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THE UNBOUNDED GROWTH OF SOLUTIONS OF LINEAR PARABOLIC DIFFERENTIAL EQUATIONS (*)

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Introduction.

The object of this paper is to study the behavior for large t of positive solutions $u(x, t)$ of the general second order linear parabolic inequality

$$(0) \qquad Lu + cu - u_t \leq 0,$$

where

$$Lu \equiv a_{ij} u_{ij} + b_i u_i .$$

Here the summation convention is used, and subscripts on u denote derivatives: $u_t = \frac{\partial u}{\partial t}$, $u_i = \frac{\partial u}{\partial x_i}$, etc. The coefficients are functions of $x = (x_1, x_2, \ldots, x_n)$ and t. The parabolicity is assumed to be uniform in t but not necessarily in x.

When $c \leq 0$ and the above inequality is reversed, a variety of theorems (see, for example [3-5]) are available concerning the limiting behavior as $t \to \infty$ of solutions defined in $E^n \times (0, \infty)$. An example from [4] is the following. Suppose $\Sigma (a_{ii} + b_i x_i) > \alpha > 0$ for all x and t, and suppose $u(x, t)$ satisfies $Lu - u_t = 0$ for $t > 0$. If $\lim_{|x| \to \infty} u(x, 0) = 0$, then $\lim_{t \to \infty} u(x, t) = 0$ uniformly in x.

In the present paper, on the other hand, we require that $c > 0$ for certain values of its arguments, and investigate under what conditions positive solutions will approach ∞ as $t \to \infty$. The results show that the behavior of positive solutions as $t \to \infty$ is intimately related to the possible

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behavior as $|x| \to \infty$ of positive solutions $V(x)$ of the corresponding elliptic inequality (with $c = 0$).

More specifically, the results in a large part framed in terms of barriers and antibarriers of L. These concepts were used in [6] and elsewhere; they are positive functions $V(x)$ satisfying $LV \leq 0$ for large $|x|$ and for each t. Barriers approach 0 and antibarriers approach ∞ as $|x| \to \infty$. It is known from [6] that if an operator has a barrier, it cannot have an antibarrier. Explicit conditions on the coefficients are given in [6] (see our corollaries to Theorems 1 and 4) which insure that one or the other of these functions exists. In the case of the Laplace operator, a barrier exists when $n \geq 3$, and an antibarrier when $n \leq 2$. If an antibarrier exists, $c \geq 0$, and $c \not\equiv 0$, then positive solutions of the parabolic problem treated here tend to infinity exponentially (Theorem 1). If a barrier exists, this is not always true (Theorem 4), but is true in any case if $c \geq 0$ and c is large enough for x in some domain (Theorem 2). Theorem 3 shows that exponential growth is possible even when $c(x, t) \to -\infty$ as $|x| \to \infty$ provided that c is large enough for x in a fixed domain and a growth condition is placed on the coefficients. Furthermore an explicit lower bound for $u(x, t$ can be obtained in this case.

A regular solution of (0) will be taken to mean a function continuous for $t \geq 0$ whose second spacial derivatives and first time derivatives are continuous for $t > 0$, and which satisfies (0) for $t > 0$.

The following functions will be used extensively in the argument:

$$A(x, t) = a_{ij}(x, t)\, x_i x_j / |x|^2$$

$$B(x, t) = a_{ii}(x, t) + b_i(x, t)\, x_i .$$

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1. Main Theorems.

THEOREM 1. *Assume there exist positive continuous functions* $\varkappa(x)$ *and* $M(x)$ *such that for all* $\xi_1, \xi_2, \dots, \xi_n$,

$$a_{ij}(x, t)\, \xi_i \xi_j \geq \varkappa(x) \sum_1^n \xi_i^2 \; ; \tag{1}$$

$$|a_{ij}(x, t)|, \; |b_i(x, t)| \leq M(x). \tag{2}$$

Assume there exists an antibarrier $V(x)$ *defined in a neighborhood of infinity such that* $LV \leq 0$ *for each* t, *and* $V(x) \to \infty$ *as* $|x| \to \infty$.

Let $c(x, t)$ *be a function satisfying* $c(x, t) \geq 0$ *and* $c(x, t) \geq c_1$ *for* $|x| < \delta$ *where* c_1 *and* δ *are some positive numbers. Let* $u(x, t) \geq 0$ *be a regular solution of* $Lu + cu - u_t \leq 0$ in $E^n \times (0, \infty)$. *If* $u(x, 0) \not\equiv 0$ *then*

$$\lim_{t \to \infty} u(x, t) = \infty;$$

in fact, there is a function $\psi(x) > 0$ *and a number* $\gamma > 0$ *such that*

$$u(x, t) \geq \psi(x) e^{\gamma t} \tag{3}$$

for $t \geq 1$.

