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Mixed problems for higher order elliptic equations in two variables, II

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MIXED PROBLEMS FOR HIGHER ORDER ELLIPTIC EQUATIONS IN TWO VARIABLES, II (*)

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CONTENTS.

INTRODUCTION. The purpose of the present paper is to give some complements to the results obtained in our previous paper [10]. We there considered mixed problems in two independent variables of the form:

$$\begin{cases} Au = f & \text{in } \Omega, \\ B_j^+ u = g_j^+ & \text{on } \Gamma_+ \quad (j = 1, 2, \dots, l) \\ B_j^- u = g_j^- & \text{on } \Gamma_- \quad (j = 1, 2, \dots, l) \end{cases}$$

Here Ω is a bounded domain in R^2 with C^∞ boundary Γ; Γ_+ and Γ_- are two disjoint open portions of Γ such that $\gamma = \overline{\Gamma}_+ \cap \overline{\Gamma}_-$ consists of precisely two points p' and p'' · A, B_j^+, B_j^- are differential operators with C^∞ coefficients in Ω, on $\overline{\Gamma}_+$, on $\overline{\Gamma}_-$ respectively, such that A is elliptic, B_j^+ cover A on $\overline{\Gamma}_+$, B_j^- cover A on $\overline{\Gamma}_-$. It was proved that the inequality

(1) $$\| u, \Omega \|_s \leq C \big(\| Au, \Omega \|_{s-m} + \sum_{j, \pm} \| B_j^\pm u, \Gamma_\pm \|_{s-m_j^\pm-1/2} + \| u, \Omega \|_{s-1} \big), \quad u \in H^s(\Omega);$$

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holds for all $s > s_0$, where

$$s_0 = \sup \mu_j^{\pm} + 1/2,$$

$\mu_j^{\pm}$ being the normal order of $B_j^{\pm}$, except when $s = \sigma_v \bmod 1$ ($v = 1, 2, \dots, q$) where $\sigma_1, \dots, \sigma_q$ are some well-defined real numbers and $q \leq 2l$. ($\sigma_1, \dots, \sigma_q$ will be called the *exceptional values* of s.) Consider now the mapping

$$T : u \longrightarrow (Au,\ B_1^+ u, \dots, B_l^+ u,\ B_1^- u, \dots, B_l^- u) \tag{2}$$

of $H^s(\Omega)$ into $K^s(\Omega) = H^{s-m}(\Omega) \times \Pi H^{s-m_j^{\pm}-1/2}(\Gamma_+)$. Let N be the null-space of T and R the range of T. It follows that, except when $s = \sigma_v \bmod 1$, we have:

1) N is of finite dimension,

2) R is closed.

In the present paper we will provide a proof of the following statement, which was announced in [10]:

3) R is of finite codimension.

Following [9], we consider the conjugate mapping $\overline{T}$ of T:

$$\overline{T}: (F, G_1^+, \dots, G_l^+, G_1^-, \dots, G_l^-) \longrightarrow \overline{A} F + \sum_{j,\pm} \overline{B}_j^{\pm} G_j^{\pm} \tag{3}$$

of $K_\Omega^{-s} = H_\Omega^{-s+m} \times \Pi H_{\Gamma_\pm}^{-s+m_j^{\pm}+1/2}$ into H_Ω^{-s}. Then 3) will be a consequence of the following inequality (« dual » to (1)):

$$\| F, R^n \|_{-s+m} + \sum_{j,\pm} \| G_j^{\pm}, \Gamma \|_{-s+m_j^{\pm}+1/2} \leq \tag{4}$$

$$\leq C (\| \overline{A} F + \sum_{j,\pm} \overline{B}_j^{\pm} G_j^{\pm},\ R^n \|_{-s} + \| F, R^n \|_{-s+m-1} +$$

