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A CLASS OF RINGS WHICH ARE THE ENDOMORPHISM RINGS OF SOME TORSION-FREE ABELIAN GROUPS

ADALBERTO ORSATTI (*)

Introduction.

A well known result of A. L. S. Corner ([2], Theorem A) states that every countable, reduced, torsion-free ring A is isomorphic with the endomorphism ring $E(G)$ of some countable, reduced, torsion-free group G. (A ring is called reduced and torsion-free if such is its additive group).

In this paper we establish a similar result for a wider class of rings, precisely for the class Δ consisting of *locally countable, reduced, torsion-free* rings.

We say that a torsion-free ring A is locally countable if for every prime number p not dividing A (i. e. $pA \neq A$) the ring $A/p^{\infty} A$ is countable, where $p^{\infty} A$ is the intersection of the ideals $p^n A$ for every natural number n. (Observe that this definition involves only the additive structure of A).

The rings of class Δ are characterized as follows (see Proposition 1). A ring A belongs to Δ if and only if A is isomorphic with a pure subring of a direct product $\prod_p R_p$, $p \in P^*$, where P^* is any given set of distinct prime numbers and R_p a countable, reduced, torsion-free Z_p-algebra. ($Z_p =$ ring of rationals whose denominators are prime to p).

The following generalization of Theorem A is proved:

THEOREM A*. *Let A be a locally countable, reduced, torsion-free ring. Then there exists a locally countable, reduced, torsion-free group G, of the same cardinal as A, whose endomorphism ring $E(G)$ is isomorphic with A.*

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The proof of this theorem relies on Corner's methods and ideas, but does not make use of Theorem A : in fact it is obtained by modifying the proof of Theorem A by means of some localization and globalization techniques as exposed in [5].

It is clear from the above characterization that every ring of class Δ is of cardinal $\leq 2^{\aleph_0}$, so that one might wonder if every such ring belongs to the class of the endomorphism rings of *countable,* reduced, torsion-free groups. This last class of rings has been characterized by Corner himself in Theorem 1.1 of [3]. The answer is negative, as proved in Proposition 2, where we construct a ring $A \in \Delta$ with the following properties : $|A| = 2^{\aleph_0}$ and if $A = E(G)$, with G a reduced torsion-free group, then $|G| \geq 2^{\aleph_0}$.

It is still an open question if the rings of class Δ satisfy the hypotheses of Theorem 2.2 of [3], which generalizes Theorem 1.1.

1. Preliminaries.

All groups considered in this paper are abelian and additively written; rings are associative and with an identity, modules are unitary. If $f: B \longrightarrow A$ is a ring homomorphism, f always maps the identity of B into the identity of A : if B is a subring of A, B contains the identity of A. We will regard sometimes the ring possessing only the zero element as a ring with an identity.

Let $\{H_i\}$ be an indexed family of groups (rings). We denote by $\prod_i H_i$ the direct product (= cartesian product) of the H_i and, for every $x \in \prod_i H_i$, by x_i the i-component of x ($x_i \in H_i$). $\sum_i H_i$ will be the subgroup (ideal) of $\prod_i H_i$ consisting of those elements whose components are almost all zero.

We often attribute to a ring some properties of its additive group; for istance we say that the ring A is reduced, torsion-free, etc... ; or that a subring of A is pure in A. By a subgroup of A we mean a subgroup of the additive group of A.

Let N be the set of positive integers and P the set of prime numbers ($P \subset N$). For every group (ring) H and for every $p \in P$, $p^\infty H$ will denote the intersection of all subgroups (ideals) $p^n H$, $n \in N$.

Every group (ring) is a topological group (ring) in the *natural topology* obtained by taking the subgroups (ideals) nH, $n \in N$, as a basis of neighbourhoods of 0. This topology will be our main tool; for its principal properties see [2]. We recall here some of them. Let H be a reduced torsion-free group (ring): then H is Hausdorff in the natural topology. Let L be a subgroup (subring) of H and endow H with the natural topology.