Remark 1. It will be clear from the proof that in place of (2) one need merely assume that $A(x, t)$ and $B(x, t)$ are locally bounded from above, uniformly in t:

$$A(x, t) \leq \bar{A}(x); \quad B(x, t) \leq \bar{B}(x) \tag{2'}$$

Remark 2. For $\varrho > 0$ let $\Gamma_\varrho = \{x: |x| < \varrho\}$. The theorem may be strengthened slightly by requiring only that u be a positive solution of $Lu + cu - u_t \leq 0$ in $\Gamma_\varrho \times (0, \infty)$. Then the conclusion (3) holds for $x \in \Gamma_\varrho$ provided that ϱ is large enough (depending on L and c). In fact, it will be clear from the proof that ϱ may be chosen $\geq R$, where R is defined following (7) below.

Corollary. *Let* $c(x, t)$ *and* $u(x, t)$ *be as in Theorem* 1. *If* $\dfrac{B(x, t)}{A(x, t)} \leq 2 + \varepsilon(|x|)$ *for large enough* x *and for all* t, *where* ε *is such that*

$$\int^{\infty} \exp\left\{-\int^{r} \varepsilon(s)\, ds / s\right\} dr / r = \infty,$$

then the conclusion (3) *follows.*

The corollary follows because under these conditions an antibarrier is constructed explicity in [6].

Proof of Theorem 1. It follows from Nirenberg's strong maximum principle that $u(x, t) > 0$ for $t > 0$. By shifting the origin of the t axis if necessary, we may assume with no loss of generality that $u > 0$ for $t = 0$ as well.

We say that a function $f(x, t)$ has *Property P* if it is continuous, has bounded piecewise continuous first derivatives, and is twice continuously differentiable in x except on a finite number of smooth surfaces. Furthermore, the directional derivative in the direction of traversal of such a surface suffers a *nonpositive* jump discontinuity. The following simple extension of the maximum principle holds, as noted for example by Il'in [4, 5].

Maximum principle: Let S be a bounded domain in $x - t$ space contained between the planes $t = 0$ and $t = t_1$. Let $u(x, t)$ be a function defined in S satisfying Property P and $Lu + cu - u_t \leq 0$ wherever u is regular. If $u \geq 0$ on the boundary of S exclusive of points with $t = t_1$, then $u \geq 0$ throughout S.

We shall need the following lemma, whose proof will be given in section 3.

LEMMA 1. *Under the hypotheses of Theorem* 1, *there is a function* $V_1(x)$ *defined for* $|x| \geq \delta$ *satisfying Property P and also*:

$$LV_1(x) < 0 \tag{4}$$

for all t and all regular points of V_1;

$$V_1(x) = 0 \text{ for } |x| = \delta; \tag{5}$$

$$V_1(x) \to \infty \text{ as } |x| \to \infty.$$

$$\text{Set } w(x) = \begin{cases} \beta_1 - |x|^2 & \textit{for } |x| \leq \delta, \\ \beta_1 - \delta^2 - \beta_2 V_1(x) & \textit{for } |x| \geq \delta, \end{cases}$$

where β_1 will be given below, and β_2 is taken small enough so that $(-w)$ will satisfy Property P.

For $|x| < \delta$,

$$Lw + cw = -2B(x, t) + cw \geq -2B + c_1(\beta_1 - \delta^2). \tag{6}$$

Now choose β_1 so large that $\beta_1 - \delta^2 > 0$, and also

$$Lw + cw \geq 1 \tag{7}$$

for $|x| < \delta$. Since $V_1 \to \infty$, there exists a number $R > 2\delta$ such that $w < 0$ for $|x| \geq R/2$.

Let Ω be the set of points in E^n for which $w(x) > 0$.

LEMMA 2. *Let* $m(t) = \inf_{\Omega} \dfrac{u(x,t)}{w(x)}$. *There are constants* $K > 0$, $\gamma > 0$ *such that* $m(t) \geq K e^{\gamma t}$.

