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AN EXTENDED INTERPOLATION INEQUALITY (*)

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1. Several years ago E. Gagliardo [1] and the author [3] independently derived a general class of elementary interpolation inequalities. Let $\mathcal{D}$ be a bounded domain in R^n having the « cone property » (see [1]). For $-\infty < p < \infty$, and for functions u defined in $\mathcal{D}$ we introduce norms and seminorms :

For $p > 0$ set

$$| u |_p = \left(\int_{\mathcal{D}} | u |^p \, dx \right)^{\frac{1}{p}}.$$

For $p < 0$ set $s = \left[-\frac{n}{p} \right]$, $-\alpha = s + \frac{n}{p}$ and define

$$| u |_p = \sup | D^s u | \qquad \text{if} \quad \alpha = 0$$

$$| u |_p = \max \, [D^s u]_\alpha \qquad \text{if} \quad \alpha > 0$$

where

$$[f]_\alpha = \sup_{x \neq y \in \mathcal{D}} \frac{| f(x) - f(y) |}{| x - y |^\alpha}. \tag{1}$$

Here D^s represents any partial derivative of order s, and the maximum and sup are also taken with respect to all such partial derivatives. $| D^j u |_p$ will denote the maximum of the $| \; |_p$ norms of all the j — th order derivatives of u. The result proved in [1] and [3] is

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THEOREM 1: *For $1 \leq q,\ r \leq \infty$ suppose u belongs to L_q and its derivatives of order m belong to L_r in $\mathcal{D}$. Then for the derivatives $D^j u$, $0 \leq j < m$ the following inequalities hold* (with constant C_1, C_2 depending only on $\mathcal{D}$, *m, j, q, r*).

(2) $$| D^j u |_p \leq C_1 | D^m u |_r^a \, | u |_q^{1-a} + C_2 | u |_q ,$$

where

(3) $$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q},$$

for all a in the interval

(4) $$\frac{j}{m} \leq a \leq 1,$$

unless $1 < r < \infty$ and $m - j - n/r$ is a nonnegative integer, in which case (1) *holds only for a satisfying $j/m \leq a < 1$.*

In [1] the result is proved for $j/m < a \leq 1$ while in [3] it is only stated for domains with smooth boundaries. However the result holds for bounded domains with the cone property since, as described in [1], such a domain may be covered by a finite number of subsets, each of which is the union of a set of parallelepipeds which are translates of each other.

The range of values of p in the theorem given by (3) and (4) is sharp. In a recent paper [2] C. Miranda proved that if u satisfies a Hölder condition then the derivatives $D\ u$ belong to L_p for a wider range of values of p. However he does not give the sharp range and it is the aim of this paper to do so, i. e., to extend Theorem 1 to negative values of q:

THEOREM 1′: *Suppose that $1/q = -\beta/n$, $\beta > 0$, then* (2) *holds for $\beta \leq j < m$, with p given by* (3), *for all a in the interval*

(4)′ $$\frac{j-\beta}{m-\beta} \leq a \leq 1$$

with the same exception as in Theorem 1.

The value $\dfrac{j-\beta}{m-\beta}$ is the smallest possible value for a. This may be seen by taking $u = \sin \lambda x_1$; for large λ we have $| u |_q = 0(\lambda^\beta)$, $| D^j u |_p = 0(\lambda^j)$, $| D^m u |_r = 0(\lambda^m)$ where no 0 can be replaced by o.

We remark, as in [3], that Theorem 1′ also holds in an infinite domain of the form $-\infty < x_s < \infty$, $0 < x_t < \infty$; $s = 1, \dots, k$, $t = k+1, \dots, n$ — with the constant $C_2 = 0$. This follows by applying (2) to a large cube and letting its side length tend to infinity.

2. In this section we reduce the proof of Theorem 1′ to an inequality in one dimension with the aid of some simplifying remarks. (i) It suffices to consider the case $0 \leq \beta \leq 1$; for the general result follows by applying this case to derivatives of u. (ii) Furthermore we need only consider the extreme values of a since the intermediate values are then handled by elementary interpolation inequalities (as in [1] and [3]). Thus we need only treat the extreme value

$$a = \frac{j - \beta}{m - \beta}, \qquad p = r \frac{m - \beta}{j - \beta}, \tag{4}$$

for which we shall prove the inequality (recall (1));

$$| D^j u |_p \leq C_1 \, | D^m u |_r^a \, [u]_\beta^{1-a} + C_2 \, [u]_\beta . \tag{5}$$

In case $\beta = 1$ this is slightly sharper than (2).

