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Classe di Scienze

MARTIN SCHECHTER

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BASIC THEORY OF FREDHOLM OPERATORS (*)

MARTIN SCHECHTER

1. Introduction.

A linear operator A from a Banach space X to a Banach space Y is called a Fredholm operator if

1. A is closed
2. the domain $D(A)$ of A is dense in X
3. $\alpha(A)$, the dimension of the null space $N(A)$ of A, is finite
4. $R(A)$, the range of A, is closed in Y
5. $\beta(A)$, the codimension of $R(A)$ in Y, is finite.

The terminology stems from the classical Fredholm theory of integral equations. Special types of Fredholm operators were considered by many authors since that time, but systematic treatments were not given until the work of Atkinson [1], Gohberg [2, 3, 4] and Yood [5]. These papers considered bounded operators. Generalizations to unbounded operators were given by Kreĭn-Krasnoselskiĭ [6], Nagy [7] and Gohberg [8]. More complete treatments were given by Gohberg-Krein [9] and Kato [10]. A general account of the history of the theory is given in [9]. For a very good general account of the theory cfr. Goldberg [19]. See also Mikhlin [20] and the references quoted there.

In Section 2 of this paper we give a simple, unified treatment of the theory which covers all of the basic points while avoiding some of the involved concepts employed by previous authors. Most of the theorems are known, but in several instances we have been able to greatly simplify the

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proofs. In particular, we have been able to avoid completely the concept of the « opening » between two subspaces. The deepest theorem we use is that of the closed graph.

In Section 3 we generalize some of the standard theorems of Section 2. Although easily proved, the results of this section are, to the best of our knowledge, new. They are of interest in several types of applications. Some applications are given in Section 4. Remarks concerning specific results and methods are given in Section 5.

2. Basic properties.

Let $\Phi(X, Y)$ denote the set of Fredholm operators from X to Y. If $A \in \Phi(X, Y)$ we can decompose X into

$$X = N(A) \oplus X', \tag{2.1}$$

where X' is a closed subspace of X. Then A restricted to $D(A) \cap X'$ has an inverse defined everywhere on $R(A)$, which is a Banach space. By the closed graph theorem this inverse is bounded. Hence we have

$$\| x \|_X \leq \text{const.} \; \| Ax \|_Y , \quad x \in D(A) \cap X'. \tag{2.2}$$

LEMMA 2.1. *Let A be a closed linear operator from X to Y. If $\alpha(A) < \infty$ and* (2.2) *holds, then $R(A)$ is closed in Y.*

Proof. If $y_n \in R(A)$ and $y_n \to y$ in Y, there are $x_n \in D(A) \cap X'$ such that $A x_n = y_n$. Hence by (2.2)

$$\| x_n - x_m \|_X \leq \text{const.} \; \| y_n - y_m \|_Y \to 0 .$$

Since X' is closed, there is an element $x \in X'$ such that $x_n \to x$ in X. Since A is a closed operator, $x \in D(A)$ and $Ax = y$.

LEMMA 2.2. *If $A \in \Phi(X, Y)$ let Y' be a complement of $R(A)$ in Y, i.e.,*

$$Y = R(A) \oplus Y'. \tag{2 3}$$

Then there is a bounded operator A' from Y to $D(A) \cap X'$ such that

a) *A' vanishes on Y'*

b) *$A'A = I$ on $D(A) \cap X'$*

c) *$AA' = I$ on $R(A)$,*

where I denotes the identity operator.

Proof. On $R(A)$ we define A' to be the inverse of A. On Y' we have it vanish. By (2.3) this defines A' completely. That it is bounded follows from (2.2).

LEMMA 2.3. *The operator A' also satisfies*

$$A'A = I + F_1 \text{ on } D(A) \tag{2.4}$$

$$AA' = I + F_2 \text{ on } Y, \tag{2.5}$$

where F_1 (resp. F_2) is a bounded operator in X (resp. Y) having range in $N(A)$ (resp. Y').

Proof. Consider the operator F_1 defined to be $-I$ on $N(A)$ and vanish on X'. By (2.1) F_1 is bounded and by definition it has a finite dimensional range. Now the operator $A'A - I$ equals F_1 an $D(A) \cap X'$ and on $N(A)$. Since

$$D(A) = N(A) \oplus D(A) \cap X', \tag{2.6}$$

we have (2.4). Similar reasoning gives (2.5). We take F_2 to be $-I$ on Y' and vanish on $R(A)$.

LEMMA. 2.4. *Let A be a densely defined closed operator from X to Y. Suppose there are bounded operators A_1, A_2 from Y to X and compact operators K_1 on X, K_2 on Y such that*

$$A_1 A = I + K_1 \text{ on } D(A) \tag{2.7}$$

$$AA_2 = I + K_2 \text{ on } Y. \tag{2.8}$$

Then $A \in \Phi(X, Y)$.

Proof. Since $N(A) \subseteq N(A_1 A)$, we have $\alpha(A) \leq \alpha(I + K_1)$, and the latter is finite by the classical theory of F. Riesz. Similarly, since $R(A) \supseteq R(AA_2)$ we have $\beta(A) \leq \beta(I + K_2) < \infty$. We must show that $R(A)$ is closed, or equivalently, that (2.2) holds. If it did not, there would be a sequence $\{x_n\} \subset D(A) \cap X'$ such that $\| x_n \| = 1$ while $Ax_n \to 0$ in Y.

Since A_1 is bounded, $(I + K_1) x_n \to 0$. Since $\{x_n\}$ is bounded, there is a subsequence (also denoted by $\{x_n\}$) such that $K_1 x_n$ converges to some element $x \in X$. Thus $x_n \to -x$ in X. Since A is closed, $x \in D(A)$ and $Ax = 0$, i.e., $x \in N(A)$. Since X' is closed, $x \in X'$ and hence $x = 0$. But $\| x \| = \lim \| x_n \| = 1$. This provides the contradiction showing that (2.2) holds.

