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**On $\mathfrak{p}$-equations and normal extensions of finite $\mathfrak{p}$-type. (II)
The analogy of the Riemann's problem**

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ON $\mathfrak{p}$-EQUATIONS AND NORMAL EXTENSIONS OF FINITE $\mathfrak{p}$-TYPE (II) THE ANALOGY OF THE RIEMANN'S PROBLEM

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§ 1. Introduction.

1.1 Let $\mathcal{M}$ be a closed Riemann surface and Σ be the direct system of all the finite subsets in $\mathcal{M}$, where the order in Σ is defined by the set theoretical inclusion. If $S \subset S'$ $(S, S' \in \Sigma)$, there exists the canonical homomorphism $\varphi_{S, S'}$ of the fundamental group $\pi_1(\mathcal{M} - S')$ onto the fundamental group $\pi_1(\mathcal{M} - S)$. We denote by $G(\mathcal{M})$ the inverse limit of $\{\pi_1(\mathcal{M} - S) \mid S \in \Gamma\}$ with respect to the homomorphisms $\{\varphi_{S, S'} \mid S \subset S'\}$ and by φ_S the canonical homomorphism of $G(\mathcal{M})$ onto $\pi_1(\mathcal{M} - S)$. We denote by $K(\mathcal{M})$ the field of meromorphic functions on $\mathcal{M}$ and by $\mathcal{D}(\mathcal{M})$ the set of all the linear homogeneous ordinary differential equations with coefficients in $K(\mathcal{M})$. We denote by $\Omega(\mathcal{M})$ the set of all the solutions of certain non-zero elements in $\mathcal{D}(\mathcal{M})$. Then it easily checked that $\Omega(\mathcal{M})$ is a commutative $K(\mathcal{M})$-algebra by the usual sum, the product and the multiplication of the elements of $K(\mathcal{M})$. The topological group $G(\mathcal{M})$ operates continuously on the discrete ring $\Omega(\mathcal{M})$ as follows: Let f be any element in $\Omega(\mathcal{M})$ and σ be any element in $G(\mathcal{M})$. Let S be the set of all the singularities of f on $\mathcal{M}$ and $\gamma(\sigma)$ be the closed path on $\mathcal{M} - S$ of which homotopy class in $\pi_1(\mathcal{M} - S)$ is the image $\varphi_S(\sigma)$ of σ by the canonical homomorphism φ_S. Then the image f^σ of f by σ is defined by the analytic continuation of f along the closed path $\gamma(\sigma)$. Therefore we can regard $\Omega(\mathcal{M})$ as a $C[G(\mathcal{M})]$-module, where $\mathbf{C}$ is the field

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of complex numbers and we mean by a $G(\mathfrak{M})$-module a discrete module on which $G(\mathfrak{M})$ operates continuously.

In these notations and terminologies the classical Riemann problem is formulated as follows (1):

Does there exist a $\mathbf{C}[G(\mathfrak{M})]$-submodule in $\Omega(\mathfrak{M})$ which is isomorphic to a given $\mathbf{C}[G(\mathfrak{M})]$-module of finite dimension over $\mathbf{C}$?

1.2 We shall explain the analogy of the Riemann's problem in the ring of Witt vectors. Let p be a prime number and Δ be field of characteristic p. Let $\bar{\Delta}'$ be the separable algebraic closure of Δ and $G(\Delta)$ be the Galois group of $\bar{\Delta}'/\Delta$, where $G(\Delta)$ is considered as a discrete group. We mean by a Witt vector with coefficients in $\bar{\Delta}'$ an infinite ordered set $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ of elements $\alpha_l (l=0,1,2,\ldots)$ in $\bar{\Delta}'$. Putting $\mathbf{0}=(0,0,0,\ldots)$, $\mathbf{1}=(1,0,0,\ldots)$, $\mathbf{p}=(0,1,0,\ldots)$, $\mathbf{p}^n=(\overbrace{0,\ldots}^{n},0,1,0,\ldots)$, we write $\sum_{l=0}^{\infty}\alpha_l^{p^{-l}}\mathbf{p}^l$ instead of $(\alpha_0, \alpha_1, \alpha_2, \ldots)$. E. Witt introduced the sum, the difference and the product of two Witt vectors by means of systems of infinite polynomials with coefficients in the prime field $GF(p)$

$$\{\Phi_{+,l}(x_0,\ldots,x_{l-1};y_0,\ldots,y_{l-1})\},\quad \{\Phi_{-l}(x_0,\ldots,x_{l-1};y_0,\ldots,y_{l-1})\},$$

$\{\Phi_{0,l}(x_0,\ldots,x_{l-1},y_0,\ldots,y_{l-1})\}$ as follows (2):

$$\left(\sum_{l=0}^{\infty}\alpha_l^{p^{-l}}\mathbf{p}^l\right)+\left(\sum_{l=0}^{\infty}\beta_l^{p^{-l}}\mathbf{p}^l\right)=\sum_{l=0}^{\infty}\gamma_{+,l}^{p^{-l}}\mathbf{p}^l,$$

$$\left(\sum_{l=0}^{\infty}\alpha_l^{p^{-l}}\mathbf{p}^l\right)-\left(\sum_{l=0}^{\infty}\beta_l^{p^{-l}}\mathbf{p}^l\right)=\sum_{1=0}^{\infty}\gamma_{-,l}^{p^{-l}}\mathbf{p}^l,$$

$$\left(\sum_{l=0}^{\infty}\alpha_l^{p^{-l}}\mathbf{p}^l\right)\cdot\left(\sum_{l=0}^{\infty}\beta_l^{p^{-l}}\mathbf{p}^l\right)=\sum_{l=0}^{\infty}\gamma_{\cdot,l}^{p^{-l}}\mathbf{p}^l,$$

(1) $$\gamma_{+,l}=\alpha_l+\beta_l+\Phi_{+,l}(\alpha_0,\ldots,\alpha_{l-1};\beta_0,\ldots,\beta_{l-1}),$$

(2) $$\gamma_{-,l}=\alpha_l-\beta_l+\Phi_{-,l}(\alpha_0,\ldots,\alpha_{l-1};\beta_0,\ldots,\beta_{l-1}),$$

(3) $$\gamma_{\cdot,l}=\alpha_l^{p^l}\beta_l+\alpha_l\beta_0^{p^l}+\Phi_{\cdot,l}(\alpha_0,\ldots,\alpha_{l-1};\beta_0,\ldots,\beta_{l-1}).$$

(1) The Riemann's problem formulated in tne classical terminology can be seen [2], II_2, 365 p. p. 383-384.

(2) See [3].

