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ON A DEFINITION OF ABELIAN VARIETY

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1. In the present note we shall give the following criterion of abelian variety which does not contain the associative law:

THEOREM. Let V be an irreducible projective variety defined over a field k. Let f be an everywhere defined regular map of $V \times V$ onto V, 1 be an k-rational point on V such that $f(a, 1) = f(1, a) = a$, $(a \in V)$, and g be an everywhere defined regular map of V into V such that $f(a, g(a)) = 1$, $(a \in V)$, where f and g are also defined over k. Then if for every a in V the regular map $T_a : x \longrightarrow f(x, a)$ of V into V is a biregular map of V onto V, it follows that V is an abelian variety with the law of composition $(f, 1, g)$.

This theorem suggests that it is seldom possible to give a nice law of composition on a given irreducible projective variety.

2. PROOF OF THEOREM. Let $(f, 1, g)$ be a law of a composition on an irreducible projective variety V satisfying the condition in Theorem and k be the field of definition of V and the law of composition $(f, 1, g)$. For the sake of simplicity we put $a \circ b = f(a, b)$ and $a_r^{-1} = g(a)$, $(a, b \in V)$. By virtue of the condition in Theorem there exists a map $T : a \longrightarrow T_a$ of V into the group G of automorphism of the variety V. Since $(T_b\, T_a^{-1})(a) = b$ for every a and b in V, the group G operates on V transitively. Hence V is a non-singular irreducible projective variety, and thus by virtue of Matsusaka's result (1) G contains the largest irreducible algebraic group G_0 defined over k. We denote by K and H the subgroups $\{\sigma \in G \mid \sigma(1) = 1\}$ and $\{\sigma \in G_0 \mid \sigma(1) = 1\}$, respectively. Put $\xi(\sigma) = \sigma(1)$ and $\eta(\sigma) = T^{-1}_{\sigma(1)}\, \sigma$, $(\sigma \in G_0)$. Then ξ and η are regular maps of G_0 into V and H, respectively, because G_0 is also transi-

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(1) See [1] p. 45-48.

tive on V and for a generic point σ in G_0 over k the unit element e in G_0 is the unique specialization of $\eta(\sigma)$ over the specialization $\sigma \longrightarrow e$. The maps ξ and η are also defined over k. Since $\sigma = T_{\xi(\sigma)}\eta(\sigma)$, $(\sigma \in G_0)$ and $T_a(1) = a$, the maps : $\sigma \longrightarrow \xi(\sigma) \times \eta(\sigma)$, $\sigma \longrightarrow T_{\xi(\sigma)} \times \eta(\sigma)$ are biregular maps of G_0 onto $V \times H$ and $T(V) \times H$, respectively, where $T(V)$ means the image of V in G_0 by T. By virtue of the structure theorem of algebraic group (2) there exists an irreducible linear group L in G_0 such that L is defined over k and the quotient $A = G_0/L$ is an abelian variety. We shall next prove that $T(V) \cap L$ is zero-dimensional. Assume for a moment that $T(V) \cap L$ contains an irreducible subvariety W of dimension at least one and w be a generic point of W over a common field k' of definition of G_0, $T(V)$, W and L. Then, since the linear group L is an affine variety and $T(V)$ is a complete variety, the variety W is not complete and there exists a specialization t of w over k' such that $t \in T(V)$ and $t \notin L$. This is a contradiction, because $t \in G_0$ and L is closed in G_0. This shows that $T(V)L = \{T_{c_1}, T_{c_2}, \ldots, T_{c_N}\}$ with points $c_1, c_2, \ldots, c_N$ in V. Let φ be the natural homomorphism of G_0 onto A and put $\lambda(a) = \varphi(T_a)$, $(a \in V)$, then λ is a regular map of V into A such that $\lambda(1) = 0$, where 0 is the origin of A. We denote by μ the regular map of $V \times V$ into A defined by $\mu(a \times b) = \lambda(a \circ b)$, $(a, b \in V)$, then by the property of a map of a product variety into an abelian variety (3) there exist two regular maps ϱ_1 and ϱ_2 of V into A such that $\varrho_1(1) = 0$ and $\mu(a \times b) = \varrho_1(a) + \varrho_2(b)$, $(a, b \in V)$. Since $\varrho_1(1) + \varrho_2(1) = \mu(1 \times 1) = \lambda(1 \circ 1) = \lambda(1) = 0$, $\varrho_2(1)$ is also the origin of A. We shall show $\varrho_1(a) = \varrho_2(a) = \lambda(a)$, $\lambda(a \circ b) = \lambda(a) + \lambda(b)$, $\lambda(a_r^{-1}) = -\lambda(a)$ as follows:

$$\varrho_1(a) = \varrho_1(a) + \varrho_2(1) = \mu(a \times 1) = \lambda(a \circ 1) = \lambda(a),\ \varrho_2(a) = \varrho_1(1) + \varrho_2(a)$$

$$= \mu(1 \times a) = \lambda(1 \circ a) = \lambda(a),\ \lambda(a \circ b) = \varrho_1(a) + \varrho_2(b) = \lambda(a) + \lambda(b),$$

$$\lambda(a) + \lambda(a_r^{-1}) = \varrho_1(a) + \varrho_2(a_r^{-1}) = \mu(a \times a_r^{-1}) = \lambda(a \circ a_r^{-1}) = \lambda(1) = 0.$$

This shows that $H \supseteq L$ implies Theorem. Next we shall show that λ is a finite regular map of V onto the image $\lambda(V)$ of V in A by λ. Since $\lambda(a) = \varphi(T_a)$, the relation $\lambda(a) = \lambda(b)$ implies $0 = \lambda(a) - \lambda(b) = \lambda(a) + \lambda(b_r^{-1}) = \lambda(a \circ b_r^{-1}) = \varphi(T_{a \circ b_r^{-1}})$ and $T_{a \circ b_r^{-1}} \in T(V) \cap L$. Hence $a \circ b_r^{-1} = c_i$ with a c_i in $\{c_1, \ldots, c_N\}$ and $a = T_{b_r^{-1}}^{-1} c_i$. Namely $\lambda(a) = \lambda(b)$ if and only if $a = T_{b_r^{-1}} c_i$ with a c_i in $\{c_1, \ldots, c_N\}$. This pro-

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(2) See [2] p. 425.

(3) See [3] II.

ves the finiteness of λ. Finaly we shall prove $H \supseteq L$. Assume for a moment $H \not\supseteq L$. Then, since L and H are irreducible, the image $\bar{L}$ of L in V by the natural map: $G_0 \longrightarrow V \times H \longrightarrow V$ is at least one-dimensional. Moreover, since λ is a finite regular map of V onto $\lambda(V)$, the image $\lambda(\bar{L})$ is also at least one dimensional. This is a contradiction, because the image of linear group in an abelian variety is always zero-dimensional. This completes the proof of Theorem.

3. COROLLARY. If a law of composition $(f, 1, g)$ on an irreducible projective variety V satisfies $f(f(b, a), g(a)) = b$ for a and b in V, then V is an abelian variety with the law of composition $(f, 1, g)$.

PROOF. Since $(T_{a_r^{-1}} T_a)(x) = (x \circ a) \circ a_r^{-1} = x, (a, x \in V)$, T_a is a biregular map of V onto V for every a in V. Hence by virtue of Theorem V is an abelian variety with the law of composition $(f, 1, g)$.

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