If $A \in \Phi(X, Y)$, the index $i(A)$ of A is defined by

$$i(A) = \alpha(A) - \beta(A). \tag{2.9}$$

Of fundamental importance is

THEOREM 2.5. *If* $A \in \Phi(X, Y)$ *and* $B \in \Phi(Z, X)$, *then* $AB \in \Phi(Z, Y)$ *and*

$$i(AB) = i(A) + i(B). \tag{2.10}$$

Proof. We first prove that $D(AB)$ is dense in Z. Let Z' be a closed subspace of Z such that

$$Z = N(B) \oplus Z'.$$

Since the projection of Z onto Z' is continuous, it follows that $D(B) \cap Z'$ is dense in Z'. Since $N(B) \subseteq D(AB)$, it suffices to show that each element $z \in D(B) \cap Z'$ can be approximated as closely as desired by an element in $D(AB) \cap Z'$. Set

$$X = R(B) \oplus X'', \tag{2.11}$$

where X'' is a finite dimensional subspace of X. Since $D(A)$ is dense in X, we may take $X'' \subset D(A)$ by shifting each basis vector of X'' a small distance to get in $D(A)$ without disturbing (2.11). Thus $R(B) \cap D(A)$ is dense in $R(B)$. Now if $z \in D(B) \cap Z'$, then for every $\varepsilon > 0$ we can find an $x \in R(B) \cap D(A)$ such that $\| x - Bz \| < \varepsilon$. There is a $z' \in D(B) \cap Z'$ such that $Bz' = x$. Hence $z' \in D(AB) \cap Z'$ and $\| z' - z \| \leq$ const. ε.

Next we show that AB is a closed operator. Suppose $\{z_n\} \subset D(AB)$, $z_n \to z$, $AB z_n \to y$. Write

$$Bz_n = x_n^{(0)} + x_n^{(1)},$$

where $x_n^{(0)} \in N(A)$ and $x_n^{(1)} \in X'$. Thus $Ax_n^{(1)} = ABz_n \to y$, and hence by (2.2) there is an $x^{(1)} \in X'$ such that $x_n^{(1)} \to x^{(1)}$. We shall show that $\| x_n^{(0)} \| \leq$ const. Assuming this for the moment, we know by the finite dimensionality of $N(A)$ that there is a subsequence of $\{x_n\}$ (also denoted by $\{x_n\}$) for which $x_n^{(0)}$ converges to some element $x^{(0)} \in N(A)$. Thus $Bz_n \to x^{(0)} + x^{(1)}$, and since B is closed, we have $z \in D(B)$ and $Bz = x^{(0)} + x^{(1)}$. Since A is closed, $x^{(0)} + x^{(1)} \in D(A)$ and $A(x^{(0)} + x^{(1)}) = y$. Hence $z \in D(AB)$ and $ABz = y$. To show that $\{x_n^{(0)}\}$ is bounded, suppose that $\zeta_n = \| x_n^{(0)} \| \to \infty$. Set $u_n = \zeta_n^{-1} x_n^{(0)}$. Then $\| u_n \| = 1$. Since $N(A)$ is finite dimensional, there is a subsequence (also denoted by $\{u_n\}$) such that u_n converges to some $u \in N(A)$. Moreover

$$B(\zeta_n^{-1} z_n) - u_n = \zeta_n^{-1}(Bz_n - x_n^{(0)}) = \zeta_n^{-1} x_n^{(1)} \longrightarrow 0,$$

since the sequence $x_n^{(1)}$ is convergent and hence bounded.

Hence $B(\zeta_n^{-1} z_n) \longrightarrow u$. Since $\zeta_n^{-1} z_n \longrightarrow 0$ and B is closed, we must have $u = 0$. But this is impossible, since $\| u \| = \lim \| u_n \| = 1$.

Next set

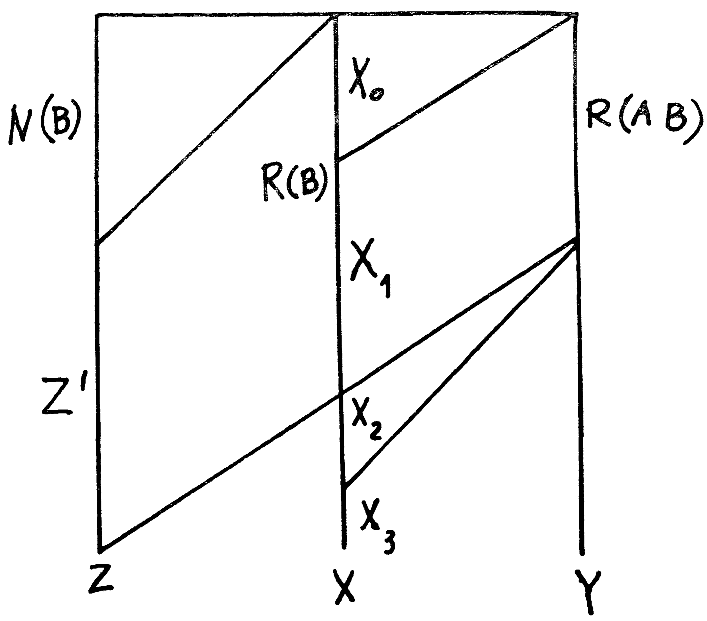
$$X_0 = R(B) \cap N(A) \tag{2.12}$$

$$R(B) = X_0 \oplus X_1 \tag{2.13}$$

$$N(A) = X_0 \oplus X_2 \tag{2.14}$$

$$X = R(B) \oplus X_2 \oplus X_3 . \tag{2 15}$$

Note that X_0, X_2, X_3 are finite dimensional and that X_1 is closed. Since $D(A)$ is dense in X we may take X_3 to be contained in $D(A)$. Let d_i denote the dimension of X_i, $i = 0, 2, 3$.



Diagram

Then (note diagram)

$$\alpha(AB) = \alpha(B) + d_0$$

$$\beta(AB) = \beta(A) + d_3$$

$$d_0 + d_2 = \alpha(A)$$

$$d_2 + d_3 = \beta(B).$$

These relations show that $\alpha(AB)$ and $\beta(AB)$ are finite and that (2.10) holds.

Finally we must show that $R(AB)$ is closed. By (2.13)-(2.15), $X = N(A) \oplus X_1 \oplus X_3$. We may therefore take $X' = X_1 \oplus X_3$ in (2.1). Now $R(AB)$ is just the range of A on $D(A) \cap X_1$. If $Ax_n \to y$ in Y for $x_n \in D(A) \cap X_1$, we have by (2.2)

$$\| x_n - x_m \|_X \leq \text{const.} \| A(x_n - x_m) \|_Y \to 0,$$

and since X_1 is closed, x_n converges to some $x \in X_1$. Since A is closed, $x \in D(A)$ and $Ax = y$. Hence $y \in R(AB)$ showing that $R(AB)$ is closed. This completes the proof of the theorem.

