

ANNALI DELLA  
SCUOLA NORMALE SUPERIORE DI PISA  
*Classe di Scienze*

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*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3<sup>e</sup> série*, tome 19,  
n° 1 (1965), p. 107-111

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# GENERATING CURVES ON ABELIAN VARIETIES AND RIEMANN'S THETA-FUNCTION

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## I. Introduction.

We shall show how to prove (and possibly illuminate) two well known theorems of Riemann and Poincaré about theta-functions, using Weil's ([9]) « intrinsic » approach to classical abelian varieties. Some variations of Matsusaka's criterion for jacobians ([2], [5]) are incidentally obtained.

## II. Generating curves.

Let  $Z$  be a positive 1-cycle on an  $n$ -dimensional abelian variety  $A$ . Since we work modulo numerical equivalence, we may suppose that  $Z$  is a (possibly reducible) curve, all of whose components pass through the identity  $0$  of  $A$ . Let  $\{Z\}$  be the smallest abelian subvariety containing  $Z$ , and call  $Z$  a generating curve if  $\{Z\} = A$ .

Let  $Z^{(n)}$  denote the  $n$ -fold Pontrjagin product of  $Z$  with itself (see [10]). There is a non-negative integer  $k(Z)$  such that, as a cycle,  $Z^{(n)} = k(Z) n! A$ .

**PROPOSITION 1.**  $Z$  is a generating curve  $\iff k(Z) \neq 0$ .

**PROOF.** The implication  $\Leftarrow$  is obvious. For the converse, first consider the case of an irreducible  $Z$ , and let  $r$  be the smallest integer for which  $Z^{(r+1)} = 0$ . Then  $|Z^{(r)}|$  (the support of  $Z^{(r)}$ ) is invariant under translations by points of  $Z$ , and since it contains  $0$ , must be a subgroup, so clearly must be equal to  $\{Z\}$ , and the proposition follows immediately in this case. Now for any  $Z$ , let its irreducible components be the  $Z_i$ , and  $n_i = \dim \{Z_i\}$ , so that by the preceding, there are positive integers  $k_i$  for which  $Z_i^{(n_i)} = n_i! k_i \{Z_i\}$ . Furthermore, one sees that the  $\{Z_i\}$  generate  $\{Z\}$ .

If the  $\{Z_i\}$  are direct summands of  $\{Z\}$  then  $n = \sum_i n_i$ , and  $k(Z) = \prod_i k_i \neq 0$ . If  $n = \sum_i n_i$ , but the sum is not direct, then the direct sum of the  $\{Z_i\}$  form a cover of degree  $d > 1$  of  $\{Z\}$ , and  $k(Z) = d \prod_i k_i > 1$ . If, finally,  $n < \sum_i n_i$ , for some  $i \neq j$   $\dim \{Z_i\} \cap \{Z_j\} > 0$ . Choose the smallest  $r_i \leq n_j$  so that  $Z_i^{(n_i)} \oplus Z_j^{(r_i+1)} = 0$  (where  $\oplus$  means Pontrjagin product). Then by an argument similar to that above, we see that  $|Z_i^{(n_i)} \oplus Z_j^{(r_j)}|$  is an abelian variety, in fact, it equals  $\{Z_i + Z_j\}$ . By symmetry, we can also find an  $r_i \leq n_i$  so that  $\{Z_i + Z_j\} = |Z_i^{(r_i)} \oplus Z_j^{(n_j)}|$ .

Hence, in the formula

$$Z^{(n)} = \sum_{g_1 + \dots + g_q = n} \frac{n!}{g_1! \dots g_q!} Z_1^{(g_1)} \oplus \dots \oplus Z_q^{(g_q)}$$

the right-hand side has at least two nonvanishing terms, so in fact,  $k(Z) \geq 2$ .

**PROPOSITION 2.** If  $k(Z) = 1$ , the  $Z_i$  are non-singular curves of genus  $n_i$ , and  $A$  is the direct sum of the  $\{Z_i\}$ , which are the jacobians of the  $Z_i$ .