Proof:

Let α be a real number such that $0 < \alpha \leq 1$ and α is small enough that

$$w + 2\alpha \leq 0 \text{ for } |x| \geq R; \tag{8}$$

$$Lw = -\beta_2 L V_1 \geq \alpha \text{ for } \delta < |x| < R. \tag{9}$$

Let $v(x,t) = w(x) + \alpha t$, and let $S = \{(x,t): v(x,t) > 0;\ 0 < t \leq 2\}$. One sees that in S,

$$Lv + cv - v_t \geq 0. \tag{10}$$

In fact, by (7) we know that for $|x| < \delta$, $Lv + cv - v_t = Lw + cw + c\alpha t - \alpha \geq 1 - \alpha \geq 0$. And by (9), $Lv + cv - v_t \geq Lw - \alpha \geq \alpha - \alpha = 0$ in the remainder of S as well (note that (8) insures that S is contained in the cylinder $\{|x| < R\}$).

Let $\partial_1 S$ be the lateral boundary of S; i. e., the part of the boundary for which $0 < t < 2$; and let $\partial_0 S$ be the base $\Omega \times \{t = 0\}$.

Let $\zeta(x,t) = \dfrac{u(x,t)}{m(0)} - v(x,t)$. By the definition of $m(0)$, $\zeta \geq 0$ on $\partial_0 S$. On $\partial_1 S$, $\zeta = u/m(0) \geq 0$ by assumption. Finally in S, we know from (10) that $L\zeta + c\zeta - \zeta_t \leq 0$ except at the irregular points of ζ. However $-v$, hence ζ, has Property P, so we conclude that $\zeta \geq 0$ in S; i. e.

$$\frac{u}{m(0)} \geq w + \alpha t,$$

or

$$\frac{u(x,t)}{w(x)} \geq m(0)\left(1 + \frac{\alpha t}{w(x)}\right) \geq m(0)\left(1 + \frac{\alpha t}{\beta_1}\right). \text{ Setting } t \geq 1$$

and taking the infemum with respect to x in Ω, we obtain the conclusion

$$m(t) \geq m(0)\left(1 + \frac{\alpha}{\beta_1}\right) = km(0) \text{ for } 1 \leq t \leq 2.$$

Applying this result successively yields $m(t) \geq k^N m(0)$, for $N \leq t \leq N+1$. The conclusion of the lemma follows immediately, for some K and γ which could be found in terms of k and $m(0)$. We now state a lemma which will be proved in section 3.

LEMMA 3. *Let L satisfy* (1) *and* (2′), *and let $u(x,t) \geq 0$ be a solution of $Lu - u_t \leq 0$ in $E^n \times [0,1]$. Let $\delta > 0$. Then there is a positive function $\chi(x)$ depending only on $\varkappa$, $\bar{A}$, $\bar{B}$, and δ such that*

$$u(x,1) \geq u_0\,\chi(x)$$

provided $u(x,0) \geq u_0$ for $|x| \leq \delta$.

The proof of Theorem 1 is now completed by combining Lemmas 2 and 3 as follows. For fixed t set $u_0 = (\beta_1 - \delta^2)\,Ke^{\gamma t}$. Thus for $|x| \leq \delta$, $u(x,t) \geq \geq Ke^{\gamma t}\,w(x) \geq u_0$. Hence by Lemma 3, $u(x, t+1) \geq \psi(x)\,e^{\gamma(t+1)}$, where we have set $\psi(x) = (\beta_1 - \delta^2)\,Ke^{-\gamma}\,\chi(x)$. This finishes the proof.

THEOREM 2. *Assume the hypotheses of Theorem 1 to hold, except that it is no longer required for an antibarrier to exist. There is a number C depending only on $\varkappa$, M, and δ such that the conclusion* (3) *of Theorem* 1 *holds provided that $c_1 \geq C$.*

REMARK: The remarks following Theorem 1 apply here as well. In this case ϱ may be chosen arbitrarily, except that $\varrho > \delta$; then C will depend on ϱ also. To see this one chooses β_1 below so that $0 < \beta_1 - \delta^2 < \beta_2\,\varphi(\varrho)$.

PROOF of THEOREM 2. The proof utilizes the following lemma.