LEMMA 2.6. *Let $\tilde{X}$ be a Banach space which is continuously embedded in X such that $D(A)$ is dense in $\tilde{X}$. Then $A \in \Phi(X, Y)$ implies $A \in \Phi(\tilde{X}, Y)$ with $\alpha(A)$ and $\beta(A)$ the same.*

Proof. Obvious

LEMMA 2.7. *Let $\tilde{X}$ be a Banach space continuously embedded in X and containing $D(A)$. Then $A \in \Phi(\tilde{X}, Y)$ implies $A \in \Phi(X, Y)$ with $\alpha(A)$* and *$\beta(A)$ unchanged.*

Proof. Let P be the operator from X to $\tilde{X}$ with $D(P) = \tilde{X}$ and defined by $Px = x$ for $x \in D(P)$. Obviously $P \in \Phi(X, \tilde{X})$. Hence by Theorem 2.5 $AP \in \Phi(X, Y)$. But $AP = A$.

THEOREM 2.8. *If $A \in \Phi(X, Y)$ and K is a compact operator from X to Y, then $(A + K) \in \Phi(X, Y)$ and*

$$i(A + K) = i(A). \tag{2.16}$$

Proof. By lemma 2.3 there is a bounded operator A' from Y to $D(A) \cap X'$ such that (2.4) and (2 5) hold. Thus

$$A'(A + K) = I + F_1 + A'K \quad \text{on} \quad D(A)$$

$$(A + K)\, A' = I + F_2 + KA' \quad \text{on} \quad Y. \tag{2.17}$$

Since A' is bounded, the operators $A'K$ and KA' are compact. Hence $(A + K) \in \Phi(X, Y)$ by Lemma 2.4. Since A is closed, we can make $D(A)$ into a Banach space $\tilde{X}$ by equipping it with the graph norm

$$\| x \|_{D(A)} = \| x \| + \| Ax \| . \tag{2.18}$$

By Lemma 2.6, $A \in \Phi(\tilde{X}, Y)$ with $\alpha(A)$ and $\beta(A)$ the same. Moreover $A' \in \Phi(Y, \tilde{X})$ by Lemma 2.2. Hence by (2.5) and (2.10)

$$i(A) + i(A') = i(I + F_1) = 0, \tag{2.19}$$

where the last equality follows from the classical Riesz theory. Again by Lemma 2.6 we have $(A + K) \in \Phi(\tilde{X}, Y)$ and thus by (2.18) and (2.10)

$$i(A + K) + i(A') = i(I + F_2 + KA') = 0.$$

This together with (2.19) show that $i(A + K) = i(A)$ when both are considered in $\Phi(\tilde{X}, Y)$. But this is the same when they are considered as operators in $\Phi(X, Y)$. This completes the proof.

THEOREM. 2.9. *For $A \in \Phi(X, Y)$ there is an $\varepsilon > 0$ such that for any bounded operator T from X to Y with $\| T \| < \varepsilon$ one has $(A + T) \in \Phi(X, Y)$,*

$$i(A + T) = i(A) \tag{2.20}$$

and

$$\alpha(A + T) \leq \alpha(A). \tag{2.21}$$

Proof. For the operator A' given by Lemma 2.3 we have

$$A'(A + T) = I + F_1 + A'T \quad \text{on} \quad D(A)$$

$$(A + T)\, A' = I + F_2 + TA' \quad \text{on} \quad Y. \tag{2.22}$$

Take $\varepsilon = \| A' \|^{-1}$. Then $\| A'T \| < 1$ and $\| TA' \| < 1$. Thus the operators $I + A'T$ and $I + TA'$ are invertible and

$$(I + A'T)^{-1} A'(A + T) = I + (I + A'T)^{-1} F_1 \quad \text{on} \quad D(A)$$

$$(A + T) A'(I + TA')^{-1} = I + F_2 (I + TA')^{-1} \quad \text{on} \quad Y.$$

This shows that $(A + T) \in \Phi(X, Y)$ (Lemma 2.4). By (2.10) and (2.22)

$$i(A + T) + i(A') = i(I + F_2 + TA') = i(I + TA') = 0.$$

Combining this with (2.19) we obtain (2.20). It remains to prove (2.21). By Lemma 2.2

$$A'(A + T) = I + A'T \quad \text{on} \quad D(A) \cap X',$$

and hence this operator is one-to-one on $D(A) \cap X'$. Moreover, $N(A + T) \cap X' = \{0\}$. For if x is in this set, it is in $D(A) \cap X'$, and $(A + T)x = 0$. Hence $(I + A'T)x = 0$ showing that $x = 0$. Since

$$N(A + T) \oplus X' \subseteq X = N(A) \oplus X',$$

we see that $\dim N(A + T) \leq \dim N(A)$ and the proof is complete.

A linear operator B from X to Y is called A-compact if $D(B) \supseteq D(A)$ and for every sequence $\{x_n\} \subset D(A)$ such that

$$\| x_n \|_{D(A)} = \| x_n \| + \| Ax_n \| \leq \text{const.}$$

the sequence Bx_n has a convergent subsequence.

THEOREM 2.10. *If $A \in \Phi(X, Y)$ and B is A-compact, then $(A + B) \in \Phi(X, Y)$ and*

$$i(A + B) = i(A). \tag{2.23}$$

Proof. If we equip $D(A)$ with the graph norm (2.18) it becomes a Banach space satisfying the hypotheses of lemmas 2.6 and 2.7. By the former, $A \in \Phi(\widetilde{X}, Y)$ and since B is a compact operator from $\widetilde{X}$ to Y we have by Theorem 2.8 that $(A + B) \in \Phi(\widetilde{X}, Y)$. This completes the proof.

THEOREM. 2.11. *For each $X \in \Phi(X, Y)$ there is an $\varepsilon > 0$ such that*

$$\| Bx \| \leq \varepsilon (\| x \| + \| Ax \|), \quad x \in D(A)$$

holding for any operator B from X to Y with $D(B) \supseteq D(A)$ implies that $(A+B) \in \Phi(X, Y)$, $i(A+B) = i(A)$ and

$$\alpha(A+B) \leq (A). \tag{2.24}$$

Proof. Similar to that of Theorem 2.10.

