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ON CERTAIN CLASSES OF — AND AN EXISTENCE THEOREM FOR — PRIMAL CLUSTERS

by ADIL YAQUB

A primal cluster is essentially a class $\{U_i\}$ of universal algebras U_i of the same species, where each U_i is primal (= strictly functionally complete), and such that every finite subset of $\{U_i\}$ is « independent ». The concept of independence is essentially a generalization to universal algebras of the Chinese remainder Theorem in number theory. The following result is proved: suppose that, for each positive integer $n \geq 2$, $(R_n, \times, +)$ is an arbitrary but fixed commutative ring with zero radical and with exactly n elements. Then there exists a permutation, $^\cap$, of R_n such that $\{(R_n, \times, ^\cap)\}$ forms a primal cluster. Furthermore, a *constructive* method for obtaining, $^\cap$, is given. It is further shown that $\{(R_n, \times, ^\cap)\} \cup \{(P_n, \times, ^\cap)\}$ forms a primal cluster, where $(P_n, \times, ^\cap)$ is the basic Post algebra of order n (species (2,1)). In section 4, we establish an *existence theorem* which shows the existence (but does not give the constructability) of a rather large class of primal clusters (of species (2,1)).

The concept of a primal cluster, first introduced by Foster [1; 2], embraces *classes* of algebras of such diverse nature as the classes of all (i) prime fields, (ii) « n-fields », (iii) basic Post Algebras, and (iv) the union of the primal clusters (ii) and (iii) above. Each cluster, $\widetilde{U}$, equationally defines — in terms of the identities jointly satisfied by the various finite subsets of $\widetilde{U}$ — a class of « $\widetilde{U}$-algebras », and a structure theory for these $\widetilde{U}$-algebras was established in [1; 2], — a theory which contains well known results for Boolean algebras, p-rings, and Post algebras. In order to expand the domain of applications of this theory, one should then always look for primal

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clusters. The primal clusters above furnish an essential extension of the primal clusters which were previously given in [1 ; 2].

It is noteworthy to observe that, *with one single exception*, the permutation, $^{\cap}$, for which $\{(R_n, \times, ^{\cap})\}$ forms a primal cluster, can always be so chosen as to further satisfy the normalizing condition : $0^{\cap} = 1$. The exceptional algebra arises when $R_n = R_4 = GF(2) \oplus GF(2)$. For, as noted in [3], there exists *no* $0 \rightarrow 1$ permutation, $^{\cap}$, of R_4 for which $(R_4, \times, ^{\cap})$ is primal. However, a permutation (satisfying $(0^{\cap})^{\cap} = 1$) is given such that $(R_4, \times, ^{\cap})$ is indeed primal (Theorem 6). The next, and most essential, step is to show that, relative to *this* permutation, $(R_4, \times, ^{\cap})$ is « independent » of any other finite subset of $\{(R_n, \times, ^{\cap})\} \cup \{(P_n, \times, ^{\cap})\}$ (Theorem 9). This done, the exceptional role played by $(R_4, \times, ^{\cap})$ now disappears altogether.

1. Fundamental Concepts. In this section, we recall the following concepts of [2]. Let $S = (n_1, n_2, \dots)$ be a finitary species, where the n_i are positive integers, and let $o_1, o_2, \dots$ denote the primitive operation symbols of S. Here, o_i is n_i-ary, $o_i = o_i(\zeta_1, \dots, \zeta_{n_i})$. By an *expression* $\varphi(\zeta, \dots)$ of species S we mean a primitive composition of one or more indeterminate-symbols $\zeta, \dots$ via the primitive operations o_i. As usual, we use the same symbols o_i to denote the primitive operations of the algebras $U_1, U_2, \dots$ when these algebras are of species S. We write « $\Phi(\zeta, \dots)(U)$ » to mean that the S — expression φ is interpreted in the S-algebra U. This simply means that the primitive operation symbols are identified with the corresponding primitive operations of U, and the indeterminate-symbols $\zeta, \dots$ are now viewed as indeterminates over U. « $\Phi(\zeta, \dots)(U)$ » is also called a *strict U-function.* An identity between the strict U-functions f, g — holding throughout U — is written as $f(\zeta, \dots) = g(\zeta, \dots)(U)$. A finite algebra U with more than one element is called *primal* if every mapping of $U \times \dots \times U$ into U is expressible by a *strict* U-function. Examples of primal algebras are widespread, and include [2]:

(i) The prime field $(F_p, \times, ^{\cap})$, $F_p = \{0, 1, 2, \dots, p-1\}$, $p =$ prime, $\zeta^{\cap} = \zeta + 1 \pmod{p}$.

(ii) The basic Post algebra $(P_n, \times, ^{\cap})$, n arbitrary [5].

