C. Procesi

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NON COMMUTATIVE JACOBSON-RINGS

C. PROCESI (*)

The Hilbert-Nullstellensatz has been generalized in various steps first to commutative rings and then to finitely generated non commutative algebras with polynomial identities (see [3] for further reference). In this paper we want to present a very general theorem which includes the previous ones as particular cases and which cannot be improved at least in a certain direction as we show with a counterexample. For the proof of the theorem we make use of corollary 1.3 of [3]; the remaining sections of [3] are not used and are in fact generalized by our results. For the various definitions and a more complete Bibliography we refer the reader to [7] 2nd edit. and in particular to appendices A, B.

1. Preparatory material.

All the rings which will be considered in this paper are supposed to satisfy a proper polynomial identity (1), i. e. an identity $p(x_1, \dots, x_n)$ with coefficients $\alpha_1, \dots, \alpha_m$ such that $\alpha_i x = 0$ for all i implies $x = 0$. This hypothesis will remain valid without any further mention.

The basic structure theorem that we will use is the following due essentially to Posner [9]:

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(1) Or an identity with coefficients ± 1 if one prefers, this is not restrictive as can be proved that a ring satisfying a proper identity satisfies also an identity with coefficients ± 1. This has been proved by Amitsur and it is not yet published, but it is not too difficult to verify. Here we are mainly interested in the existence of the identity rather than its nature.

Theorem 1.1. If R is a prime ring then:

a) R is an order (left and right) in a simple ring Q with descending chain condition.

b) If Z is the center of Q then Q is finite dimensional over Z and $RZ = Q$.

c) If R is an algebra over a commutative ring A then also Q is an algebra over A. Moreover Q and R satisfy the same identities with coefficients in A.

We also recall that the polynomial $S_n(x_1, \dots, x_n) = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma x_{\sigma(1)} \dots x_{\sigma(n)}$ ($\mathcal{S}_n$ denoting the symmetric group on n elements) is called the standard identity in n variables, and if R is an order in a simple algebra of degree n^2 over its center then R satisfies S_{2n} and no polynomial identity of lower degree [7]. These facts will be used without any further reference.

2. Ring extensions.

If R is a subring of a ring S we will denote by $C_S(R)$ the centralizer of R in S, i.e. $C_S(R) = \{x \in S \mid rx = xr \text{ for every } r \in R\}$. If A and B are two subrings of a ring R and if $B \subset C_R(A)$ we will denote by AB the subring of R generated by A and B. If A and B contain 1 this last subring is the set of all elements of the form $\sum_i a_i b_i$, $a_i \in A$, $b_i \in B$. If $x_1, \dots, x_n \in C_R(A)$ we will denote by $A(x_1, \dots, x_n)$ the subring of R generated by A and the x_i's.

Definition 2.1. A ring S containing a subring R is said to be an *extension* of R if $S = RC_S(R)$. S is said to be a *finitely generated extension* of R if $S = R(x_1, \dots, x_n)$ with $x_i \in C_S(R)$.

The polynomial ring $R[t_1, \dots, t_k]$ over R in k variables is closely related to this concept, it is in fact a finitely generated extension of R, but it is not a suitable concept for our non commutative theory; instead we have replaced it with the above notion in which the elements x_i are not supposed to commute among themselves. Of course one might try to consider more general kinds of extensions in which the elements adjoined to the ring R are not supposed to commute with the elements of R, however it is very hard to relate properties of R with properties of a general overring S. In fact all our theorems fail to be true in the more general situation; this seems to depend very strongly on the fact that S might be prime while R might have nilpotent ideals. In the next few lemmas we collect some easy results which point out the usefulness of our concept of extension.

LEMMA 2.2. Assume that the ring S is an extension of the ring R. We have then:

a) If S is prime R is prime.

b) If I is a two sided ideal of R, $IS = SI = IC_S(R)$ is a two sided ideal of S.

Proof. *a*) if $x, y \in R$ and $xRy = 0$ we have $xSy = xRC_S(R)y = xRyC_S(R) = 0$, therefore as S is a prime ring either x or y is 0 and R is also a prime ring. *b*) $S = RC_S(R)$ therefore $IS = IRC_S(R) = C_S(R)I = SI$ which is enough to prove *b*).