LEMMA 4. *Assume hypotheses* (1) *and* (2′) *of Theorem* 1. *Then for any $\delta > 0$ there exists a function $\varphi(r)$ defined for $r \geq \delta$, satisfying $\varphi(\delta) = 0$, $\varphi'(r) > 0$ for $r \geq \delta$, and $L\varphi(|x|) < 0$.*

PROOF. Let $\tau(r) = \operatorname*{Max}_{\substack{|x|=r\\ \text{all } t}} \dfrac{B(x,t)}{A(x,t)}$, and set

$$\varphi'(r) = \exp\left[-\int_\delta^r (\tau(\varrho)/\varrho)\,d\varrho\right] > 0,$$

from which $\varphi(r)$ may be determined. We find that

$$L\varphi(|x|) = A(x,t)\left[\varphi'' + \left(\frac{B}{A} - 1\right)\varphi'/|x|\right] < A(x,t)\left[\varphi'' + \frac{\tau(|x|)}{|x|}\right]\varphi' = 0,$$

which establishes Lemma 4.

Let

$$w(x) = \begin{cases} \beta_1 - |x|^2 & \text{for } |x| \leq \delta, \\ \beta_1 - \delta^2 - \beta_2\, \varphi(|x|) & \text{for } |x| \geq \delta, \end{cases}$$

where β_2 is chosen small enough so that $(-w)$ satisfies Property P, and β_1 is chosen so that $0 < \beta_1 - \delta^2 < \beta_2 \lim_{r \to \infty} \varphi(r)$. Now observe that (6) holds. It remains only to choose c_1 large enough so that (7) holds for $|x| < \delta$. By virtue of the choice of β_1, there still exists a finite R such that $w < 0$ for $|x| \geq R/2$. The remainder of the proof of Theorem 1 is repeated verbatim.

THEOREM 3. *Let $u(x, t)$ be a regular solution of*

$$L_1 u \equiv Lu + cu - u_t \leq 0 \tag{11}$$

in $S_0 = E^n \times (0, \infty)$. We assume that

(i) *there exist $A', B', C' > 0$, $0 \leq \alpha \leq 2$ such that*

$$|a_{ij}| \leq A'(|x|^{2-\alpha} + 1),\ |b_i| \leq B'(|x| + 1),\ |c| \leq C'(|x|^\alpha + 1),$$

(ii) $$a_{ij}\, \xi_i\, \xi_j \geq \varkappa\, |\xi|^2, \qquad \varkappa = \text{const} > 0,$$

(iii) *there exist constants β, γ ($\gamma > 0$, $\beta > \beta_0$, β_0 being a positive number which depends among others on A', B', C' and $\varkappa$) and a point $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ such that for large t's, say $t \geq t_0 \geq 0$, the following inequality is satisfied:*

$$\beta^2\, \Sigma a_{ij}(x_i - \bar{x}_i)(x_j - \bar{x}_j) - \beta\, [\Sigma a_{ii} + \Sigma b_i (x_i - \bar{x}_i)] + c - \gamma \geq 0,$$

iv) $u(x, t) \geq -M \exp(K|x|^\alpha)$, *if* $0 < \alpha \leq 2$,

and

$$u(x, t) \geq -M_0(|x|^{K_0} + 1), \text{ if } \alpha = 0,$$

for some

$$M, K, M_0, K_0 > 0,$$

(v) $u(x, 0) \geq 0$,

(vi) $u(x, 0) \not\equiv 0$.

Then $\lim_{t \to \infty} u(x, t) = +\infty$, *the convergence being of exponential order and uniform on every compact x-set. More precisely, there exists $\lambda > 0$ such that*

$$u(x, t) \geq \lambda \exp\left[-\frac{\beta}{2}|x - \bar{x}|^2 + \gamma(t - t_0)\right] \text{ for } t \geq t_0. \tag{12}$$

REMARK: The following example shows the necessity of assumption (iii): the equation

$$u_{xx} + (2 - 4x^2)\, u - u_t = 0$$

has the bounded solution $u = e^{-x^2}$. In this example (iii) is not fulfilled.

PROOF. We shall make use of the following Theorem T (see [1] Theorem 1): If $u(x, t)$ is a regular solution of $L_1 u \leq 0$ in S_0 satisfying (iv) and (v) and if the coefficients satisfy (i) and $a_{ij}\, \xi_i\, \xi_j \geq 0$, then $u(x, t) \geq 0$ in S. (Note that the both-sided growth condition for u in the first part of Theorem I in [1] and the stronger growth condition in case of $\alpha = 0$ are not essential. Furthermore, Theorem T is valid for any domain D contained in S_0; this again follows from Bodanko's paper).

By this theorem $(u(x, t) \geq 0$ in S_0. Now by (VI) and by Nirenberg's strong maximum principle, $u(x, t) > 0$ in S_0.

