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Pointwise convergence of singular convolution integrals

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POINTWISE CONVERGENCE OF SINGULAR CONVOLUTION INTEGRALS

by JAAK PEETRE

0. Introduction.

This article is in a way of survey character. What we attempt to do is to relate known work on singular convolution integrals in R^n associated with the names Calderòn, Zygmund, Michlin, Hörmander, Cotlar and many others. Although we obtain no strictly new results we hope that this investigation will have some value, since, up to our knowledge, not much has been done previously in this sense, at least not in print.

We shall in the first place be concerned with pointwise convergence (almost everywhere, a. e.), one of our points being that pointwise convergence can be derived fairly easily directly from norm convergence, in fact a rather weak form of it, using only rather straight forward estimates and the maximal theorem of Hardy-Littlewood, thus roughly Lebesgue's theorem on differentiation of integrals. We believe that this method might be useful also in other more complicated cases.

To fix the ideas let us start with the simple case of the Hilbert transform on the real line $R = R^1$:

$$Hf(x) = \lim_{\varepsilon \to 0} H_\varepsilon f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy$$

There are basically two ways of interpreting $\lim\limits_{\varepsilon \to 0}$: *a*) Norm convergence (usually the L_p norm), *b*) Pointwise convergence a. e. The classical results in this direction are the following: In case *a*) convergence in the L_p norm holds for every $f \in L_p$ provided $1 < p < \infty$. In particular H maps into L_p

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(M. Riesz). For $p = 1$ this is obviously not true but the following weaker substitute holds: H maps L_p into the weak Marcinkiewicz space $\widetilde{L}_1$ (Kolmogorov). In case *b*) convergence a. e holds for every $f \in L_p$ provided $1 \leq p < \infty$ (Plessner and others). Our thesis is thus that the result in case *b*) follow from the results in case *a*) by the intermediary of the maximal theorem. This is particularly simple if $1 < p < \infty$. Then our basic idea is to consider the difference

$$\varphi_\varepsilon * Hf(x) - H_\varepsilon f(x), \quad \varphi_\varepsilon(x) = \frac{1}{\varepsilon}\,\varphi\left(\frac{x}{\varepsilon}\right), \quad \int \varphi(x)\,dx = 1.$$

We show that this is a convolution with a « nice » function a_ε, « nice » from the point of view of the maximal theorem, and thus tends to 0 a. e. On the other hand $\varphi_\varepsilon * Hf(x)$ tends to $Hf(x)$ a. e., again by the maximal theorem. This simple proof might also have some didactic value. It is also possible to modify the proof so as to cover the case $p = 1$, but then it is not any longer so elementary, and by the way very close to known ones; see, in particular, Cotlar [5], chapter III.

The theory of the Hilbert transform as outlined above can be extended in several directions. In the first place comes the so-called Calderòn-Zygmund transform which is convolution with a function $a(x)$ in R^n, homogeneous of degree $-n$ and with vanishing spherical meanvalues (see [3], [4], [20]). It is the study of this case (« homogeneous case ») and, in particular, the case of still more general functions $a(x)$ which will be the object of this article.

The plan is the following. In Section 1 we collect some preliminaries mainly concerned with the maximal theorem. In Section 2 we then give the proof of pointwise convergence a. e. in the case of the Hilbert transform, as outlined above. This proof we then (Section 3) extend to the case of convolution by a general fucntion $a(x)$ in R^n. We find that pointwise convergence a. e. holds if $a(x)$ satisfies certain Hypothesis 0, 1 and 2 of which Hypothesis 2 is the deeper one. In Section 4 we discuss this Hypothesis 2 and variants of it, Hypothesis 2′ and 2″ — the latter of interest of its on because it is precisely the condition for norm convergence given by Hörmander [7] — and relate them in particular to other conditions given by Cotlar [5], [6] on one hand and Michlin [10],[11], Hörmander [7] on the other hand. Finally (Section 5) we show that Hypothesis 2′ (stronger than Hypothesis 2) actually implies convergence a. e. also when $p = 1$. As we said above in this case our method is close to that of Cotlar [5]. We remark that for the understanding of this Section the reading of Section 4 is not necessary.

In conclusion I wish to thank Guido Weiss and Sven Spanne for several helpful suggestions in connection with this work.

