Annali della Scuola Normale Superiore di Pisa
Classe di Scienze

J. J. Kohn

Potential theory and several complex variables

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 17, nº 4 (1963), p. 373-386

<http://www.numdam.org/item?id=ASNSP_1963_3_17_4_373_0>

POTENTIAL THEORY AND SEVERAL COMPLEX VARIABLES (1)

by J. J. KOHN (2)

A. Introduction.

The purpose of this paper is to present an outline of the potential theoretic method on open complex manifolds and to describe some of its applications. A detailed account of these results will be found in [7]. The type of problem discussed here was first formulated by Garabedian and Spencer in [2]. Spencer and the author in [5] investigated the problem by means of singular integral equations. Morrey in [9] established the *a priori* estimate (I) of section *G* in a special case; which was the author's starting point to the solution of the problem (see [6], [7] and [8]). Ash in his thesis and in [1] developed a method for deriving (I) with respect to moving frames which enabled him to establish (I) in greater generality. The introduction of moving frames also simplifies the statement and derivation of (III). Estimates of the same type as (III) have been established by A. Andreotti and E. Vesentini.

We wish to point out that, with the exception of complex dimension one (see section I), the boundary value problems discussed here are not « coercive ». Thus the methods for solving the Dirichlet problem cannot be generalized, instead we have adapted the techniques developed in Hörmander's book [4].

Pervenuto alla Redazione il 28 Settembre 1963.

(1) This material was presented in a lecture at the Conference on Analytic functions held in Krakow, Poland from August 30 to September 4, 1962.

(2) This research has been partially supported by the National Science Foundation through contracts held at the Institute for Advanced Study, Princeton and at Brandeis University.

B. Notation.

Let M be a hermitian complex analytic manifold of complex dimension n. We denote by $\mathcal{A}$ the space of C^∞ complex-valued forms, then we have the following direct sum decomposition:

$$\mathcal{A} = \Sigma\, \mathcal{A}^{p,q},$$

if $\varphi \in \mathcal{A}^{p,q}$ we write, in terms of holomorphic local coordinates:

$$\varphi = \Sigma\, \varphi_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q}\, dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q} .$$

The complex gradient $\bar{\partial}: \mathcal{A} \to \mathcal{A}$ is defined by:

$$\bar{\partial}\varphi = \Sigma\, \frac{\partial}{\partial \bar{z}^k} (\varphi_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q})\, d\bar{z}^k \wedge dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q},$$

where

$$\frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + \sqrt{-1}\,\frac{\partial}{\partial y^k}\right),$$

$$x^k = Re\,(z^k) \text{ and } y^k = Im\,(z^k).$$

Observe that $\bar{\partial}^2 = 0$ and that if u is a function then it is holomorphic if and only if $\bar{\partial}u = 0$.

The hermitian metric on M induces an inner product on the forms at each point of M, thus if φ and ψ are in $\mathcal{A}$, $\langle \varphi, \psi \rangle$, their inner product, is a C^∞ function on M. If the integrals of $\langle \varphi, \varphi \rangle = |\varphi|^2$ and $\langle \psi, \psi \rangle = |\psi|^2$ over M are finite we define (φ, ψ) and $\|\varphi\|$ by:

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle\, dV \quad \text{and} \quad \|\varphi\|^2 = (\varphi, \varphi),$$

we denote by $\mathcal{L}$ the hilbert space which is obtained by completion under this inner product.

Now we define $\vartheta: \mathcal{A} \to \mathcal{A}$, the formal adjoint of $\bar{\partial}$, by requiring that $\vartheta\varphi$ be that form which satisfies the equation

$$(\vartheta\varphi, \psi) = (\varphi, \bar{\partial}\psi)$$

for all compactly supported forms ψ. Observe that $\vartheta\mathcal{A}^{p,q} \subset \mathcal{A}^{p,q-1}$ (thus $\vartheta\mathcal{A}^{p,0} = \{0\}$) and that $\vartheta^2 = 0$.

Finally we define $\square : \mathcal{A} \to \mathcal{A}$ the complex Laplace-Beltrami operator by $\square = \bar{\partial}\vartheta + \vartheta\bar{\partial}$ and note that $\square$ preserves type, that is $\square\mathcal{A}^{p,q} \subset \mathcal{A}^{p,q}$.

C. Summary of potential theory when M is compact.

If M is compact we say that a form $\varphi \in \mathcal{A}$ is *harmonic* if it satisfies the equation $\square\varphi = 0$ and we denote by $\mathcal{H}$ the space of harmonic forms. Now observe that:

$$(\square\varphi, \varphi) = \| \bar{\partial}\varphi \|^2 + \| \vartheta\varphi \|^2,$$

hence φ is harmonic if and only if $\bar{\partial}\varphi = 0$ and $\vartheta\varphi = 0$. In particular if $\varphi \in \mathcal{A}^{p,0}$ then φ is harmonic if and only if $\bar{\partial}\varphi = 0$.