$$+ \sum_{j,\pm} \| G_j^{\pm},\ \Gamma \|_{-s+m_j^{\pm}+1/2-1}),$$

$$(F, G_1^+, \dots, G_l^+,\ G_1^-, \dots, G_l^-) \in K_\Omega^{-s}.$$

We have thus to prove that (4) holds for all $s > s_0$ except when $s = \sigma_v \bmod 1$ ($v = 1, 2, \dots, q$). In the proof it is of course sufficient to consider the « canonical » situation of constant coefficients and a half-space. In the proof of (1), as given in [10], the main trick was to transform (1) into the corresponding inequality for — what was there called — a Wiener-Hopf type problem:

$$\| h \| \leq C (\| Y_+ K^+ h \| + \| Y_- K^- h \|),\ h \in H, \tag{5}$$

where h is a vector whose eutries are functions and K^+ and K^- are matrices whose entries are homogeneus convolution operators of degree 0. Here we show that (4) can be transformed into the following inequality (« dual » to (5)):

$$\| W^+ \| + \| W^- \| \leq C \| K^+ W^+ + K^- W^- \| , \tag{6}$$

$$W^+ \in H_+ , \ W^- \in H_- .$$

The proof is analogous to the one of (5) given in [10] but slightly more technical (cf. Remark on p. 6). It is now easily seen that (5) and (6) are simultaneously true, so that the above statement about (4) follows. Let $\nu = \nu(s)$ be the dimension of N and $\varrho = \varrho(s)$ the codimension of R. We will also show that ν and ϱ, as functions of s, are constant in any interval that does not contain any exceptional values of s. But, as was shown in [10] by interpolation, at an exceptional value ϱ or ν must have a jump. This means in particular that no regularity can hold true.

The plan of this paper is the following. In Section 1 we carry out the reduction of (4) to (6) and establish the equivalence of (5) and (6). In Section 2, the regularity is studied. In Section 3 we consider the change of the index at an exceptional value. The different Sections are, at least what methods concern, independent of each other.

1. Deduction of the « dual » inequality.

We consider first the case when A, B_j^+, B_j^- have constant coefficients and $\Omega = R_+^2 , \Gamma_+ = R_+ , \Gamma_- = R_-$, as in [10], Section 1. We make the following

Hypothesis. A is elliptic. Both B_j^+ and B_j^- cover A. and we will consider the inequality

$$(\bar{\mathcal{J}}) \qquad \| F, R^2 \|_{-s+m} + \sum_{j,\pm} \| G_j^{\pm}, R \|_{-s+m_j^{\pm}+1/2} \leq$$

$$\leq C (\| \bar{A} F + \sum_{j,\pm} \bar{B}_j^{\pm} \bar{G}_j^{\pm} , R^2 \|_{-s} + \| F, R^2 \|_{-s+m-1} +$$

$$+ \sum_{j,\pm} \| G_j^{\pm} , R \|_{-s+m_j^{\pm}+1/2-1}),$$

$$F \in H_{R_+^2}^{-s+m}, \ G_j^{\pm} \in H_{R_{\pm}}^{-s+m_j^{\pm}+1/2}.$$

(For the definition of the relevant spaces and norms, we refer to [9], [10]).

Our main concern will be the following

Problem. For which values of s does $(\bar{\mathcal{J}})$ hold true?

We will show that $(\bar{\mathcal{J}})$ holds true for all $s > s_0$, where

$$s_0 = \sup \mu_j^{\pm} + 1/2,$$

$\mu_j^{\pm}$ being the normal order of $B_j^{\pm}$, except when $s = \sigma_v \bmod 1$ $(v = 1, 2, \ldots, q)$ where $\sigma_1, \ldots, \sigma_q$ are certain well-defined numbers, $q \leq l$, in fact the same as in [10], Section 3. It is of course no restriction to assume that A, B_j^+, B_j^- are homogeneous. The first step is now to show that $(\bar{\mathcal{J}})$ is equivalent to the corresponding inequality when the norms are all replaced by the corresponding homogeneous norms, i. e. the inequality

$$(\bar{\mathcal{J}}') \qquad \| F, R^2 \|^*_{-s+m} + \sum_{j,\pm} \| G_j^{\pm}, R \|^*_{-s+m_j^{\pm}+1/2} \leq$$