(iii) It suffices to prove (5) for $j = 1$. The result for larger j then follows by interpolating between 1 and m with the aid of Theorem 1 applied to the first derivatives of u-as one readly verifies.

(iv) For $j = 1$, $a = \dfrac{1 - \beta}{m - \beta}$ we have $ap = r$; thus, taking p^{th} power of (5), the inequality may be formulated as, — with some constant C —.

$$\int_{\mathcal{D}} |Du|^p \, dx \leq C \, [u]_\beta^{p-r} \int_{\mathcal{D}} |D^m u|^r \, dx + C \, [u]_\beta^p . \tag{5)$'$}$$

We see from the form of (5)′ that it suffices to prove it for the individual parallelepipeds whose unions cover $\mathcal{D}$, and then sum over them. There is no loss of generality in proving (5)′ for the unit cube — to which the general parallelepiped may be trunsformed after a change of variable.

Finally we remark that the inequality (5)′ for any first order derivative parallel to a cube side follows from the one dimensional result by integrating with respect to the other variables.

(v) Thus we have reduced the proof of (5)′ to the one dimensional case on the unit interval I. We shall prove

$$\int_I |Du|^p \, dx \leq K^p \, [u]_\beta^{p-r} \int_I | D^m u |^r \, dx + K^p \, [u]_\beta^p \tag{6}$$

where K is a constant depending only on m. This was also the situation in [1] and [3] where the result was reduced to the critical case in one dimension, and our proof of (6) will be similar to the proof of the critical inequality (2.6) in [3]. We shall use K to denote various constants depending only on m. Since the cases $r = 1$ and ∞ follow as limiting cases we shall confine ourselves to the case $1 < r < \infty$.

3. PROOF OF (6): We make use of the following inequality (analogous to (2.7) in [3]). Because $p > mr$ the inequality follows readily in fact from Theorem 1.

On an interval λ, whose length we also denote by λ, we have

$$\int_\lambda |Du|^p\,dx \leq K^p \lambda^{1-p+mp-p/r}\left(\int_\lambda |D^m u|^r\,dx\right)^{p/r} + K^p \lambda^{1-p} \max_\lambda |u|^p$$

where K is a constant depending only on m.

Since we may add a costant to u we may replace $\max\limits_\lambda |u|$ by $C[u]_\beta \lambda^\beta$ to obtain:

$$\int_\lambda |Du|^p\,dx \leq K^p \lambda^{1-p+mp-p/r}\left(\int_\lambda |D^m u|^r\,dx\right)^{p/r} + K^p [u]_\beta^p \lambda^{1-p+p\beta}. \tag{7}$$

In proving (6) we may suppose that $\int_I |D^m u|^r\,dx = 1$.

We shall cover the interval $[0,1/2]$ by a finite number of successive intervals $\lambda_1, \lambda_2, \ldots$ starting at the left, each having as initial point the end point of the preceding. Consider, first, inequality (7) on the interval $\lambda : 0 \leq x \leq 1/2$. If the second term on the right of (7) is greater than the first, set $\lambda_1 = \lambda$. We then have

$$\int_{\lambda_1} |Du|^p\,dx \leq 2K^p [u]_\beta^p\, 2^{p-1-p\beta}. \tag{8}$$

If the first term is the greater, decrease the interval (keeping its left end point fixed) until the two terms on the right of (7) are equal. Since the terms on the right of (7) are respectively increasing and decreasing functions of λ, and the second tends to ∞ as $\lambda \to 0$, equality of these terms must hold for some λ. In fact since $\| D^m u \|_r = 1$ the length λ of the resulting interval λ will satisfy

$$\lambda^{m-1/r-\beta} \geq [u]_\beta .$$

Call the resulting interval λ_1. The right side of (7) for the interval is then twice the first term raised to the power r/p times the second raised to the power $(1 - r/p)$, and one finds

$$\int_{\lambda_1} |Du|^p \, dx \leq 2K^p \, [u]_\beta^{p-r} \int_{\lambda_1} |D^m u|^r \, dx \,. \tag{9}$$

If λ_1 does not cover $[0,1/2]$ we continue this process until $[0,1/2]$ is covered, considering first the interval $[\lambda_1, \lambda_1 + 1/2]$ and shrinking it if necessary, and so on. On every interval of the covering we will have the estimate (9) except for the last interval on which (8) will hold. Repeat this construction for the interval $[1/2,1]$ starting at the right end point. In the end, I will be covered by a finite number of intervals with each point contained in the interior of at most two covering intervals. For each covering interval (9) holds except for two of them, on each of which, however (8) holds. Summing therefore over all the intervals we obtain the desired inequality (6).

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