Here, $P_n = \{0, \varrho_{n-2}, \varrho_{n-3}, \dots, \varrho_1, 1\}$, $\zeta \times \eta = \min(\zeta, \eta)$, where «min» refers to the above ordering, and where $0^{\cap} = 1$, $1^{\cap} = \varrho_1$, $\varrho_1^{\cap} = \varrho_2, \dots,$ $\varrho_{n-2}^{\cap} = 0$.

We now procede to define the concept of independence. Let $\{U_i\} = \{U_1, \dots, U_r\}$ be a finite set of algebras of species S. We say that $\{U_i\}$ satisfies the *Chinese residue condition,* or is *independent* if, corresponding to each set of expressions $\varphi_1, \dots, \varphi_r$ af species S, there exists a single expression Ψ such that $\Psi = \varphi_i(U_i)$ $(i = 1, \dots, r)$.

A primal cluster of species S is defined to be a class $\widetilde{U} = \{\dots, U_i, \dots\}$ of primal algebras of species S any finite subset of which is independent.

2. **Primal Algebras.** Let n be any positive integer, $n \geq 2$. Then there exist commutative rings with zero radical and with exactly n elements. Thus, the direct sum $GF(p_1^{k_1}) \oplus \dots \oplus GF(p_t^{k_t})$, where $n = p_1^{k_1} \dots p_t^{k_t}$ is such a ring. Now, for each integer $n \geq 2$, let R_n be an arbitrary but fixed commutative ring with zero radical and consisting of exactly n elements. It is well known that

(2.1) $$R_n \cong GF(n_1) \oplus \dots \oplus GF(n_t),\ 2 \leq n_1 \leq \dots \leq n_t,\ n = n_1 \dots n_t.$$

We now have the following

DEFINITION 1. For each integer $n \geq 2$, let $R_n \cong GF(n_1) \oplus \dots \oplus GF(n_t)$, $n = n_1 \dots n_t$, $2 \leq n_1 \leq \dots \leq n_t$, be an arbitrary but fixed commutative ring with zero radical and with exactly n elements. We say that R_n is (i) of *type* 1 if $t = 1$, (ii) of *type* 2 if $t > 1$ and $n \neq 4$, (iii) of *type* 3 if $t > 1$ and $n = 4$ (and hence $R_n \cong GF(2) \oplus GF(2)$).

REMARK: In the above definition, each n_i is, of course, a prime power divisor of n which, once chosen, becomes fixed. Thus, for each $n \geq 2$, we choose *only one* ring R_n satisfying the above properties.

We now procede to define a permutation, $\cap$, of R_n. This we do in several stages.

Case 1: If R_n is of *type* 1. Define, $\cap$, to be an arbitrary but fixed $0 \to 1$ cyclic permutation of R_n:

(2.2) $$\cap := \text{def} = 0,\ 1, \alpha_1, \dots, \alpha_{n-2}\ (\text{i.e., } 0^\cap = 1,\ 1^\cap = \alpha_1, \dots, (\alpha_{n-2})^\cap = 0).$$

Case 2: If R_n is of *type* 2. We distinguish three subcases.

Case 2(a): If $t > 2$. In this case, let $\{1, \alpha_1, \dots, \alpha_\sigma\}$ be the set of all elements of R_n whose t *th* components in the direct sum representation (2.1) are different from zero, and choose the notation such that $\alpha_1 \dots \alpha_\sigma = \alpha_1$. Define, $\cap$, by the ordering

(2.3) $$\cap := \text{def} = 0, 1, \beta_1, \beta_2, \dots, \beta_\tau, \alpha_\sigma, \alpha_{\sigma-1}, \dots, \alpha_1, (\alpha_1 = \alpha_1 \dots \alpha_\sigma),$$

i.e., $0^\cap = 1$, $1^\cap = \beta_1, \dots, \alpha_1^\cap = 0$, where $\beta_1, \beta_2, \dots, \beta_\tau$ are the remaining elements of R_n arranged in an arbitrary but fixed way *except that*

(2.3)′ $$\beta_1 = (1, 0, 0, \dots, 0) \text{ and } \beta_2 = (0, 1, 0, 0, \dots, 0),\ (t \geq 3).$$

Case 2(b): If $t = 2$, $n_1 = n_2 = n_t$. In this case, $n_1 > 2$ since $n \neq 4$. Now, let $\{1, \alpha_1, \dots, \alpha_c\}$ be as above but choose the notation now to *further* satisfy

$$(2.3)'' \qquad \alpha_2 = (1, \zeta), \ \alpha_3 = (\zeta, \zeta), \qquad (\zeta \neq 0, \zeta \neq 1).$$

Define, $^\cap$, as in (2.3) and (2.3)'' (but no restrictions on β_1, β_2).

Case 2(c): If $t = 2$ and $n_1 < n_2 = n_t$ (see (2.1)). In this case, define, $^\cap$, as in (2.3) (but no restrictions on $\beta_1, \beta_2, \alpha_2, \alpha_3$).