By means of these operations all the Witt vectors with coefficients in $\bar{\Delta}'$ form a commutative integral domain $\mathbf{W}(\bar{\Delta}')$. We call $\mathbf{W}(\bar{\Delta}')$ the ring of Witt vectors with coefficients in $\bar{\Delta}'$. For any subring A in $\bar{\Delta}'$ the ring $\mathbf{W}(A)$ of Witt vectors with coefficients in A is naturally considered as a subring of $\mathbf{W}(\bar{\Delta}')$. Since the ring $\mathbf{Z}_p$ of p-adic integers can be regarded as the ring of Witt vectors with coefficients in the prime field $GF(p)$, $\mathbf{Z}_p$ is considered as a subring of $\mathbf{W}(\bar{\Delta}')$. We denote by $\mathbf{K}(\bar{\Delta}')$ (resp. $K(\Delta)$) the quotient field of $\mathbf{W}(\bar{\Delta}')$ (resp. $\mathbf{W}(\Delta)$). More generally for any subfield A in $\bar{\Delta}'$ we denote by $\mathbf{K}(A)$ the quotient field of the ring $\mathbf{W}(A)$ of Witt vectors with coefficients in A. The discrete group $G(\Delta)$ operates continuously on $\mathbf{W}(\bar{\Delta}')$ as follows:

$$\left(\sum_{l=0}^{\infty} \alpha_l^{p^{-l}} \mathbf{p}^l\right)^{\sigma} = \sum_{l=0}^{\infty} \left(\alpha_l^{p^{-l}}\right)^{\sigma} \mathbf{p}^l, \qquad (\sigma \in G(\Delta)). \tag{4}$$

Hence $\mathbf{W}(\bar{\Delta}')$ is regarded as a $\mathbf{Z}_p[G(\Delta)]$-module.

In these notations and terminologies the analogy of Riemann's problem is formulated as follows:

Does there exist a $\mathbf{Z}_p[G(\Delta)]$-submodule in $\mathbf{W}(\bar{\Delta}')$ which is isomorphic to a given $\mathbf{Z}_p[G(\Delta)]$-module of finite rank over Z_p?

In the present paper we shall solve this problem. Our main theorem is as follows:

MAIN THEOREM. If $\mathbf{K}(\Delta)$ is transcendental over $\mathbf{Q}_p$, there exists a $\mathbf{Z}_p[G(\Delta)]$-module in $\mathbf{W}(\bar{\Delta}')$ which is isomorphic to a given $\mathbf{Z}_p[G(\Delta)]$-module of finite rank over $\mathbf{Z}_p$.

§ 2. Proof of the main theorem.

2.1. We shall begin by the theorem on normal base [3]:

Let L/K be a finite separable normal extension of a field K. Then there exists a base (called normal base) of L/K which consists of all the conjugates of an element in L over K.

For a finite separable extension of L/K we denote by $G(L/K)$ the Galois group of L/K and by $Tr_{L/K}$ the trace map of L into K, i. e. $Tr_{L/K}(\alpha) = \sum_{\sigma \in G(L/K)} \alpha^{\sigma}, (\alpha \in L)$.

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(3) See some standard texthooks on algebra.

LEMMA 1. Let L/K be a finite separable normal extension. Then the trace map $Tr_{L/K}$ is surjective.

PROOF. Let $\{\omega^\sigma \mid \sigma \in G(L/K)\}$ be a normal base of L/K and a be any element in K. Then there exists a unique system $\{c_\sigma \mid \sigma \in G(L/K)\}$ of elements in K such that $a = \sum_\sigma c_\sigma \omega^\sigma$. Since $a^\tau = a$ for τ in $G(L/K)$, we have $c_\sigma = c_{\sigma\tau}$ for σ, τ in $G(L/K)$. This shows that $c_\sigma = c$ for every σ in $G(L/K)$ with an element c in K. Namely

$$a = \sum_\sigma (c\omega)^\sigma = Tr_{L/K}(c\omega).$$

LEMMA 2. Let L/K be a finite separable normal extension. Let L_1 and L_2 be normal subfields of L over K such that $L_1 \cap L_2 = K$. If elements α in L_1 and β in L_2 satisfy $Tr_{L_1/K}(\alpha) = Tr_{L_2/K}(\beta)$, then there exists an element γ in L such that $Tr_{L/L_1}(\gamma) = \alpha$ and $Tr_{L/L_2}(\gamma) = \beta$.

PROOF. In view of Lemma 1 it is enough to prove Lemma 2 for the case $L = L_1 L_2$. By the condition in Lemma 2 the Galois group $G(L_1 L_2/K)$ is the direct product $G(L_1/K) \times G(L_2/K)$. We choose normal basis $\{\omega^\sigma \mid \sigma \in G(L_1/K)$ and $\{\lambda^\tau \mid \tau \in G(L_2/K)\}$ of L_1/K and L_2/K, respectively. Then $\{\omega^\sigma \lambda^\tau \mid \sigma \in G(L_1/K), \tau \in G(L_2/K)\}$ form a normal base of $L_1 L_2/K$. Put $\alpha = \sum_\sigma a_\sigma \omega^\sigma$ and $\beta = \sum_\tau b_\tau \lambda^\tau$ with coefficients in K. Let us consider the following system of linear equations in $\{X_{\sigma,\tau}\}$:

$$Tr_{L_1L_2/L_1}(\sum_{\sigma,\tau} X_{\sigma,\tau}\,\omega^\sigma \lambda^\tau) = \sum_\sigma a_\sigma \omega^\sigma,$$

$$Tr_{L_1L_2/L_2}(\sum_{\sigma,\tau} X_{\sigma,\tau}\,\omega^\sigma \lambda^\tau) = \sum_\tau b_\tau \lambda^\tau,$$

where $G(L_1 L_2/K)$ operates on $\{X_{\sigma,\tau}\}$ trivially. This system is equivalent to

$$\sum_\tau X_{\sigma,\tau} = (Tr_{L_2/K}(\lambda))^{-1} v_\sigma\,,\ (\sigma \in G(L_1(K)),$$

$$\sum_\sigma X_{\sigma,\tau} = (Tr_{L_1/K}(\omega))^{-1} b_\tau\,,\ (\tau \in G(L_2/K)).$$

Since $Tr_{L_1/K}(\alpha) = Tr_{L_2/K}(\beta)$, we have

$$(\sum_\sigma a_\sigma)\, Tr_{L_1/K}(\omega) = (\sum_\tau b_\tau)\, Tr_{L_2/K}(\lambda)$$

and

$$Tr_{L_2/K}(\lambda)^{-1} a_\varepsilon - Tr_{L_1/K}(\omega)^{-1} (\sum_{\tau \neq \varepsilon} b_\tau) = (Tr_{L_1/K}(\omega))^{-1} b_\varepsilon - (Tr_{L_2/K}(\lambda))^{-1} \sum_{\sigma \neq \varepsilon} a_\sigma\,,$$

where ε is the unit element in $G(L_1 L_2/K)$.