LEMMA 2.3 (Amitsur) If R is a prime ring and c is a regular element of R then cR contains a two sided ideal $U \neq 0$.

Proof. Let $S_n(y_1, \dots, y_n)$ be the minimal standard identity satisfied by R. We can assume that $n > 2$ otherwise R is commutative and the lemma is trivial. We know that this identity is also the minimal standard identity of $Q(R)$, the ring of quotients of R. Now cR has $Q(R)$ as ring of quotients (2) therefore by the same remark cR cannot satisfy $S_{n-1}(y_1, \dots, y_{n-1})$. Consider the set $T = \{S_{n-1}(cr_1, \dots, cr_{n-1}) \mid r_i \in R\}$, $T \neq 0$ by what we have just observed, we claim that $0 \neq RTR \subset cR$. In fact if $r \in R$ we have $S_n(r, cr_1, \dots$ $\dots, cr_{n-1}) = 0$. If we expand this identity we obtain:

$$0 = rS_{n-1}(cr_1, \dots, cr_{n-1}) + \Sigma(-1)^i cr_i S_{n-1}(r, cr_1, \dots, \widehat{cr_i}, \dots, cr_{n-1})$$

therefore $rS_{n-1}(cr_1, \dots, cr_{n-1}) \in cR$ which implies $RTR \subset cR$. The fact that $RTR \neq 0$ is an easy consequence of the assumption that R is prime. This concludes the proof of the lemma, taking $U = RTR$.

LEMMA 2.4. Let S be a prime ring and an extension of a ring R. If $c \in R$ is regular in R, c is regular in S.

Proof. Let Q be the quotient ring of S and Z its center. We have $Q = ZS = ZC_S(R)R$ therefore Q is an extension of R. Now cR contains a two sided ideal U of R and so UQ is a two sided ideal of Q contained in cQ. As Q is simple we must have $cQ = Q$. This implies clearly that c is invertible in Q hence in particular regular in S.

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(2) This is a trivial property of left orders depending on the following identity $q^{-1}p = q^{-1}c^{-1}cp = (cq)^{-1}cp$.

3. Jacobson rings and Hilbert algebras.

DEFINITION 3.1. *a*) A ring R is called a *Jacobson ring* if every prime ideal P of R is an intersection of maximal ideals.

b) An algebra R over a field F is called a *Hilbert algebra* if for every maximal ideal M the algebra R/M is finite dimensional over F.

The interest of this definition lies in the strict relation between these concepts and the Hilbert Nullstellensatz, see [3] and [8].

The main theorem of this paper is a generalization of the following commutative theorem:

If R is a Jacobson ring then:

a) $R[x]$ is a Jacobson ring.

b) if M is a maximal ideal in $R[x]$ then $R \cap M$ is maximal in R and $[R[x]/M : R/R \cap M]$ is finite.

c) If R is also a Hilbert algebra then $R[x]$ is a Hilbert algebra.

Before we state and prove our main theorem we need one more general lemma.

LEMMA 3.2. *If R is a prime semisimple ring and $d \in R$ is a regular element of R there is a maximal ideal M such that $\bar{d}$ is invertible in R/M.*

Proof. Let $U \subset dR$ be a non zero two sided ideal of R. As R is semisimple there is a maximal ideal M such that $U \not\subset M$. In $\bar{R} = R/M$ we have that $\bar{U} \neq 0$ is a two sided ideal. Therefore $\bar{U} = \bar{R}$ as $\bar{R}$ is simple. Now $\bar{d}\bar{R} \supset \bar{U} = \bar{R}$ hence $\bar{d}$ is invertible.