Case 3: If R_n is of *type* 3. In this case, (2.1) readily reduces to $R_n \cong GF(2) \oplus GF(2)$. Define, $^\cap$, by the ordering

$$(2.4) \qquad {}^\cap : = \mathrm{def} = 0, \alpha, 1, \beta \ \ (\text{i.e.. } 0^\cap = \alpha, \alpha^\cap = 1, \dots, \beta^\cap = 0).$$

We now recall the definition of the *characteristic function* [4]:

$$(2.5) \qquad \Delta(\zeta) = \begin{cases} 0 & \text{if } \zeta = 0 \\ 1 & \text{if } \zeta \neq 0 \end{cases}, \quad \delta(\zeta) = \begin{cases} 1 & \text{if } \zeta = 0 \\ 0 & \text{if } \zeta \neq 0 \end{cases}.$$

We also recall the following [2]

DEFINITION 2. A *frame* is an algebra $(U, \times, {}^\cap; 0, 1)$ of species (2,1) possessing distinguished elements $0, 1$ $(0 \neq 1)$ such that

$$0 \times \zeta = \zeta \times 0 = 0, \ 1 \times \zeta = \zeta \times 1 = \zeta \ (\zeta \in U),$$

where $\zeta^\cap$ is a *cyclic* permutation of the elements of U such that $0^\cap = 1$.

We shall also need the following result which is an immediate consequence of [3; Theorem 3].

LEMMA 3. *Let* $(U, \times, {}^\cap)$ *be a finite frame. A sufficient* (*and necessary*) *condition for* $(U, \times, {}^\cap)$ *to be primal is that* $\Delta(\zeta)$, *or equivalently* $\delta(\zeta)$, *is expressible as a strict U-function.*

We are now in a position to consider the algebra $(R_n, \times, {}^\cap)$ in regard to primality, where, $^\cap$, is as in cases 1, 2, 3, according to the type of R_n.

THEOREM 4. *Let* $(R_n, \times, +)$ *be a commutative ring with zero radical and with exactly n elements,* $n \geq 2$, *and let* R_n *be of type* 1. *Then* $(R_n, \times, {}^\cap)$ *is primal, where,* $^\cap$, *is as in* (2.2).

PROOF. In this case, $R_n = GF(n)$. It is readily verified that (compare with [4])

$$\delta(\zeta) = \{\zeta^{\cup} \zeta^{\cup_2} \zeta^{\cup_3} \dots \zeta^{\cup_{n-1}}\}^{n-1},$$

(see (2.5)), where $\zeta^{\cup}$ is the inverse of $\zeta^{\cap}$, and $\zeta^{\cup_k} = (\dots(\zeta^{\cup})^{\cup}\dots)^{\cup}$ (k iterations). The result now follows readily from Lemma 3.

THEOREM 5. *Let* $(R_n, \times, +)$ *be a commutative ring with zero radical and with exactly* n *elements,* $n \geq 2$, *and let* R_n *be of type* 2. *Then* $(R_n, \times, {}^{\cap})$ *is primal, where,* ${}^{\cap}$, *is as in* (2.3).

PROOF. The ring R_n, as is well known, is isomorphic to the direct sum of $GF(n_1), \dots, GF(n_t)$, for some $n_1 \leq n_2 \leq \dots \leq n_t$. Clearly, $n = n_1 n_2 \dots n_t$. Now, define r_n as follows:

$$r_n = \text{def} = n - \frac{n}{n_t}. \tag{2.6}$$

Since R_n is of type 2, therefore,

$$r_n = n_1 \dots n_t - n_1 \dots n_{t-1} = n_1 \dots n_{t-1}(n_t - 1) \geq 4, \tag{2.7}$$

since, if $t \geq 3$, $n_1 \dots n_{t-1} \geq 4$, while, if $t = 2$, then $n_t \geq 3$ since $n > 4$. Again, the result readily follows. Now, we claim that

$$\Delta(\zeta) = \{(\zeta^{\cup} \zeta^{\cup_2} \zeta^{\cup_3} \dots \zeta^{\cup_{r_n-1}})(\zeta^{\cap})\}^{\cap} = \begin{cases} 0 & \text{if } \zeta = 0 \\ 1 & \text{if } \zeta \neq 0 \end{cases}. \tag{2.8}$$