3. Some generalizations.

We now show how some of the theorems of the preceding section can be strengthened. At first, some of these genalizations may appear unnecessary, but in some applications they turn out to be either essential or extremely convenient. Such application are given in the next section.

We first show how the hypotheses of Lemma 2.4 can be satisfied under apparently weaker conditions.

REMARK. 3.1. In applying Lemma 2.4, it suffices to verify (2.7) on a set S dense in $D(V)$ with respect to the graph norm and to verify (2.8) on a set U dense in Y. Moreover, $R(A_2) \subseteq D(A)$.

Proof. If $x \in D(A)$, there is a sequence $\{x_n\} \subset S$ such that $x_n \to x$, $Ax_n \to Ax$. Since A_1 is bounded $A_1 Ax_n \to A_1 Ax$. Hence (2.7) holds on all of $D(A)$. Similarly, if $y \in Y$, there is a sequence $\{y_n\} \subset U$ such that $y_n \to y$. Since A_2 is bounded, $A_2 y_n \to A_2 y$. Moreover, $A A_2 y_n = (I + K_2) y_n \to (I + K_2) y$. Since A is a closed operator, we see that $A_2 y \in D(A)$ and $A A_2 y = (I + K_2) y$. This gives the desired result.

The next generalization is a generalization of Lemma 2.7.

LEMMA 3.2. *Let A be a densely defined linear operator from X to Y, and let $\widetilde{X}$ be a Banach space continuosly embedded in X such that $D(A) \cap \widetilde{X}$ is dense in $D(A)$ with respect to the graph norm. Let $\widetilde{A}$ be the restriction of A to $D(A) \cap \widetilde{X}$, and assume that $\widetilde{A} \in \Phi(\widetilde{X}, Y)$. Then $D(A) \subseteq \widetilde{X}$ and $A \in \Phi(X, Y)$.*

Proof. Let $\widetilde{X}'$ be a closed subspace of $\widetilde{X}$ such that $\widetilde{X} = N(\widetilde{A}) \oplus \widetilde{X}'$. Since $\widetilde{A} \in \Phi(\widetilde{X}, Y)$ we have by (2.2)

$$\| x \|_{\widetilde{X}} \leq C \| Ax \|_{Y}, \quad x \in D(\widetilde{A}) \cap \widetilde{X}'. \tag{3.3}$$

Now if $x \in D(A)$, there is a sequence $\{x_n\}$ of elements of $D(\widetilde{A})$ such that $x_n \to x$, $Ax_n \to Ax$ in X. We decompose each x_n into $x_n' + x_n''$, where

$x_n' \in \widetilde{X}' \cap D(\widetilde{A})$ and $x_n'' \in N(\widetilde{A})$. Then by (3.3) x_n' converges in $\widetilde{X}$ to an element $x' \in D(\widetilde{A}) \cap \widetilde{X}'$. Thus $x_n'' = x_n - x_n'$ converges in X to some element $x'' \in N(\widetilde{A})$. Hence $x = x' + x'' \in D(\widetilde{A})$. The rest follows from Lemma 2.7.

A linear operator B will be called A-closed if $\{x_n\} \subset D(A) \cap D(B)$, $x_n \to x$, $Ax_n \to y$, $Bx_n \to z$ imply that $x \in D(B)$ and $Bx = z$.

We now give a generalization of Theorem 2.10.

THEOREM 3.3. *Suppose that $A \in \Phi(X, Y)$ and that B is an A-closed linear operator from X to Y such that $D(A) \cap D(B)$ is dense in $D(B)$ with respect to the graph norm. Assume that there is a linear manifold S in $D(A) \cap D(B)$ which is dense in $D(A)$ in the graph topology and such that*

$$\| x_n \|_{D(A)} \leq C, \quad x_n \in S$$

implies that $\{(A - B)x_n\}$ has a convergent subsequence. Then $B \in \Phi(X, Y)$ and $i(B) = i(A)$.

Proof. By Lemma 2.3 there is a bounded operator A' from Y to $D(A)$ such that

$$A'A = I + F_1 \quad \text{on} \quad D(A), \quad AA' = I + F_2 \quad \text{on} \quad Y,$$

where F_1 has finite rank on $D(A)$, F_2 on Y. Now the restriction of $A - B$ to S is bounded from $D(A)$ to Y. For otherwise there would be a sequence $\{x_n\} \subseteq S$ such that $\| x_n \|_{D(A)} \leq C$ and $\| (A - B)x_n \| \to \infty$. This is impossible by hypothesis. Hence there is a compact operator K_3 from $D(A)$ to Y such that $A - B = K_3$ on S. Hence

$$A'B = I + F_1 - A'K_3 \quad \text{on} \quad S \tag{3.4}$$

$$BA' = I + F_2 - K_3 A' \quad \text{on} \quad A'^{-1} S. \tag{3.5}$$

Let $\widetilde{B}$ be the restriction of B to $D(A) \cap D(B)$ and consider it as an operator from $D(A)$ to Y. The operator $A'K_3$ in (3.4) is compact in $D(A)$ while $K_3 A'$ in (3.5) is compact in Y. Hence if $A'^{-1} S$ is dense in Y we can apply Remark 3.1 to conclude that $\widetilde{B} \in \Phi(D(A), Y)$. Assuming this for the moment, we note that we now can apply Lemma 3.2 if we set $\widetilde{X} = D(A)$. We can thus conclude $B \in \Phi(X, Y)$ with $N(B) = N(\widetilde{B})$ and $R(B) = R(\widetilde{B})$. Now $A' \in \Phi(Y, D(A))$ while $\widetilde{B} \in \Phi(D(A), Y)$. Hence $BA' \in \Phi(Y, Y)$. Since it equals a bounded operator on a dense set, it must be bounded and defined everywhere. Hence $BA' = I + F_2 - K_3 A'$ on all of Y and hence has index zero. Thus

$$i(B) + i(A') = 0.$$

Hence $i(B) = i(A)$. It thus remains to show that $A'^{-1}S$ is dense in Y. Now $Y = R(A) \oplus Y'$, where Y' is a finite dimensional subspace of Y. Thus if $y \in Y$, $y = y' + y''$, where $y'' \in R(A)$, $y' \in Y'$. Thus there is an $x \in D(A) \cap X'$ such that $Ax = y''$, and for any $\varepsilon > 0$ there is an $x_0 \in S \cap X'$ such that $\| x - x_0 \|_{D(A)} < \varepsilon$. In fact by (2.1) $X = N' \oplus X'$, where N' is finite dimensional and contained in S. This is accomplished by shifting the basis vectors of $N(A)$ slightly to get them in S. Thus $\| Ax - Ax_0 \| < \varepsilon$ and hence $\| Ax_0 + y' - y \| < \varepsilon$. But $A'(Ax_0 + y') = x_0 \in S \cap X'$ showing that $A'^{-1}S$ is dense in Y, and the proof is complete.