At first we shall show that there are $\lambda, \beta_0 > 0$ such that

$$u(x, t_0) \geq \lambda \exp\left(-\frac{\beta_0}{2} \mid x - \bar{x} \mid^2\right) \text{ for } x \in E^n. \tag{13}$$

For this purpose consider the function

$$v(x, t) = \exp\left(-\frac{\mu \mid x - \bar{x} \mid^2 - \nu}{t - t_0 + \eta}\right), \; 0 < \eta < t_0.$$

The positive constants μ, ν can be chosen so that $L_1 v \geq 0$ in $E^n \times [t_0 - \eta, t_0]$. Indeed, we have

$$L_1 v = \frac{v}{(t - t_0 + \eta)^2}\, [4\mu^2\, \Sigma a_{ij}\, (x_i - \bar{x}_i)(x_j - \bar{x}_j) - 2\mu(t - t_0 + \eta)\, \Sigma a_{ii}$$

$$- 2\mu\, (t - t_0 + \eta)\, \Sigma b_i (x_i - \bar{x}_i) + c\, (t - t_0 + \eta)^2 - \mu \mid x - \bar{x} \mid^2 + \nu].$$

Now we use the inequalities

$$t - t_0 + \eta \leq \eta, \mid x_i \mid \leq \mid x \mid, \mid \bar{x} \mid \cdot \mid x - \bar{x} \mid \leq \frac{1}{2}\, (\mid \bar{x} \mid^2 + \mid x - \bar{x} \mid^2),$$

$$\mid x \mid^2 \leq 2 \mid x - \bar{x} \mid^2 + 2 \mid \bar{x} \mid^2$$

to derive

$$L_1 v \geq \frac{v}{(t - t_0 + \eta)^2} \Big\{ [4\varkappa\, \mu^2 - (8\eta\, nA' + 4\eta\, nB' + 1)\, \mu - 4\eta^2\, C'] \mid x - \bar{x} \mid^2 -$$

$$- (8\eta\, n\mu A' + \eta n\, \mu B' + 4\eta^2\, C') \mid \bar{x} \mid^2 - 4\eta\, \mu A' - \eta n\, \mu B' - 2\eta^2\, C' + \nu \Big\}.$$

Let $\mu_0(\eta)$ be the largest root of the equation

$$4\varkappa\,\mu^2 - (8\eta\, nA' + 4\eta\, nB' + 1)\,\mu - 4\eta^2\, C' = 0.$$

Putting

$$\nu_0 = (8\eta\, n\mu_0\, A' + \eta n\, \mu_0\, B' + 4\eta^2\, C^2)\,|\,\bar{x}\,|^2 + 4\eta\, n\mu_0\, A' + \eta n\mu_0\, B' + 2\pi^2\, C',$$

we see that the function

$$\nu_0(x, t) = \exp\left(-\frac{\mu_0\,|\,x - \bar{x}\,|^2 - \nu_0}{t - t_0 + \eta}\right)$$

satisfies $L_1\, v_0 \geq 0$ in $E^n \times [t_0 - \eta, t_0]$. Let $R > \sqrt{\frac{\mu_0}{\nu_0}}$ be fixed. We define

$$\lambda = \min_{\substack{|\,x - x\,| \leq R \\ t \in [t_0 - \eta,\, t_0]}} u(x, t).$$

Since $u(x, t) > 0$ in S, λ is a positive number.

The function

$$w(x, t) = u(x, t) - \lambda v_0(x, t)$$

satisfies the inequality $L_1\, w \leq 0$ in $E^n \times [t_0 - \eta, t_0]$. Furthermore $w(x, t) \geq 0$ for $|\,x - \bar{x}\,| = R$, $t \in [t_0 - \eta, t_0]$ and for $|\,x - \bar{x}\,| \geq R$, $t = t_0 - \eta$. Now we are able to use Theorem T again to conclude that $w(x, t) \geq 0$ in the region $|\,x - \bar{x}\,| \geq R$, $t_0 - \eta \leq t \leq t_0$. Since

$$v_0(x, t) \geq \exp\left(\frac{\mu_0\,|\,x - \bar{x}\,|^2}{t - t_0 + \eta}\right),$$

we have

$$(14) \qquad u(x, t) \geq \lambda \exp\left(-\frac{\mu_0\,|\,x - x\,|^2}{t - t_0 - \eta}\right) \text{ for } |\,x - \bar{x}\,| \geq R,\ t_0 - \eta \leq t \leq t_0.$$

From the definition of λ it follows that inequality (14) remains true in the whole strip $E^n \times [t_0 - \eta, t_0]$. Substituting, in particular, $t = t_0$ and putting $\frac{\mu_0}{\eta} = \frac{\beta_0}{2}$ we get (13).