Putting

$$c_{\sigma,\tau}=\begin{cases}(Tr_{L_2/K}(\lambda))^{-1}a_\sigma\ , & \text{for}\quad \sigma\neq\varepsilon \quad\text{and}\quad \tau=\varepsilon\\ (Tr_{L_1/K}(\omega))^{-1}b_\tau\ , & \text{for}\quad \tau\neq\varepsilon \quad\text{and}\quad \sigma=\varepsilon\\ \qquad 0 \qquad\quad , & \text{for}\quad \sigma\neq\varepsilon \quad\text{and}\quad \tau\neq\varepsilon\\ (Tr_{L_2/K}(\lambda))^{-1}a_\varepsilon-(Tr_{L_1/K}(\omega))^{-1}\sum\limits_{\varrho\neq\varepsilon}b_\varrho\ , & \text{for}\quad \sigma=\varepsilon \quad\text{and}\quad \tau=\varepsilon,\end{cases}$$

we have a solution $(c_{\sigma,\tau})$ of the above equations in K. Hence the element $\gamma=\sum\limits_{\sigma,\tau}c_{\sigma,\tau}\,\omega^\sigma\lambda^\tau$ satisfies the condition in Lemma 2.

We shall formulate Lemma 2 to the problem in the rings of Witt vectors:

LEMMA 3. Let Λ be a finite separable normal extension of Δ. Let Λ_1 and Λ_2 be normal subfields of Λ over Δ such that $\Lambda_1\cap\Lambda_2=\Delta$. If elements $\boldsymbol{\alpha}$ in $\mathbf{W}(\Lambda_1)$ and $\boldsymbol{\beta}$ in $\mathbf{W}(\Lambda_2)$ satisfy $Tr_{\mathsf{K}(\Lambda_1)/\mathsf{K}(\Delta)}(\boldsymbol{\alpha})=Tr_{K(\Lambda_2)/K(\Delta)}(\boldsymbol{\beta})$, then there exists an element $\boldsymbol{\gamma}$ in $W(\Lambda)$ such that $Tr_{\mathsf{K}(\Lambda)/\mathsf{K}(\Lambda_1)}(\boldsymbol{\gamma})=\boldsymbol{\alpha}$ and $Tr_{\mathsf{K}(\Lambda)/\mathsf{K}(\Lambda_2)}(\boldsymbol{\gamma})=\boldsymbol{\beta}$.

PROOF. It is sufficient to show that the coefficients $\gamma_0,\gamma_1,\dots$ in the expansion $\sum\limits_{l=0}^{\infty}\gamma_l^{p^{-l}}\mathbf{p}^l$ of $\boldsymbol{\gamma}$ in Lemma 3 are successively constructed. Put $\boldsymbol{\alpha}=\sum\limits_{l=0}^{\infty}\alpha_l^{p^{-l}}\mathbf{p}^l$ and $\boldsymbol{\beta}=\sum\limits_{l=0}^{\infty}\beta_l^{p^{-l}}\mathbf{p}^l$. Then, since $Tr_{\mathsf{K}(\Lambda_1)/\mathsf{K}(\Delta)}(\boldsymbol{\alpha})=Tr_{\mathsf{K}(\Lambda_2)/\mathsf{K}(\Delta)}(\boldsymbol{\beta})$, we have $Tr_{\mathsf{K}(\Lambda_1)/\mathsf{K}(\Delta)}(\alpha_0\,1)\equiv Tr_{\mathsf{K}(\Lambda_2)/\mathsf{K}(\Delta)}(\beta_0\,1)$ mod $\mathbf{p}$ and $Tr_{\Lambda_1/\Delta}(\alpha_0)=Tr_{\Lambda_2/\Delta}(\beta_0)$. Hence, by virtue of Lemma 2, we have γ_0 in Λ such that $Tr_{\Lambda/\Lambda_1}(\gamma_0)=\alpha_0$ and $Tr_{\Lambda/\Lambda_2}(\gamma_0)=\beta_0$, namely

$$\left.\begin{aligned}Tr_{\mathsf{K}(\Lambda)/\mathsf{K}(\Lambda_1)}(\gamma_0\,\mathbf{1})&\equiv\alpha_0\,\mathbf{1}\equiv\boldsymbol{\alpha}\\ Tr_{K(\Lambda)/K(\Lambda_2)}(\gamma_0\,\mathbf{1})&\equiv\beta_0\,\mathbf{1}\equiv\boldsymbol{\beta}\end{aligned}\right\}\ \text{mod } \mathbf{p}.$$

Assume we have already $\gamma_0,\dots,\gamma_{n-1}$ in Λ such that

$$\left.\begin{aligned}Tr_{\mathsf{K}(\Lambda)/\mathsf{K}(\Lambda_1)}\left(\sum_{l=0}^{n-1}\gamma_l^{p^{-l}}\mathbf{p}^l\right)&\equiv\boldsymbol{\alpha}\\ Tr_{\mathsf{K}(\Lambda)/\mathsf{K}(\Lambda_2)}\left(\sum_{l=0}^{n-1}\gamma_l^{p^{-l}}\mathbf{p}^l\right)&\equiv\boldsymbol{\beta}\end{aligned}\right\}\ \text{mod } \mathbf{p}^n.$$

Put

$$\alpha - Tr_{K(\Delta)/K(\Delta_1)}\left(\sum_{l=0}^{n-1} \gamma_l^{p^{-l}}\, \mathfrak{p}^l\right) = \sum_{l=0}^{\infty} \alpha_l'^{\,p^{-n-l}}\ \mathfrak{p}^{n+l}$$

and

$$\beta - Tr_{K(\Delta)/K(\Delta_2)}\left(\sum_{l=0}^{n-1} \gamma_l^{p^{-l}}\, \mathfrak{p}^l\right) = \sum_{l=0}^{\infty} \beta_l'^{\,p^{-n-l}}\ \mathfrak{p}^{n+l}.$$