$$\leq C \| \bar{A} F + \sum_{j,\pm} \bar{B}_j^{\pm} G_j^{\pm}, R^2 \|^*_{-s},$$

$$F \in \overset{*}{H}{}^{-s+m}_{R^2_+}, \quad G_j^{\pm} \in \overset{*}{H}{}^{-s+m_j^{\pm}+1/2};$$

where

$$\| F, R^2 \|^*_{-s+m} = \left(\int (|\xi_1| + |\xi_2|)^{-2s+2m} \, | F(\xi_1, \xi_2) |^2 \, d\xi_1 \, d\xi_2 \right)^{1/2} \quad \text{etc.}$$

and $\overset{*}{H}{}^{-s+m}_{R^2_+}$ etc. stands for the subspace of $H^{-s+m}_{R^2_+}$ etc. such that the norm is finite. (Cf. the remark below.) In order to prove $(\bar{\mathcal{J}}')$ one just has to replace in $(\bar{\mathcal{J}})$ F by $F_\varepsilon = F_\varepsilon(x_1, x_2) = \varepsilon^m F(x_1/\varepsilon, x_2/\varepsilon)$ and $G_j^{\pm}$ by $(G_j^{\pm})_\varepsilon = (G_j^{\pm})_\varepsilon (x_1, x_2) = \varepsilon^{m_j^{\pm}} G_j^{\pm}(x_1/\varepsilon, x_2/\varepsilon)$ and let ε tend to 0. Conversely $(\bar{\mathcal{J}})$ follows from $(\bar{\mathcal{J}}')$ by utilizing «partial regularity» (cf. [9], in particular the proof of theorem 1) in an appropriate manner. We omit the details. Having thus established the equivalence of $(\bar{\mathcal{J}})$ and $(\bar{\mathcal{J}}')$, let us make use of the analogue of $(\bar{\mathcal{J}}')$ for the Dirichlet problem

$$(7) \qquad \| F^0, R^2 \|^*_{-s+m} + \sum_j \| G_j^0, R \|^*_{-s+(j-1)+1/2} \leq$$

$$\leq C \| \bar{A} F^0 + \sum_j D_2^{j-1} G_j^0, R^2 \|^*_{-s},$$

$$F^0 \in \overset{*}{H}{}^{-s+m}_{R^2_+}, \quad G_j^0 \in \overset{*}{H}{}^{-s+(j-1)+1/2}_{R},$$

which is proved in the same way starting from the analogue of $(\mathcal{J})$ (cf. [9]), as well as of its trivial converse. Let us determine F^0 and G_j^0 such that

$$\bar{A}\, F + \sum_{j,\pm} \bar{B}_j^{\pm}\, G_j^{\pm} = \bar{A}\, F^0 + \sum_j D_2^{j-1}\, G_j^0 ,$$

which is obviously possible. Then we have from $(\bar{\mathcal{J}}')$ and the converse of (7)

$$(\bar{\mathcal{J}}'') \qquad \| F, R^2 \|^*_{-s+m} + \sum_{j,\pm} \| G_j^{\pm} , R \|^*_{-s+m_j^{\pm}+1/2} \leq$$

$$\leq C \,(\sum_j \| G_j^0 , R \|^*_{-s+(j-1)+1/2} + \| F^0, R^2 \|^*_{-s+m}),$$

which in view of (7) is equivalent to $(\bar{\mathcal{J}}')$. In particular if $F^0 = 0$, we obtain

$$(\bar{\mathcal{J}}''') \qquad \sum_{j,\pm} \| G_j^{\pm} , R \|^*_{-s+m_j^{\pm}+1/2} \leq C \sum_j \| G_j^0 , R \|^*_{-s+(j-1)+1/2} ,$$

which is also equivalent to $(\bar{\mathcal{J}}')$. In fact, we have

$$\|F, R^2 \|^*_{-s+m} \leq C \,\| \bar{A}\, F, R^2 \|^*_{-s} \leq$$

$$\leq C \,(\| \bar{A}\, F^0 + \sum_j D_2^{j-1}\, G_j^0 , R^2 \|^*_{-s+m} + \sum_{j,\pm} \| G_j^{\pm} , R \|^*_{-s+m_j^{\pm}+1/2})$$

and, if $(\bar{\mathcal{J}}'')$ holds, all terms on the right hand side can be estimated in terms of the right hand side of $(\bar{\mathcal{J}}''')$. Let us now set

$$W_j^{\pm} = (D_1 \pm i\, 0)^{-s+m_j^{\pm}+1/2}\; G_j^{\pm} .$$

Then we obtain

$$(8) \qquad \sum_{j,\pm} \| W_j^{\pm}, R \| \leq C \sum_j \| \,| D_1 |^{-s+(j-1)+1/2}\, G_j^0 , R \|.$$