1. Some preliminaries.

If $1 \leq p \leq \infty$ and f is a measurable function in R^n let us set

$$\|f\|_{L_p} = \left(\int |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

and

$$\|f\|_{\tilde{L}_p} = \sup_{\sigma} \sigma \left(\int\limits_{|f(x)| \geq \sigma} dx \right)^{\frac{1}{p}}$$

(If $p = 1$ we shall write simply $\|f\| = \|f\|_{L_1}$.) Denote by L_p and $\tilde{L}_p$ respectively the corresponding spaces. They are both linear. Moreover it is well-known that $\|f\|_{L_p}$ is a norm and that L_p is complete in this norm (F. Riesz-Fischer theorem). However $\|f\|_{\tilde{L}_p}$ is no norm but only a *quasi-norm* (i. e. in place of the ordinary triangle inequality we have the weaker inequality $\|f+g\|_{\tilde{L}_p} \leq k(\|f\|_{\tilde{L}_p} + \|g\|_{\tilde{L}_p})$ with $k > 1$.) For $1 < p \leq \infty$, but not for $p = 1$, it is equivalent to a norm. In any case $\tilde{L}_p$ is complete. We note also that $f \in L_p$ or $\tilde{L}_p$ if and only if $|f|^p \in L_1$ or $\tilde{L}_1$ respectively.

Let E be a Banach space A mapping T of E into $\Lambda = L_p$ or $\tilde{L}_p$ is said to be *quasi-linear* if

$$|T(f+g)(x)| \leq k(|Tf(x)| + |Tg(x)|),$$

$$|T(cf)(x)| \leq k\,|c|\,|Tf(x)|$$

for some k and all f, g, c, and *bounded* if

$$\|Tf\|_\Lambda \leq C\,\|f\|_E$$

for some C.

Consider a family of quasi-linear mappings T_ε, $\varepsilon > 0$, from E into Λ. We define (maximal function)

$$Mf(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|$$

It is obviously quasi-linear.

LEMMA 1. *Suppose M is bounded, as mapping from E into Λ, and $T_\varepsilon f(x)$ converges a. e. in some dense subset of E. Then $T_\varepsilon f(x)$ converges a. e. in the whole of E.*

Next we recall

LEMMA 2 (Marcinkiewicz interpolation theorem). *Suppose T is a bounded quasi-linear mapping from L_{p_0} into $\widetilde{L}_{p_0}$ and from L_{p_1} into $\widetilde{L}_{p_1}$, $1 \leq p_0 < p_1 \leq \infty$. Then T is also bounded from L_p into L_p or from $\widetilde{L}_p$ into $\widetilde{L}_p$ for any p with $1 < p < \infty$.*

We define also (Hardy-Littlewood maximal function)

$$\Theta f(x) = \sup_{r} \int_{|x-y| \leq r} |f(y)| \, dy / r^n \times \text{(volume of the unit sphere)}$$

It is obviously quasi-linear.

LEMMA 3 (Hardy-Littlewood maximal theorem). *θ is a bounded mapping from L_1 into $\widetilde{L}_1$ and, thus by Lemma 1 from L_p into L_p or from $\widetilde{L}_p$ into $\widetilde{L}_p$ $1 < p \leq \infty$.*

Finally we define

$$\| a \|^* = \sup_{f,\, x} | a * f(x) | / \theta f(x) = \sup_{f} | a * f(0) | / \theta f(0)$$

Obviously, since $\theta f(x) \geq f(x)$ (by Lemma 1), we have $\| a \|^* \geq \| a \|$.

LEMMA 4. *The norm $\| a \|^*$ is equivalent to the following one:*

$$\sum_{k=-\infty}^{\infty} 2^{kn} \sup_{2^k < |x| \leq 2^{k+1}} | a(x) |$$

We remark that it follows from Lemma 3 and Lemma 4 that Lemma 1 can be applied to

$$T_\varepsilon f(x) = a_\varepsilon * f(x)$$

provided

$$\sup_{\varepsilon > 0} \sum_{k=-\infty}^{\infty} 2^{kn} \sup_{2^k < |x| \leq 2^{k+1}} | a_\varepsilon(x) | < \infty$$

For the proof of Lemma 1-3 see e. g. Cotlar [5], chapter III. For the one of Lemma 4 see e. g. Alexits [1], p. 240-246.