The following is the basic theorem of potential theory on compact manifolds. It asserts that there exists a « Green's operator » N for the Laplace-Beltrami operator, which commutes with $\bar{\partial}$ and which has certain regularity properties. More precisely we have:

THEOREM. There exists a unique bounded operator $N : \mathcal{L} \to \mathcal{L}$, whose null space is $\mathcal{H}$ and whose range is orthogonal to $\mathcal{H}$ such that:

a) If $\varphi \in \mathcal{A}$ then $\varphi = \square N\varphi + H\varphi$, where H is the orthogonal projection on $\mathcal{H}$.

Furthermore N has the following properties:

b) N preserves differentiability, i. e. $N\mathcal{A} \subset \mathcal{A}$.

c) If $\varphi \in \mathcal{A}$ then $N\bar{\partial}\varphi = \bar{\partial}N\varphi$.

d) N is completely continuous.

The following are immediate consequences of properties a), b) and c),

I) $\mathcal{H}$ represents the $\bar{\partial}$-cohomology, that is

$$\mathcal{H} \approx \frac{\text{null space of } \bar{\partial}}{\bar{\partial}\mathcal{A}}$$

II) Given $\alpha \in \mathcal{A}$ there exists $\varphi \in \mathcal{A}$ such that $\square\varphi = \alpha$ if and only if α is orthogonal to $\mathcal{H}$; and then $\varphi = N\alpha$ satisfies the equation.

III) Given $\alpha \in \mathcal{A}^{p,q}$, $q > 0$, there exists a $\varphi \in \mathcal{A}^{p,q-1}$ such that $\bar{\partial}\varphi = \alpha$ if and only if (i) α is orthogonal to $\mathcal{H}$ and (ii) $\bar{\partial}\alpha = 0$; and then $\varphi = \vartheta N\alpha$ satisfies the equation.

IV) If f is a function then we have

$$Hf = f - \vartheta N \bar{\partial} f.$$

D. Some applications of potential theory on open manifolds.

The expression for the holomorphic projection of a function given by IV) is useful in establishing existence theorems on non-compact manifolds. The formula is not so interesting on compact manifolds since in that case all holomorphic functions are constant.

As the first example we outline the solution of the Levi problem on manifolds — the problem was first solved by entirely different methods by H. Grauert (see [3]). We restrict our attention to a manifold M which is an open submanifold of a hermitian manifold M' such that $\overline{M}$ (the closure of M) is compact and bM (the boundary of M) is a C^∞ submanifold of M' of real dimension $2n - 1$. Under these assumptions we say that M is a *finite manifold.*

DEFINITION. A finite manifold M is called *strongly pseudo-convex* if every $P \in bM$ has a neighborhood U (in M') on which there is a real C^∞ function f such that:

a) $f(Q) > 0$ if $Q \notin \overline{M} \cap U$ and $f(Q) < 0$ if $Q \in M \cap U$.

b) $df \neq 0$.

c) There is a holomorphic coordinate system $\{z^1, \dots, z^n\}$ on U such that the n by n matrix $(f_{z^i \bar{z}^j})$ is positive definite.

The Levi problem can be stated as follows. If M is strongly pseudo-convex and if $P \in bM$ to show that there exists a holomorphic function h on M such that $\lim_{Q \to P} h(Q) = \infty$ uniformly. It is not hard to construct a holomorphic function u in a small neighborhood U of P, whose zeros intersect $\overline{M}$ only at P (in fact we may choose for u a polynomial of degree two). Now let V and W be neighborhoods of P such that $U \supset \overline{V} \supset V \supset \overline{W}$ and let ϱ be a C^∞ function on U which is one on W and zero outside of V. Further let f be the function defined on M by:

$$f(Q) = \begin{cases} \dfrac{\varrho(Q)}{u(Q)} & \text{if } Q \in U \cap M \\ 0 & \text{if } Q \notin U \cap M. \end{cases}$$

Then we define the required function h by:

$$h = f - \vartheta N \bar{\partial} f.$$

Now observe that $\bar{\partial} f$ is C^∞ on $\overline{M}$ and hence (by the regularity theorem discussed in the last section) $N\bar{\partial} f$ is also C^∞ on $\overline{M}$, therefore h has the same singularity at P as f and hence h is the required solution, since by IV) it is also holomorphic.