We also give a partial converse of Theorem 2.5.

THEOREM 3.4. *Let $A \in \Phi(X, Y)$ and let E be a densely defined closed linear operator from Y to a Banach space W. If $EA \in \Phi(X, W)$, then $E \in \Phi(Y, W)$.*

Proof. Let Y' be a finite dimensional subspace of Y satisfying (2.3). By shifting the basis vectors of Y' slightly we may arrange that $Y' \subseteq D(E)$. Let A' be defined as in Lemma 2.2. Then

(3.6) $$AA' = I + F_2 \quad \text{on} \quad Y,$$

where F_2 vanishes on $R(A)$ and equals $-I$ on Y'. Thus on $D(E)$ we have

(3.7) $$EAA' = E + EF_2 .$$

Note that EAA' is defined on $D(E)$ while EF_2 is compact from $D(E)$ to itself. Now one checks easily that $A' \in \Phi(Y, D(A))$ while $EA \in \Phi(D(A), W)$ by Lemma 2.6. In order that this last statement be true we must verify that $D(EA)$ is dense in $D(A)$ in the graph norm. This is indeed so. For if $x \in D(A)$, $x = x' + x''$, where $x'' \in N(A) \subseteq D(EA)$ and $x' \in D(A) \cap X'$. Since $Y' \subseteq D(E)$, we see that $R(A) \cap D(E)$ is dense in $R(A)$. In particular for $\varepsilon > 0$ we can find an $\tilde{x} \in D(EA) \cap X'$ such that $\| A\tilde{x} - Ax \| < \varepsilon$. Thus $\| \tilde{x} - x' \| <$ const. ε showing that $\tilde{x}$ is close to x' in the graph norm. Thus we may conclude that $EAA' \in \Phi(Y, W)$ (Theorem 2.5) and hence it is in $\Phi(D(E), W)$ (Lemma 2.6). From the compactness of EF_2 on $D(E)$ we see by (3.7) that $E \in \Phi(D(E), W)$ (Theorem 2.8) and hence it is in $\Phi(Y, W)$ (Lemma 2.7).

When the roles of A and E are interchanged, all that can be said is the following.

THEOREM 3.5. *If $E \in \Phi(Y, W)$, A is closed and densely defined from X to Y and $EA \in \Phi(X, W)$, then the restriction of A to $D(EA)$ is in $\Phi(X, D(E))$. Thus if in addition E is bounded from Y to W, we have $A \in \Phi(X, Y)$.*

Proof. By Lemma 2.3 there is a bounded operator E' from W to Y such that

$$E'E = I + F_3 \quad \text{on} \quad D(E),$$

where F_3 is $-I$ on $N(E)$ and vanishes on a closed complement of $N(E)$ in Y. Thus

$$E'EA = A + F_3 A \quad \text{on} \quad D(EA).$$

The operator $F_3 A$ is compact from $D(A)$ to $D(E)$ while the operator $E'EA$ with domain $D(EA)$ is in $\Phi(D(A), D(E))$ (Theorem 2.5). Thus A restricted to $D(EA)$ is in $\Phi(D(A), D(E))$ and hence in $\Phi(X, D(E))$ (Lemma 2.7).

To illustrate that we cannot expect to do better, let C^0 denote the space of continuous functions $x(t)$ in the interval $0 \leq t \leq 1$ with norm

$$\| x \|_0 = \max_{0 \leq t \leq 1} | x(t) |.$$

Let C_0^1 denote the set of functions $y(t)$ on the some interval which vanish at $t = 0$ and have continuous first derivatives in the interval. The norm in C_0^1 is

$$\| y \|_1 = \| y \|_0 + \| y' \|_0 .$$

The operator $Ax = \int_0^t x(s)\, ds$ is clearly seen to be in $\Phi(C^0, C_0^1)$. Let E be differentiation with respect to t of functions in C_0^1. Considered as an operator in C^0, E is easily seen to be in $\Phi(C^0, C^0)$. Moreover, EA is the identity operator in C^0 and hence is in $\Phi(C^0, C^0)$. However, one cannot conclude that $A \in \Phi(C^0, C^0)$.

In connection with the above, one does have the following.

LEMMA 3.6. *Let $\tilde{Y}$ be a Banach space continuously embedded in Y. Assume that $A \in \Phi(X, Y)$ and let $\tilde{A}$ be the restriction of A to those $x \in D(A)$ such that $Ax \in \tilde{Y}$. Then $\tilde{A} \in \Phi(X, Y)$.*

Proof. Let P be the operator from Y to $\tilde{Y}$ with $D(P) = \tilde{Y}$ and defined by $Py = y$, $y \in D(P)$. It is easily checked that $P \in \Phi(Y, \tilde{Y})$. Thus $PA \in \Phi(X, \tilde{Y})$ But $\tilde{A} = PA$.

THEOREM 3.7. *Let A be a densely defined, closed linear operator from X to Y. Suppose that there is a bounded linear operator E from Y to W with $\alpha(E) < \infty$ and such that $EA \in \Phi(X, W)$. Then $A \in \Phi(X, Y)$.*

Proof. We have $R(E) \supseteq R(EA)$, which is closed and of finite codimension (cf. Lemma 4.4). Thus $E \in \Phi(Y, W)$. We now apply Theorem 3.5 making use of the fact that E is bounded.

COROLLARY 3.8. *Let A be a bounded linear operator from X to Y. Suppose there is a bounded operator E from Y to X with $\alpha(E) < \infty$ and a compact operator K on X such that*

$$EA = I + K. \tag{3.8}$$

Then $A \in \Phi(X, Y)$ and $i(A) \geq -\alpha(E)$.