Then, since $Tr_{K(\Delta_1)/K(\Delta)}(\alpha) = Tr_{K(\Delta_2)/K(\Delta)}(\beta)$, we have

$$Tr_{K(\Delta_1)/K(\Delta)}\left(\sum_{l=0}^{\infty} \alpha_l'^{\,p^{-n-l}}\ \mathfrak{p}^{n+l}\right) = Tr_{K(\Delta_2)/K(\Delta)}\left(\sum_{l=0}^{\infty} \beta_l'^{\,p^{-n-l}}\ \mathfrak{p}^{n+l}\right),$$

and thus $Tr_{\Delta_1/\Delta}(\alpha_0') = Tr_{\Delta_2/\Delta}(\beta_0')$. Therefore by virtue of Lemma 2 we have γ_n in Δ such that

$$\left.\begin{aligned} \alpha - Tr_{K(\Delta)/K(\Delta_1)}\left(\sum_{l=0}^{\infty} \gamma_l^{p^{-l}}\, \mathfrak{p}^l\right) &\equiv Tr_{K(\Delta)/K(\Delta_1)}(\gamma_n^{p^{-n}}\, \mathfrak{p}^n) \\ \beta - Tr_{K(\Delta)/K(\Delta_1)}\left(\sum_{l=0}^{n-1} \gamma_l^{p^{-l}}\, \mathfrak{p}^l\right) &\equiv Tr_{K(\Delta)/K(\Delta_1)}(\gamma_n^{p^{-n}}\, \mathfrak{p}^n) \end{aligned}\right\} \mod \mathfrak{p}^{n+1},$$

namely

$$\left.\begin{aligned} \alpha &\equiv Tr_{K(\Delta)/K(\Delta_1)}\left(\sum_{l=0}^{n} \gamma_l^{p^{-l}}\, \mathfrak{p}^l\right) \\ \beta &\equiv Tr_{K(\Delta)/K(\Delta_2)}\left(\sum_{l=0}^{n} \gamma_l^{p^{-l}}\, \mathfrak{p}^l\right) \end{aligned}\right\} \mod \mathfrak{p}^{n+1}.$$

2.2. We shall first prove the following Lemma and apply it together with Lemma 2 to construct of the matric solution of $\mathbf{A}^{\sigma} = \mathbf{AM}(\sigma)$, $(\sigma \in G(\Delta))$ in $\mathbf{W}(\bar{\Delta}')$.

LEMMA 4. Let L/K be a finite separable normal extension and $\{N(\sigma) \mid \sigma \in G(L/K)\}$ be a representation of the Galois group by non-singular matrices with coefficients in K. Let ω be an element in L such that all the conjugates of ω over K form a normal base. Then the matrix $\sum_{\sigma \in G(L/K)} N(\sigma^{-1})\,\omega^{\sigma}$ is non-singular.

PROOF. It is sufficient to prove Lemma 4 for every irreducible representation in K. Since every irreducible representation appears in the regular representation $\{R(\sigma) \mid \sigma \in G(L/K)\}$ as an irreducible component, it is sufficient to prove $\det(\sum_{\sigma} R(\sigma^{-1})\,\omega^{\sigma}) \neq 0$. Giving an order $\sigma_1 > \dots > \sigma_n$ in

$G(L/K)$, we shall calculate $(i \times j)$-element of $\sum_\sigma R(\sigma^{-1})\omega^\sigma$:

$$\sum_\sigma \delta(\sigma_i \sigma \sigma_j^{-1})\omega^\sigma = \omega^{\sigma_i^{-1}\sigma_j},$$

where $\delta(\varepsilon) = 1$ for the unit element ε and $\delta(\tau) = 0$ for $\tau \neq \varepsilon$. Since $\{\omega^\sigma \mid \sigma \in G(L/K)\}$ form a normal base of L/K, the matrix of which $(i \times j)$ element is $\omega^{\sigma_i^{-1}\sigma_j}$ is not singular. This proves Lemma 4.

Let $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ be a representation of $G(\Delta)$ by non singular $r \times r$-matrices with coefficients in $\mathbf{Z}_p$. We denote by $\Gamma(m)$ the subgroup $\{\sigma \in G(\Delta) \mid \mathbf{M}(\sigma) \equiv \text{identity mod } \mathfrak{p}^m\}$ in $G(\Delta)$ and $\Delta(m)$ the subfield of $\overline{\Delta}'$ consisting of all the elements fixed by every element in $\Gamma(m)$, $(m = 1, 2, \ldots)$. Then $\Gamma(m)$ are normal subgroups of finite indecis and $\Delta(m)/\Delta$ are finite separable normal extensions of Δ.

For each m we choose a system of representatives of $G(\Delta)/\Gamma(m)$ in $G(\Delta)$ and we understand by $\sum_{\sigma \bmod \Gamma(m)}$ that the sum is taken over all σ running through the representatives of $G(\Delta)/\Gamma(m)$.

THEOREM 1. Let $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ be a representation of $G(\Delta)$ by non-singular $r \times r$-matrices with coefficients in $\mathbf{Z}_p$. Then there exists a non-singular matrix $\mathbf{A}$ with coefficients in $\mathbf{W}(\overline{\Delta}')$ such that $\mathbf{A}^\sigma = \mathbf{M}(\sigma)\mathbf{A}$, $(\sigma \in G(\Delta))$.

PROOF. We use the notations $\Gamma(m)$, $\Delta(m)$, $\sum_{\sigma \bmod \Gamma(m)}$ in the above. In view of Lemma 1 and the theorem of normal base, there exists a system $(\omega_1, \omega_2, \ldots)$ of elements in $\mathbf{W}(\overline{\Delta})'$ such that

1) $\omega_m \in \mathbf{W}(\Delta(m))$, $(m = 1, 2, \ldots)$, $\omega_1 = \omega, \mathbf{1}$.
2) all the conjugates of ω_1 over Δ form a normal base of $\Delta(1)/\Delta$,
3) $Tr_{\mathrm{K}(\Delta(m+1))/\mathrm{K}(\Delta(m))}(\omega_{m+1}) = \omega_m$, $(m = 1, 2, \ldots)$.