As another application of IV) we sketch a proof of the existence of holomorphic coordinates on an integrable almost-complex manifold; this theorem was first proved by Newlander and Nirenberg in [10] and the proof given here is along the lines suggested by Spencer in [11]. If W is an integrable almost-complex manifold and if $P \in W$ we want to construct a holomorphic coordinate system with origin at P. Observe that the tangent space at P has a complex structure and denote by $u^1, \dots, u^n$ the coordinates on this space. Let M be the unit ball in the tangent space. Let $\Phi : M \rightarrow W$, be a diffeomorphism which sends the origin into P and which induces the identity map on the tangent space at P. For each t, $0 \leq t \leq 1$ we define $\Phi_t : M \rightarrow W$ by:

$$\Phi_t (u^1, \dots, u^n) = \Phi (tu^1, \dots, tu^n).$$

Thus for each t we obtain an integrable almost-complex manifold M_t, whose underlying differentiable manifold is M and whose almost-complex structure is induced by I_t. Note that the structure on M_0 is the same as the complex structure on the tangent space at P.

The manifolds M_t (for small t) are strongly pseudo-convex integrable almost-complex manifolds on which there exists an operator N_t which has property IV). Furthermore if $\varphi \in \mathscr{A}$, then $N_t \varphi$ and all its derivatives are continuous in t in the sense of the sup norm on compact sets. Now let u_t^k be the functions on M_t defined by:

$$u_t^k = u^k - \vartheta_t N_t \bar{\partial}_t u^k .$$

Then the u_t^k are holomorphic functions on M_t, and for small t there exists a neighborhood V of the origin on which the u_t^k are a coordinate system. Now for fixed t, small enough, we define the functions z^k on $\Phi_t(V)$ by:

$$z^k (Q) = u_t^k \, \Phi_t^{-1} (Q),$$

and the z^k are holomorphic coordinates in a neighborhood of P.

E. Formulation of the problem.

In this section we formulate the problem of finding an operator N on an arbitrary complex hermitian manifold M satisfying the properties which are the appropriate generalizations of those listed in section C.

First note that in general the null space of $\square$ is in general much bigger than the intersection of the null spaces of $\bar{\partial}$ and ϑ. Hence the definition of $\mathcal{H}$ must be changed. Furthermore observe that every holomorphic function on M is an element of the $\bar{\partial}$-cohomology; thus, in general, the $\bar{\partial}$-cohomology of M is infinite dimensional so if we want I) of section C to hold we cannot expect that N will be completely continuous.

We will define a closed operator L with domain $\mathcal{D}_L \subset \mathcal{L}$, which for smooth forms in $\mathcal{D}_L$ will coincide with $\square$ and $\mathcal{H}$ will be the null space of L. Let T be the closure of $\bar{\partial}$ and let $\mathcal{D}_T$ denote the domain of T. Let T^* be the hilbert space adjoint of T and $\mathcal{D}_{T^*}$ the domain of T^*. Now we define L by:

$$L = TT^* + T^*T$$

and we observe that

$$\mathcal{D}_L = \{\varphi \in \mathcal{D}_T \cap \mathcal{D}_{T^*} \mid T\varphi \in \mathcal{D}_{T^*}, T^* \varphi \in \mathcal{D}_T\}.$$

Then we have:

PROPOSITION. L is self-adjoint.

This proposition depends on the fact that $T^2 = 0$ and $(T^*)^2 = 0$, we then obtain:

$$(L + I)^{-1} = (I + TT^*)^{-1} + (I + T^*T)^{-1} - I$$

is a bounded symmetric operator and hence L is self-adjoint. The following is an immediate consequence.

COROLLARY. $\mathcal{L} = \overline{L\mathcal{D}_L} \oplus \mathcal{H}$

The operator L restricted to $\mathcal{D}_L \ominus \mathcal{H}$ (the orthogonal complement of $\mathcal{H}$ in $\mathcal{D}_L$) is one to one, we wish to prove the existence of a bounded operator N on $\mathcal{L}$ whose restriction to $\mathcal{L} \ominus \mathcal{H}$ is the inverse of L. *The operator N exists if and only if $L\mathcal{D}_L$ is closed.* If $L\mathcal{D}_L$ is closed then for $\varphi \in$ we have

$$\varphi = L\xi + H\varphi,$$

where $\xi \in \mathcal{D}_L$ and H is the orthogonal projection on $\mathcal{H}$. Then we define N by $N\varphi = \xi - H\xi$. So that if $L\mathcal{D}_L$ is closed then N exists and if N exists then, by the closed graph theorem, $L\mathcal{D}_L$ is closed.

Whenever $L\mathcal{D}_L$ is closed then the operator N exists and satisfies the following:

a) If $\varphi \in \mathcal{L}$ then $\varphi = LN\varphi + H\varphi$.

b) N preserves differentiability.

c) If $\varphi \in \mathcal{D}_T$ then $TN\varphi = NT\varphi$.