2. The case of the Hilbert transform, $1 < p < \infty$.

Consider thus again

$$Hf(x) = \lim_{\varepsilon \to 0} H_\varepsilon f(x) = \lim_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy$$

We assume that it is known that $\lim\limits_{\varepsilon \to 0}$ exists in some sense, say, distribution sense, for every $f \in L_p$ and defines a bounded linear mapping H from L_p into L_p (weak form of M. Riesz theorem). (Here as in the rest of this and the following Section we take $1 < p < \infty$.) We claim that for every $f \in L_p$ holds

$$\varphi_\varepsilon * Hf(x) - H_\varepsilon f(x) = a_\varepsilon * f(x)$$

where

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\, \varphi\left(\frac{x}{\varepsilon}\right), \qquad \int \varphi(x)\, dx = 1, \quad \varphi(x) = 0 \text{ if } |x| > \frac{1}{2},$$

$$\varphi \text{ and } \varphi' \text{ bounded}$$

and

$$a_\varepsilon(x) = \begin{cases} H\varphi_\varepsilon(x) \text{ if } |x| \leq \varepsilon \\ \\ H\varphi_\varepsilon(x) - \dfrac{1}{x} \text{ if } |x| > \varepsilon. \end{cases}$$

Indeed this is certainly true when f is a « smooth » function, say, continuously differentiable and with compact support, so the general case follows by a density argument. We want to evaluate the norm $\| a \|^*$.

First let $|x| \leq \varepsilon$. Then we may write

$$a_\varepsilon(x) = \frac{1}{\varepsilon} \lim_{\delta \to 0} \int\limits_{|x-y|>\delta} \frac{1}{x-y}\, \varphi\left(\frac{y}{\varepsilon}\right) dy =$$

$$= \frac{1}{\varepsilon} \lim_{\delta \to 0} \int\limits_{\frac{3\varepsilon}{2} \geq |x-y| > \delta} \frac{1}{x-y}\, \varphi\left(\frac{y}{\varepsilon}\right) dy = \frac{1}{\varepsilon} \int\limits_{\frac{3\varepsilon}{2} \geq |x-y|} \frac{1}{x-y} \left(\varphi\left(\frac{y}{\varepsilon}\right) - \varphi\left(\frac{x}{\varepsilon}\right)\right) ax$$

for $|x| \leq \varepsilon$, $|y| \leq \frac{\varepsilon}{2}$ implies $|x-y| \leq \frac{3\varepsilon}{2}$ and moreover holds

$$\lim_{\delta \to 0} \int\limits_{\frac{3\varepsilon}{2} \geq |x-y| > \delta} \frac{1}{x-y}\, dy = 0$$

Hence by the mean value theorem, since φ' is bounded

$$(1) \qquad |a_\varepsilon(x)| \leq \frac{1}{\varepsilon} \int \frac{C\left|\frac{x}{\varepsilon} - \frac{y}{\varepsilon}\right|}{|x-y|}\, dx \leq \frac{C}{\varepsilon} \text{ if } |x| \leq \varepsilon .$$

Next let $|x| > \varepsilon$. Now

$$a_\varepsilon(x) = \frac{1}{\varepsilon} \lim_{\delta \to 0} \int\limits_{|x-y| \geq \delta} \frac{1}{x-y}\, \varphi\left(\frac{y}{\varepsilon}\right) dy - \frac{1}{x} =$$

$$= \frac{1}{\varepsilon} \int\limits_{|y| \leq \frac{\varepsilon}{2}} \left(\frac{1}{x-y} - \frac{1}{x}\right) \varphi\left(\frac{y}{\varepsilon}\right) dy = \frac{1}{\varepsilon} \int\limits_{|y| \leq \frac{\varepsilon}{2}} \frac{y}{x(x-y)}\, \varphi\left(\frac{y}{\varepsilon}\right) dy$$

But $|x| > \varepsilon$, $|y| \leq \frac{\varepsilon}{2}$ implies $|x-y| \geq \frac{|x|}{2}$. Hence, since φ is bounded,

$$(2) \qquad |a_\varepsilon(x)| \leq \frac{C}{\varepsilon} \frac{1}{|x|^2} \int\limits_{|y| < \frac{\varepsilon}{2}} |y|\, dy \leq \frac{C}{\varepsilon} \left(\frac{\varepsilon}{|x|}\right)^2 \text{ if } |x| > \varepsilon .$$

From (1) and (2) we conclude that

$$|a_\varepsilon(x)| \leq C \frac{\frac{1}{\varepsilon}}{1 + \left(\frac{|x|}{\varepsilon}\right)^2} \text{ for all } x$$

