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PATHOLOGICAL SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS

by JAMES SERRIN

In this note we consider the simplest kind of divergence structure elliptic equation, namely

$$\Sigma \frac{\partial}{\partial x_i}\left(a_{ij}\,\frac{\partial u}{\partial x_j}\right) = 0 \tag{1}$$

where the coefficients a_{ij} are bounded measurable functions of $x = (x_1, \dots, x_n)$ satisfying the ellipticity condition

$$\lambda \xi^2 \leq a_{ij}\,\xi_i\,\xi_j \leq \Lambda \xi^2, \quad \lambda, \Lambda > 0). \tag{2}$$

Because of the general assumptions made on the coefficients it is natural that the equation be interpreted in a weak sense. In particular, let $u = u(x)$ be a function having strong derivatives $\partial u/\partial x_i$ which are locally summable over a domain D. Then u may be called a *weak solution* of (1) over D if

$$\int a_{ij}\,\frac{\partial u}{\partial x_j}\,\frac{\partial \Phi}{\partial x_i}\,dx = 0 \tag{3}$$

for any continuously differentiable function $\Phi = \Phi(x)$ with compact support in D.

In developing the theory of equations (1) or (3) it is generally assumed that the derivatives $\partial u/\partial x_i$ are locally of class L_2 in D, and this additional requirement is accordingly built into the definition of weak solutions. This being done, one can then prove that solutions are locally bounded, have a local maximum principle, and that the Dirichlet problem for

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smooth data on smooth boundaries has one and only one solution. If, however, one does not introduce this additional requirement, the definition of weak solution still remains meaningful as we have given it. In fact, the above definition is in a sense the most « natural » one, in that it requires of the function $u(x)$ just exactly enough to make relation (3) well defined.

Nevertheless, the generality allowed in this definition is too great to allow a comprehensive theory to be developed. We shall show here that there exist weak solutions in the sense defined above that are neither locally bounded nor have a local maximum principle; and that, moreover, the Dirichlet problem need not have unique solutions in the class considered. In addition, the violation of local boundedness will be shown to hold even in the class of solutions whose derivatives are locally in $L_{1+\mu}$, where μ is any fixed number less than one. Thus the usual requirement that the derivatives be in L_2 forms an important and essential part of the theory.

Consider now equation (1) with the particular coefficients

$$a_{ij} = \delta_{ij} + (a-1)\frac{x_i x_j}{r^2}, \qquad (r = |x|),$$

where a is a constant greater than one. It is easily checked that these coefficients are bounded and that (2) holds with $\lambda = 1$, $\Lambda = a$. Moreover one finds that the function

$$u = x_1 r^{1-n-\varepsilon}$$

is a classical solution in $|x| > 0$ provided a and ε are related by

$$a = \frac{n-1}{\varepsilon(\varepsilon + n - 2)}, \qquad 0 < \varepsilon < 1.$$

We shall now show that u is in fact a weak solution throughout E^n. First of all, since $u = 0(r^{2-n-\varepsilon})$, $u_x = 0(r^{1-n-\varepsilon})$, it is easily verified that u is strongly differentiable with $u_x \in L_\beta$ for any $\beta < n/(n+\varepsilon-1)$. Moreover, if Φ is any continuously differentiable function with compact support in E^n, then obviously

$$\int a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i}dx = \lim_{\varrho\to 0}\int_{r>\varrho} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i}dx$$

$$= -\lim_{\varrho\to 0}\oint_{r=\varrho}\Phi\, a_{ij}\frac{\partial u}{\partial x_j}\frac{x_i}{r}ds.$$

But by direct calculation

$$-\oint_{r=\varrho} \Phi\, a_{ij} \frac{\partial u}{\partial x_j} \frac{x_i}{r}\, ds = \frac{\omega_n (n-1)}{\varepsilon} \varrho^{-1-\varepsilon} \oint_{r=\varrho} \Phi\, x_1\, d\omega.$$

Since Φ is continuously differentiable the last integral is easily seen to be $0(\varrho^2)$ as $\varrho \to 0$, whence combining these results it follows that

$$\int a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \Phi}{\partial x_i}\, dx = 0.$$

This proves that u is a weak solution in E^n.

Clearly the function u is neither locally bounded nor has a local maximum principle. It can be used, moreover, to show that the Dirichlet problem need not have unique solutions. In fact, let v be the unique weak solution of (1) with L_2 derivatives, taking the same values on $r = 1$ as the function u. Then $u - v$ has zero data on $r = 1$, but is not identically zero. This shows clearly that the Dirichlet problem can have two solutions corresponding to the same data, provided we give up the requirement that these solutions have L_2 derivatives.

In case $n = 2$, we have $u_x \in L_\beta$ for any $\beta < 2/(1+\varepsilon)$, whence by choosing ε sufficiently near zero it is clear that any relaxation of the requirement that derivatives be in L_2 will lead to difficulties. When $n > 2$ the function

$$u(x) = x_1 (x_1^2 + x_2^2)^{-(1+\varepsilon)/2}$$

may be thought of as a solution (by descent) of another uniformly elliptic equations whose coefficients a_{11}, a_{12}, a_{22} are those associated with the case $n = 2$ above. Since this solution is not locally bounded, the proof of our assertions is complete.

REMARKS. Although the above example shows that in general it is reasonable to require solutions to have strong derivatives which are locally in L_2, nevertheless, for any given equation (1) it can be shown that regularity properties remain valid in a somewhat more extended class of functions, depending in particular on the moduli of ellipticity [1]. From this point of view, the above example shows that there are definite limits beyond which one cannot go without losing the outlines of the theory.

In conclusion, we should like to conjecture that if the coefficients a_{ij} are Hölder continuous, then any weak solution as defined here must be in fact a classical solution.

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[1] N. G. MEYERS, *An L^p estimate for the gradient of solutions of second order elliptic divergence structure equations.* Ann. Scuola Norm. Pisa XVII (1963), pp. 189-206.