These properties are the generalizations of properties a), b) and c) of section *C*. Thus the properties I) II), III) and IV) are also suitably generalized.

F. Statement of the main results.

We restrict our attention to the case where M is a strongly pseudo-convex manifold, as defined in section *D*. The convexity assumption is needed since if M is a bounded domain in $\mathbb{C}^n$ and if the matrix $(f_{z^i \bar{z}^j})$ is negative definite at some point of the boundary then the range of T is not closed and hence $L\mathcal{D}_L$ is not closed.

We denote by $\dot{\mathcal{A}}$ the subspaces of $\mathcal{A}$ consisting of forms which are C^∞ on $\overline{M}$ (that is up to and including the boundary). The main result is then given by the following theorem.

THEOREM. If M is a strongly pseudo-convex manifold then there exists a bounded operator N on $\mathcal{L}$ which has the following properties:

a) If $\varphi \in \mathcal{L}$ then $\varphi = LN\varphi + H\varphi$.

b) N preserves differentiability and differentiability up to the boundary, i.e $N\mathcal{A} \subset \mathcal{A}$ and $N\dot{\mathcal{A}} \subset \dot{\mathcal{A}}$.

c) If $\varphi \in \mathcal{D}_T$ then $NT\varphi = TN\varphi$.

d) The restriction of N to $\mathcal{L}^{p,q}$ for $q > 0$ is completely continuous and thus $\mathcal{H}^{p,q}$, $q > 0$ is finite dimensional.

In order to establish this theorem we first give another description of the operator L. Let $\mathcal{D} = \mathcal{D}_T \cap \mathcal{D}_{T^*}$ we define an inner product D on $\mathcal{D}$ by:

$$D(\varphi, \psi) = (T\varphi, T\psi) + (T^* \varphi, T^* \psi) + (\varphi, \psi),$$

then $\mathcal{D}$ is a hilbert space under D. Now it is easy to verify that L is the unique self-adjoint operator whose domain is contained in $\mathcal{D}$ and such that if $\varphi \in \mathcal{D}_L$ then

$$D(\varphi, \psi) = (L\varphi + \varphi, \psi)$$

for all $\psi \in \mathcal{D}$.

The existence of the operator N follows from the following proposition which is established by means of the *a priori* estimates discussed in the next section.

PROPOSITION. If M is strongly pseudo-convex and if $q > 0$ then D restricted to $\mathcal{D}^{p,q}$ is completely continuous, i.e. if $\{\varphi_m\}$ is a sequence in $\mathcal{D}^{p,q}$ such that $D(\varphi_m, \varphi_m) \leq 1$ then it has a subsequence which converges in $\mathcal{L}$.

Note that this proposition implies that $\mathcal{H}^{p,q}$ is finite dimensional when $q > 0$ and since $\mathcal{H}^{p,0}$ is in general infinite dimensional the restriction $q > 0$ is essential.

The above proposition implies that if $q > 0$ then there exists $C > 0$ such that for all $\varphi \in \mathcal{D}^{p,q} \ominus \mathcal{H}^{p,q}$ we have

$$\| T\varphi \|^2 + \| T^* \varphi \|^2 \geq C \| \varphi \|^2.$$

Furthermore if $\varphi \in \mathcal{D}_L$ then the left side of this inequality equals $(L\varphi, \varphi)$ and by applying Schwarz's inequality we obtain $\| L\varphi \| \geq C \| \varphi \|$ which proves that if $q > 0$ then $L\mathcal{D}_L^{p,q}$ is closed. Now to show that $L\mathcal{D}_L^{p,0}$ is closed we want to prove that there exists $C > 0$ such that if $\varphi \in \mathcal{D}_L^{p,0} \ominus \mathcal{H}^{p,0}$ we have $\| L\varphi \| \geq C \| \varphi \|$. Since T^* restricted to $\mathcal{D}^{p,0}$ is zero, L restricted to $\mathcal{D}_L^{p,0}$ equals $T^* T$. If $\psi \in \mathcal{D}_L^{p,0}$ then $T\psi \in \mathcal{D}^{p,1} \ominus \mathcal{H}^{p,1}$ and thus, applying the above inequality we obtain $\| L\psi \| \geq C \| T\psi \|$. Recall that $\mathcal{L}^{p,0} = \overline{L\mathcal{D}_L^{p,0}} \oplus \mathcal{H}^{p,0}$, thus if $\varphi \in \mathcal{D}_L^{p,0} \ominus \mathcal{H}^{p,0}$, it can be approximated by $L\alpha$; that is we can choose $\alpha \in \mathcal{D}_L^{p,0}$ so that $\| \varphi - L\alpha \|$ is as small as we wish. Then we have:

$$\begin{aligned} \| \varphi \|^2 &\backsim | (\varphi, L\alpha) | \\ &= | (T\varphi, T\alpha) | \leq \| T\varphi \| \; \| T\alpha \| \leq \frac{1}{C} \| L\varphi \| \; \| L\alpha \| \backsim \frac{1}{C} \| L\varphi \| \; \| \varphi \| . \end{aligned}$$