To prove (2.8), we first suppose that, ${}^{\cap}$, satisfies (2.3) and (2.3)′ (see case 2(a) above). Then $\Delta(0) = \{(\alpha_1 \dots \alpha_\sigma)(1)\}^{\cap} = \alpha_1^{\cap} = 0$, since $\sigma = r_n - 1$. Next, consider $\Delta(\gamma)$, $\gamma \neq 0$, $\gamma \in R_n$. If *any* of $\gamma^{\cup}, \gamma^{\cup_2}, \dots, \gamma^{\cup_{r_n-1}}, \gamma^{\cap}$ is zero, then, clearly, $\Delta(\gamma) = 0^{\cap} = 1$. Suppose then that none of these elements is zero. Then either (case (i)) $\gamma^{\cup_{r_n-1}} = 1$ or (case (ii)) $\gamma^{\cup_{r_n-1}} = \beta_i$ or α_j for some i, j. Consider case (i). Now, by (2.7), $r_n - 1 \geq 3$, and hence *both* of $\gamma^{\cup_{r_n-2}} (= 1^{\cap} = \beta_1)$ and $\gamma^{\cap_{r_n-3}} (= \beta_1^{\cap} = \beta_2)$ belong to the set of «factors» of $\Delta(\gamma)$ appearing in the right-side of (2.8). Since, by (2.3)′, $\beta_1 \beta_2 = 0$, therefore, $\Delta(\gamma) = 1$ if $\gamma^{\cup_{r_n-1}} = 1$. Now, consider case (ii). Here, $\gamma^{\cup_{r_n-1}} \neq 1 (= (1, 1, \dots, 1))$. Recalling the direct sum representation (2.1), any element α of R_n can be written in the form $\alpha = (\alpha^{(1)}, \dots, \alpha^{(t)})$. Now, let $1 \leq r \leq t$, and let $z(r)$ be the number of zeros in $\{\alpha^{(r)}\}$ as α ranges over the *complementary set* S of $\{\gamma^{\cup}, \gamma^{\cup_2}, \dots, \gamma^{\cup_{r_n-1}}, \gamma^{\cap}\}$ in R_n. Since, in our present case, $(1, 1, \dots, 1) \in S$, and since the number of elements of S is equal to $n - r_n = \frac{n}{n_t}$, therefore,

$$(2.9) \qquad z(r) \leq \frac{n}{n_t} - 1 < \frac{n}{n_r} \, (1 \leq r \leq t), \text{ since } n_r \leq n_t .$$

But, there are exactly $\frac{n}{n_r}$ elements (i.e., vectors) of R_n with zero in the r *th* component. Hence, (2.9) now implies that for at least one element in the complement of $S \, (= \{\gamma^{\cup}, \gamma^{\cup_2}, \ldots, \gamma^{\cup_{r_n-1}}, \gamma^{\cap}\})$, the r *th* component of such an element is zero. And since this is true for *each* $r, 1 \leq r \leq t$, therefore,

$$(2.9)' \qquad (\gamma^{\cup} \gamma^{\cup_2} \ldots \gamma^{\cup_{r_n-1}})(\gamma^{\cap}) = 0.$$

Hence, once more, $\Delta(\gamma) = 1$, $\gamma \neq 0$, and (2.8) is proved in this case. The proofs that (2.8) holds when, $\cap$, satisfies (2.3), (2.3)'', (see case 2(b), case 2(c), above) are similar and will here be omitted.

Returning to the proof of Theorem 5, we observe that the result now readily follows from Lemma 3 and (2.8).

Next, we consider the ring $(R^n, \times, +)$ where R_n is of *type* 3. As observed in case 3 (immediately preceding (2.4)), $R_n \cong GF(2) \oplus GF(2)$, and the permutation, $\cap$, is now necessarily *not* a $0 \rightarrow 1$ permutation. For, it was shown in [3] that there exist *no* $0 \rightarrow 1$ permutation, $\cap$, such that $(GF(2) \oplus GF(2), \times, \cap,)$ is primal. However,, we do have the following.

THEOREM 6. *The algebra* $(GF(2) \oplus (GF(2), \times, \cap)$ (*species* (2,1)) *is primal, where,* $\cap$, *is as in* (2.4).

PROOF. First, we remark that, since $0^{\cap} \neq 1$, Lemma 3 does not readily apply (see Definition 2). We shall give a direct proof of Theorem 6 which follows the outline of the proof of Lemma 3 given in [4], and, in addition, suggests a generalization of the latter. To this end, define

$$(2.10) \qquad a \times_{\cap_2} b = (a^{\cap_2} \times b^{\cap_2})^{\cup_2},$$

where $a^{\cap_2} = (a^{\cap})^{\cap}$, $a^{\cup_2} = (a^{\cup})^{\cup}$. It is readily verified that

$$(2.11) \qquad a \times_{\cap_2} 0 = 0 \times_{\cap_2} a = a.$$

Furthermore, let $\overset{\times_{\cap_2}}{\sum\limits_{a_i \in U}} \alpha_i = \text{def} = \alpha_1 \times_{\cap_2} \alpha_2 \times_{\cap_2} \alpha_3 \times_{\cap_2} \cdots$, where $\alpha_1, \alpha_2, \alpha_3, \ldots$ denote all the elements of the algebra U. Finally, let $\delta_\alpha(\zeta) = 1$ if $\zeta = \alpha$

and $\delta_a(\zeta) = 0$ if $\zeta \neq \alpha$. Using (2.11), it is easily verified, if $U = GF(2) \oplus GF(2)$, and, $^{\cap}$, is as in (2.4), that *every* mapping $f: U \times \ldots \times U \rightarrow U$, satisfies

$$f(\zeta, \eta, \ldots) = \overset{\times_{\cap_2}}{\underset{\alpha, \beta, \ldots \in U}{\Sigma}} f(\alpha, \beta, \ldots)\, \delta_\alpha(\zeta)\, \delta_\beta(\eta) \ldots . \tag{2.12}$$

