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RING-LOGICS AND CERTAIN CLASSES OF RINGS

ADIL YAQUB

Introduction. Boolean rings $(B, \times, +)$ and Boolean logics (= Boolean algebras) $(B, \cap, *)$ though historically and conceptionally different, are equationally interdefinable in a familiar way [7]. With this equational interdefinability as motivation, Foster [1; 2] introduced and studied the theory of ring-logics. Indeed, let $(R, \times, +)$ be a commutative ring with unit 1, and let $K = \{\varrho_1, \varrho_2, \dots\}$ be a transformation group in R . The K -logic of the ring $(R, \times, +)$ is the (operationally closed) system $(R, \times, \varrho_1, \varrho_2, \dots)$ whose class R is identical with the class of ring elements, and whose operations are the ring product « \times » together with the unary operations $\varrho_1, \varrho_2, \dots$ of K . The ring $(R, \times, +)$ is called a *ring-logic*, mod K if (1) the « $+$ » of ring is *equationally* definable in terms of its K -logic $(R, \times, \varrho_1, \varrho_2, \dots)$, and (2) the « $+$ » of the ring is *fixed* by its K -logic. The Boolean theory results from the special choice, for K , of the «Boolean group», C , generated by $x^* = 1 - x$ (order 2, $x^{**} = x$). Furthermore, by choosing K to be the «natural group», N , generated by $x^\wedge = 1 + x$, Foster showed [1] that a p -ring with unit is a ring-logic, mod N . Again, by choosing K to be the «normal group», D , where the generator x^\cap of D is now no longer linear, Foster [2] was able to show that a p^k -ring with unit is a ring-logic, mod D . These results naturally suggest the following question: are the groups C, N, D , in any way related, and are they the only possible transformation groups with respect to which the corresponding rings are ring-logics? It turns out that for the class of all p^k -rings (and hence, in particular, for p -rings and Boolean rings) *any* transitive $0 \rightarrow 1$ permutation of $GF(p^k)$ induces a transformation group in the corresponding p^k -ring R with respect to which R is a ring-logic.

Indeed, x^*, x^\wedge, x^\cap above are merely examples of some transitive $0 \rightarrow 1$ permutations of $GF(2), GF(p), GF(p^k)$, respectively, and these in turn induce the above transformation groups C, N, D , with respect to which the corresponding rings are ring-logics.

1. The Finite Field Case. Let $(F_{p^k}, \times, +)$ be a (finite) Galois field with exactly p^k elements (p prime). Then, as is well known, $F_{p^k} = \{0, \zeta, \zeta^2, \dots, \zeta^{p^k-1} (=1)\}$ for some ζ in F_{p^k} . We now have the following

THEOREM 1. *Let F_{p^k} be a Galois field, and let ζ be a generator of F_{p^k} . Let $\circ : x \rightarrow x^\circ$ be any permutation of F_{p^k} . Then \circ is expressible as a polynomial in x over F_{p^k} .*

PROOF. Denote the elements of F_{p^k} by x_1, \dots, x_n ($n = p^k$), and denote x_i° by x'_i ($i = 1, \dots, n$). We shall show that x° can be written as

$$(1.1) \quad x^\circ = a_0 + a_1 x + \dots + a_{n-1} x^{n-1} \quad (n = p^k)$$

for some a_0, a_1, \dots, a_{n-1} in F_{p^k} . Since $x_i^\circ = x'_i$ ($i = 1, \dots, n$), therefore, (1.1) gives n linear equations in the n unknowns a_0, a_1, \dots, a_{n-1} . Now, the determinant of the coefficients of the a_i is the familiar VanderMonde determinant which, except possibly for sign, is equal to $\prod_{i,j=1, i>j}^n (x_i - x_j)$, and hence does not vanish since the x_i are *distinct* elements of F_{p^k} . Hence the above equations are solvable, and the theorem is proved.