We claim that

$$(9) \qquad G_j^0 = \sum_{k,\pm} \Phi_{jk}^{\pm}\, G_k^{\pm}$$

where $\{\Phi_{jk}^{\pm}\}$ is the « characteristic matrix » (cf. [4], [8], [9]) of $B_j^{\pm}$ (with respect to A). But, in the notation of [10], we have

$$K_{jk}^{\pm} = (D_1 \pm i0)^{s-m_j^{\pm}-1/2}\, \Phi_{jk}^{\pm}\, | D_1 |^{-s+(1-k)+1/2}.$$

Thus we shall obtain

$$\sum_{j,\pm} \| W_j^{\pm}, R \| \leq C \sum_{j,\pm} \| \sum_k K_{jk}^{\pm} W_k^{\pm}, R \|, \; W_j^{\pm} \varepsilon H_{R_{\pm}},$$

or in « matrix form »

$$\sum_{\pm} \| W^{\pm} \| \leq C \| \bar{K}^{+} W^{+} + \bar{K}^{-} W^{-} \|, \; W^{\pm} \varepsilon H_{\pm}. \tag{10}$$

To prove (9) let us take Fourier transforms; we obtain

$$\bar{A}(\xi_1, \xi_2) F(\xi_1, \xi_2) + \sum_{j,\pm} \bar{B}_j^{\pm}(\xi_1, \xi_2) G_j^{\pm}(\xi_1) = \sum \xi_2^{j-1} G_j^0(\xi_1).$$

Hence

$$F(\xi_1, \xi_2) = \frac{\sum_j \xi_2^{j-1} G_j^0(\xi_1) - \sum_{j,\pm} \bar{B}_j^{\pm}(\xi_1, \xi_2) G_j^{\pm}(\xi_1)}{\bar{A}(\xi_1, \xi_2)}.$$

But, by the Paley-Wiener theorem, $F(\xi_1, \xi_2)$ is analytic in ξ_2 in the half-plane Im $\xi_2 < 0$. Let $\{\varrho_p(\xi_1)\}_{p=1}^{l}$ be the roots of the equation $A(\xi_1, \xi_2) = 0$ with positive imaginary parts. Hence, pretending for the moment that they are distinct, we get

$$\sum_j (\varrho_k(\xi_1))^{j-1} G_j^0(\xi_1) = \sum_{j,\pm} \bar{B}_j^{\pm}(\xi_1, \varrho_k(\xi_1)) G_j^{\pm}(\xi_1)$$

which apparently leads to (9) in view of the definition $\{\Phi_{jk}^{\pm}\}$:

$$\{\Phi_{jk}^{\pm}(\xi_1)\} = \{B_j^{\pm}(\xi_1, \varrho_k(\xi_1))\} \{(\varrho_k(\xi_1))^{j-1}\}^{-1}$$

(cf. [8], [9]). The case of non-distinct roots can be easily handeled in a similar manner (cf. [4]).

Remark. One can give the above calculations as well as the corresponding ones in [10], Section 2, a more precise formulation if one uses systematically the spaces obtained by completion in the norms $\| F, R^2 \|_{-s+m}^{*}$ etc. (« homogeneous norms »). One is then lead to spaces whose elements in general are ***not*** distributions in the sense of L. Schwartz (f. [5]), but as long as one is concerned with differential operators with constant coefficients only, they are for most purposes as useful as the spaces corresponding to the norm $\| F, R^2 \|_{-s+m}$ etc., i. e. $H_{R_+^2}^{-s+m}$ etc.