Hence by Lemma 4

$$\| a_\varepsilon \|^* \leq C \sum_{k=-\infty}^{\infty} \frac{\frac{1}{\varepsilon}}{1 + \left(\frac{2^k}{\varepsilon}\right)^2}\, 2^k \leq C$$

where the last bound is independent of ε. Thus we obtain

$$|H_\varepsilon f(x)| \leq C(\theta H f(x) + \theta f(x)).$$

Since the mapping defined by the right hand side is quasi-linear and by Lemma 3 bounded from L_p into L_p it follows by Lemma 1 that $H_\varepsilon f(x)$ converges a. e. to $Hf(x)$ for every $f \in L_p$. Thus we have proven Plessner's theorem in this special case.

REMARK 1. By Lebesgue's theorem on dominated convergence it follows now also that $H_\varepsilon f(x)$ converges in the L_p norm to $Hf(x)$ i. e. the stronger form of M. Riesz' theorem.

3. The general case, $1 < p < \infty$.

We now consider

$$Hf(x) = \lim_{\varepsilon \to 0} H_\varepsilon f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} a(x-y) f(y)\, dy$$

where $a(x)$ is an « arbitrary » function. We assume the analogue of M. Riesz' theorem, i. e. that $\lim_{\varepsilon \to 0}$ exists in, say, distribution sense and defines a bounded linear mapping H from L_p into L_p. (We recall that $1 < p < \infty$ by assumption.) Then holds again

$$\varphi_\varepsilon * Hf(x) - H_\varepsilon f(x) = a_\varepsilon * f(x)$$

where now

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right), \qquad \int \varphi(x)\, dx = 1,\ \varphi(x) = 0 \text{ if } |x| > \frac{1}{2}$$

φ and φ' (the gradient !) bounded

and

$$a_\varepsilon(x) = \begin{cases} H\varphi_\varepsilon(x) & \text{if } |x| \leq \varepsilon \\ H\varphi_\varepsilon(x) - a(x) & \text{if } |x| > \varepsilon \end{cases}$$

We assume

Hypothesis 0. $\displaystyle\int_{R' < |x| \leq R''} a(x)\, dx = 0$ whenever $0 < R' < R'' < \infty$.

Then by the argument if Section 2 we get

$$| a_\varepsilon (x) | \leq \frac{C}{\varepsilon^{n+1}} \int\limits_{|x| \leq \frac{3\varepsilon}{2}} | x | \, | a (x) | \, dx \text{ if } | x | \leq \varepsilon.$$

If we also assume

$$\textit{Hypothesis 1} \quad \int\limits_{|x| \leq \varepsilon} | x | \, | a (x) | \, dx \leq C\varepsilon$$

we thus obtain

$$| a_\varepsilon (x) | \leq \frac{C}{\varepsilon^n} \text{ if } | x | \leq \varepsilon. \tag{3}$$

By the argument of Section 2 we also get

$$| a_\varepsilon (x) | \leq \frac{C}{\varepsilon^n} \int\limits_{|y| \leq \frac{\varepsilon}{2}} | a (x - y) - a (x) | \, dy \text{ if } | x | > \varepsilon$$

or

$$| a_\varepsilon (x) | \leq \frac{C}{\varepsilon^n} \int\limits_{|y| \leq \frac{\varepsilon}{2}} | b_{\varepsilon, y} (x) | \, dy \text{ if } | x | > \varepsilon$$

if we set

$$b_{\varepsilon, y} (x) = \begin{cases} a (x - y) - a (x) & \text{if } | x | > \varepsilon \\ 0 & \text{if } | x | \leq \varepsilon \end{cases}$$

Let us now also assume

$$\textit{Hypothesis 2.} \quad \left\| \int\limits_{|y| \leq \frac{\varepsilon}{2}} | b_{\varepsilon, y} (x) | \, dy \right\|^* \leq C \, \varepsilon^n.$$

Then it follows from (3) and (4) that

$$\| a_\varepsilon \|^* \leq C$$

and as in Section 2 we obtain thus the following

THEOREM 1. *Assume that H is a bounded linear mapping from L_p into L_p $1 < p < \infty$, and assume also Hypothesis* 0, 1 *and* 2. *Then for every $f \in L_p$, $1 < p < \infty$, $H_\varepsilon f (x)$ convergences a. e. to $Hf (x)$.*

REMARK 2. By Lebesgue's theorem on dominated convergence it follows again that $H_\varepsilon f(x)$ converges in the L_p norm to $Hf (x)$.