Put

$$\mathbf{A}_m = \sum_{\sigma \bmod \Gamma(m)} \mathbf{M}(\sigma^{-1})\omega_m^\sigma, \qquad (m = 1, 2, \ldots),$$

where $\mathbf{1}$ is the identity in $\mathbf{W}(\overline{\Delta}')$. Since $\omega_m \in \mathbf{K}(\Delta(m))$ and $\mathbf{M}(\varrho) \equiv$ identity mod $\mathfrak{p}^m$ for ϱ in $\Gamma(m)$, the class of $\mathbf{A}_m$ mod $\mathfrak{p}^m$ is independent of the choice of the representatives of $G(\Delta)/\Gamma(m)$. Moreover we have the following set important relations:

$$\mathbf{A}_{m+1} \equiv \mathbf{A}_m \bmod \mathfrak{p}^m, \qquad (m = 1, 2, \ldots),$$

because

$$\mathbf{A}_{m+1} \equiv \sum_{\sigma \bmod \Gamma(m+1)} \mathbf{M}(\sigma^{-1})\, \omega_m^{\sigma}$$

$$\equiv \sum_{\tau \bmod \Gamma(m)} \mathbf{M}(\tau^{-1}) \sum_{\substack{\varrho \bmod \Gamma(m+1) \\ \varrho \in \Gamma(m)}} \mathbf{M}(\varrho^{-1})\, \omega_{m+1}^{\varrho\tau}$$

$$\equiv \sum_{\tau \bmod \Gamma(m)} \mathbf{M}(\tau^{-1}) \sum_{\substack{\varrho \bmod \Gamma(m+1) \\ \varrho \in \Gamma(m)}} \omega_{m+1}^{\varrho\tau} \bmod \mathfrak{p}^m$$

$$\equiv \sum_{\tau \bmod \Gamma(m)} \mathbf{M}(\tau^{-1})\, Tr_{K(\Delta(m+1))/K(\Delta(m))}(\omega_{m+1})^{\tau}$$

$$\equiv \sum_{\tau \bmod \Gamma(m)} \mathbf{M}(\tau^{-1})\, \omega_m \equiv \mathbf{A}_m \bmod \mathfrak{p}^m.$$

Therefore there exists the limit $\mathbf{A} = \lim_{m\to\infty} \mathbf{A}_m$ such that $\mathbf{A} \equiv \mathbf{A}_m \bmod \mathfrak{p}^m$. $(m = 1, 2, \ldots)$. On the other hand $\mathbf{A}_m^{\sigma} = \sum_{\tau \bmod \Gamma(m)} \mathbf{M}(\tau^{-1})\, \omega_m^{\tau\sigma} \equiv$ $\equiv \mathbf{M}(\sigma) \sum_{\tau \bmod \Gamma(m)} \mathbf{M}(\tau^{-1})\, \omega_m^{\tau} \equiv \mathbf{M}(\sigma)\, \mathbf{A}_m \bmod \mathfrak{p}^m$, $(\sigma \in G(\Delta)\,;\ m = 1, 2, \ldots)$, hence we have $\mathbf{A}^{\sigma} = \mathbf{M}(\sigma)\, \mathbf{A}$, $(\sigma \in G(\Delta))$. By virtue of Lemma 4 $\mathbf{A}_1$ is non-singular, so $\mathbf{A}$ is also non-singular. This completes the proof of Theorem 1.

By an argument based on the same ideas as in the proof of Theorem 1 we have the following theorem :

THEOREM 2. Let $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ be a representation of $G(\Delta)$ by non-singular $r \times r$-matrices with coefficients in $\mathbf{Z}_p$ and B_1 be a $r \times r$-matrix with coefficients in $\Delta(1)$ such that $B_1^{\sigma} = \overline{\mathbf{M}}(\sigma)\, B_1$, $(\sigma \in G(\Delta))$, where $\overline{\mathbf{M}}(\sigma)$ is the reduction of $\mathbf{M}(\sigma)$ modulo $\mathfrak{p}$ and $\Delta(1)$ the subfield of $\overline{\Delta}'$ consisting of all the elements in $\overline{\Delta}'$ fixed by the element σ such that $\mathbf{M}(\sigma) \equiv$ identity **mod** $\mathfrak{p}$. Then there exists a matrix $\mathbf{B}$ with coefficients in $\mathbf{W}(\overline{\Delta}')$ such that

1) B_1 is the reduction of $\mathbf{B}$ modulo $\mathfrak{p}$,
2) $\mathbf{B}^{\sigma} = \mathbf{M}(\sigma)\, \mathbf{B}$, $(\sigma \in G(\Delta))$.

PROOF. On this proof $\omega_1 = \omega_1 \mathbf{1}$, $\omega_2 \ldots$ denote the same elements of $(\overline{\Delta}')$ as in the proof of Theorem 1. Since $\sum_{\tau \bmod \Gamma(1)} \overline{M(\tau^{-1})}\, \omega_1^{\tau}$ is non-singular (Lemma 4), we can put

$$C = \Big(\sum_{\tau \bmod \Gamma(1)} \overline{M}(\tau^{-1})\, \omega_1^{\tau}\Big)^{-1} B_1\,,$$

$$B_m = \Big(\sum_{\sigma \bmod \Gamma(m)} \mathbf{M}(\sigma^{-1})\, \omega_m^{\sigma}\Big)\,(C\,\mathbf{1}), \qquad (m = 1, 2, \ldots).$$

Then C is a matrix with coefficiente in Δ and $C\cdot\mathbf{1}$ is a matrix with coefficients in $\mathbf{W}(\Delta)$. Moreover $\mathbf{B}_m \equiv \mathbf{A}_m(C\cdot\mathbf{1}) \bmod \mathbf{p}^m$ and B_1 is the reduction of $\mathbf{B}_m$ modulo $\mathbf{p}$, where A_m is the same as in the proof of Theorem 1. Hence putting $\mathbf{B} = \lim_{m\to\infty} \mathbf{B}_m$, we have a $\mathbf{B}$ satisfying the conditions in Theorem 2.

2.3 In order to solve the problem affirmatively, we need to prove the existence of a vector ${}^t\mathfrak{a} = (\alpha_1, \dots, \alpha_r)$ with coefficients in $\mathbf{W}(\bar{\Delta}')$ such that 1) $\mathfrak{a}^\sigma = \mathbf{M}(\sigma)\,\mathfrak{a}$, $(\sigma \in G_\Delta)$ and 2) $\alpha_1, \dots, \alpha_r$ are linearly independent over $\mathbf{Q}_p$. The existence of non-zero vector satisfying 1) is guaranteed by Theorem 1. We shall first notice the existence of vectors satisfying 1) and 2) for the irreducible representations $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ of $G(\Delta)$ and, then under the assumption that $\mathbf{K}(\Delta)/\mathbf{Q}_p$ is transcendental, we shall prove the existence of vectors satisfying 1) and 2) by the induction on the number of irreducible components in the representations.

THEOREM 3. Let $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ be an irreducible representation of $G(\Delta)$ by $r \times r$-matrices with coefficients in $\mathbf{Z}_p$. Then there exists a system of elements $\xi_1, \dots, \xi_r$ in $\mathbf{W}(\bar{\Delta}')$ such that 1)

$$\begin{pmatrix}\xi_1\\ \vdots\\ \xi_r\end{pmatrix}^\sigma = \mathbf{M}(\sigma)\begin{pmatrix}\xi_1\\ \vdots\\ \xi_r\end{pmatrix}, \quad (\sigma \in G(\Delta)),$$

2) $\xi_1, \dots, \xi_r$ are linearly independent over $\mathbf{Q}_p$.