**PROOF.** The proof of Proposition 1 shows that  $A$  is the direct sum of the  $\{Z_i\}$ , and that  $Z_i^{(n_i)} = n_i! \{Z_i\}$ , so we are reduced to the case of  $Z$  irreducible. In that case, let  $J$  be the jacobian of its normalization  $\bar{Z}$ . Since  $Z$  is generating, the map  $\bar{Z} \rightarrow Z$  gives rise to a surjection  $J \rightarrow A$ , so  $\bar{Z}$  has genus  $g \geq n$ . But  $k(Z) = 1$  means that the  $n$ -fold symmetric product of  $\bar{Z}$  is birationally equivalent to  $A$ , so by a remark of Weil ([11] p. 37)  $n = g$ , and  $J \rightarrow A$  is an isomorphism. In particular,  $Z = \bar{Z}$ .

### III. The associated Kähler metric

Assume we are in the classical case, and define an hermitian form on  $H^0(A, \Omega^1)$  (the holomorphic differentials) by

$$H(\alpha, \alpha') = \frac{\sqrt{-1}}{2} \int_Z \alpha \wedge \bar{\alpha}'$$

**PROPOSITION 3.**  $Z$  is generating  $\iff H$  is positive definite.

**PROOF.** The implication  $\Leftarrow$  is obvious. Conversely, let  $Z$  generate  $A$ , and  $J_j$  be the jacobians of the normalizations  $\bar{Z}_j$  of the components  $Z_j$  of  $Z$ . The maps  $\varphi_j: \bar{Z}_j \rightarrow Z_j$  induce a surjection  $\varphi: \prod_j J_j \rightarrow A$ , whose dual map

$\widehat{\varphi}: \widehat{A} \rightarrow \Pi_j \widehat{J}_j$  on the Picard varieties thus has finite kernel. The lifting of  $\widehat{\varphi}$  to the universal covering spaces is the map  $\Pi_j \varphi_j^*: H^0(A, \Omega^1) \rightarrow \Pi_j H^0(J_j, \Omega^1)$  gotten by pulling back differentials, which, being linear and having a discrete kernel, is an injection. So if  $\alpha \neq 0$  is a differential on  $A$ , some  $\varphi_j^*(\alpha) \neq 0$ , and

$$H(\alpha, \bar{\alpha}) \geq \frac{\sqrt{-1}}{2} \int_{\bar{z}_j} \varphi_j^*(\alpha) \wedge \overline{\varphi_j^*(\alpha)} = \int_{\bar{z}_j} \|\operatorname{Re}(\varphi_j^*(\alpha))\|^2 > 0$$

(where  $\|\cdot\|$  is the Dirichlet norm) by a standard result about harmonic forms on Riemann surfaces.

Now let  $\alpha_1, \dots, \alpha_n$  be an orthonormal basis for  $H$ . Then  $(\sum_i \alpha_i)^2$  gives a hermitian form on the tangent space to  $A$  at 0, and hence an invariant Kähler metric. Associated with this is the closed 2-form  $u = \frac{\sqrt{-1}}{2} \sum_i \alpha_i \wedge \bar{\alpha}_i$ , and Hodge's adjoint operation  $*$  on differential forms of cohomology. An easy computation (see Weil [9] p. 20) shows that  $*u = \left(\frac{\sqrt{-1}}{2}\right)^{n-1} \sum_i \Omega_i$ , where  $\Omega_i = \prod_{i \neq j} \alpha_j \wedge \bar{\alpha}_j$ . Let  $\gamma = \int_A u^n/n!$  be the volume of  $A$  for this metric.