Therefore $\| L\varphi \| \geq C \| \varphi \|$ for all $\varphi \in \mathcal{D}_L^{p,0} \ominus \mathcal{H}^{p,0}$ which proves that $L\mathcal{D}_L^{p,0}$ is closed.

G. The basic a priori estimates.

Let $\dot{\mathcal{D}} = \mathcal{D} \cap \dot{\mathcal{A}}$, then we have,

THEOREM. If M is strongly pseudo-convex and if $q > 0$ then there exists $C > 0$ such that for all $\varphi \in \dot{\mathcal{D}}^{p,q}$ we have

$$D(\varphi, \varphi) \geq CE(\varphi)^2, \tag{I}$$

where

$$E(\varphi)^2 = \| \varphi \|_{\bar{z}}^2 + \int_{bM} | \varphi |^2 \, dS + \| \varphi \|^2,$$

$\| \varphi \|_{\bar{z}}$ depends on the choice of a finite covering $\{U_\alpha\}$ of $\overline{M}$ by coordinate neighborhoods and is given by:

$$\| \varphi \|_{\bar{z}}^2 = \Sigma \int_{M \cap U_\alpha} | \varphi_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q , \bar{z}^k} |^2 \, dV$$

and dS is the volume element on the boundary.

The estimate (I) is used to prove that D restricted to $\dot{\mathcal{D}}^{p,q}, q > 0$, is completely continuous. Given a sequence $\{\varphi_k\}$ such that $D(\varphi_k, \varphi_k) \leq 1$ then, by Rellich's lemma, for any compact set $K \subset M$ there exists a subsequence of $\{\varphi_k\}$ which converges in the $\| \ \|$-norm restricted to K. To be able to choose a subsequence which converges on all of M we must show that the $\| \ \|$-norm of φ_k over a boundary strip can be made small, independently of k, if the width of the strip is small. This is accomplished by using the following estimate in conjunction with (I).

PROPOSITION. There exists $C > 0$ and $\varepsilon > 0$ such that for all $\varphi \in \dot{\mathcal{A}}$ and all $0 \leq a \leq \varepsilon$ we have:

$$\text{(II)} \qquad \int_{bM_a} |\varphi|^2 \, dS \leq C E(\varphi)^2$$

where

$M_a = \{P \in M \mid \text{distance of } P \text{ to } bM \text{ is greater than } a\}$.

It is not enough to prove that D is completely continuous on $\dot{\mathcal{D}}^{p,q}$ we must show that D is completely continuous on $\mathcal{D}^{p,q}$. This is done by showing that $\dot{\mathcal{D}}^{p,q}$ is dense in $\mathcal{D}^{p,q}$ in the D-norm. The denseness follows from the regularity theorem and this theorem uses the estimate described below.

Let $R(P)$ be the distance of P to bM and let r be a real C^∞ function such that in some neighborhood of bM we have:

$$r(P) = \begin{cases} \sqrt{2}\, R(P) & \text{if } P \notin M \\ -\sqrt{2}\, R(P) & \text{if } P \in M. \end{cases}$$

For $\tau \geq 0$ let $w_\tau(P) = \exp \tau \mid r(P) \mid$. Then if $\varphi, \psi \in \mathcal{L}$ we define:

$$(\varphi, \psi)_\tau = \int_M \langle \varphi, \psi \rangle w_\tau \, dV \quad \text{and} \quad \| \varphi \|_\tau^2 = (\varphi, \varphi)_\tau .$$

The norms $\| \ \|_\tau$ are all equivalent in the sense that they induce the same topology on $\mathcal{L}$. Now if $\varphi, \psi \in \mathcal{D}$ we define $D_\tau(\varphi, \psi)$ by:

$$D_\tau(\varphi, \psi) = (T\varphi, T\psi)_\tau + (T^* \varphi, T^* \psi)_\tau + (\varphi, \psi)_\tau$$

and again for all τ, D_τ defines the same topology on $\mathcal{D}$.