If L is pure in H, then the natural topology of L coincides with the relative topology. L is dense in H if and only if the group H/L is divisible. If H is divisible by every positive integer prime to some fixed $p \in P$, then the natural topology of H coincides with the p-adic topology. The (group or ring) homomorphisms are uniformly continuous mappings with respect to the natural topologies of the corresponding structures.

Z will denote the ring of integers, $Z_p\,(p \in P)$ the ring of rationals whose denominators are prime to p, $\widehat{Z}_p$ the ring of p-adic integers. Maps are written on the left.

The group theoretical terminology is that of Fuchs's book [4].

Let A be a reduced torsion-free ring. For every $p \in P$, consider the ring $A_p^* = (A/p^\infty A) \otimes Z_p$ (tensor product of Z-algebras) and the ring homomorphism $\varphi_p : A \longrightarrow A_p^*$ resultant of the canonical maps $A \longrightarrow A/p^\infty A$ and $A/p^\infty A \longrightarrow A_p^*$. $A/p^\infty A$ is torsion-free and without elements $(\neq 0)$ of infinite p-height; then the map $A/p^\infty A \longrightarrow A_p^*$ is injective — so that the kernel of φ_p is $p^\infty A$ — and A_p^* is a reduced torsion-free Z_p-algebra. Define $A^* = \prod_p A_p^*,\, p \in P$, and let $\varphi : A \longrightarrow A^*$ be the canonical map given by

$$\varphi(a)_p = \varphi_p(a) \qquad\qquad (a \in A,\ p \in P)$$

where $\varphi(a)_p$ is the p-component of $\varphi(a)$. Then A^* is a reduced torsion-free ring and φ is a ring homomorphism.

In [5] we defined for every group G the groups $G_p^* = (G/p^\infty G) \otimes Z_p$, $G^* = \prod_p G_p^*,\, p \in P$, and the canonical homomorphisms $G \longrightarrow G_p^*$, $G \longrightarrow G^*$. G_p^* and G^* were called respectively the *Hausdorff p localization* and the *natural pre-completion* of G. This terminology will be used also for A.

From the embedding lemma of [5] we obtain the following

LEMMA 1. *Let A be a reduced torsion-free ring. Then the canonical homomorphism $\varphi : A \longrightarrow A^*$ is injective; $\varphi(A)$ is a pure subring of A^*; the group $A^*/\varphi(A)$ is divisible, i. e. $\varphi(A)$ is dense in A^* endowed with the natural topology.*

Denote by A^p the image of $\varphi(A)$ under the canonical projection $A^* \longrightarrow A_p^*$; A^p will be called the *p-proiection* of A.

From the definition of φ we get $A^p = \varphi_p(A)$. Since $A_p^*/\varphi_p(A)$ is a divisible torsion group with trivial p-primary component ([5], pag. 5) and A_p^* is torsion-free, we have

LEMMA 2. *Let A be a reduced torsion-free ring. Then A^p is a p-pure subring of A_p^*; the pure subring (subgroup) of A_p^* generated by A^p coincides with A_p^*; A^p is dense in A_p^* endowed with the natural topology.*

The natural topology of A_p^* coincides with the p-adic topology. Let $\widehat{A_p^*}$ be the natural ($=p$-adic) completion of A_p^*; then $\widehat{A_p^*}$ is a reduced torsion-free ring which contains A_p^* as a pure and dense subring ([2], Lemma 1.4). Extending by continuity the Z_p-algebra structure of A_p^*, $\widehat{A_p^*}$ becomes a $\widehat{Z_p}$-algebra. Moreover $\widehat{A_p^*}$ is torsion-free over $\widehat{Z_p}$, otherwise the additive group of $\widehat{A_p^*}$ would contain some cyclic p-group.