Let  $[X]$  denote the cohomology class of either a form or cycle,  $X$ , denote either intersection or cup products by juxtaposition, and preserve the notation  $X \oplus Y$  and  $X^{(r)}$  for Pontrjagin products of cohomology classes, as well as cycles. (It is not hard to see that the two are compatible). Then we have:

PROPOSITION 4.  $*[u] = \gamma[Z]$

PROOF. Let  $\eta$  be a closed (1,1) form on  $A$ , hence cohomologous to  $\frac{\sqrt{-1}}{2} \sum_{ij} a_{ij} \alpha_i \wedge \bar{\alpha}_j$ , for constant  $a_{ij}$ . Now

$$\frac{\sqrt{-1}}{2} \alpha_i \wedge \bar{\alpha}_j \wedge \Omega_k = \begin{cases} u^n/n! & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases}$$

so  $[\eta \wedge *u] = [\sum_j a_{jj} u^n/n!]$ . But on the other hand,

$$\int_Z \eta = \sum_{ij} \frac{\sqrt{-1}}{2} a_{ij} \int_Z \alpha_i \wedge \bar{\alpha}_j = \sum_{ij} a_{ij} H(\alpha_i, \alpha_j) = \sum_j a_{jj}$$

and the proposition follows immediately.

PROPOSITION 5.  $[Z^{(n-1)}] = \gamma^{1-n} (n-1)! [u]$ .

PROOF. By a general result ([4]) one has  $*(XY) = (*X) \oplus (*Y)$  for  $X$  and  $Y$  any cohomology classes. Hence

$$[Z]^{(n-1)} = (*[u]/\gamma)^{(n-1)} = \gamma^{1-n} *[u]^{(n-1)}.$$

But by [9] p. 25  $[u]^{n-1} = (n-1)! * [u]$ , and our assertion follows.

COROLLARY 1.  $k(Z) = \gamma^{-n}$ , so if  $\gamma \geq 1$  (in particular, if the metric is Hodge,  $k(Z) = \gamma = 1$  and  $A$  is a product of jacobians).

PROOF.  $[Z^{(n)}] = k(Z) n! [A]$ , but also  $[Z]^{(n)} = (*[u]/\gamma)^{(n)} = \gamma^{-n} *[u]^{(n)} = \gamma^{-n} n! [A]$ . The last assertion follows from Proposition 2.

Assume we are dealing with a product of jacobians, and let, as usual,  $r! W_r = Z^{(r)}$  and  $\Theta = W_{n-1}$ .

COROLLARY 2. (Poincaré [6]. For numerical equivalence, in the abstract case, see [5]).  $(n-r)! [W_r] = [\Theta^{n-r}]$ .

$$\text{PROOF. } (n-r)! [W_r] = \frac{(n-r)!}{r!} Z^{(r)} = \frac{(n-r)!}{r!} * [Z]^r = (n-r)!/r! * [u^r].$$

But by an elementary algebraic identity ([1] p. 170)  $*[u^r] = r!/(n-r)! [u^{n-r}]$ , from which the corollary follows.

COROLLARY 3. (Riemann [7]. See e. g. [3] for a rigorous presentation). Let  $\vartheta$  be « Riemann's » theta-function. Then a translate of  $\Theta$  is « cut out by »  $\vartheta$ .

PROOF. Let  $\vartheta$  cut out the positive divisor  $X$ . Examining its « factors of automorphy » we see that  $\vartheta$  « belongs to » the hermitian form  $H$ , in the sense of Weil [9] ch. VI. Hence ([9] p. 112)  $[X] = [u] = [\Theta]$ , and so ([9] p. 115)  $X$  is linearly equivalent to a translate of  $\Theta$ . But since  $\gamma = 1$ ,  $\lambda(X) = 1$  by the (Frobenius) Riemann-Roch theorem, and the corollary follows.

Note that  $\vartheta$  is even, so  $X = X^-$ , the image of  $X$  under the endomorphism  $x \rightarrow -x$ . On the other hand,  $\Theta = \Theta_c$ , where  $c$  is the « canonical point » (see Weil [8] p. 73). So if  $\Theta = X_r$ , then  $2r = c$ .

The coordinates of the point  $r$  are the traditional « Riemannian constants ».

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