PROPOSITION. If M is a strongly pseudo-convex manifold and if $q > 0$ then there exists $C > 0$ such that for all $\tau \geq 0$ and all $\varphi \in \dot{\mathcal{D}}^{p,q}$ we have:

$$\text{(III)} \qquad \tau \| \varphi \|_\tau^2 \leq C (D_\tau (\varphi, \varphi) + \tau^2 \| A(\varphi) \|^2),$$

where $A(\varphi)$ is a combination of components of φ which vanishes on bM.

H. Remarks about the proofs of the estimates.

Before describing how the estimate (III) is used to establish the regularity properties and how these are used to prove that $\dot{\mathcal{D}}$ is dense in $\mathcal{D}$ we will outline the ideas behind the proofs of the estimates.

First we remark that, by using a partition of unity, it suffices to prove the estimates for forms whose supports lie in a coordinate neighborhood in M' which intersects bM. So let $P_0 \in bM$ and U a coordinate neighborhood with origin at P_0. Let $\zeta^1, \dots, \zeta^n$ be an orthonormal basis for the forms of type $(1, 0)$ at each point of U, such that

$$\zeta^k = \Sigma\, a_j^k \, dz^j$$

where the a_j^k are C^∞ functions on U and $a_j^n = r_{z^j}$ so that $\zeta^n = \partial r$.

If $\varphi \in \mathcal{A}^{p,q}$ then in $U \cap M$ we have: $\varphi = \varphi_{I\bar{J}}\, \zeta^{I\bar{J}}$ where I and J are ordered sets of p-tuples and q-tuples respectively and

$$\zeta^{I\bar{J}} = \zeta^{i_1} \wedge \dots \wedge \zeta^{i_p} \wedge \bar{\zeta}^{j_1} \wedge \dots \wedge \bar{\zeta}^{j_q}.$$

Then we have:

LEMMA. If $\varphi \in \dot{\mathcal{A}}^{p,q}$ then $\varphi \in \dot{\mathcal{D}}^{p,q}$ if and only if $\varphi_{I\bar{J}} = 0$ on bM whenever $n \in J$.

If u is a differentiable function on u we define u_{ζ^i} and $u_{\bar{\zeta}^i}$ by:

$$u_{\zeta^i} = \langle du, \zeta^i \rangle \text{ and } u_{\bar{\zeta}^i} = \langle du, \bar{\zeta}^i \rangle$$

where the $\bar{\zeta}^i$ are conjugates of the ζ^i. Consider the following commutator:

$$u_{\zeta^i \bar{\zeta}^j} - u_{\bar{\zeta}^j \zeta^i} = \Sigma\, (C_{ij}^k u_{\zeta^k} - \bar{C}_{ji}^k u_{\bar{\zeta}^k}).$$

The strong pseudo-convexity is equivalent to the positive definiteness of the $(n-1)$ by $(n-1)$ matrix $(C_{ij}^n)_{i, j<n}$ evaluated on bM. The inequality

(I) is obtained as follows:

$$\bar{\partial}\varphi = \varphi_{I\bar{J}\,\bar{\zeta}^k}\,\overline{\zeta^k} \wedge \zeta^{I\bar{J}} + \dots$$

where the remaining terms are combinations of components which do not involve derivatives; then

$$\|\bar{\partial}\varphi\|^2 = \sum_{k,\,I,\,J} \int_M |\varphi_{I\bar{J}\,\bar{\zeta}^k}|^2\, dV + a^{IJk}_{KLm} \int_M \varphi_{I\bar{J}\,\zeta^k}\, \overline{\varphi}_{\bar{K}L\,\zeta^m}\, dV + \dots,$$

where,

$$a^{IJk}_{KLm} = \begin{cases} -\operatorname{sgn}(Hk)\operatorname{sgn}(Hm) \text{ if } I = K,\ J = \langle Hm \rangle,\ L = \langle Hk \rangle \\ 0 \text{ otherwise,} \end{cases}$$

H is some $(q-1)$-tuple, $\langle Hk \rangle$ is the order q-tuple consisting of H and k, $\operatorname{sgn}(Hk)$ is zero if $k \in H$ and is the sign of the permutation which maps (Hk) onto $\langle Hk \rangle$ if $k \notin H$. The first term of the above expression is bounded from below by $C\|\varphi\|^2_{\bar{z}}$ and the terms indicated by dots contain products in which only one factor is differentiated with respect to the $\bar{\zeta}$, thus they can be estimated by (small const.) $\|\varphi\|^2_{\bar{z}}$ + (large const.) $\|\varphi\|^2$. Integration by parts on the second term gives:

$$\int_M \varphi_{I\langle \overline{Hm} \rangle\,\bar{\zeta}^k}\, \overline{\varphi}_{\bar{I}\langle Hk \rangle\,\zeta^m}\, dV = \int_M \varphi_{I\langle \overline{Hm} \rangle\,\zeta^m}\, \overline{\varphi}_{\bar{I}\langle Hk \rangle\,\bar{\zeta}^k}\, dV$$