4. Discussion of the previous hypotheses.

Hypothesis 0 and 1 are quite easy to verify in praxis. Therefore we shall concentrate upon Hypothesis 2. We shall mainly work with the following (obviously) stronger one.

Hypothesis 2'. $\| b_{\varepsilon, y}(x) \|^* \leq C$ if $|y| \leq \frac{\varepsilon}{2}$.

We shall now show that Hypothesis 2' is fulfilled in several more special situations. We shall here use, though this is of course not essential, some ideas of the abstract theory of interpolation spaces (see in particular [13] as well as references given there).

a'. Let us introduce the (semi-)norms

$$\| a \|_{X_0} = \sup | x |^n \, | a(x) |$$

and

$$\| a \|_{X_1} = \sup | x |^{n+1} \, | a'(x) | \qquad (a', \text{ gradient of } a)$$

We get at once

$$\sup_{2^k < |x| \leq 2^{k+1}} | b_{\varepsilon, y}(x) | \leq \sup_{2^k < |x| \leq 2^{k+1}} | a(x) | + \sup_{\substack{2^k < |x| \leq 2^{k+1} \\ |y| \leq |x|/2}} | a(x - y) | \leq$$

$$\leq C 2^{-kn} \| a \|_{X_0}$$

as well as

$$\sup_{2^k < |x| \leq 2^{k+1}} | b_{\varepsilon, y}(x) | \leq \sup_{\substack{2^k < |x| \leq 2^{k+1} \\ |z| \leq |x|/2}} | a'(x - z) | \, | y | \leq C 2^{-kn} \frac{\varepsilon}{2^k} \| a \|_{X_1}$$

where we have made use of $|y| < \frac{\varepsilon}{2}$, $b_{\varepsilon, y}(x) = 0$ if $|x| \leq \varepsilon$. Let now $a = a_0 + a_1$. Then we get

$$2^{kn} \sup_{2^k < |x| \leq 2^{k+1}} | b_{\varepsilon, y}(x) | \leq C \left(\| a_0 \|_{X_0} + \frac{\varepsilon}{2^k} \| a_1 \|_{X_1} \right).$$

If we set (see [13])

$$K(t, a) = K(t, a, X_0, X_1) = \inf_{a = a_0 + a_1} (\| a_0 \|_{X_0} + t \, \| a \|_{X_1})$$

it follows that

$$2^{kn} \sup_{2^k < |x| \leq 2^{k+1}} | b_{\varepsilon, y}(x) | \leq CK\left(\frac{\varepsilon}{2^k}, a\right)$$

Taking the sum we get

$$\| b_{\varepsilon, y} \|^* \leq C \sum_{\varepsilon \leq 2^{k+1}}^{\infty} K\left(\frac{\varepsilon}{2^k}, a\right) \leq$$

$$\leq C \int_0^1 K(t, a) \frac{dt}{t} \text{ if } | y | \leq \frac{\varepsilon}{2}$$

We have thus proven

THEOREM 2'. *If*

$$\int_0^1 K(t, a, X_0, X_1) \frac{dt}{t} < \infty \text{ (Dini condition)}$$

then Hypothesis 2' is fulfilled.

In the homogeneous case, $a(x) = \dfrac{\omega\left(\frac{x}{|x|}\right)}{|x|^n}$ where $\omega(x)$ is a function defined for $|x| = 1$ only, it is not hard to see that this is the consequence of an ordinary Dini condition for $\omega(x)$, i. e. precisely the case considered by Calderòn and Zygmund [3].

We interrupt the presentation by temporarily considering the following (obviously) weaker.

Hypothesis 2'' $\| b_{\varepsilon, y}(x) \| \leq C$ if $| y | < \frac{\varepsilon}{2}$

REMARK 3. This Hypothesis 2'' has also interest of its own because, as Hörmander [7] (see also Benedek-Calderòn-Panzone [2], J. Schwartz [16] where the vector valued case is treated) has shown, it implies that H is a bounded linear mapping from L_1 into $\widetilde{L}_1$ and thus, by Lemma 2, from L_p into L_p, $1 < p < \infty$, *provided* this latter property holds for some fixed p_0, say, $p_0 = 2$. Hörmander's proof which is a development of the one given by Calderòn-Zygmund [3] is the homogeneous case, depends in an essential way on a covering lemma given by these authors, which again goes back to F. Riesz. For the Hilbert transform there are, apart from several complex variable proofs, including the original one of M. Riesz, also some rather elementary real variable ones (see Stein-Weiss [17], Weiss

[18], O′ Neil-Weiss [12], Loomis [9]). On the other hand, according to an idea of Calderòn and Zygmund [4], the homogeneous case in several variables can be reduced to the one dimensional case, as a matter of fact under the even weaker assumption $a(x) \in L \log L$. Such weak conditions are of course excluded from the present discussion.