In (2.12), each of $\alpha, \beta, \ldots$ ranges independently over all the elements of $U (= GF(2) \oplus GF(2))$. Furthermore, for *any* $\zeta \in U$, we have

$$\zeta\zeta^{\cap}\ \zeta^{\cap_2}\zeta^{\cap_3} = 0, \ (\zeta\zeta^{\cap}\ \zeta^{\cap_2}\zeta^{\cap_3})^{\cap} = 0^{\cap} = \alpha, \text{ etc.}$$

Hence, *all* elements of U are expressible as *strict* U-functions. Moreover, it is easily verified that with, $^{\cap}$, as in (2.4), we have

$$\begin{cases} \delta_0(\zeta) = (\zeta^{\cap}\ \zeta^{\cap_2})^{\cap}\ (\zeta^{\cup}\ \zeta^{\cup_2})^{\cup}, \ \delta_\beta(\zeta) = \delta_0(\zeta^{\cap}), \ \delta_1(\zeta) = \delta_0(\zeta^{\cap_2}), \\ \delta_\alpha(\zeta) = \delta_0(\zeta^{\cap_3}). \end{cases} \tag{2.13}$$

Hence, the entire right-side of (2.12) is expressible as a *strict* U-function. Therefore, U is primal, and the theorem is proved.

It was proved in [4] that the basic Post algebra $(P_n, \times, {}^{\cap})$ of order n (see (ii) of section 1) is primal. This, together with Theorems 4, 5, 6, yields the following

THEOREM 7. *For each positive integer $n \geq 2$, let R_n be an arbitrary but fixed commutative ring with exactly n elements and with zero radical, and let P_n be the basic Post algebra of order n. Then every element-algebra in* $\{(R_n, \times, {}^{\cap})\} \cup \{(P_n, \times, {}^{\cap})\}$ *is primal, where,* $^{\cap}$, *is determined by* (2.2)-(2.4) *for* R_n, *and by* (ii) *of section* 1 *for* P_n.

3. **Independence.** In this section, we investigate the independence of the algebras considered in Theorem 7. It is noteworthy to observe that the results of this section could have been made to precede the theorems of the preceding section (i. e., independence could be established prior to primality). We begin with constructing *expressions* $|_{12}(\zeta)$ and $|_{21}(\zeta)$ such that

$$|_{12}(\zeta) = \begin{cases} 1 \ (U_1) \\ 0 \ (U_2), \end{cases} \quad \text{and} \quad |_{21}(\zeta) = \begin{cases} 0 \ (U_1) \\ 1 \ (U_2) \end{cases} \tag{3.1}$$

for each pair of distinct algebras U_1, U_2 under consideration.

THEOREM 8. *Let* $(R_n, \times, {}^{\cap})$, $(P_n, \times, {}^{\cap})$ *be as in Theorem* 7, *and let* $\widetilde{U} = \{(R_n, \times, {}^{\cap})\} \cup \{(P_m, \times, {}^{\cap})\}$, $n \geq 2$, $m \geq 3$. *Then, for any two element-algebras* U_1, U_2 *of* $\widetilde{U}$, *there exist expressions* $|_{12}(\zeta)$, $|_{21}(\zeta)$ *which satisfy* (3.1).

PROOF. We distinguish several cases depending upon the choices of U_1 and U_2.

Case 1 : $U_1 = R_n = GF(n)$ is of type 1, $U_2 = R_m = GF(m)$ is of type 1. Assume, without any loss of generality, that $m < n$. Define

$$E = \zeta\zeta^{\cap}\,\zeta^{\cap_2} \ldots \zeta^{\cap_{mn-1}}. \tag{3.2}$$

It is readily verified that

$$|_{12}(\zeta) = (E^{\cup}\,E^{\cup_2} \ldots E^{\cup_{n-1}})^{n-1} = \begin{cases} 1 & (U_1) \\ 0 & (U_2) \end{cases} \tag{3.3}$$

$$|_{21}(\zeta) = \{\,|_{12}(\zeta)\cdot(\,|_{12}(\zeta))^{\cup_2}\}^{\cap} = \begin{cases} 0 & (U_1) \\ 1 & (U_2). \end{cases} \tag{3.4}$$

Case 2 : $U_1 = R_n = GF(n)$ is of type 1, $U_2 = R_m$ is of type 2. Again, let E be as in (3.2). Then,

$$|_{12}(\zeta) = (E^{\cap_2}\,E^{\cup})^{n-1} = \begin{cases} 1 & (U_1) \\ 1 & (U_2), \end{cases} \quad \text{if} \quad n > 2.$$

$$|_{12}(\zeta) = \{(E^{\cup}\,E^{\cup_2} \ldots E^{\cup_{r_{m-1}}})\,(E^{\cap})\}^{\cap} = \begin{cases} 1 & (U_1) \\ 0 & (U_2), \end{cases} \quad \text{if} \quad n = 2.$$

$|_{21}(\zeta)$ is as in (3.4).