Let now S be a prime ring extension of R. By what we have proved in section 2, R is a prime ring and every regular element of R is regular in S. Therefore if we denote by $Q(R)$ and $Q(S)$ the quotient rings of R and S we can extend the injection map $i: R \to S$ to a map $i^*: Q(R) \to Q(S)$ so that the diagram:

$$\begin{array}{ccc} R & \xrightarrow{i} & S \\ \downarrow & & \downarrow \\ Q(R) & \xrightarrow{i^*} & Q(S) \end{array}$$

in which the vertical maps are the natural imbeddings, is commutative. Moreover i^* is a monomorphism and $Q(S)$ is an extension of $Q(R)$. That i^* is a monomorphism is clear as $Q(R)$ is a simple ring, as for the second statement we have $Q(S) = ZS$ where Z is the center of $Q(S)$ so that

$Q(S) = ZRC_S(R) = Q(R)\, C_{Q(S)}(Q(R))$. We are now ready to state our main theorem.

THEOREM 3.3. If S is a finitely generated extension of a Jacobson ring R we have:

a) S is a Jacobson ring.

b) If M is a maximal ideal of S, then $R \cap M$ is a maximal ideal in R and S/M is of finite length over $R/R \cap M$.

If moreover R is a Hilbert algebra over a field F we also have:

c) S is a Hilbert algebra over F.

Proof. a) Let P be a prime ideal of S, we want to prove that $\bar{S} = S/P$ is semisimple. If we denote by $\bar{R}$ the image of R in $\bar{S}$ we have that $\bar{S}$ is a finitely generated prime extension of $\bar{R}$ and $\bar{R}$ is semisimple. *a*) will clearly follow from the more general result:

LEMMA 3.4. If S is a finitely generated prime extension of a semisimple ring R then S is semisimple.

Proof. Let $S = R(x_1, \dots, x_k)$, denote by $Q(R)$ and $Q(S)$ the quotient rings of R and S. By what we have previously remarked $Q(R) \subset Q(S)$ in a natural way. We can therefore consider the ring $T = Q(R)(x_1, \dots, x_k)$. T is an extension of $Q(R)$ and $S \subset T \subset Q(S)$. Therefore T is a prime ring. We claim that T is semisimple, in fact $Q(R)$ is a finite dimensional algebra over a field F, therefore T is a finitely generated algebra over F and the claim follows from corollary 1.3 of [3]. Let us now assume by contradiction that the Jacobson radical $J(S)$ of S is not 0 and let $0 \neq f \in J(S)$. $S \subset T$ and as T is semisimple there is a maximal ideal M of T with $f \notin M$. Consider the ring $\bar{T} = T/M$; we can identify $Q(R)$ with a subring of $\bar{T}$ and under this identification $\bar{T} = Q(R)(\bar{x}_1, \dots, \bar{x}_k)$ is an extension of $Q(R)$. Again by corollary 1.3 of [3] $\bar{T}$ is of finite length over $Q(R)$. Let F be the center of $Q(R)$ and C the centralizer of $Q(R)$ in T, we claim that C is a simple algebra over F and that $\bar{T} \cong Q(R) \otimes_F C$ under the obvious map. First of all let Z be the center of T; $F \subset Z$ and $Q(R) \otimes_F Z$ is a simple algebra, with center Z, isomorphic to $Q(R)Z$ under the obvious map. $Q(R)Z$ is then a central simple algebra over Z and its centralizer in T is exactly C, therefore C is a central simple algebra over Z and:

$$\bar{T} \cong Q(R)Z \otimes_Z C \cong (Q(R) \otimes_F Z) \otimes_Z C \cong Q(R) \otimes_F C.$$

From the fact that $\overline{T}$ is of finite length over $Q(R)$ we can deduce that the dimension of C over F is a finite number m. We have via the regular representation a homomorphism $\overline{\gamma}: C \longrightarrow F_m$ ($m \times m$ matrices over F); tensoring this with $1_{Q(R)}$ we have a map $\gamma: \overline{T} \cong Q(R) \otimes_F C \longrightarrow Q(R) \otimes_F F_m \cong Q(R)_m$. Moreover if $a \in Q(R)$, we have:

$$\gamma(a) = \begin{pmatrix} a & 0 & & \dots & 0 \\ 0 & a & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \dots & a \end{pmatrix}.$$