A^* is a pure and dense subring of $\prod_p \widehat{A_p^*}, p \in P$, which is complete in the natural topology, as easily verified. By means of the injection φ, we identify A with $\varphi(A)$: by Lemma 1, A becomes a pure and dense subring of A^*. Then the natural completion $\widehat{A}$ of A coincides with $\widehat{A^*}$, hence with $\prod_p \widehat{A_p^*}$. (See [5], P. 5. and Teorema 1). Now the following pure and dense inclusions hold:

$$A_p^* \subseteq \widehat{A_p^*};\ A \subseteq A^* = \prod_p A_p^* \subseteq \prod_p \widehat{A_p^*} = \widehat{A}. \tag{1}$$

Let $\widehat{Z} = \prod_p \widehat{Z_p}$, $p \in P$, be the natural completion of Z and identify Z with the (pure and dense) subring of $\widehat{Z}$ generated by the identity of $\widehat{Z}$. Extending by continuity the obvious Z-algebra structure of A, $\widehat{A}$ becomes a $\widehat{Z}$-algebra. The product of an element $\pi \in \widehat{Z}$ by an element $a \in \widehat{A}$ is given by the following relations on p-components:

$$(\pi a)_p = \pi_p a_p \qquad (p \in P, \pi_p \in \widehat{Z_p}, a_p \in \widehat{A_p^*}). \tag{2}$$

This is an immediate consequence of the principle of the extension of identities, [1], because (2) holds for $\pi \in Z$ and $a \in A$.

We conclude this section with the following remark.

LEMMA 3. *Let L be a p-pure subgroup of the reduced torsion-free Z_p-module H. Then the group $L \otimes Z_p$ is canonically isomorphic with the pure subgroup of H generated by L.*

PROOF. Let L_p be the pure subgroup of H generated by L. L_p/L is a divisible torsion group with trivial p-primary component. Then the canonical isomorphism is obtained from the exact sequence $0 \to L \to L_p \to L_p/L \to 0$ (where the maps are the natural ones) by tensor multiplication by Z_p.

2. The proof of Theorema A*.

Let $A\ (\neq 0)$ be a ring satisfying the hypotheses of Theorem A^*. Since A is reduced and torsion-free, the above remarks hold, in particular inclusions (1) are verified.

Let P^* be the non void subset of P consisting of the primes p such that $pA \neq A$. If $p \notin P^*$ we have $A_p^* = \widehat{A}_p^* = 0$, hence the p-component of every element of $\widehat{A}$ is zero and we may take $\widehat{A} = \prod_p \widehat{A}_p^*$, $p \in P^*$. Note that, if $p \in P^*$, A_p^* is countable.

The first part of the proof is a localization process: for every $p \in P^*$ we construct a countable pure subgroup G_p of $\widehat{A}_p^*$, $G_p \supset A_p^*$, whose endomorphism ring is isomorphic with A_p^*; in this part we will follow exactly, except for small details, the proof of Theorem A of [2].

For a given $p \in P^*$, choose in $\widehat{A}_p^*$ a maximal family $\{f_i, i \in I\}$ of elements of A_p^* linearly independent over $\widehat{Z}_p$. Then for every $v \in A_p^*$ there exist a non negative integer n_v and elements $\pi_v^i, i \in I$, of $\widehat{Z}_p$ such that, in $\widehat{A}_p^*$,

$$p^{n_v} v = \sum_i \pi_v^i f_i$$

where almost all the π_v^i vanish. If we take always the smallest possible n_v, then v uniquely determines the π_v^i, since $\widehat{A}_p^*$ is torsion-free over $\widehat{Z}_p$. Let Π_p be the pure subring of $\widehat{Z}_p$ generated by these $\pi_v^i\ (i \in I, v \in A_p^*)$. Since A_p^* is countable, so is Π_p. Moreover we have

Lemma 4. *If in* $\widehat{A}_p^*\ (p \in P^*)$:

$$\sum_{j=1}^{n} \gamma_j v_j = 0 \qquad (n \in N)$$

where the γ_j *are elements of* $\widehat{Z}_p$ *linearly independent over* Π_p *and the* $v_j \in A_p^*$, *then the* v_j *all vanish.*

The proof of this lemma is the same as the one of Lemma 2.1. of [2].