Case 3 : $U_1 = R_n = GF(n)$ is of type 1, $U_2 = GF(2) \oplus GF(2)$ is of type 3. Then,

$$|_{12}(\zeta) = (E^{\cap}\,E^{\cup})^{n-1} = \begin{cases} 1 & (U_1) \\ 0 & (U_2) \end{cases}$$

$$|_{21}(\zeta) = \{\,|_{12}(\zeta)\cdot(\,|_{12}(\zeta))^{\cup_3}\}^{\cap_2} = \begin{cases} 0 & (U_1) \\ 1 & (U_2). \end{cases} \tag{3.5}$$

Case 4 : $U_1 = R_n$ is of type 2 and $U_2 = R_m$ is of type 2. Let r_n be defined as in (2.6) and let E be as in (3.2). We now distinguish the following subcases.

Case 4 (A): $r_m > r_n$. Then, by (2.3), (2.8),

$$|_{12}(\zeta) = \{(E^{\cup} E^{\cup_2} \dots E^{\cup_{r_m-1}})(E^{\cap})\}^{\cap} = \begin{cases} 1 & (U_1) \\ 0 & (U_2). \end{cases}$$

$|_{21}(\zeta)$ is as in (3.4).

Case 4 (B): $r_m < r_n$. By symmetry, this is essentially the same as *Case* 4 (A). Observe that (3.4), too, is symmetric in $|_{12}(\zeta)$ and $|_{21}(\zeta)$. Indeed, so long as $0^{\cap} = 1$, we have,

$$(3.4)' \qquad |_{12}(\zeta) = \{\, |_{21}(\zeta) \cdot (\, |_{21}(\zeta))^{\cup_2}\}^{\cap} .$$

Case 4 (C): $r_m = r_n$. Assume, without loss of generality, that $m < n$. Using (2.8), it is readily verified that

$$|_{12}(\zeta) = \{(E^{\cup_{m+1}} E^{\cup_{m+2}} \dots E^{\cup_{m+r_m-1}})(E^{\cup_{m-1}})\}^{\cap} = \begin{cases} 1 & (U_1) \\ 0 & (U_2). \end{cases}$$

$|_{21}(\zeta)$ is as in (3.4).

Case 5: $U_1 = R_n$ is of type 2 and $U_2 = GF(2) \oplus GF(2)$ is of type 3 Then, by (2.4), (2.3), (3.2), it is easily seen that

$$|_{12}(\zeta) = (E^{\cap} E^{\cap_2})^{\cup} = \begin{cases} 1 & (U_1) \\ 0 & (U_2); \end{cases} \qquad |_{21}(\zeta) \text{ is as in (3.5).}$$

Case 6: $U_1 = P_n$ and $U_2 = P_m$. Assume, without any loss of generality, that $m > n$. Then, by (ii) of section 1 and (3.2), it is easily verified that

$$|_{12}(\zeta) = (E^{\cup} E^{\cup_2} \dots E^{\cup_{m-1}})^{\cap} = \begin{cases} 1 & (U_1) \\ 0 & (U_2); \end{cases} \qquad |_{21}(\zeta) \text{ is as in (3.4).}$$

Case 7: $U_1 = P_n$ and $U_2 = R_m = GF(m)$ is of type 1. Then,

$$|_{21}(\zeta) = ((E^{\cup})^{m-1})^{\cap})^{m-1} = \begin{cases} 0 & (U_1) \\ 1 & (U_2), \end{cases} \quad \text{if} \quad m \neq 2,$$

$$|_{21}(\zeta) = (E^{\cup} E^{\cup_2})^{\cap} = \begin{cases} 0 & (U_1) \\ 1 & (U_2), \end{cases} \quad \text{if} \quad m = 2.$$

(Observe that, if $m = 2$, then $n \geq 3$).

$|_{12}(\zeta)$ is as in (3.4)$'$.

Case 8: $U_1 = P_n$ and $U_2 = R_m$ is of type 2. Then,

$$|_{12}(\zeta) = (E^{\cap_2} E^{\cup})^{\cap} = \begin{cases} 0 & (U_1) \\ 1 & (U_2), \end{cases} \quad \text{if} \quad n \neq 2.$$

(Observe that, if $n = 2$, then $U_1 = P_2 = R_2$, which yields *Case* 2).
$|_{12}(\zeta)$ is as in (3.4)′.