We shall from now on be primarily concerned only with *transitive* $0 \rightarrow 1$ permutations of F_{p^k} . This simply means a permutation, \circ , of F_{p^k} such that (i) $0^\circ = 1$, and (ii) for any given elements α, β in F_{p^k} , there exists an integer r such that $\alpha^{\circ r} = \beta$, where $\alpha^{\circ r} = (\dots ((\alpha^\circ)^\circ) \dots)^\circ$ (r -iterations). We now have the following

THEOREM 2. *Let, \circ , be any transitive $0 \rightarrow 1$ permutation of the Galois field F_{p^k} , and let K be the transformation group in F_{p^k} generated by, \circ . Then the elements of F_{p^k} are equationally definable in terms of the K -logic (F_{p^k}, \times, \circ) .*

PROOF. Since, \circ , is a transitive permutation of F_{p^k} , therefore, $F_{p^k} = \{0, 0^\circ, 0^{\circ 2}, \dots, 0^{\circ p^k-1}\}$. A similar argument shows that, for any x in F_{p^k} , $xx^\circ x^{\circ 2} \dots x^{\circ p^k-1} = 0$. Hence 0 (and with it $0^\circ, 0^{\circ 2}, \dots, 0^{\circ p^k-1}$) is expressible in terms of the K -logic, and the theorem is proved.

We recall from [4] the *characteristic function* $\delta_\mu(x)$, defined as follows: for any given $\mu \in F_{p^k}$, $\delta_\mu(x) = 1$ if $x = \mu$, and 0 if $x \neq \mu$.

We now have the following

THEOREM 3. *Let F_{p^k}, K, \circ , be as in Theorem 2. Then the characteristic functions $\delta_\mu(x)$, $\mu \in F_{p^k}$, are equationally definable in terms of the K -logic (F_{p^k}, \times, \circ) .*

PROOF. Since, α , is a *transitive* $0 \rightarrow 1$ permutation of F_{p^k} , therefore, $\mu^{\alpha r} = 0$ for some integer r . Now, one readily verifies that, since $y^{p^k-1} = 1$, $y \neq 0$, $y \in F_{p^k}$, $\delta_\mu(x) = ((x^{\alpha r})^{p^k-1})^{\alpha^{p^k-1}}$, and the theorem is proved.

Now, let, \cup , denote the inverse of the $0 \rightarrow 1$ transitive permutation, α , and as in [2], define $a \times_\alpha b = (a^\alpha \times b^\alpha)^\cup$. Then, $a \times_\alpha 0 = a = 0 \times_\alpha a$. Hence, we have the following « normal expansion formula » [4]

$$(1.2) \quad f(x, y, \dots) = \sum_{\alpha, \beta, \dots \in F_{p^k}}^{\times_\alpha} f(\alpha, \beta, \dots) (\delta_\alpha(x) \delta_\beta(y) \dots).$$

In (1.2), α, β, \dots range independently over all the elements of F_{p^k} while x, y, \dots are indeterminates over F_{p^k} . $\sum_{\alpha_i \in F}^{\times_\alpha} \alpha_i$ denotes $\alpha_1 \times_\alpha \alpha_2 \times_\alpha \dots$, where $\alpha_1, \alpha_2, \dots$ are all the elements of F .

THEOREM 4. *Let, α , be any transitive $0 \rightarrow 1$ permutation of the Galois field F_{p^k} , and let K be the transformation group in F_{p^k} generated by, α . Then $(F_{p^k}, \times, +)$ is a ring-logic, mod K .*

PROOF. By (1.2),

$$x + y = \sum_{\alpha, \beta \in F_{p^k}}^{\times_\alpha} (\alpha + \beta) (\delta_\alpha(x) \delta_\beta(y)).$$

Now, by Theorem 2 and Theorem 3, the right-side of the above equation is equationally definable in terms of the K -logic $(F_{p^k}, \times, \alpha)$. Hence the « + » of F_{p^k} is *equationally* definable in terms of the K -logic. Next, we show that $(F_{p^k}, \times, +)$ is *fixed* by ist K -logic. Suppose that $(F_{p^k}, \times, +')$ is another ring with the same class of elements F_{p^k} and the same « \times » as $(F_{p^k}, \times, +)$ and which has the *same logic* as $(F_{p^k}, \times, +)$. To prove that $+ ' = +$. But this follows since, up to isomorphism, there is *only one* Galois field with exactly p^k elements.