4. Some applications.

In this section we illustrate some of the theorems of preceding sections, We consider the case $X = Y$ and set $\Phi(X) = \Phi(X, X)$. Let A be a closed, densely defined linear operator on X. A point $\lambda_0 \in \sigma(A)$ is called *isolated* if there is an $\varepsilon > 0$ such that $\lambda \in \varrho(A)$ for all λ satisfying $0 < |\lambda - \lambda_0| < \varepsilon$. The set of all complex λ for which $A - \lambda \in \Phi(X)$ is denoted by Φ_A and called the Φ — set of A. Set

$$r(A) = \lim_{k \to \infty} \alpha(A^k), r^*(A) = \lim_{k \to \infty} \beta(A^k).$$

If $r(A - \lambda_0) < \infty$, the point λ_0 is said to be of finite multiplicity.

We shall first prove the following.

THEOREM 4.1. *A point λ_0 is in Φ_A with $r(A - \lambda_0)$ and $r^*(A - \lambda_0)$ both finite if and only if one can find an integer $m \geq 1$, a bounded operator E on X and compact operator K on X such that*

$$(A - \lambda_0)^m E = E(A - \lambda_0)^m = I + K \text{ on } D(A^m). \tag{4.1}$$

It can be shown easily (cf. [9,11]) that any isolated point $\lambda_0 \in \sigma(A)$ with finite multiplicity is in Φ_A with both $r(A - \lambda_0)$ and $r^*(A - \lambda_0)$ finite. Below we shall prove the converse. Hence Theorem 4.1 can be reformulated to read as follows:

THEOREM 4.1′. *A necessary and sufficient condition that a point $\lambda_0 \in \sigma(A)$ be isolated and of finite multiplicity is that for some integer $m \geq 1$ there exist operators E and K satisfying the conditions of Theorem* 4.1.

In proving Theorem 4.1 we shall make use of the following simple lemmas.

LEMMA 4.2. *If $r(A) < \infty$, then there is an integer n such that $N(A_k) = N(A^n)$ for all $k \geq n$.*

Proof. $\alpha(A^k)$ is a non-decreasing sequence of integers bounded from above.

LEMMA 4.3 *If $A \in \Phi(X)$ and $r^*(A) < \infty$, then there is an integer m such that $R(A^k) = R(A^m)$ for all $k \geq m$.*

Proof. $\beta(A^k)$ is also such a sequence.

LEMMA 4.4. *Suppose $X = X_1 \oplus M$, where X_1 is a closed subspace of X and M is finite dimensional. If X_2 is a linear manifold in X containing X_1, then X_2 is closed.*

Proof. Clearly $X_2 = X_1 \oplus M \cap X_2$.

LEMMA 4.5. *If $A = A_1 A_2 \dots A_k$ is in $\Phi(X)$ and the A_j commute on $D(A)$, then each A_j is in $\Phi(X)$.*

Proof. Clearly $D(A_j) \supseteq D(A)$, $N(A_j) \subseteq N(A)$, $R(A_j) \supseteq R(A)$ for each j. Apply Lemma 4.4.

COROLLARY 4.6. *$A^m \in \Phi(X)$ if and only if $A \in \Phi(X)$.*

LEMMA 4.8. *If $A \in \Phi(X)$ and both $r(A)$ and $r^*(A)$ are finite, then $i(A) = 0$. Hence any n satisfying Lemma* 4.2 *also satisfies Lemma* 4.3.

Proof. By Lemmas 4.2 and 4.3 there is an integer n such that $\alpha(A^k) = \alpha(A^n)$, $\beta(A^k) = \beta(A^n)$ for all $k \geq n$. Thus for any $k \geq n$ $i(A^k) = \alpha(A^k) - \beta(A^k) = i(A^n)$. But $i(A^k) = ki(A)$ (Theorem 2.5). Let $k \to \infty$.

LEMMA 4.8. *If $A \in \Phi(X)$ and both $r(A)$ and $r^*(A)$ are finite, then for any integer n satisfying Lemma* 4.2 *we have $X = N(A^n) \oplus R(A^n)$.*

Proof. We shall show that $N(A^n) \cap R(A^n)$ consists only of the element 0. Since $R(A^n)$ in closed (Theorem 2.5) and $i(A^n) = 0$ (Lemma 4.7) it follows that $\beta(A^n) = \alpha(A^n)$. If $N(A^n) \oplus R(A^n)$ did not contain the whole of X, the codimension of $R(A^n)$ would be greater than $\alpha(A^n)$. It thus remains to show that $N(A^n) \cap R(A^n) = \{0\}$. If x_0 is an element of this set we have on one hand that $A^n x_0 = 0$ and on the other $x_0 = A^n x_1$ for some $x_1 \in D(A^n)$. Thus $A^{2n} x_1 = 0$ showing that $x_1 \in N(A^{2n}) = N(A^n)$. But then $x_0 = A^n x_1 = 0$ as was to be proved.

Lemma 4.9. *Under the same hypotheses, there is an $\varepsilon > 0$ such that $\lambda \in \varrho(A)$ for $0 < |\lambda| < \varepsilon$.*

Proof. By Theorem 2.9 Φ_A contains a neighborhood of the origin and since $i(A) = 0$ (Lemma 4.7), we have $i(A - \lambda) = 0$ in this neighborhood. The result will follow if we can show that $N(A - \lambda) = 0$ for $\lambda \neq 0$ in some neighborhood of the origin. By Lemmas 4.2, 4.3 and 4.8 there is an integer n such that $X = N(A^n) \oplus R(A^n)$ and $N(A^k) = N(A^n)$, $R(A^k) = R(A^n)$ for all $k \geq n$. Thus $N(A^n)$ and $R(A^n)$ are invariant subspaces for A. Thus it suffices to show that $(A - \lambda) u = 0$ implies $u = 0$ for $u \in N(A^n)$ and $(A - \lambda) v = 0$ implies $v = 0$ for $v \in R(A^n)$. If $u \in N(A^n)$ and $(A - \lambda) u = 0$, then $A^{n-1} u = \lambda^{-1} A^n u = 0$, $A^{n-2} u = \lambda^{-1} A^{n-1} u = 0, \dots, u = \lambda^{-1} Au = 0$. We now show that A is continuously invertible on $R(A^n)$. It thus follows that the same is true of $A - \lambda$ for λ is a neighborhood of the origin. This will complete the proof. If $w \in R(A^n)$ and $Aw = 0$, then $w = A^n g$ and $A^{n+1} g = 0$. Since $N(A^{n+1}) = N(A^n)$ $w = A^n g = 0$. Moreover, if $f \in R(A^n)$, then $f = A^n h$ for some $h \in D(A^n)$. By Lemma 4.8 we may take $h \in R(A^n)$. Hence $h = A^n v$ and $f = A^{2n} v$. Set $x = A^{2n-1} v$. Then $Ax = f$ and $x \in R(A^n)$. Thus we see that A is a closed linear operator which is one-to-one and onto on $R(A^n)$. Hence A is continuously invertible on $R(A^n)$, and the proof is complete.