Case 9: $U_1 = P_n$ and $U_2 = GF(2) \oplus GF(2)$ is of type 3. It is easily verified, by (3.2), (2.4), and (ii) of section 1, that

$$|_{12}(\zeta) = (E^{\cap} E^{\cap_2})^{\cup} = \begin{cases} 1 & (U_1) \\ 0 & (U_2); \end{cases} \quad |_{21}(\zeta) \quad \text{is as in (3.5).}$$

The above cases exhaust all possibilities for U_1 and U_2, and the theorem is proved.

THEOREM 9. *Let* $\tilde{U} = \{(R_n, \times, {}^{\cap})\} \cup \{(P_m, \times, {}^{\cap})\}$, $n \geq 2$, $m \geq 3$, *where* $(R_n, \times, {}^{\cap})$ *and* $(P_m, \times, {}^{\cap})$ *are as in Theorem* 7. *Then any finite subset of* $\tilde{U}$ *is independent.*

PROOF. We first remark that $(R_n, \times, {}^{\cap}) \cong (P_m, \times, {}^{\cap})$ if and only if $n = m = 2$, and hence no two element-algebras of $\tilde{U}$ are isomorphic. Now, suppose $\{U_1, \dots, U_r\}$ is any finite subset of $\tilde{U}$, where all the U_i are distinct, and imbed $\{U_1, \dots, U_r\}$, if necessary, to $\{U_0, U_1, \dots, U_r\}$, where $U_0 = (U_0, \times, {}^{\cap}) = (GF(2) \oplus GF(2), \times, {}^{\cap})$ (if some U_i, $1 \leq i \leq r$, is already U_0, no such imbedding is needed). Suppose that $\Phi_0, \Phi_1, \dots, \Phi_r$ are any expressions (of species (2,1)). We shall construct an expression Ψ such that $\Psi = \Phi_i(U_i)$ $(i = 0, 1, \dots, r)$. First, define

(3.6) $$a \times_{\cap} b = (a^{\cap} \times b^{\cap})^{\cup}, \quad \text{where } \zeta^{\cup} \text{ is the inverse of } \zeta^{\cap}.$$

Let $\zeta^{\cap_2} = (\zeta^{\cap})^{\cap}$, $\zeta^{\cup_2} = (\zeta^{\cup})^{\cup}$, etc. It is easily seen that

(3.7) $$a \times_{\cap} 0 = 0 \times_{\cap} a = a\,(U_i), \quad i = 1, \dots, r; \quad \text{and} \quad 0 \times_{\cap} 0 = 0\,(U_0).$$

Now, define

(3.8) $$|_{ij}(\zeta) = \begin{cases} 1 & (U_i) \\ 0 & (U_j), \end{cases} \quad 0 \leq i, \quad j \leq r, \quad i \neq j,$$

(3.9) $$|_i(\zeta) = |_{i0}(\zeta) \cdot |_{i1}(\zeta) \dots |_{ir}(\zeta) \;(\textit{no } |_{ii}(\zeta) \text{ term}), \quad i = 0, 1, \dots, r.$$

Then

$$|_i(\zeta) = \begin{cases} 1 & (U_i) \\ 0 & (U_j), \quad j \neq i, \end{cases} \qquad (0 \leq i,\ j \leq r).$$

Let

$$\Psi_1(\zeta) = (\{\Phi_1^{\cup_2} \cdot |_1(\zeta)\} \times_\cap \dots \times_\cap \{\Phi_r^{\cup_2} \cdot |_r(\zeta)\})^{\cap_2},$$

$$\Psi_2(\zeta) = \{\Phi_0^{\cup} \cdot |_0(\zeta)\}^{\cap}, \quad \Psi(\zeta) = \Psi_1(\zeta)\ \Psi_2(\zeta).$$

It is readily verified, using (3.7), (2.2)-(2.4), that

$$\text{(3.10)} \qquad \Psi(\zeta) = \Phi_i(U_i)\ (i = 0, 1, \dots, r).$$

Now, by Theorem 8, each $|_{ij}(\zeta)$, $0 \leq i, j \leq r$, $i \neq j$, and hence each $|_i(\zeta)$ (see (3.9)), is an *expression*. Therefore $\Psi_1(\zeta)$, $\Psi_2(\zeta)$, $\Psi(\zeta)$ are expressions (of species (2,1)). Hence, by (3.10), $\{U_0, U_1, \dots, U_r\}$ is independent. Therefore, a-*fortiori*, $\{U_1, \dots, U_r\}$ is independent, and the theorem is proved. An easy combination of Theorem 7 and Theorem 9 yields (compare with [6])

THEOREM 10. *Let* $\widetilde{U} = \{R_n, \times, {}^\cap)\} \cup \{P_m, \times, {}^\cap)\}$, $n \geq 2$, $m \geq 3$, *where* $(R_n, \times, {}^\cap)$ *and* $(P_m, \times, {}^\cap)$ *are as in Theorem 7. Then* $\widetilde{U}$ *is a primal cluster* (*of species* (2,1)).

