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A theorem on generalized absolute Riesz summability


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A THEOREM ON GENERALIZED ABSOLUTE RIESZ SUMMABILITY

by S. M. MAZHAR

1.1. Let $\Sigma a_n$ be a given infinite series and $\{\lambda_n\}$ be an increasing sequence of positive numbers tending to infinity with $n$. We write

$$A_1(\omega) = A_1^\omega(\omega) = \sum_{\lambda_n < \omega} a_n,$$
$$A_1^\omega(\omega) = 0,$$
$$A_1^\omega(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^\alpha a_n,$$
$$\int_{\lambda_1}^\omega (\omega - \ell)^\alpha \, dA_1(\ell)$$

and

$$C_1(\omega) = A_1^\omega(\omega) / \omega^\alpha.$$

A series $\Sigma a_n$ is said to be summable by Riesz means of «type» $\lambda$ and «order» $\alpha$ or, simply, summable $(R, \lambda, \alpha)$, $\alpha \geq 0$ to the sum $s$ if

$$\lim_{\omega \to \infty} C_1^\alpha(\omega) = s,$$

where $s$ is any finite number [8].

The series $\Sigma a_n$ is said to be summable $|R, \lambda, \alpha|$, $\alpha \geq 0$, if the function $C_1^\alpha(\omega) \in BV(h, \infty)$, that is to say, if

$$\int_{h}^{\infty} \left| \frac{d}{d\omega} C_1^\alpha(\omega) \right| \, d\omega < \infty,$$

where $h$ is a finite positive number [6, 7].

Similarly the series $\Sigma a_n$ is said to be summable $| R, \lambda, \alpha |_k$, $\alpha > 0$, $k \geq 1$, $\alpha k' > 1$, $\frac{1}{k} + \frac{1}{k'} = 1$, if the integral
\[
\int_{0}^{\infty} \omega^{k-1} \frac{d}{d\omega} C_{2}^{\alpha}(\omega) \, d\omega
\]
is convergent [4].

In 1915 Hardy and Riesz [2] proved the following interesting theorem concerning the Riesz summability of an infinite series.

**Theorem A.** If $\lambda > 0$ and $\Sigma a_n$ is summable $| R, \lambda, \alpha |$, then the series $\Sigma a_n \lambda_n^\alpha$ is summable $| R, \lambda, \alpha |$, where $\lambda_n = e^{\lambda n}$.

Analogous problem was considered by Tatchell [9] for absolute Riesz summability. He proved the following theorem:

**Theorem B.** If $\alpha \geq 0$ and $\Sigma a_n$ is summable $| R, \lambda, \alpha |$, then $\Sigma a_n \lambda_n^{-\alpha}$ is summable $| R, \lambda, \alpha |$, where $\lambda_n = e^{\lambda n}$.

The object of the present note is to establish the corresponding result for the generalized absolute Riesz summability, namely summability $| R, \lambda, \alpha |_k$ for integral values of $\alpha$. In a subsequent note it is proposed to discuss the non-integral case.

2.1. In what follows we shall prove the following theorem:

**Theorem.** If $\alpha$ is a positive integer and $\Sigma a_n$ is summable $| R, \lambda, \alpha |_k$, then $\Sigma a_n \lambda_n^{-\alpha+1/k'}$ is summable $| R, \lambda, \alpha |_k$, where $\lambda_n = e^{\lambda n}$, $k \geq 1$, $\lambda > 0$ and $\frac{1}{k} + \frac{1}{k'} = 1$.

It is evident that for $k = 1$ our theorem includes the above theorem of Tatchell for integral values of $\alpha$.

2.2. We require the following lemmas for the proof of this theorem:

**Lemma 1** [3]. If $\alpha > 0$ and $R_n(\omega)$ is the Rieszian sum of type $\lambda$ and order $\alpha$ of the series $\Sigma a_n \lambda_n$, then
\[
\omega^{\alpha+1} \frac{d}{d\omega} C_{2}^{\alpha}(\omega) = \alpha R_{\alpha-1}(\omega) = \frac{d}{d\omega} R_{\alpha}(\omega).
\]

(1) See also Borwein [1] who defined the summability $| R, \lambda, \alpha |_k$.

(2) This theorem for the case $\alpha = 1$ is due to Mohanty [5].
LEMMA 2 [2]. If \( l \) is a positive integer, then

\[
A_l(t) = \frac{1}{l!} \left( \frac{d}{dt} \right)^l A^l(t).
\]

3.1. PROOF OF THEOREM. Under the hypothesis of the theorem we have by Lemma 1

\[
\int_{\lambda_1}^{\infty} \omega^{-(l+\alpha k)} |E_{\alpha-1}(\omega)|^k d\omega < \infty
\]

and we have to establish the convergence of the integral

\[
\int_{\lambda_1}^{\infty} \omega^{-(l+\alpha k)} |E_{\alpha-1}(\omega)|^k d\omega,
\]

where \( E_{\alpha-1}(\omega) \) is the Rieszian sum of order \((\alpha - 1)\) and of type \( l \), of the series \( \sum a_n \lambda_n^{-(\alpha+1)/k} e^{i\lambda n} \).