2. The General Case. In this section we shall extend the results of Theorem 4 to p -rings and p^k -rings by use of the familiar subdirect structure of these rings [6; 5]. Thus, suppose R is a commutative ring with unit 1, and suppose that p is a *prime* integer. R is called a *p-ring* [6] if $a^p = a$, $pa = 0$ for all a in R . Furthermore, R is called a *p^k-ring* [2] if (i) $a^{p^k} = a$, $pa = 0$ for all a in R , and (ii) R has a subring (= field) F which is isomorphic to the Galois field F_{p^k} and where $1 \in F$. (Under a somewhat broader definition, p^k -rings were first introduced by McCoy [5]). Clearly, every

p -ring R with unit is a p^k -ring ($k = 1$) (in this case (i) implies (ii) in the above definition, since F can be chosen as the prime field of R). From [5], we now recall the following fundamental subdirect structure

THEOREM 5. *A p^k -ring is isomorphic to a subdirect power of the Galois field F_{p^k} .*

We are now in a position to prove the following

THEOREM 6. *Any p^k -ring R with unit is a ring-logic, mod K , where K is the transformation group in R induced by any transitive $0 \rightarrow 1$ permutation, α , of F_{p^k} .*

PROOF. By Theorem 5, R is isomorphic to a (not necessarily finite) subdirect power $F_{p^k}^m$ of F_{p^k} . Now, suppose $x = (x_1, x_2, \dots)$ is any element in $R (= F_{p^k}^m)$. Define $(x_1, x_2, \dots)^\alpha = (x_1^\alpha, x_2^\alpha, \dots)$, and let K be the transformation group generated by, α . We shall now show that $F_{p^k}^m$ is a ring-logic, mod K . Indeed, by Theorem 4, there exists a « logical expression » $\varphi(a, b; \times, \alpha)$ such that $a + b = \varphi(a, b; \times, \alpha)$ for all a, b in F_{p^k} . Since the operations are component-wise in $F_{p^k}^m$, therefore, for all x, y in $F_{p^k}^m (= R)$, we have $x + y = \varphi(x, y; \times, \alpha)$. Hence the « + » of $F_{p^k}^m$ is equationally definable in terms of the K -logic. Next, we show that $F_{p^k}^m$ is fixed by its K -logic. Suppose that $(F_{p^k}^m, \times, +')$ is another ring with the same class of elements and the same « \times » as $(F_{p^k}^m, \times, +)$ and which has the same logic as $(F_{p^k}^m, \times, +)$. To prove $+ = +'$. Now, a new « +' » in $F_{p^k}^m$ defines and is defined by a new « +_i' » in F_{p^k} ($= i$ -th component in $F_{p^k}^m$) such that $(F_{p^k}, \times, +'_i)$ is a ring, for each i . Furthermore, the assumption that $(F_{p^k}^m, \times, +')$ has the same logic as $(F_{p^k}^m, \times, +)$ is equivalent to the assumption that each $(F_{p^k}, \times, +'_i)$ has the same logic as $(F_{p^k}, \times, +)$. Since, by Theorem 4, $(F_{p^k}, \times, +)$ is a ring-logic, and hence with its « + » fixed, therefore, $+'_i = +$ for each i . Hence $+ = +'$, and the theorem is proved.

COROLLARY 7. *Any p -ring R with unit is a ring-logic, mod K , where K is the transformation group in R induced by any transitive $0 \rightarrow 1$ permutation of F_p .*

This is the case $k = 1$ of Theorem 6.

It is noteworthy to observe that, since there is *only one* $0 \rightarrow 1$ (transitive) permutation of F_2 , the level of generality given in Theorem 6 and Corollary 7 is not apparent in the Boolean case.

Now, by choosing $a_0, a_1, \dots, a_{p^k-1}$, in (1.1), in all of the $(p^k - 2)!$ available ways to get transitive $0 \rightarrow 1$ permutations of F_{p^k} , we obtain the

corresponding transformation groups with respect to which a p^k -ring is a ring-logic. Thus, if in (1.1) we choose, $x^n = 1 - x$ ($p^k = 2^1$), we recover the generator x^* of the Boolean group C (see introduction). Similarly, if we set $x^n = 1 + x$ ($p^k = p$) in (1.1), we obtain the generator x^\wedge of the natural group N . Finally, by selecting the a_i in (1.1) so that $0^n = 1, 1^n = \zeta, \zeta^n = \zeta^2, \dots, (\zeta^{p^k-3})^n = \zeta^{p^k-2}, (\zeta^{p^k-2})^n = 0$, where ζ is a generator of F_{p^k} , we obtain the generator x^n of the normal group D (see [2]). Hence, we have proved, as a further corollary of Theorem 6, the following theorem which contains Foster's results [1; 2] (see also [8]):

COROLLARY 8. (i) *Any Boolean ring with unit is a ring-logic, mod C ; (ii) any p -ring with unit is a ring-logic, mod N ; (iii) any p^k -ring with unit is a ring-logic, mod D ; where C, N, D , are the Boolean group, natural group, and normal group, respectively.*

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