As $f \notin M$, $0 \neq \overline{f} \in \overline{T}$; $\overline{T}$ is a simple ring and so $\overline{T} \overline{f} \overline{T} = \overline{T}$. Consequently we can find elements $t_i, s_i \in \overline{T}$ such that $\Sigma_i s_i \overline{f} t_i = 1$. Let us consider the sequence of canonical maps:

$$R \longrightarrow S \longrightarrow T \longrightarrow \overline{T} \longrightarrow Q(R)_m$$

and let $\overline{S}$ be the image of S in $\overline{T}$. $0 \neq \overline{f} \in J(\overline{S})$ moreover $Q(R)\overline{S} = \overline{T}$, therefore we can find regular elements $d, e \in R$ such that $dt_i \in \overline{S}$ and $s_i e \in \overline{S}$. If we set $c = de$ we get $c = \sum_i dt_i \overline{f} s_i e \in J(\overline{S})$. Let $\widetilde{x}_i$ be the image of x_i in $Q(R)_m$, the $\widetilde{x}_i$ are matrices over $Q(R)$ hence we can find a regular element $b \in R$ such that $b\widetilde{x}_i \in R_m$ for all i (b is identified with the matriy $\gamma(b)$). Replacing b with a multiple gb if necessary we can assume furthermore that $b \in Rc \subset J(\overline{S})$. If $\widetilde{x}_{i_1} \dots \widetilde{x}_{i_r}$ is a monomial in the $\widetilde{x}_i$ we have

$$b^r \widetilde{x}_{i_1} \dots \widetilde{x}_{i_r} = b\widetilde{x}_{i_1} b\widetilde{x}_{i_2} \dots b\widetilde{x}_{i_r} \in R_m$$

as b commutes with the $\widetilde{x}_i$ and $b\widetilde{x}_i \in R_m$. By lemma 3.2 there is a maximal ideal M in R such that $\overline{b}$ is invertible in R/M. We claim that $MS \neq S$. If $MS = S$ we would have $1 = \sum_k m_k X_k$ where the X_k are monomials in the $\widetilde{x}_i$. Let u be an integer greater or equal than the degrees of the various X_k's, then $b^u = \Sigma m_k X_k b^u \in MR_m = M_m$. This is a contradiction since $\overline{b}$ is invertible in R/M. Thus MS is a proper ideal of S and as M is maximal in R, $M = MS \cap R$. In S/MS, $\overline{b}$ is invertible, but $\overline{b} \in J(S/MS)$ which is again a contradiction. So the lemma is proved.

b) We now assume that S is simple and we are going to show that R is also simple under the only assumption that R is semisimple. This will imply in particular the first half of *b*). As in *a*) we can construct a map $S \longrightarrow Q(R)_m$ now S is simple so this map is a monomorphism and $S = \overline{S}$. If we let b and M be as above we have $MS = 0$ because S is a simple ring; therefore $M = 0$ and R is also a simple ring. To finish the proof of *b*) we have to show that S is of finite length over R. Now R is a finite dimensional algebra over its center F and S is a finitely generated simple algebra over F therefore S is finite dimensional over F by corollary 1.3 of [3], and *b*) is completely proved.

c) This is now a triviality from *b*) and the definitions.

We now want to show with an example that the condition that the x_i's commute with R is really necessary.

Let $R = \left\{ \begin{pmatrix} n & m \\ 0 & n \end{pmatrix} / n \in Z,\ m \in Z_{(2)} ,\ \text{i. e. } Z \text{ localized at the prime ideal } (2) \right\}$, R is a commutative ring, $J(R)^2 = 0$ and $R/J(R) \cong Z$, therefore R is a Jacobson ring. Let $S = R(e_{2,1}) = (Z_{(2)})_2$ be the ring of 2×2 matrices over $Z_{(2)}$; clearly S is not a Jacobson ring; on the other hand $e_{2,1} \notin C_S(R)$.

Along the lines of the previous theorem we want to prove now a theorem of a more geometric interest.