By writing \( \omega = e^x \) in the above integral (3.1.2) we find that the required condition can also be written in the form

\[
\int_{\lambda_1}^{\infty} e^{-\alpha x k} |E_{\alpha-1}(e^x)|^k dx < \infty.
\]

We have

\[
E_{\alpha-1}(e^x) = \int_{\lambda_1}^{e^x} (e^u - u)^{\alpha-1} dE(u)
\]

\[
= \int_{\lambda_1}^{e^x} (e^u - e^x)^{\alpha-1} dE(e^u)
\]

\[
= \int_{\lambda_1}^{e^x} (e^u - e^x)^{\alpha-1} e^t t^{\alpha-1}/k dB(t)
\]

\[
= [(e^x - e^x)^{\alpha-1} e^t t^{\alpha-1}/k B(t)]_{\lambda_1}^{e^x}
\]

\[
- \int_{\lambda_1}^{e^x} dB(t) \frac{d}{dt} [(e^x - e^x)^{\alpha-1} e^t t^{\alpha-1}/k] dt.
\]
Applying Lemma 2 and integrating \((x - 1)\) times we have

\[
E_{n-1}(e^x) = \left[(e^x - e^{i\theta}) e^{x \cdot t-a^{1/k}} B(t)^n_1 + \right.
\]

\[
+ O(n! \sum_{n=1}^{\infty} (-1)^i \left( \frac{d}{dt} \right)^{n-i-1} B_{n-1}(t) \left( \frac{d}{dt} \right)^i \left[(e^x - e^{i\theta}) e^{x \cdot t-a^{1/k}} \right]_1^x
\]

\[
+ C \int_0^x B_{n-1}(t) \left( \frac{d}{dt} \right)^n \left[(e^x - e^{i\theta}) e^{x \cdot t-a^{1/k}} \right] dt
\]

\[
= C B_{n-1}(x) e^{ax} x^{-a-1/k} + C \int_0^x B_{n-1}(t) \left( \frac{d}{dt} \right)^n \left[(e^x - e^{i\theta}) e^{x \cdot t-a^{1/k}} \right] dt
\]

\[
= L_1 + L_2, \text{ say.}
\]

Since

\[
\int_0^\infty e^{-ax} |L_1|^k dx \leq C \int_0^\infty x^{-(1+ak)} |B_{n-1}(x)|^k dx < \infty,
\]

it is, therefore, by virtue of Minkowski's inequality sufficient to prove that

\[
\int_0^\infty e^{-ax} |L_2|^k dx < \infty.
\]

Now

\[
L_2 = O \left\{ \int_0^x |B_{n-1}(t)| t^{-a-1/k} e^{at} dt \right\} + O \left\{ \int_0^x \left( \sum_{i=1}^{\infty} e^{ax} e^{(a-\delta)t} t^{-a-1/k} \right) dt \right\}
\]

\[
= L_{21} + L_{22}.
\]

Applying Hölder's inequality, we observe that

\[
\int_0^\infty e^{-ax} |L_{21}| dx = O \left\{ \int_0^\infty e^{-ax} \int_0^x |B_{n-1}(t)| ^k t^{-(1+ak)} e^{at} e^{ak|k-1|} dt dx \right\}
\]

\((^*)\) Where \(C\) denotes a constant not necessarily the same at each occurrence.
Also, in order to show that it is sufficient to prove the convergence of the integral

$$\int_{\lambda_i}^{\infty} e^{-x_{ik}} \left| I_{\mu_{ik}} \right|^k \, dx < \infty$$

it is sufficient to prove the convergence of the integral

$$\int_{\lambda_i}^{\infty} e^{-x_{ik}} \left| B_{u_{ik}} \right|^k \, dx \leq \int_{\lambda_i}^{\infty} e^{-x_{ik}} \left| B_{u_{ik}} \right|^k \left| B_{a_{ik}} \right|^k \, dx$$

for $1 \leq i \leq a - 1$.

Using Hölder's inequality we find that the above integral is

$$\leq C \int_{\lambda_i}^{\infty} t^{-(1+\alpha)} \left| B_{u_{ik}} \right|^k \left| B_{a_{ik}} \right|^k \int_{t}^{\infty} t^{(a-1)(k-1)} \, dt$$

by hypothesis.

This completes the proof of the theorem.

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