We can now give the.

Proof of Theorem 4.1. Without loss of generality, we may take $\lambda = 0$. Assume that $A \in \Phi(X)$ and that both $r(A)$ and $r^*(A)$ are finite. Let $x_1, \dots, x_s$ be a basis for $N(A^n)$, where n satisfies Lemma 4.2. By Lemma 4.8 we can choose bounded linear functionals $x_1^*, \dots, x_s^*$ on X which vanish on $R(A_n)$ and such that $x_j^*(x_k) = \delta_{jk}$, the Kronecker delta. Set

$$Vx = \sum_{k=1}^{s} x_k^*(x)\, x_k.$$

Since V is of finite rank and hence compact, we see that $A^n + V \in \Phi(X)$ and $i(A^n + V) = 0$ (Theorem 2.8). If $(A^n + V)x = 0$, then $Vx \in R(A^n) \cap N(A^n)$ and hence Vx must vanish (Lemma 4.8). This gives $A^n x = 0$ showing that $x \in N(A^n)$. But $Vx = 0$ implies $x_j^*(x) = 0$ for each j, and hence x must vanish. This means that $N(A^n + V) = \{0\}$ showing that $A^n + V$ must have a bounded inverse E. Now $A^n V = VA^n = 0$ since $R(V) = N(A^n)$ and $N(V) = R(A^n)$. Hence

$$(A^n + V)V = V(A^n + V) = V^2,$$

showing that

$$V = EV^2 = V^2 E,$$

or

$$VE = EV^2 E = EV,$$

whence

$$EA^n = A^n E = I - EV$$

on $D(A^n)$. Since E is bounded, EV is a compact operator on X. Thus (4.1) holds with $m = n$. Conversely, assume that (4.1) holds for some bounded E and compact K. Set $W = A^m$. By Remark 3.1, $W \in \Phi(X)$, $R(E) \subseteq D(W)$ and

$$WE = I + K \text{ on } X.$$

By the Riesz theory, there is an integer l such that

$$N[(I+K)^j] = N[(I+K)^l],\ R[(I+K)^j] = R[(I+K)^l]$$

for all $j \geq l$. Thus for such j we have

$$N[(I+K)^l] = N[(I+K)^j] \supseteq N[(EW)^j] = N(E^j W^j) \supseteq N(W^j),$$

since E and W commute on $D(W)$. Hence $\alpha(W^j)$ is bounded from above showing that $r(A) < \infty$. Similarly, $R[(I+K)^l] = R[(I+K)^j] = R[WE)^j] = R(W^j E^j) \subseteq R(W^j)$ showing that $\beta(W^j)$ is bounded from above. Hence $r^*(A) < \infty$, and the proof is complete.

Next we give an application of Theorem 3.3. We define the essential spectrum $\sigma_e(A)$ of A as the set of those $\lambda \in \sigma(A)$ such that $\lambda \in \sigma(A + K)$ for every compact operator K. An easy application of Theorem 2.8 shows that a point λ is not in $\sigma_e(A)$ if and only if $\lambda \in \Phi_A$ and $i(A - \lambda) = 0$ (cf. [13]).

In the remaining theorems we assume that A and B are linear operators on X such that there is a $\lambda \in \Phi_A \cap \Phi_B$ with $i(A - \lambda) = i(B - \lambda)$. By A'_λ we shall denote any particular operator given by Lemma 2.2 corresponding to $A - \lambda$. Of course if $\lambda \in \varrho(A)$, then $A'_\lambda = (A - \lambda)^{-1}$. We define B'_λ similarly.

THEOREM 4.10. *If there is a linear manifold S dense in X such that*

$$\| x_n \| \leq C,\ x_n \in S \tag{4.2}$$

implies that $\{(A'_\lambda - B'_\lambda)x_n\}$ has a convergent subsequence, then

$$\sigma_e(B) = \sigma_e(A). \tag{4.3}$$

Proof. Without loss of generality we may assume that $\lambda = 0$ and write A' and B' in place of A'_λ and B'_λ respectively. By Lemma 2.3 we have

$$(4.4) \qquad (A - \mu) A' - (B - \mu) B' = F_2 - F_4 + \mu (B' - A'),$$

where

$$(4.5) \qquad B' B = I + F_3 \text{ on } D(B), BB' = I + F_4 \text{ on } X.$$

Now suppose $\mu \in \Phi_B$. Since $B' \in \Phi(X, D(B))$ we see that $T = (B - \mu) B' \in \Phi(X)$. Moreover, since A' is bounded and A is closed, the operator $U = (A - \mu) A'$ is closed. We now apply Theorem 3.3 to (4.4) to conclude that $U \in \Phi(X)$. But $A' \in \Phi(X, D(A))$. Hence $A - \mu \in \Phi(D(A), X)$ (Theorem 3.4) and consequently in $\Phi(X)$ (Lemma 2.7). Moreover, by (4.4)

$$i(A - \mu) + i(A') = i(B - \mu) + i(B').$$

Since $i(A') = - i(A) = - i(B) = i(B')$, we have

$$(4.6) \qquad i(A - \mu) = i(B - \mu).$$

This shows that $\sigma_e(A) \subseteq \sigma_e(B)$. Conversely, if $\mu \in \Phi_A$ we apply Theorem 3.3 in the opposite direction to show that $\mu \in \Phi_B$. Thus (4.6) holds and $\sigma_e(B) \subseteq \sigma_e(A)$. This completes the proof

Corollary 4.11. *If there is a linear manifold $S \subseteq D(AB'_\lambda)$ dense in X such that* (4.2) *implies that $\{A'_\lambda (A - B) B'_\lambda x_n\}$ has a convergent subsequence, then* (4.3) *holds. The same is true if, instead, $S \subseteq D(BA'_\lambda)$ and* (4.2) *implies that $\{B'_\lambda (A - B) A'_\lambda x_n\}$ has a convergent subsequence.*

Proof. On such sets S the identities

$$A' - B' = A' (B - A) B' + F_1 B' - A' F_4$$

$$A' - B' = B' (A - B) A' + B' F_2 - F_3 A'$$

hold. We now apply Theorem 4.10.