Now the function

$$z(x, t) = u(x, t) - \lambda \exp\left[-\frac{\beta}{2}\,|\,x - \bar{x}\,|^2 + \gamma\,(t - t_0)\right],\ \beta \geq \beta_0\,,\ t \geq t_0\,,$$

satisfies, by assumption (iii), $L_1 z \leq 0$ for $t \geq t_0$. Furthermore, from (13), it follows that $z(x, t_0) \geq 0$. Consequently, by Theorem T, $z(x, t) \geq 0$ for $t \geq t_0$, $x \in E^n$, which was to be proved.

2. A Countertheorem.

THEOREM 4. *Let the operator L satisfy* (1), (2′) *of Theorem* 1. *Also assume it has a barrier near infinity; i. e. a positive function $W(x)$ defined for $|x| \geq X$ such that $LW < 0$, and $W(x) \to 0$ as $|x| \to \infty$. Finally assume the coefficients a_{ij} and b_i are Hölder continuous functions of all their arguments. Then there exists a smooth function $c(x) \geq 0$ with $c(x) \not\equiv 0$, and a positive but bounded solution $u(x, t)$ of $Lu + cu - u_t = 0$.*

REMARK. This theorem shows a result by Szybiak [7] to be incorrect.

COROLLARY. *Let L satisfy* (1), (2′) *and have Hölder continuous coefficients as in Theorem* 4. *If $\dfrac{B(x, t)}{A(x, t)} \geq 2 + \varepsilon(|x|)$ for large enough x and for all t, where $\int^{\infty} \exp\left\{-\int^{r} \varepsilon(s)\, ds/s\right\} dr/r < \infty$, then the conclusion of Theorem 4 holds.*

The corollary follows because under these conditions a barrier is constructed explicitly in [6].

The proof of Theorem 4 will employ the following lemma.

LEMMA 5. *Assume hypotheses* (1) *and* (2′). *Then for any $\delta > 0$ there exists a function $\varphi_1(r)$ defined for $r \geq \delta$, satisfying $\varphi_1(\delta) = 0$, $\varphi_1'(r) > 0$ for $r \geq \delta$, and $L\varphi_1(|x|) \geq 0$.*

PROOF: The proof is the same as that of Lemma 4, except that now we set

$$\tau(r) = \operatorname*{Min}_{\substack{|x|=r \\ \text{all } t}} \left(\frac{B(x, t)}{A(x, t)} - 1\right). \tag{15}$$

PROOF OF THEOREM 4. Let $m = \operatorname*{Min}_{|x|=X} W(x)$ and $R_0 > X$ a number such that $W(x) \leq m/2$ for $|x| = R_0$. Let $\delta < X$ and σ be positive numbers so small that $L(|x|^2) = 2(a_{ii} + b_i x_i) \geq \sigma > 0$ for $|x| < \delta$.

Let

$$W_1(x) = \begin{cases} \alpha - \beta |x|^2, & |x| \leq \delta, \\ \operatorname{Min}\{\alpha - \beta\delta^2 - \varepsilon\varphi_1(|x|),\ W(x)\}, & |x| \geq \delta, \end{cases}$$

where φ_1 is the function obtained in Lemma 5, and $\alpha, \beta, \varepsilon$ are positive numbers chosen as follows. First ε is chosen so that $\varepsilon \varphi_1(R_0) = m/2$; then β is chosen so small that W_1 has Property P near $|x| = \delta$; finally α is chosen so that $\alpha - \beta\delta^2 = m$. Thus by construction $W_1(x)$ has Property P for all x, and satisfies

$$LW_1 \leq -\beta\sigma < 0, \; |x| < \delta;$$

$$LW_1 \leq 0, \qquad |x| \geq \delta.$$

Let $c(x) \geq 0$ be a function such that $c \equiv 0$ for $|x| \geq \delta$. Clearly

$$LW_1 + cW_1 - (W_1)_t \begin{cases} \leq -\beta + c\alpha & \text{for } |x| < \delta \\ \leq 0 & \text{for } |x| > \delta. \end{cases} \tag{16}$$

We require that $c \not\equiv 0$, but that c be small enough so that $LW_1 + cW_1 - (W_1)_t \leq 0$ for all x where defined.