We now give the discussion of Hypothesis 2″ analogous to the interrupted one for Hypothesis 2′

a″. Let us introduce the (semi-)norms

$$\| a \|_{Y_0} = \sup_R \int\limits_{R \leq |x| < 2R} | a(x) | \, dx$$

and

$$\| a \|_{Y_1} = \sup R \int\limits_{R \leq |x| < 2R} | a'(x) | \, dx$$

(Note that $\| a \|_{Xi} \leq \| a \|_{Y_1}$, $i = 0, 1$!) Then we have

THEOREM 2″. *If* $\int_0^1 K(t, a, Y, Y_1) \frac{dt}{t}$ (Dini condition) *then Hypothesis* 2″ *is fulfilled.*

The proof is parallel to the one of Theorem 2′. Therefore we shall omit it. Note that, in the homogeneous case, we thus obtain norm convergence under weaker assumptions than in Calderòn-Zygmund [3].

b″. In a number of publications (see [5], [6]) Cotlar has considered the case when $a(x)$ admits *a* decomposition of the type (Cotlar decomposition)

$$a(x) = \sum_{j=\infty}^{\infty} u_j(x) \tag{5}$$

where

$$\int\limits_{R' < |x| \leq R''} u_j(x) \, dx = 0, \quad 0 < R' < R'' < \infty \tag{6}$$

and

$$\| \, |x| \, u_j(x) \| \leq C 2^j, \quad \| u_j'(x) \| \leq C 2^{-j} \tag{7}$$

(or, slightly more generally, with $0 < \alpha < 1$,

$$\| \, |x|^\alpha u_j(x) \| \leq C 2^j, \quad \| u_j(x+h) - u_j(x) \| \leq C |h|^\alpha 2^{-j}) \tag{8}$$

The following result holds true.

THEOREM 3''. *If a admits a Cotlar decomposition of the above type then Hypothesis 2'' is fulfilled.*

We only consider the stronger case of (7). Indeed, in this case, we easily convince ourselves that

$$\| b_{\varepsilon, y}(x) \| \leq C \frac{1}{\varepsilon} \| \, | x | \, a(x) \|$$

and

$$\| b_{\varepsilon, y}(x) \| \leq C_\varepsilon \| a'(x) \|$$

Applying this to $a(x) = u_j(x)$ we get

$$\| v_{j, \varepsilon, y}(x) \| \leq C \min \left(\varepsilon 2^j, \frac{1}{\varepsilon 2^j} \right)$$

where $v_{j, \varepsilon, y}(x)$ has a similar meaning (with respect to $u_j(x)$) as $b_{\varepsilon, y}(x)$ (with respect to $a(x)$). Finally upon taking the sum we get

$$\| b_{\varepsilon, y}(x) \| \leq C \sum_{k=-\infty}^{\infty} \min \left(\varepsilon 2^j, \frac{1}{\varepsilon 2^j} \right) \leq C$$

where the last bound is independent of ε

REMARK 4. We note also that under the some assumptions one can prove, as was done by Cotlar, that H is a bounded mapping from L_2 into L_2. Thus by Remark 3 H is a bounded mapping from L_p into L_p, $1 < < p < \infty$.