Let us now consider (10) more closely. Set, as in [10], Section 2, $M = K^+ (K^-)^{-1}$. Then (10) can be written as

$$\| W^+ \| + \| W^- \| \leq C \| \bar{M} W^+ + W^- \| . \tag{11}$$

In particular if $Y_- \bar{M} W^+ = W^-$ we get

$$\| W^+ \| \leq C \| Y_+ \bar{M} W^+ \| , \quad W^+ \in H_+ . \tag{12}$$

Conversely, if (12) holds we have

$$\| W^- \| \leq \| Y_- \bar{M} W^+ + W^- \| + \| Y_- \bar{M} W^+ \| \leq$$
$$\leq \| Y_- \bar{M} W^+ + W^- \| + C \| Y_+ \bar{M} W^+ \|$$

which together with (12) implies (11) so that (11) and (12) are equivalent. But (12) is identical with formula (16) in [10], except for the fact that we have $\bar{M}$ instead of M. Therefore the results of [10], Section 3, which — we recall — depend on a result about the spectral properties of the « reduced » Hilbert transform (cfr. [7], [11], [12]), imply that (12) holds if and only if the matrix

$$c^* = (\bar{m}_+)^{-1} \bar{m}_- ,$$

where m_+ and m_- ars defined as in [10], Section 3, has no negative eigenvalues. But obviously the eigenvalues of c^* and c, where as in [10], Section 3, $c = m_+^{-1} m_-$, are conjugates of each other so that (12) holds if and only if c has no negative eigenvalues. We have thus proved:

The inequality ($\bar{\mathcal{J}}$) holds if and only if the iuequality ($\mathcal{J}$) of [10] *holds, i. e. if and only if $s > s_0$ and $s \neq \sigma_v$* mod 1 *($v = 1, 2, \dots, \sigma_q$) where $\sigma_1, \sigma_2, \dots, \sigma_q$ ($q \leq l$) are the exceptional values determined in* [10].

Having thus settled the case of constant coefficients in a half-space, it is now not difficult to treat the case of variable coefficients in a bounded domain. For the details we refer the reader to [9], [10] where the same (routine) transition in analogous cases is considered. We have the following result :

The inequality (4) *holds if and only if the inequality* (1) *holds, i. e. if and only if $s > s_0$ and $s \neq \sigma_v$* mod 1 *($v = 1, 2, \dots, q$) where $\sigma_1, \sigma_2, \dots, \sigma_q$ ($q \leq 2l$) are the exceptional values determined in* [10].

In particular we have thus finally also established (cf. introduction):

3) *R is of finite codimension.*

In the rest of the paper we shall more closely study the dependence of ϱ and ν of the parameter s.

2. **Study of the regularity.** The object of this Section is to prove the following statement:

ϱ and ν are constant in any intervall that does not contain any exceptional value.

We need the following lemmas.

LEMMA 1. *If $u \in H^{s_1}(\Omega)$ for all $s_1 < s$ and if moreover $\sup_{s_1 < s} \| u, \Omega \|_{s_1} < \infty$ then $u \in K^s(\Omega)$.*

LEMMA 2. *If $V \in K_\Omega^{-s_1}$ for all $s_1 < s$ and if moreover $\sup_{s_1 < s} ||| V, R^2 |||_{s_1} < \infty$ then $V \in K_\Omega^{-s}$.*

We first observe that the analogue of Lemma 1 when (a bounded domain) is replaced by R^2 is certainly true. For

$$\sup_{s_1 < s} \| u \|_{s_1} \leq M$$

implies that for every A and every $s_1 < s$

$$\int_{|\xi_1| + |\xi_2| \leq A} (1 + |\xi_1|^2 + |\xi_2|^2)^{s_1} |u(\xi_1, \xi_2)|^2 \, d\xi_1 \, d\xi_2 \leq M$$

from which we get, letting $s_1 \rightarrow s$,

$$\int_{|\xi_1| + |\xi_2| \leq A} (1 + |\xi_1|^2 + |\xi_2|^2)^{s} |u(\xi_1, \xi_2)|^2 \, d\xi_1 \, d\xi_2 \leq M$$

and then, letting $A \rightarrow \infty$,

$$\int (1 + |\xi_1|^2 + |\xi_2|^2)^{s} |u(\xi_1, \xi_2)|^2 \, d\xi_1 \, d\xi_2 \leq M$$

which means that $\| u \|_s \leq M$ and $u \in H^s$.