Let $\varphi_0(x)$ be a smooth function satisfying $0 < \varphi_0(x) \leq W_1(x)$. We shall construct a solution $u(x, t)$ satisfying $0 < u(x, t) \leq W_1(x)$, $u(x, 0) = \varphi_0(x)$. For any $R > 0$, let Γ_R be the ball $\{|x| < R\}$, and let $u_R(x, t)$ be the solution in $\Gamma_R \times (0, \infty)$ of $Lu_R + cu_R - (u_R)_t = 0$; $u_R(x, 0) = \varphi_0(x)$; $u_R(x, t) = 0$ for $|x| = R$. Since $(W_1 - u_R)$ satisfies Property P and is nonnegative on the boundary, we have that $u_R(x, t) \leq W_1(x)$ for all R, and for $(x, t) \in \Gamma_R \times (0, \infty)$. Let $R \to \infty$; then the u_R form an increasing bounded sequence approaching some limit $u(x, t) \leq W_1(x)$. By the Schauder estimates [2], for each bounded set the derivatives $(u_R)_{ij}$ and $(u_R)_t$ are equicontinuous. A subsequence of the u_R is therefore termwise differentiable to these orders of differentiation. If follows that $Lu + cu - u_t = 0$, $u(x, 0) = \varphi_0(x)$, and $0 < u < W_1$. This completes the proof.

3. Proofs of the Lemmas.

Proof of Lemma 1. Suppose the antibarrier $V(x)$ is defined for $|x| \geq X > \delta$. Let $\varphi(|x|)$ be the function constructed in Lemma 4, and let $m_1 = \varphi(X)$, $m_2 = \varphi(X+1) > m_1$. Let $V_2(x) = a_1 V(x) + a_2$, where $a_1 > 0$ and a_2 are constants chosen so that

$$m_1 < V_2(x) < m_2 \tag{17}$$

for $X \leq |x| \leq X+1$. $V_2(x)$ is likewise an antibarrier. Let

$$V_3(x) = \begin{cases} \varphi(|x|), & \delta \leq |x| \leq X, \\ \text{Min}\,[\varphi(|x|),\ V_2(x)], & X \leq |x| \leq X+1, \\ V_2(x), & |x| \geq X+1. \end{cases}$$

Condition (17) assures that $V_3(x)$ will be continuous, and the fact that it is the minimum of two regular functions guarantees the proper jump relation for $V_3(x)$ to satisfy Property P. Finally let $V_1(x) = V_3(x) + \varphi(|x|)$; (4) follows since $L\varphi < 0$.

PROOF OF LEMMA 3. It follows from the strong maximum principle that there exists such a function χ depending on L; our task will be to find one depending only on $\varkappa, \bar{A}, \bar{B}$, and δ. For this we use two auxiliary lemmas.

LEMMA 6. *Let δ and u be as in Lemma 3. There is a number $\sigma > 0$ depending only on $\varkappa$, $\bar{A}, \bar{B}$, and δ such that*

$$u(x,t) \geq \sigma u_0 \tag{18}$$

for $|x| \leq \sigma\delta,\ 0 \leq t \leq 1$.

PROOF: We shall construct a function $v(x,t) = h(|x|)e^{-\lambda t}$ satisfying $Lv \geq 0$, $v = 0$ for $|x| = \delta$, and $v > 0$ for $|x| < \delta$. Setting $r = |x|$, we calculate

$$Lv - v_* = e^{-\lambda t}\{A(x,t)(h'' - h'/r) + (h'/r)B(x,t) + \lambda h\},$$

Hence

$$Lv - v_* \geq e^{-\lambda t}\{\varkappa(x)h'' + (\bar{B}(x) - \varkappa(x))h'/r + \lambda h\}$$

wherever $h'' \geq 0$, $h' \leq 0$; and

$$Lv - v_* \geq e^{-\lambda t}\{\bar{A}(x)h'' + (\bar{B}(x) - \varkappa(x))h'/r + \lambda h\}$$

wherever $h'' \leq 0$, $h' \leq 0$. We set $N_1 = \sup_{|x| \leq \delta}(\bar{B} - \varkappa)$, $N_2 = \sup_{|x| \leq \delta}\bar{A}(x)$, $N_3 = \inf_{|x| \leq \delta}\varkappa(x)$,
and define

$$h(r) = \begin{cases} \alpha - \beta r^2, & 0 \leq r \leq r_0 \\ (\delta - r)^2, & r_0 \leq r \leq \delta, \\ 0 & r \geq \delta, \end{cases}$$