For every $v \in A_p^*$, choose two elements $\alpha_p(v), \beta_p(v) \in \widehat{Z}_p$ such that they all form a family which is algebraically independent over Π_p. This is possible because A_p^* is countable and $\widehat{Z}_p$ is of trascendence degree $2^{\aleph_0}$ over Π_p. Let A^p be the p-projection of $A : A^p \subseteq A_p^*$. For every $u \in A^p$ define the element $e_p(u) \in \widehat{A}_p^*$ by putting

$$e_p(u) = \alpha_p(u) 1_p + \beta_p(u) u \tag{3}$$

where 1_p is the identity of $\widehat{A}_p^*$, and let G_p be the pure subgroup of $\widehat{A}_p^*$ generated by A^p and by the subgroups $e_p(u) A^p$, $u \in A^p$, of $\widehat{A}_p^*$. G_p is a countable reduced torsion-free Z_p-module. Now G_p contains A_p^*: in fact A_p^* is pure in $\widehat{A}_p^*$ and so, by Lemma 2, A_p^* is the minimal pure subgroup of $\widehat{A}_p^*$ containing A^p. Moreover, for every $u \in A^p$, G_p contains the pure subgroup $H(u)$ of $\widehat{A}_p^*$ generated by $e_p(u) A^p$ and it is easily verified that $H(u)$ contains $e_p(u) A_p^*$. It is now clear that G_p coincides with the pure subgroup of $\widehat{A}_p^*$ generated by A_p^* and the $e_p(u) A_p^*$, $u \in A^p$. It follows that $G_p A_p^* = G_p$ and so every right multiplication in $\widehat{A}_p^*$ by an element of A_p^* induces an endomorphism on G_p. These multiplications are distinct because G_p contains the identity of A_p^*. We now prove that every endomorphism of G_p is obtained in this way. Let δ be an arbitrary endomorphism of G_p. Since G_p is pure and dense in $\widehat{A}_p^*$, the natural ($= p$-adic) completion of G_p coincides with the additive group of $\widehat{A}_p^*$. Consequently δ extends to a $\widehat{Z}_p$-endomorphism $\widehat{\delta}$ of $\widehat{A}_p^*$ ([2], Lemma 1.4). Let u be an arbitrary element of A^p and consider $\delta(e_p(u))$. We have by (3):

$$\delta(e_p(u)) = \widehat{\delta}(\alpha_p(u) 1_p + \beta_p(u) u) = \alpha_p(u) \delta(1_p) + \beta_p(u) \delta(u). \tag{4}$$

Now $\delta(e_p(u))$, $\delta(1_p)$, $\delta(u)$ are elements of G_p and so, by the definition of G_p, there exist $m, n \in N$ such that

$$\begin{cases} m\,\delta(e_p(u)) = b + \sum_{i=1}^{n} e_p(u_i)\, b_i \\ m\,\delta(1_p) = c + \sum_{i=1}^{n} e_p(u_i)\, c_i \\ m\,\delta(u) = d + \sum_{i=1}^{n} e_p(u_i)\, d_i \end{cases} \tag{5}$$

where the u_i are distinct elements of A^p, $u_1 = u$ and $b, c, d, b_i, c_i, d_i \in A_p^*$.

Substituting from (5) in (4) we obtain

$$b + \sum_{i=1}^{n} (\alpha_p(u_i)\,1_p + \beta_p(u_i)\,u_i)\,b_i =$$

$$= \alpha_p(u)\left[c + \sum_{i=1}^{n} (\alpha_p(u_i)\,1_p + \beta_p(u_i)\,u_i)\,c_i\right] + \beta_p(u)\left[d + \sum_{i=1}^{n} (\alpha_p(u_i)\,1_p + \beta_p(u_i)\,u_i)\,d_i\right].$$