THEOREM 3.5. Let R and S be as in theorem 3.3. We assume that the minimal primes of S (in the set of all primes) are in finite number (this condition is satisfied for instance if S is a finitely generated algebra over a field). Then there exists an element $c \in R$ such that c does not belong to any minimal prime of R and if P is a prime ideal in R with $c \notin P$, there is a prime ideal P' of S with $P' \cap R = P$.

(This result has the following geometric interpretation: in the hypothesis of the theorem Spec S is decomposed in a finite number of irreducible components. We have the map Spec $S \longrightarrow$ Spec R. Given an irreducible component V_i of Spec R there is an irreducible component V_i' of Spec S mapping generically onto V_i. In the particular hypothesis of the theorem, the element c that we find defines a closed set $V(c)$ of Spec R which does not contain any component and the theorem affirms that the image of the map Spec $S \longrightarrow$ Spec R contains the dense open set Spec $R - V(c)$).

Proof. Assume first of all that $S = R(x_1, \dots, x_k)$ is prime. We are going to imitate the reasoning in lemma 3.4. We know that in this case R is prime, $Q(R) \subset Q(S)$ in a natural way and $T = Q(R)(x_1, \dots, x_k)$ is semisimple. As in lemma 3.4, if M is a maximal ideal of T, T/M can be imbedded in the $m \times m$ matrices over $Q(R)$ in such a way that R goes into diagonal matrices. In the notation of lemma 3.4 let $b \in R$ be a regular element such that $b\widetilde{x}_i \in R_m$, let furthermore n be such that R satisfies the standard identity S_{2n} (such an n exists because R is prime). We claim that $c = b^n$ solves the problem. First of all $c \neq 0$ because b is regular, we have to prove the main assertion: if $P \subset R$ is a prime ideal and $c \notin P$ there is a prime ideal $P' \subset S$ such that $P' \cap R = P$. We prove this in two steps, the first step is to take a maximal ideal $M \subset R$ such that $c \notin M$ and try to find for this a prime ideal $P' \subset S$ with $P' \cap R = M$. We proceed as follows: MS is a two sided ideal of S, if $MS = S$ we have as in lemma 3.4 that for an integer t, $b^t \in MR_m$, therefore $b^t \in M$ and b is nilpotent modulo M. Now R/M is a simple algebra of dimension at most n^2 over its center (as it satisfies S_{2n}), therefore as b is nilpotent in R/M it must be nilpotent of index $\leq n$, this is clearly a contradiction because we were assuming that $b^n = c \notin M$. Now $MS \cap R \neq R$ and therefore $MS \cap R = M$, enlarging MS to a maximal ideal M' we still have $M' \cap R = M$ and therefore the contention is proved for maximal ideals. We pass now to the general case of a prime ideal P with $c \notin P$. Let $I_1 = \{ \bigcap M_\alpha \mid$ where M_α runs over the set of all maximal ideals such that $M_\alpha \supset P$ and $c \notin M_\alpha\}$ and $I_2 = \{ \bigcap M_\beta \mid$ where M_β runs over the set of all maximal ideals such that $M_\beta \supset P$ and $c \in M_\beta\}$. $I_1 \cap I_2$ is the intersection of all maximal ideals of R containing P and as R is a Jacobson ring we must have $P = I_1 \cap I_2$. Now $I_2 \neq P$ as $c \in I_2$ and $c \notin P$, from this, the fact that P is a prime ideal and the relation $P = I_1 \cap I_2$ it follows clearly that $P = I_1$. Therefore $PS = (\bigcap M_\alpha) S \subset \subset \bigcap (M_\alpha S)$ and $PS \cap R \subset \bigcap (M_\alpha S) \cap R = \bigcap M_\alpha = P$. We have just proved that $PS \cap R = P$. Let now $U = \{x \in R \mid x$ is regular modulo $P\}$, U is a multiplicatively closed set and $U \cap PS = \varnothing$. We extend PS to an ideal Q maximal with respect to $U \cap Q = \varnothing$. Q is a prime ideal by a standard argument. If $I = Q \cap R \neq P$ we would have in R/P the non zero ideal $\bar{I}$, this contains a regular element $\bar{a}$ (this follows either from a lemma to Goldie's theorem, see [6], or directly from the theorem of Posner), therefore we would have an element $a \in Q \cap R$ and regular modulo P which is a contradiction to the choice of Q. Therefore we must have $Q \cap R = P$ and the theorem is proved in this case. We are now ready to attack the general case. Let $Q_1, \dots, Q_r$ be the minimal primes of S, $P_i = Q_i \cap R$ is a prime ideal and $\bigcap P_i = \bigcap Q_i \cap R$ is a nil ideal. Now if L is a minimal prime of R, L contains every nil ideal and so $L \supset \bigcap P_i$, therefore we must have that $L = P_i$ for some i. We have proved therefore, under the only assumption that Spec S is decomposed in a finite number of irreducible components, and of course that S is an extension of R, that Spec R is also decomposed in such a way and that the generic point of every irreducible component of Spec R is the image of the generic point of an irreducible component of Spec S. We put ourselves now in the hypothesis that R is a Jacobson ring and S is finitely generated over R. Let $P_1 =$