4. **Existence Theorem.** In this section, we prove that the class of all non-isomorphic binary algebras $\{(B_i, \times)\}$, endowed with a suitably chosen permutation, ${}^\cap$, form a primal cluster $\{(B_i, \times, {}^\cap)\}$ (of species (2,1)). By a binary algebra $(B, \times)$ we mean a universal algebra of species (2) and with distinguished elements 0, 1 $(0 \neq 1)$ such that

$$0 \times \zeta = \zeta \times 0 = 0, \quad 1 \times \zeta = \zeta \times 1 = \zeta \qquad (\zeta \in B).$$

Following [3], we define a *primal frame* to be a frame (see Definition 2) which is also a primal algebra. From [2 ; 3], we recall the following

THEOREM 11. *Let* $(B, \times)$ *be a finite binary algebra, not isomorphic to* $(GF(2) \oplus GF(2), \times)$ *but otherwise entirely arbitrary. Then there exists a cyclic permutation,* ${}^\cap$, *of* B *such that* $(B, \times, {}^\cap)$ *is a primal frame. Furthermore, there exists no permutation,* ${}^\cap$, *of* $GF(2) \oplus GF(2)$ *such that* $(GF(2) \oplus GF(2),$ $, \times, {}^\cap)$ *is a primal frame.*

This is essentially [3 ; Theorem 2].

THEOREM 12. *The class* $\widetilde{U} = \{U_i\}$ *of all non-isomorphic primal frames* $U_i = (U_i, \times, {}^\cap)$, *forms a primal cluster.*

This is essentially [2; Theorem 10.7].

We are now in a position to prove the following

THEOREM 13. *Let* $\widetilde{B} = \{B_i\}$ *be the class of all non-isomorphic binary algebras* $B_i = (B_i, \times)$. *Then there exists a cyclic permutation,* $^{\cap}$, *of* B_i *such that* $\{(B_i, \times, ^{\cap})\}$ *forms a primal cluster.*

PROOF. Assume, without any loss of generality, that $B_0 = GF(2) \oplus \oplus GF(2) = \{0, 1, \alpha, \beta\}$. Define, $^{\cap}$, for B_0, by

$$0^{\cap} = \alpha, \quad \alpha^{\cap} = 1, \quad 1^{\cap} = \beta, \quad \beta^{\cap} = 0. \tag{4.1}$$

Furthermore, for any $B_i (\neq B_0)$, define, $^{\cap}$, to be that cyclic $0 \longrightarrow 1$ permutation of B_i whose *existence* is guaranteed by Theorem 11 above. Now, in view of Theorem 6, Theorem 11, Theorem 12, and the definition of a primal cluster, we will be through if we can prove that $\{(B_0, \times, ^{\cap}), (B_1, \times, ^{\cap}), \dots \dots, (B_n, \times, ^{\cap})]$ is independent for each positive integer n. Thus, suppose $\Phi_0, \Phi_1, \dots, \Phi_n$ are any expressions (of species (2,1)). By Theorems 11, 12, there *exists* an expression Φ such that $\Phi = \Phi_i(B_i)$ $(i = 1, \dots, n)$. Now, let q be the largest of the orders (= number of elements) of $B_0, B_1, \dots, B_n$, and let

$$E = \zeta\zeta^{\cap}\zeta^{\cap_2}\dots\zeta^{\cap_{q-1}} \quad (\text{where } \zeta^{\cap_k} = (\dots(\zeta^{\cap})^{\cap}\dots)^{\cap} \quad (k \text{ iterations})).$$

Then, it is readily verified that (see (4.1))

$$|'(\zeta) = (E^{\cap} E^{\cap_2})^{\cup} = \begin{cases} 1 & (B_i) \\ 0 & (B_0) \end{cases} \quad (i = 1, \dots, n)$$

$$|''(\zeta) = \{(|'(\zeta))(|'(\zeta))^{\cup_2}\}^{\cap_2} = \begin{cases} 0 & (B_i) \\ 1 & (B_0) \end{cases} \quad (i = 1, \dots, n).$$

Now, let

$$\Psi(\zeta) = \{\Phi^{\cup_2} \cdot |'(\zeta)\}^{\cap_2}\{\Phi_0^{\cup} \cdot |''(\zeta)\}^{\cap}.$$

Then,

$$\Psi(\zeta) = \begin{cases} \Phi (= \Phi_i) & (B_i) \\ \Phi_0 & (B_0) \end{cases} \quad (i = 1, \dots, n).$$

Hence, $\{B_0, B_1, \dots, B_n\}$ is independent, and the theorem is proved.

COROLLARY 14. *Let* $(B_1, \times)$, $(B_2, \times)$ *be any two non isomorphic binary algebras. Then there exists a cyclic permutation,* $^{\cap}$, *of* B_i $(i = 1, 2)$ *which simultaneously renders both* $(B_1, \times, ^{\cap})$, $(B_2, \times, ^{\cap})$ *primal and independent.*

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