REMARK 5''. We do not know what is the precise relation between our Dini condition and the Cotlar decomposition. However it is easy to show at least that $a \leq Y_0 \cap Y_1$ implies that $a(x)$ admits a Cotlar decomposition, simply by taking

$$u_j(x) = \varphi_j(x)\, a(x)$$

where $\varphi_j(x)$ is a suitable partition of unity. Note also that the Cotlar decomposition is somewhat related to interpolation. Namely (5), (7) express precisely that $a \in (L_{1, |x|}, W_1^1)_{\frac{1}{2}, \infty}$ while as (5), (8) say that $a \in (L_{1, |x| \alpha}, W_1^{\alpha, \infty})_{\frac{1}{2}, \infty}$. Here we use the notation of [13], $L_{1, w}$ denotes L_1 with respect to the weight function $w(x)$.

c''. Michlin [10], [11] has given conditions on the Fourier transform $\hat{a}(\xi)$ of $a(x)$. His result, which goes back to a similar result in the periodic case due to Marcinkiewicz, was later simplified and somewhat extended by Hörmander [7]. To put this in our present framework we establish

THEOREM 4''. *Suppose that*

$$\left(\int\limits_{R < |\xi| \leq 2R} |\widehat{a^{(M)}}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \leq C R^{\frac{n}{2} - M}, \quad 0 \leq M \leq N$$

for some integer $N > \frac{n}{2}$. *Then* $a(x)$ *admits a Cotlar decomposition.*

(One can also consider the case of fractional N, see [8], [14]). Let us assume slightly more: $N > \frac{n}{2} + 1$. Then we can actually work with the stronger form (7). (The general case can be treated in a similar way using (8) in place of (7).) We define $u_j(x)$ by

$$\widehat{u_j}(\xi) = \psi_j(\xi) \, \widehat{a}(\xi)$$

where $\psi_j(\xi)$ is a partition of unity such that

$$\sum_{j=-\infty}^{\infty} \psi_j(\xi) = 1$$

$$\psi_j(\xi) = 0 \text{ unless } 2^{j-1} \leq |\xi| < 2^{j+1},$$

$$|\psi_j^{(M)}(\xi)| \leq C_M \, 2^{-jM}, \quad 0 \leq M \leq N.$$

It is clear that (5) holds and, if we choose $\psi_j(\xi)$ depending on $|\xi|$ only also (6). By a (special case of) well-known theorem of S. Bernstein (see Zygmund [19], vol 1, p. 240-241, for the one-dimensional periodic case, see also [14] for the general case) we have

$$\|a(x)\| \leq C \sum_{M=0}^{N} \|\widehat{a^{(M)}}(\xi)\|_{L_2}, \quad N > \frac{n}{2} \tag{9}$$

Taking account of the assumptions on $\widehat{a}(\xi)$ and the above properties of $\psi_j(\xi)$ also (7) follows when (9) is applied to $a = xu_j$ and $a = u_j'$ respectively.

After this detour on the Hypothesis 2″ it is now easy to complete the interrupted discussion of Hypothesis 2′.

b′. We consider again functions $a(x)$ admitting a decomposition of type (Cotlar decomposition)

$$a(x) = \sum_{j=-\infty}^{\infty} u_j(x)$$

where as previously

$$\int_{R' < |x| \leq R''} u_j(x)\, dx = 0, \quad 0 < R' < R'' < \infty$$

but now

$$\| \, |x| \, u_j(x) \|^* \leq C 2^j, \quad \| u_j'(x) \|^* \leq C 2^{-j} \tag{10}$$

(or, slightly more generally, with $0 < \alpha < 1$,

$$\| \, |x|^\alpha u_j(x) \|^* \leq C 2^j, \quad \| u_j(x+h) - u_j(x) \|^* \leq \leq C |h|^\alpha 2^{-j}. \tag{11}$$

We have

THEOREM (3′). *If a admits a Cotlar decomposition of the above type then Hypothesis 2′ is fulfilled.*

The reader should have no difficulties in supplying the details of the proof which is quite similar to the one of Theorem 3″.

REMARK 5′. Remark analogous to Remark 5″.

c′. We can also give the analogue of Theorem 4″.

THEOREM 4′. *Suppose that*

$$\int_{R < |\xi| \leq 2R} | \widehat{a}^{(M)}(\xi) | \, d\xi \leq C R^{n-M}, \quad 0 \leq M \leq N$$

for some integer $N > n$. Then $a(x)$ admits a Cotlar decomposition.

Again we leave the proof of the reader. We only note that instead of (9) we have to use

$$\| a(x) \|^* \leq C \sum_{M=0}^{N} \| \widehat{a}^{(M)}(\xi) \|_{L_1}, \; N > n$$

REMARK 6. We conclude by the observation that the apparent relation between the primed and dubble primed cases can be made still more pertinent by working with a whole family of norms

$$\sum_{k=-\infty}^{\infty} 2^{k\frac{n}{q}}\Bigl(\int_{2^k \le |x| < 2^{k+1}} |a(x)|^{q'}\,dx\Bigr)^{\frac{1}{q'}}, \ \frac{1}{q}+\frac{1}{q'}=1\le q\le\infty$$

instead of just the two norms $\| a \|^*$ and $\| a \|$ corresponding to the extremal cases $q=1$ and $q=\infty$ respectively.