Since K_Ω^{-s} can be considered as a product of closed subspaces of H^s, Lemma 2 is an immediate consequence.

Let us now prove Lemma 1. Let s_n be a sequence tending to s from below. For every n we can find $u_n \in H^{s_n}$ such that

$$u_n \mid \Omega = u,$$

$$\| u_n \|_s \leq M.$$

Utilizing the fact that the bounded sets in H^{s_n} are weakly compact and the usual diagonalization procedure we may now pick up a subsequence v_n such that $v_n \in H^{s_n}$ and such that v_n $(n \geq m)$ converges weakly in H^{s_m}. Denote the common limit by v. Obviously

$$\| v \|_{s_n} \leq M$$

so that by what we proved above $v \in H^s$. Moreover

$$v \mid \Omega = u.$$

Hence $u \in H^s(\Omega)$ and Lemma 1 is proven.

Let us now show that ϱ and ν are continuous from below:

$$\varrho(s - 0) = \varrho(s), \quad \nu(s - 0) = \nu(s), \tag{13}$$

when s is not an exceptional value. Suppose that for some s that is not an exceptional value we have $\nu(s_1) > \nu(s)$ for all $s_1 > s$. Then there is u such that $Tu = 0$ and $u \in H^{s_1}(\Omega)$ for $s_1 < s$ but $u \notin H^s(\Omega)$. But since s is not an exceptional value we have for s_1 sufficiently close to s:

$$\| u, \Omega \|_{s_1} \leqq c \, \| u, \Omega \|_{s_1 - 1}$$

with a C independent of s_1, which in view of Lemma 1 leads to a contra diction. Suppose next that for some s that is not an exceptional value $\varrho(s_1) < \varrho(s)$ for all $s_1 < s$. Then there is a v such that $v \in K^s(\Omega)$ and $Tu = v$ has a solution $u \in H^{s_1}(\Omega)$ for every $s_1 < s$ but not for $s_1 = s$. By an extension of a well-known compactness argument one can then prove that, since s is not an exceptional value, for s_1 sufficiently close to s it is possible to choose u such that

$$\| u, \Omega \|_{s_1} \leq C' \, ||| v, \Omega |||_{s_1} \tag{14}$$

where C' is independent of s_1. In view of Lemma 1 we get again a contradiction. Thus ϱ and ν are continuous from below.

Let us now give the proof of (14). In fact, we know that

$$\| u, \Omega \|_{s_1} \leq C \, (||| Tu, \Omega |||_{s_1} + \| u, \Omega \|_{s_1 - 1})$$

with a constant C independent of s_1 if s_1 is sufficiently close to s. Let us split up $H^{s_1}(\Omega)$ into a direct sum $N^{s_1} + M^{s_1}$ where N^{s_1} is the nullspace of T, which by what we said above may be assumed to be independent of s_1, limiting ourselves to s_1 close to s, and $M^{s_1} \cap H^{s_2}(\Omega) = M^{s_2}$ if $s \geq s_2 \geq s_1$. Let us show that

$$\| u, \Omega \|_{s_1} \leq C' \||| Tu, \Omega \|||_{s_1}, \quad u \in M^{s_1} \tag{15}$$

with a constant independent of s_1, limiting ourselves to s_1 close to s. If this would not be true there is a sequence s_n tending to s from below such that

$$u_n \in M^{s_n}, \quad \| u_n, \Omega \|_{s_n} = 1, \qquad \||| Tu_n, \Omega \|||_{s_n} \leq 1/n.$$

Using Rellich's theorem we can pick up a subsequence u_n' such that u_n' is converging in $H^{s_n - 1}(\Omega)$ for each n to an element u. Now clearly u_n' is a Cauchy sequence in $H^{s_n}(\Omega)$ for each n. Hence $u \in H^{s_n}(\Omega)$. Hence by Lemma 1 $u \in H^s(\Omega)$ and $\| u, \Omega \|_s \geq 1$. Since obviously $Tu = 0$ we have a contradiction.