PROOF. In view of Theorem 1 there exists a non-zero vector ${}^t(\alpha_1, \dots, \alpha_r)$ with coefficients in $\mathbf{W}(\bar{\Delta}')$ such that

$$\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_r\end{pmatrix}^\sigma = \mathbf{M}(\sigma)\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_r\end{pmatrix}, \quad (\sigma \in G(\Delta)).$$

Let $\mathbf{V}$ be $\mathbf{Z}_p[G(\Delta)]$-module spanned by $\alpha_1, \dots, \alpha_r$ over $\mathbf{Z}_p$ in $\mathbf{W}(\bar{\Delta}')$. If the rank of $\mathbf{V}$ is less than r, the representation $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ is not irreducible. This shows the linearly independentness of $\alpha_1, \dots, \alpha_r$ over $\mathbf{Q}_p$.

In the induction process we shall need the following lemma:

LEMMA 5. Let ξ be a transcendental element in a field L over a subfield K and $\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_{r-s}, \gamma_1, \dots, \gamma_s$ be elements in L such that $\{\alpha_1, \dots, \alpha_s\}$ and $\{\beta_1, \dots, \beta_{r-s}\}$ are sets of linearly independent elements over

K. Then there exists a positive integer n_0 such that for every $n \geq n_0$ the elements $\alpha_1 \xi^n + \gamma_1, \dots, \alpha_s \xi^n + \gamma_{s'}$ $\beta_1, \dots, \beta_{r-s}$ are linearly independent over K.

PROOF. We shall give a proof from algebraic geometry. Put $M = \bar{K}(\xi, \alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_{r-s}, \gamma_1, \dots, \gamma_s)$, where $\bar{K}$ denotes the algebraic closure of K. We choose a normal projective variety V defined over $\bar{K}$ as a model of the function field $M/\bar{K}$. We denote by Σ the $\bar{K}$-module spanned by $\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_{r-s}, \gamma_1, \dots, \gamma_s$ in M over $\bar{K}$ and denote by d the maximal of the degrees of the polar divisors of elements in Σ. For a prime divisor Y on V we mean by v_y the valuation defined as follows: If the multiplicity of Y in the principal divisor (f) is n_y, the value $v_y(f)$ is given by $n_y \deg Y$. We choose a prime divisor X on V such that $v_x(\xi) \geq 1$. Since the degree of polar divisors of elements in Σ is at most d, we have $-d \leq v_x(g) \leq d$ $(g \neq 0$ in $\Sigma)$. We put $n_0 = 2d+1$. Let n be a positive integer not less than n_0 and assume that there exist $a_1, \dots, a_s, b_1, \dots, b_{r-s}$ in K such that $\sum_i a_i(\alpha_i \xi^n + \gamma_i) + \sum_i b_j \beta_j = 0$. Then, since $\sum_i a_i \gamma_i + \sum_i b_j \beta_j \in \Sigma$, we have the two cases:

1) $\sum_i a_i \gamma_i + \sum_j b_j \beta = 0$, 2) $-d \leq v_x(\sum_i a_i \gamma_i + \sum b_j \beta_j) \leq d$. If $\sum_i a_i \gamma_i + \sum_j b_j \beta_j = 0$, it follows that $(\sum_i a_i \alpha_i)\xi^n = 0$ and $\sum_i a_i \alpha_i = 0$. Since $\alpha_1, \dots, \alpha_r$ are linearly independent over K, we have $a_1 = \dots = a_s = 0$. By the linear independence of $\beta_1, \dots, \beta_{r-s}$ over K we have $b_1 = \dots = b_{r-s} = 0$, because $\sum_j b_j \beta_j = 0$. Let us assume $-d \leq v_x(\sum_i a_i \gamma_i + \sum_j b_j \beta_j) \leq d$. Then $v_x((\sum_i a_i \alpha_i)\xi^n) = v_x(\sum_i a_i \gamma_i + \sum_j b_j \beta_j) \leq d$. On the other hand, since $\sum_i a_i \alpha_i \in \Sigma$ and $\sum_i a_i \alpha_i \neq 0$, we have $v_x((\sum_i a_i \alpha_i)\xi^n) = n v_x(\xi) + v_x(\sum_i a_i \alpha_i) \geq n - d \geq 2d + 1 - d = d+1$. This is a contradiction. Therefore $\alpha_1 \xi^n + \gamma_1, \dots, \alpha_r \xi^n + \gamma_s, \beta_1, \dots, \beta_{r-s}$ are linearly independent over K.

The next lemma is the reduction of the problem.

LEMMA 6. Let $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ be a representation of $G(\Delta)$ by non-singular $r \times r$-matrices with coefficients in $\mathbf{Z}_p$ such that

$$\mathbf{M}(\sigma) \begin{pmatrix} \overbrace{\mathbf{N}_1(\sigma)}^{s} & \overbrace{\mathbf{\Lambda}(\sigma)}^{r-s} \\ \mathbf{0} & \mathbf{N}_2(\sigma) \end{pmatrix} \begin{matrix} \}\, s \\ \}\, r-s \end{matrix}, \quad (\sigma \in G(\Delta)).$$

Let $\Delta(m)$ be the subfield of $\bar{\Delta}'$ consisting of all the elements fixed by the subgroup $\Gamma(m) = \{\sigma \in G(\Delta) \mid \mathbf{M}(\sigma) \equiv \text{identity mod } \mathbf{p}^m\}$ $(m = 1, 2, \dots)$. Assume two systems $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots)$ and $(\mathfrak{b}_1, \mathfrak{b}_2, \ldots)$ of vectors with coefficients in $\mathbf{W}(\overline{\Delta}')$ satisfy

1) the coefficients of $\mathfrak{a}_m$ and $\mathfrak{b}_m$ belong to $\mathbf{W}(\Delta(m))$,

2) $\mathrm{Tr}_{K(\Delta(m+1))/K(\Delta(m))}(\mathfrak{a}_{m+1}) = \mathfrak{a}_m$, $\mathrm{Tr}_{K(\Delta(m+1))/K(\Delta(m))}(\mathfrak{b}_{m+1}) = \mathfrak{b}_m$,