REMARK 7. One of the points of this Section was to relate the work of Cotlar [5], [6] to the work of Hörmander [7]. Our results may indicate that the « Cotlar decomposition » might be of little pratical value. This may be true in this special but there are certainly other cases where the « Cotlar decomposition » is most useful: In the first place comes Cotlar's own extension of the Calderòn-Zygmund theory so as to include ergodic theory (see [6]). Other applications, in a different direction, of the « Cotlar decomposition » are made in [15].

5. The general case, $p=1$.

We show now how to extend the method of Section 2 and 3 to the extremal case $p=1$. In a less explicite from this is already contained in Cotlar [5] (see, in particular, Theorem 10, p. 152) who however works out the details essentially in the case of the Hilbert transform only.

THEOREM 5. *Assume that H is a bounded linear mapping from L_1 into $\tilde{L}_1$ and assume also Hypothesis 2′. Then for every $f \in L_1$, $H_\varepsilon f(x)$ converges a. e. to $Hf(x)$.*

Indeed from Hypothesis 2′ we get by the definition of $\| a \|^*$ that

$$| b_{\varepsilon, y} * f(x) | \le C\,\theta f(x), \quad |y| < \frac{\varepsilon}{2}$$

or

$$| H_\varepsilon f(x) - Hf(x-y) + H\varrho_{\varepsilon, x} f(x-y) | \le C\,\theta f(x), \ |y| < \frac{\varepsilon}{2}$$

where $\varrho_{\varepsilon, x}(z)$ is the characteristic function of the set $|x-z| < \frac{\varepsilon}{2}$

$$\varrho_{\varepsilon, x}(z) = \begin{cases} 1 & |x-z| \le \frac{\varepsilon}{2} \\ 0 & |x-z| > \frac{\varepsilon}{2} \end{cases}$$

If we set

$$Mf(x) = \sup_{\varepsilon} | H_{\varepsilon} f(x) |$$

we have thus for all x

$$Mf(x) \leq C\, \theta f(x) + | Hf(y) | + | H \varrho_{\varepsilon, x} f(y) |), \; | y - x | < \frac{\varepsilon}{2}$$

for some $\varepsilon > 0$ (depending on x). In view of Lemma 1 the theorem follows now from the following

LEMMA 5 (see Cotlar [5], Theorem 10, p. 152). *Let M, T_1, T_2, T_3 be mappings such that*

$$Mf(x) \leq c(| T_1 f(x) | + | T_{\varepsilon} f(y) | + T_3\, \varrho_{\varepsilon, x} f(y) |), \; | y - x | < \frac{\varepsilon}{2}$$

If T_1, T_2, T_3 are bounded from L_1 into $\widetilde{L}_1$ then so is M.

For completeness we briefly indicate the proof. It is easy to reduce the lemma to the case when only one of T_1, T_2, T_3 is different from 0, i. e. we may distinguish three cases:

Case 1^0. $Mf(x) \leq C\, | T_1 f(x) |$

case 2^0. $Mf(x) \leq C\, | T_2 f(y) |, \; | y - x | < \frac{\varepsilon}{2}$

case 3^0. $Mf(x) \leq C\, | T_3\, \varrho_{\varepsilon, x} f(y) |, \; | y - x | < \frac{\varepsilon}{2}$

Case 1^0 is obvious.

Case 2^0. Let $0 < \alpha < 1$. Then it follows that

$$(Mf(x))^{\alpha} \leq C^{\alpha}\, \theta\, (T_2 f)^{\alpha}(x)$$

But $T_2 f \in L_1$ implies $(T_2 f)^{\alpha} \in \widetilde{L}_1^{\frac{1}{\alpha}}$. Therefore by Lemma 3 $\theta (T_2 f)^{\alpha} \in \widetilde{L}_1^{\frac{1}{\alpha}}$. Hence $(Mf)^{\alpha} \in \widetilde{L}_1^{\frac{1}{\alpha}}$ which implies $Mf \in \widetilde{L}_1$.

Case 3^0 is similar to case 2^0 and we therefore omit details.