Applying the above arguments to $\bar{T}$ and utilizing Lemma 2 instead of Lemma 1 one readily sees that ϱ and ν are continuous from above

$$\varrho(s + 0) = \varrho(s), \quad \nu(s + 0) = \nu(s) \tag{16}$$

when s is not an exceptional value. Obviously (13) and (16) together imply that ϱ and ν are continuous in any intervall not containing any exceptional value.

REMARK. The same method works of course also in other cases than the mixed problem. E. g. it leads to a simplification of our paper [9] in the sense that one avoids the introduction of the finite difference argument in the proof of the regularity theorem.

3. **An observation concerning the index.** The *index* of the mapping T is by definition the number

$$\iota = \iota(s) = \varrho(s) - \nu(s).$$

When s is fixed, ι does not change when $A, B_j^{\pm}$ are replaced by

$$A + \sum_{|\alpha| \leq m} c_\alpha(x) D_\alpha, \quad B_j^{\pm} + \sum_{|\alpha| \leq m_j^{\pm}} d_{j\alpha}^{\pm}(x) D_\alpha,$$

the quantity

$$\sum_{|\alpha| \leq m} \sup | c_\alpha(x) | + \sum_{|\alpha| \leq m_j^{\pm}} \sup | d_{j\alpha}^{\pm}(x) |$$

being sufficiently small. This follows at once from the results of e. g. Atkinson [1] in the abstract case (cf. Gohberg and Krein [3], Kato [6] for more recent work in this field) and has in the case of boundary problems been observed by several authors. We consider here how ι changes when s changes. In view of the results of Section 2, ι is a constant in any interval that does not contain any exceptional values, so it remains only to study the behavior of ι near an exceptional value. Our guess ts that the index changes by one unit at an exceptional value of « multiplicity » one:

$$\iota(\sigma + 0) - \iota(\sigma - 0) = 1$$

if σ is an exceptional value of multiplicity one. We shall not prove this statement in all generality but just support the conjecture by proving it in an illustrative special case, namely the case when $A = \Delta$ and B^+ is differentiation $d/d\vec{s}$ along the field of tangent vectors of Γ and B^- is differentiation $d/d\vec{e}$ along an arbitrary field of vectors $\vec{e}$, which is essentially the situation studied by Fichera [2]. We assume that $\vec{e}$ forms with Γ the angels $\pi\alpha'$ and $\pi\alpha''$ at the points p' and p'' where $\overline{\Gamma^+}$ and $\overline{\Gamma^-}$ meet. As was shown in [10], example 1, p. 347, the exceptional values are in this case $s = \alpha'$ mod 1 and $s = \alpha''$ mod 1. We may also assume that $0 < \alpha' < \alpha'' < 1$. Let s_1 and s_2 be two numbers such that $s_1 < s_2$ and that there is precisely one exceptional value in between. There are (essentially) two cases:

1^0
$$k + \alpha'' < s_1 < k + 1,$$
$$k + 1 + \alpha' < s_2 < k + 1 + \alpha''$$

and

2^0
$$k + \alpha' < s_1 < k + \alpha'',$$
$$k + 1 < s_2 < k + 1 + \alpha'$$

where k is an integer. Let us consider for instance the case 1^0; the case 2^0 can be treated in an analogous way. It is possible to deform $\vec{e}$ near p' in such a manner that α' becomes 0. By what we said above $\iota(s_1)$ and $\iota(s_2)$ will remain unchanged throughout this deformation. On the other hand, if $\alpha' = 0$ then a necessary and sufficient condition for a solution $u \varepsilon H^{s_1}(\Omega)$ of $Tu = v$, where $v \varepsilon K^{s_2}(\Omega)$, to be in $H^{s_1}(\Omega)$ is that the derivatives up to a certain order of $du/d\vec{s}$ and $du/d\vec{e}$ agree at the point p', whence $\iota(s_2) - \iota(s_1) = 1$. This proves our conjecture in the special case.

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