$= Q_1 \cap R, \dots, P_h = Q_h \cap R$ be the minimal primes of R (not all the P_i's are necessarily minimal primes of course). We have an imbedding $R/\bigcap_{i=1}^{h} P_i \to \bigoplus_{i=1}^{h} R/P_i$, these two rings have the same ring of quotients $\bigoplus_{i=1}^{h} Q(R/P_i)$ [6]. Let now $0 \neq b_i \in R/P_i$ solve the problem for $R/P_i \subset S/Q_i$, from the above remark we can find elements $w, z \in R/\bigcap_{i=1}^{h} P_i, z$ regular with $w = z(b_1, .., b_n)$ in $\bigoplus R/P_i$. The claim is that a preimage $c \in R$ of w solves our problem. First of all c does not belong to any minimal prime ideal of R as the elements $b_1, \dots, b_h$ are all different from 0 and z is regular. Now let $P \subset R$ be a prime ideal of R and $c \notin P$. P contains a minimal prime of R, say P_1, in S/Q_1 we have $\bar{c} \notin \bar{P}$ and $\bar{c} = \bar{z}\, b_1$, therefore we can find a prime ideal $Q \supset Q_1$ such that $Q \cap R = P$.

We finish with a proposition of a more algebraic flavor which may be of some interest in the study of closed points.

Proposition 3.6. Let S be a finitely generated extension of a Jacobson ring R. If every maximal ideal of R is finitely generated (as a two sided ideal) then every maximal ideal of S is also finitely generated.

Proof. Let $M \subset S$ be a maximal ideal. $M \cap R$ is a maximal ideal of R and therefore it is finitely generated. It is therefore enough to prove that in $S/(M \cap R)S$ the maximal ideal $\bar{M}$ is finitely generated. Changing our notation we can therefore assume that R is a simple ring, in these conditions S is a finitely generated algebra over the center F of R. We have finally arrived to the following problem : $S = F(a_1, \dots, a_k)$. $M \subset S$ a maximal ideal and we claim that M is finitely generated. S/M is finite dimensional over F (cor. 1.3 of [3]). Let $\bar{s}_1, \dots, \bar{s}_h$ be a basis of S/M over F, with $s_i \in S$. We have a multiplication table $\bar{s}_j \bar{s}_i = \Sigma \alpha_{jit} \bar{s}_t$, and we also have $\bar{a}_q = \Sigma \beta_{qt} \bar{s}_t$, $q = 1, \dots, k$. The elements $s_j s_i - \Sigma \alpha_{jit} s_t$ and $a_u - \Sigma \beta_{ut} s_t$ are in finite number and we claim that they generate M. Let I be the ideal generated by these elements, $I \subset M$ so we have an onto mapping $S/I \to S/M$. Now if we call s_i' the image of s_i in S/I it is clear that the F subspace spanned by the s_i' is a subalgebra and as $a_u' = \Sigma \beta_{ut} s_t'$ in S/I we have that this subalgebra is actually the entire ring S/I. Now the dimension of this algebra is at most h as the s_i' span it. Therefore the map $S/I \to S/M$ must be an isomorphism.

Università di Roma

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