Corollary 4.12. *Assume that there is a linear manifold $S \subseteq D(A) \cap D(B)$ which is dense in $D(B)$ with respect to the graph norm of $D(B)$ such that*

$$(4.7) \qquad \| z_n \|_{D(B)} \leq C, \; z_n \in S$$

implies that $\{A'_\lambda (A - B) z_n\}$ has a convergent subsequence. Then (4.3) *holds.*

Proof. Write $X=R(B)\oplus M$, where $M=R(F_3)$ is finite dimensional. Set $S'=BS\oplus M$. Then S' is dense in X. In fact if $x\in R(B)$, there is, by hypothesis a sequence $\{x_n\}\subset S$ such that $x_n\rightarrow B'x$, $Bx_n\rightarrow BB'x=x$. But $Bx_n\in S'$. Now suppose $\{x_n\}\subseteq S'$ and $\|x_n\|\leq C$. Write $x_n=x_n'+x_n''$, where $x_n'\in R(B)$ and $x_n''\in M$. Then there is a $z_n\in S$ such that $Bz_n=x_n'$. Since $\|Bz_n\|\leq$ const., we can choose the z_n to be uniformly bounded. Thus by hypothesis $\{A'(A-B)z_n\}$ has a convergent subsequence. But $B'Bz_n=B'x_n$, or $z_n=B'x_n+F_3z_n$. Since F_3 has finite rank, we see that $\{A'(A-B)B'x_n\}$ has a convergent subsequence. We now apply Corollary 4.11.

In the next two theorems we assume that $D(A)\subseteq D(B)$.

THEOREM 4.13. *If there is a linear manifold S dense in $D(B)$ in the graph norm such that* (4.7) *implies that* $\{(A-B)A_\lambda' z_n\}$ *has a convergent subsequence, then*

$$\sigma_e(B)\subseteq\sigma_e(A). \tag{4.8}$$

Proof. We first note that we may assume that $N(A')=R(F_2)$ is contained in $D(B)$. For a slight adjustment of basis vectors of this set will get them into the dense set $D(B)$ while such an adjustment is tantamount to adding to A' an operator of finite rank. Now on $D(B)$ we have

$$B-\mu-B(A-\mu)A'=-\mu(A-B)A'-(B-\mu)F_2. \tag{4.9}$$

Because of the assumption $D(A)\subseteq D(B)$, the domains of all of the operators in (4.9) are the same, namely $D(B)$. If $\mu\in\Phi_A$, then $T=B(A-\mu)A'\in\Phi(X)$, and since B is closed, we may apply Theorem 3.3 to conclude that $\mu\in\Phi_B$. Note that we cannot go in the opposite direction, since knowledge that $T\in\Phi(X)$ does not do us any good concerning $A-\mu$ (Theorem 3.5).

THEOREM 4.14. *Assume that there is a linear manifold S dense in $D(BA)$ with respect to the graph norm such that*

$$\|x_n\|_{D(BA)}\leq C,\quad x_n\in S$$

implies that $\{(A-B)x_n\}$ *has a convergent subsequence. Then* (4.8) *holds.*

Proof. We can reduce this theorem to the preceding one, but it is even easier to deduce it directly from the identity

$$(B-\mu)A-B(A-\mu)=-\mu(A-B)$$

holding on $D(BA)$, and to follow the same reasoning.

5. Remarks.

5.1. In the Russian literature operators satisfying properties 1, 3-5 are called Φ-operators, with the Φ standing for Fredholm. The term Fredholm operator is reserved for Φ-operators having index 0. We have added property 2 (density of the domain) for convenience and do not find it practical to differentiate on the basis of index.

5.2. For bounded operators, Lemmas 2.3 and 2.4 are due to Atkinson [1], Yood [5] and Mikhlin (cf. [20]). Mikhlin calls A_1 (or A') a *regularizer* of A.

5.3. Theorem 2.5 is due to Atkinson [1] for bounded operators and to Gohberg [8] for unbounded operators (cf. also the A. M. S. Translation of [9]). Our proof that AB is closed was taken from Kato [10], while our proof of (2.10) is taken from [9]. Our proof that $R(AB)$ is closed seems new.

5.4. The idea for Lemma 2.7 and its proof came from Kato [10].

5.5. For the histories of Theorems 2.8 and 2.9 cf. [9]. Our proof of (2.21) appears to be new and much simpler than any found in the literature.

5.6. Theorems 2.10 and 2.11 as well as the device employed in obtaining them from Theorems 2.8 and 2.9 are due to Nagy [7].

5.7. To the best of our knowledge, all of the results of Section 3 are new.

5.8. Theorem 4.1 seems to be new. A similar result for bounded operators is given by Yood [5], from which some of the ideas of proof were borrowed.

5.9. The term essential spectrum originated in [12] where it was applied to self-adjoint problems for ordinary differential equations on a half-line. In that paper the term was applied to that part of the spectrum which remains invariant under changes in the boundary conditions. Several definitions are found in the literature. The one employed here was introduced in [13]. Other definitions are due to Wolf [14] and Browder [15]. They all coincide for the case of a self-adjoint operator in Hilbert space.

5.10. Theorem 4.10 generalizes a device employed by Birman [16], Wolf [17] and Rejto [18]. It has the advantage of not requiring a priori knowledge of the spectra of A and B. Theorems 4.13 and 4.14 generalize results of [13].

5.11. Corollary 3.8 was suggested by a result due to Mikhlin (cf. [20]). It was proved by him for the case $\alpha(E) = 0$.

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