As the $a_p(u_i), \beta_p(u_i)$ are algebraically independent over Π_p, from Lemma 4 we get

$$\text{(6)} \qquad b_1 = c, \quad ub_1 = d$$

while the other b's, c's and d's all vanish. By the last two of (5) and by (6) we have:

$$m\delta(1_p) = c, \qquad m\delta(u) = uc.$$

Since A_p^* is pure in G_p, it follows $\delta(1_p) \in A_p^*$, and since G_p is torsion-free, $\delta(u) = u\delta(1_p)$. So we see that δ coincides on A^p with the right multiplication by $\delta(1_p) \in A_p^*$. Now by Lemma 2 A^p is dense in A_p^*, and A_p^* is dense in G_p for G_p is pure in $\widehat{A_p^*}$. It follows that A^p is dense in G_p endowed with the natural topology. Then, by the principle of the extension of identities, δ coincides with the right multiplication by $\delta(1_p)$ on the whole of G_p.

The first part of the proof is now complete.

The second part consists in constructing a group G with the required properties by means of the G_p, $p \in P^*$, using a local-global argument.

For every $a \in A$ consider the elements $\alpha(a), \beta(a) \in \widehat{Z}$ defined as follows:

$$\text{(7)} \qquad \begin{cases} \alpha(a)_p = \alpha_p(a_p), & \beta(a)_p = \beta_p(a_p) \qquad \text{if } p \in P^* \\ \alpha(a)_p = \beta(a)_p = 0 & \qquad\qquad\qquad\quad \text{if } p \notin P^*. \end{cases}$$

Note that a_p, being the p-component of a, belongs to A^p. Define the elements $e(a) \in \widehat{A}$ by putting

$$e(a) = \alpha(a)\,1 + \beta(a)\,a \qquad (a \in A)$$

where 1 is the identity of $\widehat{A}$. Every element of $\widehat{A}$ is determined by its p-components with $p \in P^*$; by (2), (3) and (7) we have, for every $p \in P^*$ and $a \in A$:

$$\text{(8)} \qquad e(a)_p = \alpha_p(a_p)\,1_p + \beta_p(a_p)\,a_p = e_p(a_p).$$

Let G be the pure subgroup of $\widehat{A}$ generated by A and the $e(a)A$, $a \in A$. G is reduced torsion-free and of the same cardinal as A. In $\widehat{A}$ we have $GA = G$, so that every right multiplication by an element of A induces an endomorphism on G; these multiplications are distinct because G contains the identity of A. In order to complete the proof of the theorem, it suffices to show that every endomorphism of G is obtained in this way and G is locally countable.

For this purpose, let us determine the Hausdorff p-localizations G_p^*, $p \in P$, and the natural pre-completion G^* of G. Since G is pure in $\widehat{A}$ we have for every $p \in P$

$$p^\infty G = (p^\infty \widehat{A}) \cap G. \tag{9}$$

If $p \notin P^*$, then $p^\infty \widehat{A} = \widehat{A}$, hence $p^\infty G = G$ and $G_p^* = 0$.

Suppose then $p \in P^*$ and let ε_p be the canonical projection of $\widehat{A}$ onto $\widehat{A}_p^*$: ε_p maps every element of $\widehat{A}$ into its p-component. By (9), $p^\infty G$ consists of the elements of G whose p-component is zero; consequently $G/p^\infty G$ is canonically isomorphic with $\varepsilon_p(G)$ which, as easily verified, is p-pure in $\widehat{A}_p^*$. Hence, by Lemma 3, we identify G_p^* with the pure subgroup of $\widehat{A}_p^*$ generated by $\varepsilon_p(G)$. Next we prove that $G_p^* = G_p$ ($p \in P^*$), from which it follows that G is locally countable, because G_p is countable. G_p^* contains $\varepsilon_p(G)$ which, by the definition of G, contains $A^p = \varepsilon_p(A)$ and $e_p(a_p)A^p$ for every $a \in A$. But, when a runs over A, a_p exhausts A^p; hence, by the definition of G_p, $G_p^* \supseteq G_p$. On the other hand a straightforward calculation shows that $\varepsilon_p(G) \subseteq G_p$; this implies $G_p^* \subseteq G_p$ and so $G_p^* = G_p$.