$$- \int_{bM} C^n_{mk}\, \varphi_{I\langle \overline{Hm} \rangle}\, \overline{\varphi}_{\bar{I}\langle Hk \rangle}\, dS + \dots,$$

where the remaining terms can be estimated by (small const.) $E(\varphi)^2$ + (large const.) $\|\varphi\|^2$. It remains to observe that

$$(\vartheta\varphi)_{I\bar{H}} = \pm\, \varphi_{I\langle \overline{Hm} \rangle\,\zeta^m} + \dots,$$

where the remaining terms contain no derivatives. Thus we obtain the desired estimate by combining the above statements.

The *a priori* estimate (III) is obtained by a similar argument. The precise form of (III) is given in the following.

THEOREM. If M is strongly pseudo-convex there exists a constant C such that for all $\varphi \in \dot{\mathcal{D}}^{p,q}$, $q > 0$, φ supported in $U \cap \overline{M}$ and all $\tau \geq 0$ we have:

$$\tau \| \varphi \|_\tau^2 \leq C (D_\tau(\varphi, \varphi) + \tau^2 \sum_{n \in J} \| \varphi_{I\bar{J}} \|_\tau^2).$$

I. Regularity on the boundary.

In this section we briefly indicate the method used in obtaining the regularity theorems $N\dot{\mathcal{A}} \subset \dot{\mathcal{A}}$, $H\dot{\mathcal{A}} \subset \dot{\mathcal{A}}$ and the denseness of $\dot{\mathcal{D}}$ in $\mathcal{D}$ under the D-norm. The detailed proofs are given in part II of [7].

The technique which has been developed for establishing regularity of standard elliptic systems is based on an *a priori* bound of the L_2-norms of the first derivatives, here we only have the weaker bound of the E-norm. Now for functions (or forms) which vanish on the boundary the E-norm is equivalent to the L_2-norm of the first derivatives, hence in the case $q = n$ we can apply the usual techniques (in particular when $n = 1$ this is the only case). When $0 < q < n$ it is easy to construct a sequence of forms in $\dot{\mathcal{D}}^{p,q}$ whose D-norms and E-norms converge but the L_2-norms of the first derivatives diverges.

Let $\mathcal{D}_1^{p,q}$ be the closure, under D, of $\dot{\mathcal{D}}^{p,q}$; then if $q > 0$ the estimates (I), (II) and (III) hold for forms in $\mathcal{D}_1^{p,q}$. Note that the components appearing in the last term of (III) vanish on the boundary hence we can bound their first derivatives. Then by an adaptation of Hörmander's techniques, which is too lengthy to describe here, we obtain the following.

PROPOSITION. If M is strongly pseudo-convex then there exists $\tau > 0$ such that if $\varphi \in \mathcal{D}_1^{p,q}$, $q > 0$ and satisfies the equation $D_\tau(\varphi, \psi) = (\alpha, \psi)_\tau$ for some $\alpha \in \dot{\mathcal{A}}^{p,q}$ and all $\psi \in \mathcal{D}_1^{p,q}$ then $\varphi \in \dot{\mathcal{A}}^{p,q}$.

The following theorem implies the existence of N.

THEOREM. If M is strongly pseudo-convex then $\mathcal{D}_1 = \mathcal{D}$, i. e. $\dot{\mathcal{D}}$ is dense in $\mathcal{D}$ under the D-norm.

PROOF. Choose τ so that the above proposition holds. Let $\gamma \in \mathcal{D}^{p,q}$ be orthogonal to $\dot{\mathcal{D}}^{p,q}$ in the D_τ inner product, that is $D_\tau(\gamma, \psi) = 0$ for all $\psi \in \dot{\mathcal{D}}^{p,q}$. Let $\alpha_k \in \dot{\mathcal{A}}^{p,q}$ be a sequence such that $\gamma = \lim \alpha_k$ in the $\| \, \|$-norm. Let $\varphi_k \in \mathcal{D}_1^{p,q}$ be such that:

$$D_\tau(\varphi_k, \beta) = (\alpha_k, \beta)_\tau$$

for all $\beta \in \mathcal{D}_1^{p,q}$. Then $\varphi_k \in \dot{\mathcal{A}}^{p,q}$ and hence it follows (see [7]) that the above equation holds for all $\beta \in \mathcal{D}^{p,q}$; so that in particular $D_\tau(\varphi_k, \gamma) = (\alpha_k, \gamma)_\tau = 0$ and since $\| \gamma \|_\tau^2 = \lim (\alpha_k, \gamma)_\tau = 0$ we conclude that $\gamma = 0$ and hence $\mathcal{D}_1^{p,q} = \mathcal{D}^{p,q}$.