where $r_0 = \text{Max}\left[\frac{\delta}{2}, \delta - \frac{N_3}{N_1}\right]$, and α and β are chosen so that h and h' are continuous at $r = r_0$. Then for $r_0 \leq r \leq \delta$,

$$Lv - v_* \geq e^{-\lambda t}[2N_3 - 2N_1(\delta - r_0) + \lambda h] \geq 0,$$

and for $0 \leq r \leq r_0$,

$$Lv - v_* \geq e^{-\lambda t}[-2\beta N_2 - 2\beta N_1 + \lambda(\alpha - \beta r_0^2)].$$

Since the coefficient of λ is positive, we may choose λ large enough so that $Lv \geq 0$ throughout.

Now let $V(x, t) = \frac{u_0}{\alpha} v(x, t)$; clearly $V \leq u$ for $t = 0$ and also for $r = \delta$. Hence $u(x, t) \geq V(x, t)$ for $r \leq \delta, t \geq 0$; in particular $u(x, t) \geq \frac{u_0}{\alpha} e^{-\lambda} h(r) \geq \sigma u_0$ for $r \leq \sigma\delta$, $0 \leq t \leq 1$, and an appropriately chosen σ.

LEMMA 7. *Given any numbers $R > 0$ and $p > 0$, there is a number p_1 depending on $\varkappa$, $\bar{A}$, $\bar{B}$, R, and p, such that if $u(x, t) > u_0 e^{-p/t}$, for $|x| = R$, $0 \leq t \leq 1$, then $u(x, t) \geq u_0 e^{-p_1/t}$ for $R \leq r \leq R + 1$, $0 \leq t \leq 1$.*

PROOF: We define $v(x, t) = u_0 f(t(R + 2 - r))$, where $f(s) = e^{-p_1/s}$, and p_1 will be determined later. Then

$$Lf - f_* = A(x, t)(t^2 f'' + tf'/r) - \left[\frac{B(x, t)t}{r} + (R + 2 - r)\right] f'.$$

We assume $p_1 \geq 4$; then $f'(s) \geq 0$ and $f''(s) \geq 0$ for $0 \leq s \leq 2$, and we have, setting $s = t(R + 2 - r)$,

$$\begin{aligned} Lf - f_* &\geq \varkappa t^2 f'' + \left[(\varkappa - \bar{B})\frac{t}{r} - 2\right] f' \\ &= \frac{\varkappa}{(R + 2 - r)^2}\left[s^2 f'' - \frac{2 + (\bar{B} - \varkappa)t/r}{\varkappa}(R + 2 - r)^2 f'\right] \\ &\geq \frac{\varkappa}{(R + 2 - r)^2}[s^2 f'' - Kf'] \end{aligned}$$

where

$$K = 2 \max_{R \leq r \leq R+2} \frac{2 + (\bar{B}(x) - \varkappa(x))/r}{\varkappa(x)}.$$

But $s^2 f'' - Kf' = (p_1/s)(-2 + (p_1 - K)/s) f \geq p_1(-4 + p_1 - K) f/2s$ for $0 \leq s \leq 2$, $p_1 \geq K$. Hence $Lv - v_* \geq 0$ for $p_1 \geq K + 4$. We now set

$$p_1 = \text{Max}\,[2p, K + 4]$$

so that $Lv - v_* \geq 0$ for $R \leq r \leq R + 2$, $0 \leq t \leq 1$; $v(x, t) = u_0\, e^{-p_1/2t} \leq u(x, t)$ for $r = R$; and $v(x, t) = 0 \leq u(x, t)$ for $t = 0$, and for $r = R + 2$. Hence $u(x, t) \geq v(x, t)$ in the annular cylinder $R \leq r \leq R + 2$, $0 \leq t \leq 1$. In particular for $r \leq R + 1$,

$$u(x, t) \geq u_0\, e^{-p_1/t}.$$

This completes the proofs of Lemmas 6 and 7.

Lemma 3 is proved now as follows. By assumption $u(x, 0) > u_0 > 0$ in some interval; suppose it is the interval $r \leq \delta$. Then Lemma 6 provides a lower bound for u in the region $r \leq \sigma\delta$, $t \leq 1$. Now apply Lemma 7 successively with $R_\nu = \sigma\delta + \nu$, $\nu = 0, 1, 2, \ldots$ to obtain the conclusion.

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