From the above remarks it follows:

$$G^* = \prod_p G_p, \ p \in P^*; \qquad G \subseteq G^* \subset \widehat{A}.$$

Now, let ω be an arbitrary endomorphism of G. By P. 2. of [5] ω extends uniquely to an endomorphism ω^* of G^*. (The proof of P. 2. suggests a way for constructing ω^*). Observe that if p and q are distinct primes of P^*, Hom $(G_p, G_q) = 0$ because G_p is a Z_p-module, G_q is a Z_q-module and both are reduced and torsion-free. Recalling how we obtained the endomorphisms of G_p, we see that every endomorphism of $G^* = \prod_p G_p$, $p \in P^*$, is induced by a right multiplication in $\widehat{A}$ by an element of $A^* = \prod_p A_p^*$, $p \in P^*$. Then ω coincides on G with the right multiplication by $\omega^*(1) = \omega(1) \in A^* \cap G$. If we can prove that $\omega(1) \in A$, the conclusion is reached. In fact, we will prove that $A^* \cap G = A$ [1]. It is clear that $A^* \cap G \supseteq A$; conversely, if $g \in G \cap A^*$, then $mg = b + \sum_{i=1}^{n} e(a_i) b_i$ with $m, n \in N, b, a_i, b_i \in A$, and $\sum_{i=1}^{n} e(a_i) b_i = mg - b = c \in A^*$. As for the p-components, for every $p \in P^*$, we obtain by (8)

$$\sum_{i=1}^{n} (\alpha_p(a_{ip}) 1_p + \beta_p(a_{ip}) a_{ip}) b_{ip} = c_p .$$

But, as a_{ip}, b_{ip}, c_p belong to A_p^*, it follows from Lemma 4 that $c_p = 0$ for every $p \in P^*$. This means $c = 0$ and so $mg = b \in A$. Hence $g \in A$, because A is pure in G.

3. The rings of class Δ and an example.

Let Δ be the class of the rings satisfying the hypotheses of Theorem A*. Δ may be characterized as follows.

PROPOSITION 1. *A ring A belongs to Δ if and only if A is isomorphic with a pure subring of a ring of type $R = \prod_p R_p$, $p \in P^*$, where P^* is a set of distinct primes and, for every $p \in P^*$, R_p is a coutable reduced torsion-free Z_p-algebra.*

PROOF. The necessity follows immediately from Lemma 1. Let A be a pure subring of R: A is reduced and torsion-free. We have $A/p^\infty A = A/(p^\infty R) \cap A$ for every $p \in P$ and $A \neq pA$ if and only if $p \in P^*$. For every $p \in P^*$, $A/p^\infty A$ is isomorphic with a p-pure subring of R_p and is countable.

Finally we show that the class Δ is not contained in the class of the endomorphism rings of countable reduced torsion-free groups; these rings are characterized by Theorem 1.1. of [3].

PROPOSITION 2. *There exists in Δ a ring A of cardinal $2^{\aleph_0}$ such that every reduced torsion-free group, whose endomorphism ring is isomorphic with A, is of cardinal $\geq 2^{\aleph_0}$.*

PROOF. For every $p \in P$, let R_p be a countable pure subring of $\widehat{Z}_p$ of rank > 1. R_p properly contains Z_p as a pure and dense subring. Define

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[1] I am indebted to F. Menegazzo for this suggestion.

$R = \prod_p R_p$, $p \in P$, and consider the subring A of R:

$$A = \{\alpha \in R \mid \alpha_p \in Z_p \text{ for almost all } p\}$$

where, as usual, α_p is the p-component of α. We have the proper inclusions

$$\sum_p R_p \subset A \subset \prod_p R_p \qquad (p \in P);$$

it is easily verified that A is pure in R so that, by Proposition 1, $A \in \Delta$; A is of cardinal $2^{\aleph_0}$.