THEOREM. If M is strongly pseudo-convex then $(L+I)^{-1} \dot{\mathcal{A}}^{p,q} \subset \dot{\mathcal{A}}^{p,q}$ for $q > 0$, $N\dot{\mathcal{A}} \subset \dot{\mathcal{A}}$ and $H\dot{\mathcal{A}} \subset \dot{\mathcal{A}}$.

PROOF. First observe that for any $\psi \in \dot{\mathcal{D}}^{p,q}$ we have:

$$D_\tau(\psi, \psi) \leq C \{ \,| D(\psi, \psi w_\tau) | + \tau^2 (\sum_{n \in J} \| \psi_{I\bar{J}} \|_\tau^2 + \sum_{n \in L} \| (\partial \psi)_{K\bar{L}} \|_\tau^2)\}.$$

The equation $(L+I)^{-1} \alpha = \varphi$, $\alpha \in \dot{\mathcal{A}}^{p,q}$, is equivalent to requiring that $D(\varphi, \beta) = (\alpha, \beta)$ for all $\beta \in \mathcal{D}^{p,q}$. Since $\varphi \in \mathcal{D}_L^{p,q}$ and since $\mathcal{D}_1^{p,q+1} = \mathcal{D}^{p,q+1}$ we have that $\bar{\partial}\varphi \in \mathcal{D}_1^{p,q+1}$ and the $\| \;\; \|$-norm of the first derivatives of $(\bar{\partial}\varphi)_{K\bar{L}}$, $n \in L$, is finite — in fact bounded by $CD(\bar{\partial}\varphi, \bar{\partial}\varphi) \leq C \| \alpha \|^2$. Thus choosing τ large enough we can apply the same arguments as in the previous proposition using the above inequality in conjunction with (III). The statements $N\dot{\mathcal{A}}^{p,q} \subset \dot{\mathcal{A}}^{p,q}$ and $H\dot{\mathcal{A}}^{p,q} \subset \dot{\mathcal{A}}^{p,q}$ are obtained by similar arguments. Finally to show that $H\dot{\mathcal{A}}^{p,0} \subset \dot{\mathcal{A}}^{p,0}$ and $N\dot{\mathcal{A}}^{p,0} \subset \dot{\mathcal{A}}^{p,0}$ we note that if $\alpha \in \dot{\mathcal{A}}^{p,0}$ then $H\alpha = \alpha - \vartheta N \bar{\partial}\alpha$ so that $H\alpha \in \dot{\mathcal{A}}^{p,0}$ and $N\alpha = \vartheta N^2 \bar{\partial}\alpha \in \dot{\mathcal{A}}^{p,0}$.

Brandeis University
Waltham, Massachusetts

REFERENCES

[1] ASH, M. E., « *The basic $\bar{\partial}$-estimate in the non-kähler case* », Proc. Amer. Math. Soc., to appear.

[2] GARABEDIAN, P. R. and SPENCER, D. C. « *Complex boundary value problems* », Trans Amer. Math. Soc , vol. 73 (1952), pp. 223-242.

[3] GRAUERT, H., « *On Levi's problem and the imbedding of real-analytic manifold* », Ann. of Math., vol. 68 (1958) pp. 460-472.

[4] HÖRMANDER, L., « *Lectures on partial differential operators* », Academic Press, New York, 1963.

[5] KOHN, J. J, and SPENCER, D. C , « *Complex Neumann problems* », Ann. of Math., vol. 66 (1957) pp. 89-140.

[6] KOHN, J. J., « *Solution of the $\bar{\partial}$-Neumann problem on strongly pseudo-convex manifolds* », Proc. Nat. Acad. of Sci., U.S.A., vol. 47 (1961) pp. 1198-1202.

[7] KOHN, J. J., « *Harmonic integrals on strongly pseudo-convex manifolds I* », Ann. of Math., vol. 78 (1963) pp. 112-148.

[8] KOHN, J. J., « *Regularity at the boundary of the $\bar{\partial}$-Neumann problem* », Proc. Nat. Acad. of Sci., U.S.A., vol. 49 (1963) pp. 206-213.

[9] MORREY, C. B., « *The analytic embedding of abstract real-analytic manifolds* », Ann. of Math , vol. 68 (1958) pp. 159-201.

[10] NEWLANDER, A. and NIRENBERG, L., « *Complex analytical coordinates in almost-complex manifolds* », Ann. of Math., vol. 65 (1957) pp. 391-404.

[11] SPENCER, D. C., « *Potential theory and almost-complex manifolds* », Lectures on functions of a complex variable, Univ. of Michigan Press, 1955, pp. 15-43.