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THE GEGENBAUER FUNCTION

by W. R. SCHUCANY (*)

In the notes of the American Mathematical Monthly, [1] G. G. Bilodeau gives a new expression for the Gegenbauer polynomials by employing the concept of a fractional derivative. It is interesting to note that by employing the results obtained in [3], one can obtain a single formula for both the classical polynomials and the Gegenbauer functions \((C_p^n)\) as defined by Bateman [2].

The only definition necessary for clarity is the following:

\[
\int_a^x (x - t)^{n-a-1} f(t) \, dt \in C^n \quad \text{on} \quad (a, b),
\]

where \(n\) is a positive integer, then the Holmgren-Riesz transform of \(f\) is given by

\[
\frac{x}{a} \int_a^x f(x) = aD_x^nf(x) = \frac{D_x^n}{\Gamma(n - a)} \int_a^x (x - t)^{n-a-1} f(t) \, dt.
\]

Now one solution of the equation

(1) \((1 - x^2)y'' - (2\mu + 1) xy' + \beta (2 + 2\mu) y = 0,\)

may be written in its \(H - R\) transform form as

(2) \(y = k (1 - x^2)^{-\mu+1/2} D_x^\beta (1 - x^2)^{\beta+\mu-1/2}, \quad \beta > 0 \quad \text{and} \quad x > 1,\)

(See [3]).

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If the arbitrary constant $k$ equals

\[ \frac{(-1)^\beta \Gamma(\mu + 1/2) \Gamma(\beta + 2\mu)}{2^\beta \Gamma(\beta + 1) \Gamma(2\mu) \Gamma(\mu + \beta + 1/2)}, \]

then when $\beta = n$, equation (2) becomes the standard Rodrigues formula for the Gegenbauer polynomials. Also for positive non-integer values of $\beta$, $\mu > -1/2$ and $\mu \neq 0$, equation (2) can be shown to be equivalent to

\[ y = C_\beta^n(x) = \frac{\Gamma(\beta + 2\mu)}{\Gamma(\beta + 1) \Gamma(2\mu)} F[\beta + 2\mu, -\beta; \mu + 1/2; \frac{1-x}{2}] \]

for $x \in (1, 3)$, where $F[a, b; c; z]$ is the Hypergeometric function. (See [2]).

Consider (2) when $\beta = m$, an integer, and note that when $n = |[\beta]|$,

\[ 1D_\beta^x (1 - x)^{\beta + \mu - 1/2} = \frac{D_n^x}{\Gamma(n - \beta)} \int_1^x (x - t)^{n - \beta - 1} (1 - t)^{\beta + \mu - 1/2} \, dt. \]

Letting $z = \frac{t - 1}{x - 1}$, the right hand side of the above equation becomes,

\[ \frac{D_n^x}{\Gamma(n - \beta)} \int_0^1 [x - 1 + z(1 - x)]^{n - \beta - 1} [z(1 - x)]^{\beta + \mu - 1/2} [2 - z(1 - x)]^{\beta + \mu - 1/2} (x - 1) \, dz \]

\[ = \frac{(-1)^{n - \beta} 2^{\beta + \mu - 1/2}}{\Gamma(n - \beta)} D_\beta^n (1 - x)^{n + \mu - 1/2} \int_0^1 (1 - z)^{n + \beta - 1} z^{\beta + \mu - 1/2} \cdot \left[1 - z \left(\frac{1-x}{2}\right)\right]^{\beta + \mu - 1/2} \, dz. \]

Then setting $w = \frac{1-x}{2}$ we have,

\[ 1D_\beta^x (1 - x)^{\beta + \mu - 1/2} = \frac{(-1)^\beta 2^{\beta + 2\mu - 1} \Gamma(\beta + \mu + 1/2)}{\Gamma(n + \mu + 1/2)} \cdot D_n^w [w^{n + \mu - 1/2} F[-\beta - \mu + 1/2, \beta + \mu + 1/2; n + \mu + 1/2; w]], \]

\[ = \frac{(-1)^\beta 2^{\beta + 2\mu - 1} \Gamma(\beta + \mu + 1/2)}{\Gamma(\mu + 1/2)} \left(\frac{1-x}{2}\right)^{n - 1/2} F[-\beta - \mu + 1/2, \beta + \mu + 1/2; \mu + 1/2; \frac{1-x}{2}]. \]
Now applying a linear transformation formula we get
\[ 1D_x^\beta \left( 1 - x^2 \right)^{\beta + \mu - 1/2} \]
\[ = \frac{(-1)^\beta 2^\beta \Gamma(\beta + \mu + 1/2)}{\Gamma(\mu + 1/2)} (1 - x^2)^{\mu - 1/2} F \left[ \beta + 2\mu, -\beta; \mu + 1/2; \frac{1 - x}{2} \right]. \]

Therefore
\[ \frac{(-1)^\beta \Gamma(\mu + 1/2) \Gamma(\beta + 2\mu)}{2^\beta \Gamma(\beta + 1) \Gamma(2\mu) \Gamma(\mu + \beta + 1/2)} (1 - x^2)^{-\mu + 1/2} 1D_x^\beta (1 - x^2)^{\beta + \mu - 1/2} \]
\[ = \frac{\Gamma(\beta + 2\mu)}{\Gamma(\beta + 1) \Gamma(2\mu)} F \left[ \beta + 2\mu, -\beta; \mu + 1/2; \frac{1 - x}{2} \right] = C_\mu (x). \]

Hence the single expression (2) with the given value of \( k \) is a valid formula for the Gegenbauer polynomials, and with the generality afforded by the \( H - R \) transform the expression may be interpreted as including the associated function as well.

REFERENCES