3) putting (4)

$$\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_s\end{pmatrix} = \lim_{m\to\infty} \sum_{\sigma \bmod \Gamma(m)} \mathbf{N}_1(\sigma^{-1})\, \mathfrak{a}_m^\sigma, \quad \begin{pmatrix}\beta_1\\ \vdots\\ \beta_{r-s}\end{pmatrix} = \lim_{m\to\infty} \sum_{\sigma \bmod \Gamma(m)} \mathbf{N}_2(\sigma^{-1})\, \mathfrak{b}_m^\sigma,$$

the sets $\{\alpha_1, \ldots, \alpha_s\}$ and $\{\beta_1, \ldots, \beta_{r-s}\}$ are sets of linearly independent elements over $\mathbf{Q}_p$. Then, if $\mathbf{K}(\Delta)$ is transcendental over $\mathbf{Q}_p$, there exists a system of vectors $(\mathfrak{x}_1, \mathfrak{x}_2, \ldots)$ such that 1) the coefficients of $\mathfrak{x}_m$ belong to $\mathbf{W}(\Delta(m))$, 2) $Tr_{K(\Delta(m+1))/K(\Delta(m))}(\mathfrak{x}_{m+1}) = \mathfrak{x}_m$, 3) putting

$$\begin{pmatrix}\xi_1\\ \vdots\\ \xi_r\end{pmatrix} = \lim_{m\to\infty} \sum_{\sigma \bmod \Gamma(m)} \mathbf{M}(\sigma^{-1})\, \mathfrak{x}_m^\sigma,$$

the elements $\xi_1, \ldots, \xi_r$ are linearly independent over $\mathbf{Q}_p$.

Proof. Put

$$\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_s\end{pmatrix} = \lim_{m\to\infty} \sum_{\sigma \bmod \Gamma(m)} \mathbf{\Lambda}(\sigma^{-1})\, \mathfrak{b}_m^\sigma$$

and apply Lemma 5 to $\alpha_1, \ldots, \alpha_s$, $\beta_1, \ldots, \beta_{r-s}$, $\gamma_1, \ldots, \gamma_s$, $L = \mathbf{K}(\overline{\Delta}')$ and $K = \mathbf{Q}_p$. Then there exists a non-zero element η in $\mathbf{W}(\Delta)$ such that $\alpha_1\eta + \gamma_1, \ldots, \alpha_s\eta + \gamma_s$, $\beta_1, \ldots, \beta_{r-s}$ are linearly independent over $\mathbf{Q}_p$. Therefore, putting

$$\mathfrak{x}_m = \begin{pmatrix}\mathfrak{a}_m \eta\\ \mathfrak{b}_m\end{pmatrix}, \qquad (m = 1, 2, \ldots),$$

we get a system of vectors $(\mathfrak{x}_1, \mathfrak{x}_2, \ldots)$ satisfying the conditions in Lemma 6.

Applying Lemma 6 successively we have our main theorem:

Theorem 4 (the main theorem). If $\mathbf{K}(\Delta)$ is transcendental over $\mathbf{Q}_p$, there exists a $\mathbf{Z}_p[G(\Delta)]$-submodule in $\mathbf{W}(\Delta')$ which is isomorphic to a given $\mathbf{Z}_p[G(\Delta)]$-submodule of finite rank over $\mathbf{Z}_p$.

Proof. We shall prove the next assertion:

(*A*) Let $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ be any representation of $G(\Delta)$ by non-singular $r \times r$-matrices with coefficients in $\mathbf{Z}_p$. Let $\Gamma(m)$ be the subgroup

(4) The conditions 1) and 2) imply the existence of limits in 3).

$\{\sigma \in G(\Delta) \mid \mathbf{M}(\sigma) \equiv \text{identity} \bmod \mathfrak{p}^m\}$ and $\Delta(m)$ the subfield in Δ' consisting of all the elements fixed by every element in $\Gamma(m)$. Then, if $\mathbf{K}(\Delta)/\mathbf{Q}_p$ is transcendental, there exists a system of vectors $(\mathfrak{x}_1, \mathfrak{x}_2, \dots)$ such that

1) the coefficients of $\mathfrak{x}_m$ belong to $\mathbf{W}(\Delta(m))$,

2) $Tr_{\mathbf{K}(\Delta(m+1))/\mathbf{K}(\Delta(m))}(\mathfrak{x}_{m+1}) = \mathfrak{x}_m$,

3) putting (5)

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \end{pmatrix} = \lim_{m \to \infty} \sum_{\sigma \bmod \Gamma(m)} \mathbf{M}(\sigma^{-1}) \mathfrak{x}_m^\sigma,$$

the elements $\xi_1, \dots, \xi_r$ are linearly independent over $\mathbf{Q}_p$.

The vector ${}^t(\xi_1, \dots, \xi_r)$ in (A) satisfies the condition in Theorem 4, hence it is enough to prove (A) by the induction on the number of irreducible components. First we assume $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ is irreducible. Then, if we denote by $\mathbf{A}_1, \mathbf{A}_2, \dots$ the same matrices as in the proof of Theorem 1 and denote by $\mathfrak{a}_1, \mathfrak{a}_2, \dots$ the first column vectors of $\mathbf{A}_1, \mathbf{A}_2, \dots$, respectively, then by the argument in the proof of Theorem 3 the system $(\mathfrak{a}_1, \mathfrak{a}_2, \dots)$ satisfies the condition in (A). We assume (A) for the case in which the number of irreducible components is less than n.

Let $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ be a representation of $G(\Delta)$ by non-singular $r \times r$-matrices with coefficients in $\mathbf{Z}_p$ such that $i)$

$$\mathbf{M}(\sigma) = \begin{pmatrix} \overbrace{\mathbf{N}_1(\sigma)}^{s} & \overbrace{\mathbf{\Lambda}(\sigma)}^{r-s} \\ 0 & \mathbf{N}_2(\sigma) \end{pmatrix} \begin{matrix} \}\, s \\ \\ \}\, r-s \end{matrix}, \quad (\sigma \in G(\Delta))$$

ii) $\{\mathbf{N}_1(\sigma) \mid \sigma \in G(\Delta)\}$ is irreducible, iii) the number of irreducible components in $\{\mathbf{N}_2(\sigma) \mid \sigma \in G(\Delta)\}$ is $n-1$. We denote by $\Delta(i, m)$ the subfield of $\bar{\Delta}'$ consisting of all the elements fixed by every element in $\Gamma(i, m) = \{\sigma \in G(\Delta) \mid \mathbf{N}_i(\sigma) \equiv \text{identity} \bmod \mathfrak{p}^m\}$, $(i = 1, 2; m = 1, 2, \dots)$. By the induction assumption there exist systems of vectors $(\mathfrak{a}_1, \mathfrak{a}_2, \dots)$ and $(\mathfrak{b}_1, \mathfrak{b}_2, \dots)$ such that

1) the coefficients of $\mathfrak{a}_m$ (resp. $\mathfrak{b}_m$) belong to $\mathbf{W}(\Delta(1, m))$. (resp. $\mathbf{W}(\Delta(2, m))$),