Let G be reduced torsion-free group such that $E(G) = A$. For every $p \in P$ consider the element $\varepsilon_p \in A$ such that the p-component of ε_p is the identity 1_p of R_p whereas the other components all vanish. As ε_p is an idempotent element of A, G splits into the direct sum of the endomorphic images $\varepsilon_p(G)$ and $(1 - \varepsilon_p)(G)$, where 1 is the identity of A. Let G_p be the subgroup of G consisting of those elements which are divisible by every prime different from p; G_p is a reduced torsion-free Z_p-module. Since ε_p is divisible in (R and hence in) A by every prime different from p, while $1 - \varepsilon_p$ is divisible by every power of p, we have $\varepsilon_p(G) \subseteq G_p$ and $(1 - \varepsilon_p)(G) \subseteq p^\infty G$. On the other hand $G_p \cap p^\infty G = 0$ because G is reduced and torsion-free. Hence $\varepsilon_p(G) = G_p$, $(1 - \varepsilon_p)(G) = p^\infty G$ and

$$G = G_p \oplus p^\infty G. \tag{10}$$

We now show that the endomorphism ring $E(G_p)$ of G_p is isomorphic with R_p. By the direct decomposition (10), every endomorphism β of G_p extends to an endomorphism $\bar{\beta}$ of G such that $\bar{\beta}(G) \subseteq G_p$, $\bar{\beta}(p^\infty G) = 0$. Since ε_p induces the identity on G_p, we have $\bar{\beta} = \varepsilon_p \bar{\beta} \in \varepsilon_p A$. Conversely, every element of $\varepsilon_p A$ induces an endomorphism on G_p and vanishes on $p^\infty G$. It follows that $E(G_p)$ is isomorphic with the ring $\varepsilon_p A$, hence with R_p. Every non trivial endomorphism of G_p is injective: in fact, since G_p is in a natural way an R_p-module, $\widehat{G_p}$ is a module over $\widehat{R_p} = \widehat{Z_p}$; since G_p is a torsion-free group, $\widehat{G_p}$ is torsion-free over $\widehat{Z_p}$; then G_p is torsion-free over R_p.

It is clear that, for every $p \in P$, G_p coincides with the Hausdorff p-localization G_p^* of G and ε_p coincides with the canonical homomorphism $G \to G_p^*$. By means of the ε_p, $p \in P$, we construct the canonical homomorphism ε of G in its natural pre-completion $G^* = \prod_p G_p$ and identify G with the pure and dense subgroup $\varepsilon(G)$ of G^*. Since, if p and q are distinct primes, Hom $(G_p, G_q) = 0$, the endomorphism ring of $\prod_p G_p$ is $\prod_p R_p$. As $A \subset \prod_p R_p$, every endomorphism of G extends to an endomorphism of $\prod_p G_p$. Consequently the effect of $\alpha \in A$ on $g \in G$ is described by the following formulae on the p-components:

$$\alpha\,(g)_p = \alpha_p\,(g_p) \qquad (p \in P). \tag{11}$$

It is easily verified that G contains $\sum_p G_p$. Moreover this inclusion is proper since the endomorphism ring of $\sum_p G_p$ is $\prod_p R_p$, while $E(G) = A$ which is not isomorphic with $\prod_p R_p$. Then we can find and element $\bar{g} \in G$ and an infinite subset $\bar{P}$ of P such that $\bar{g}_p \neq 0$ for overy $p \in \bar{P}$. Let $\bar{A}$ be the ideal of A consisting of all $\alpha \in A$ such that $\alpha_p = 0$ if $p \notin \bar{P}$. $|\bar{A}| = 2^{\aleph_0}$ because $A \supset \prod_p Z_p, p \in \bar{P}$. Consider the additive homomorphism $\gamma : \bar{A} \to G$ mapping $\alpha \in \bar{A}$ into $\alpha(\bar{g}) \in G$. If $\alpha \in \bar{A}, \alpha \neq 0$, there exists $p \in \bar{P}$ such that $\alpha_p \neq 0$; since every non trivial endomorphism of G_p is injective $\alpha_p(\bar{g}_p) \neq 0$; by (11) this implies $\alpha(\bar{g}) \neq 0$, i. e. γ is injective. Hence $|G| \geq 2^{\aleph_0}$.

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