 \left( \sum_{n=1}^N |a_n| \right)^2 \leq (2\pi)^2 N^3 \sum_1^N |a_n|^2.$$

Hence, in the present case,

$$(4.2) \quad \int_{E_2} F''(z) dz \leq 8\pi^2 N^3 \delta \sum_1^N |a_n|^2.$$

Now suppose that  $E_2$  consists of at least 3 intervals, say  $(\alpha_1, \beta_1), \dots, (\alpha_s, \beta_s)$ .

By the definition of  $E_2$ , each interval  $(\alpha_j, \beta_j)$  contains some  $\xi_j$  for which

$$F'(\xi_j) = 0.$$

By Rolle's theorem, there exists some  $\eta_j$  between  $\xi_j$  and  $\xi_{j+1}$  for which

$$F''(\eta_j) = 0.$$

We make the obvious convention that  $\xi_{s+1} = \xi_1$ , and so on.

We have

$$\int_{\alpha_j}^{\beta_j} F''(z) dz = \int_{\alpha_j}^{\beta_j} \left\{ \int_{\eta_{j-1}}^z F'''(y) dy \right\} dz.$$

Both  $\eta_{j-1}$  and  $z$  are contained in the interval  $(\xi_{j-1}, \xi_{j+1})$ . Hence, since  $\beta_j - \alpha_j < \delta$ ,

$$\int_{\alpha_j}^{\beta_j} F''(z) dz \leq \delta \int_{\xi_{j-1}}^{\xi_{j+1}} |F'''(y)| dy.$$

The intervals  $(\xi_{j-1}, \xi_{j+1})$  cover the whole interval of length 1 twice. Hence

$$(43) \quad \int_{E_2} F''(z) dz \leq 2\delta \int_0^1 |F'''(y)| dy \leq 2\delta (4\pi N\delta)^3 \sum_1^N |a_n|^2$$

by Lemma 6. Comparing this with (42), we see that (43) is valid in both the two cases.

It follows from (41) that

$$\int_{E_2} \{\Phi(z) - F(z)\} dz \leq \left\{ \frac{(4\pi N\delta)^3}{12} + \sum_{k=2}^{\infty} c_k (4\pi N\delta)^{2k} \right\} \sum_1^N |a_n|^2.$$

Adding to this the estimate in (40) for the integral over  $E_1$ , and substituting in (39), we obtain

$$\sum_r |S(x_r)|^2 \leq \delta^{-1} (1 + U) \sum_1^N |a_n|,$$

where

$$U = \sum_{k=1}^{\infty} c_k (4\pi N\delta)^{2k+1} + \frac{(4\pi N\delta)^3}{12} + \sum_{k=2}^{\infty} c_k (4\pi N\delta)^{2k}.$$

Since  $c_1 = \frac{1}{24}$ , this implies

$$U \leq \frac{1}{8} (4\pi N\delta)^3 + 2 \sum_{k=2}^{\infty} c_k (4\pi N\delta)^{2k}.$$

By (36) with  $\theta = \pi$ , we have

$$\sum_{k=2}^{\infty} c_k \pi^{2k} = \frac{\pi}{2} - 1 - \frac{1}{24} \pi^2 < 0.16.$$

Hence, for  $N\delta \leq \frac{1}{4}$ ,

$$U \leq \frac{1}{8} (4\pi N\delta)^3 + 2 (4N\delta)^4 (0.16) < 270 (N\delta)^3.$$

This proves (6).

8. Proof of the second part of Theorem 2.

We take  $N = 2$ , and

$$S(x) = 1 + e^{2\pi i x},$$

so that  $|S(x)|^2 = 4 \cos^2 \pi x$ . We have

$$\sum_{n=1}^N |a_n|^2 = 2.$$

We take the points  $x_r$  at

$$0, \pm \delta, \pm 2\delta, \dots, \pm m\delta$$

where

$$(2m + 1 + \zeta) \delta = 1,$$

and  $0 < \zeta < 1$ . Note that the gap (mod 1) between  $m\delta$  and  $-m\delta$  is  $1 - 2m\delta > \delta$ .

We have

$$\begin{aligned} \sum_r |S(x_r)|^2 &= 4 \sum_{r=-m}^m \cos^2 \pi r \delta = 2 \sum_{r=-m}^m (1 + \cos 2\pi r \delta) \\ &= 2 \left( 2m + 1 + \frac{\sin (2m + 1) \pi \delta}{\sin \pi \delta} \right). \end{aligned}$$

Now  $(2m + 1) \delta = 1 - \zeta \delta$ , so  $\sin (2m + 1) \pi \delta = \sin \pi \zeta \delta$ .

Hence

$$\sum_r |S(x_r)|^2 = 2 \left( 2m + 1 + \zeta + \left( \frac{\sin \pi \zeta \delta}{\sin \pi \delta} - \zeta \right) \right) = 2\delta^{-1} (1 + V),$$

where

$$V = \delta \left( \frac{\sin \pi \zeta \delta}{\sin \pi \delta} - \zeta \right).$$

For fixed  $\zeta$ , as  $\delta \rightarrow 0$ , we have

$$V \sim \frac{1}{6} \pi^2 \zeta (1 - \zeta^2) \delta^3,$$

and on taking  $\zeta = 1/\sqrt{3}$  we get

$$V \sim \frac{\pi^2}{9\sqrt{3}} \delta^3 = \frac{\pi^2}{72\sqrt{3}} N^3 \delta^3.$$

Since  $\frac{\pi^2}{72\sqrt{3}} > \frac{1}{12}$ , this example satisfies (7).