2) $Tr_{\mathbf{K}(\Delta(1, m+1))/\mathbf{K}(\Delta(1, m))}(\mathfrak{a}_{m+1}) = \mathfrak{a}_m$,
$Tr_{\mathbf{K}(\Delta(2, m+1))/\mathbf{K}(\Delta(2, m)}(\mathfrak{b}_{m+1}) = \mathfrak{b}_m$,

3) putting (6)

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_s \end{pmatrix} = \lim_{m \to \infty} \sum_{\sigma \bmod \Gamma(1, m)} \mathbf{N}_1(\sigma^{-1}) \mathfrak{a}_m^\sigma$$

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(5), (6) see (4).

and

$$\begin{pmatrix}\beta_1\\ \vdots \\ \beta_{r-s}\end{pmatrix} = \lim_{m\to\infty} \sum_{\sigma \bmod \Gamma(2,m)} N_2(\sigma^{-1})\, \mathfrak{b}_m^{\sigma},$$

the sets $\{\alpha_1, \dots, \alpha_s\}$ and $\{\beta_1, \dots, \beta_{r-s}\}$ are sets of linearly independent elements over Q_p.

Put $\Delta^{(1)} = \bigcup_m \Delta(1, m)$ and $\Delta^{(2)} = \bigcup_m \Delta(2, m)$. Then there exists a non-decreasing arithmetic function Φ such that $\Delta(1, \Phi(m)) \supset \Delta^{(1)} \cap \Delta(m)$ and $\Delta(2, \Phi(m)) \supset \Delta^{(2)} \cap \Delta(m)$, $(m = 1, 2, \dots)$, because $\Delta(m)\,(m = 1, 2, \dots)$ are finite extensions over Δ. We shall inductively show that we can choose vectors $\mathfrak{c}_m$ and $\mathfrak{d}_m$ with coefficients in $W(\Delta(m))$ such that

$$(B_m) \quad \begin{cases} Tr_{K(\Delta(1,\Phi(m))/K(\Delta^{(1)}\cap\Delta(m))}(\mathfrak{a}_{\Phi(m)}) = Tr_{K(\Delta(m))/K(\Delta^{(1)}\cap\Delta(m))}(\mathfrak{c}_m), \\ Tr_{K(\Delta(2,\Phi(m)))/K(\Delta^{(2)}\cap\Delta(m))}(\mathfrak{b}_{\Phi(m)}) = Tr_{K(\Delta(m))/K(\Delta^{(1)}\cap\Delta(m))}(\mathfrak{d}_m), \\ Tr_{K(\Delta(m+1))/K(\Delta(m))}(\mathfrak{c}_{m+1}) = \mathfrak{c}_m, \quad Tr_{K(\Delta(m+1))/K(\Delta))}(\mathfrak{d}_{m+1}) = \mathfrak{d}_m. \end{cases}$$

Since $Tr_{K(\Delta(1,\Phi(1)))/K(\Delta^{(1)}\cap\Delta(1))}(\mathfrak{a}_{\Phi(1)})$ is a known element in $K(\Delta^{(1)} \cap \Delta(1))$ applying Lemma 3 to $\Delta_1 = \Delta_2 = \Delta^{(1)} \cap \Delta(1)$, $\Delta = \Delta_{(1)}$ and $\alpha = \beta = Tr_{K(\Delta(1,\Phi(1)))/K(\Delta^{(1)}\cap\Delta(1))}(\mathfrak{a}_{\Phi(1)})$, we have an element $\mathfrak{c}_1$ with coefficients in $W(\Delta(1))$ satisfying (B_1). Assume we have already $\mathfrak{c}_1 \mathfrak{c}_2, \dots, \mathfrak{c}_{m-1}$ such that the coefficients of $\mathfrak{c}_l$ belong to $W(\Delta(l))$ and $\mathfrak{c}_l$ satisfy (B_l), $(l = 1, 2, \dots, m-1)$. Then, since $(\Delta^{(1)} \cap \Delta(m)) \cap \Delta(m-1) = \Delta^{(1)} \cap \Delta(m-1)$ and

$$Tr_{K(\Delta^{(1)}\cap\Delta(m))/K(\Delta^{(1)}\cap\Delta(m-1))}(Tr_{K(\Delta(1,\Phi(m)))/K(\Delta^{(1)}\cap\Delta(m))}(\mathfrak{a}_{\Phi(m)}))$$

$$= Tr_{K(\Delta(m-1))/K(\Delta^{(1)}\cap\Delta(m+1))}(\mathfrak{c}_{m-1}),$$

we can use Lemma 3 and get a vector $\mathfrak{c}_m$ with coefficients in $W(\Delta(m))$ such that

$$Tr_{K(\Delta(m))/K(\Delta^{(1)}\cap\Delta(m))}(\mathfrak{c}_m) = Tr_{K(\Delta(1,\Phi(m)))/K(\Delta^{(1)}\cap\Delta(m))}(\mathfrak{a}_{\Phi(m)})$$

and

$$Tr_{K(\Delta(m+1))/K(\Delta(m))}(\mathfrak{c}_{m+1}) = \mathfrak{c}_m.$$

By the same method we have $\mathfrak{d}_1, \mathfrak{d}_2, \dots$ satisfying the conditions. By virtue of the last condition in (B_m) the limits

$$\lim_{m\to\infty} \sum_{\sigma \bmod \Gamma(m)} N_1(\sigma^{-1})\, \mathfrak{c}_m^{\sigma} \text{ and } \lim_{m\to\infty} \sum_{\sigma \bmod \Gamma(m)} N_2(\sigma^{-1})\, \mathfrak{d}_m^{\sigma}$$

exist and by the first two conditions in (B_m) we have

$$\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_s\end{pmatrix}=\lim_{m\to\infty}\sum_{\sigma \bmod \Gamma(m)} \mathrm{N}_1(\sigma^{-1})\,\mathfrak{c}_m^{\sigma}, \qquad \begin{pmatrix}\beta_1\\ \vdots\\ \beta_{r-s}\end{pmatrix}=\lim_{m\to\infty}\sum_{\sigma \bmod \Gamma(m)} \mathrm{N}_2(\sigma^{-1})\,\mathfrak{d}_m^{\sigma}.$$

Therefore, applying Lemma 6 to $(\mathfrak{c}_1, \mathfrak{c}_2, \ldots)$ and $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots)$, we have a system of vectors $(\mathfrak{x}_1, \mathfrak{x}_2, \ldots)$ satisfying the conditions in (A). This completes the proof of the main theorem.

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