

ANNALI DELLA SCUOLA NORMALE SUPERIORE DI PISA *Classe di Scienze*

RAGHAVAN NARASIMHAN

A note on Stein spaces and their normalisations

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3^e série, tome 16,
n° 4 (1962), p. 327-333

http://www.numdam.org/item?id=ASNSP_1962_3_16_4_327_0

© Scuola Normale Superiore, Pisa, 1962, tous droits réservés.

L'accès aux archives de la revue « *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze* » (<http://www.sns.it/it/edizioni/riviste/annaliscienze/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

A NOTE ON STEIN SPACES AND THEIR NORMALISATIONS

RAGHAVAN NARASIMHAN (Bombay) (*)

§ 1. Introduction.

It is well known that every open Riemann surface is a Stein manifold. But no proof has so far appeared of the corresponding statement for complex spaces of dimension one (with arbitrary non-normal singularities) viz. that *every (reduced) complex space of dimension one, which has no compact irreducible components, is a Stein space*. The object of the present note is to give a proof of the following theorem on complex spaces, of which the statement made above is a particular case in view of the fact that every normal complex space of dimension one is nonsingular (i. e. a disjoint union of Riemann surfaces).

THEOREM 1. *A (reduced) complex space X is a Stein space if and only if its normalisation X^* is a Stein space.*

A corollary to this statement is the following.

A complex space all of whose irreducible components are Stein spaces is itself a Stein space.

Of course, this statement becomes trivial if we replace «irreducible components» by «connected components».

§ 2. Preliminaries.

Let (X, \mathcal{H}) be a complex space in the sense of Grauert [3] and (X, \mathcal{O}) the corresponding *reduced* complex space; for $x \in X$, \mathcal{H}_x may contain nilpotent elements, while \mathcal{O}_x does not. If \mathcal{H}_x contains no nilpotent elements, then $\mathcal{H}_x = \mathcal{O}_x$.

(*) Supported in part by AF-EOAR Grant 62-35.

Let (X, \mathcal{O}) be a reduced complex space. We call X a Stein space if it is holomorph-convex [i. e., for any infinite discrete set $D \subset X$, there is a holomorphic function f for which $f(D)$ is unbounded] and if holomorphic functions separate points of X . The following theorem is well known [1].

THEOREM a. *Let (X, \mathcal{O}) be a paracompact reduced complex space. Then X is a Stein space if and only if for every coherent analytic subsheaf $\mathcal{F} \subset \mathcal{O}$, we have*

$$H^1(X, \mathcal{F}) = 0$$

If (X, \mathcal{O}) is Stein, then for any coherent analytic sheaf S , we have $H^q(X, S) = 0$, $q \geq 1$.

The following theorem can be deduced from Theorem a; see [3, § 2, Satz 3].

THEOREM b. *Let (X, \mathcal{H}) be an arbitrary complex space for which the corresponding reduced space (X, \mathcal{O}) is Stein. Let S be any coherent \mathcal{H} -sheaf. Then we have*

$$H^q(X, S) = 0 \text{ for } q \geq 1.$$

Let now X, Y be two reduced complex spaces and $\pi: X \rightarrow Y$ a proper holomorphic map with discrete fibres. Let S be a coherent analytic sheaf on X and let $\pi_\nu(S)$ be the ν^{th} direct image of S under π , i. e. for any open set $U \subset Y$, we have

$$H^0(U, \pi_\nu(S)) = H^\nu(\pi^{-1}(U), S).$$

Then we have [5, Satz 27]

THEOREM c. *$\pi_\nu(S) = 0$ for $\nu \geq 1$, $\pi_0(S)$ is a coherent analytic sheaf on Y . We require also the following theorem [4, Satz 6]*

THEOREM d. *Let X, Y be complex spaces, and $\varphi: X \rightarrow Y$ a holomorphic map. Let S be an analytic sheaf on X . Suppose that for $\nu \geq 1$, we have $\varphi_\nu(S) = 0$. Then, for $\nu \geq 0$, we have*

$$H^\nu(X, S) = H^\nu(Y, \varphi_0(S)).$$

Let now (X, \mathcal{O}) be a reduced complex space. X is called *normal* if for any $x \in X$, the local ring \mathcal{O}_x is integrally closed in its complete ring of quotients.

To every reduced complex space (X, \mathcal{O}) corresponds a « normalisation » (X^*, \mathcal{O}^*) . (X^*, \mathcal{O}^*) is a normal complex space, and there is a proper

holomorphic map $\pi: X^* \rightarrow X$ which is onto and has discrete fibres. If $\tilde{\mathcal{O}} = \pi_0(\mathcal{O}^*)$, then for $x \in X$, $\tilde{\mathcal{O}}_x$ is the integral closure of \mathcal{O}_x and if $A \subset X$ is the singular locus of X , then $\pi|(X^* - \pi^{-1}(A))$ is an analytic isomorphism onto $X - A$. $\tilde{\mathcal{O}}$ is a subsheaf of the sheaf of germs of meromorphic functions on X .

§ 3. Proof of Theorem 1.

Let (X, \mathcal{O}) be a complex space for which the normalisation (X^*, \mathcal{O}^*) is Stein. Let \mathcal{I} be a coherent sheaf of ideals, i. e. an analytic subsheaf of \mathcal{O} on X . Let $\tilde{\mathcal{O}} = \pi_0(\mathcal{O}^*)$ where $\pi: X^* \rightarrow X$ is the canonical map. For $x \in X$, let \mathcal{W}_x be the largest ideal in \mathcal{O}_x such that $\mathcal{W}_x \cdot \tilde{\mathcal{O}}_x \subset \mathcal{O}_x$ and let $\mathcal{W} = \bigcup_{x \in X} \mathcal{W}_x$.

Then \mathcal{W} is an analytic sheaf on X ; moreover, it is a *coherent* analytic sheaf on X ; see [6 § 2 Prop. 9 and remark which follows Prop. 9].

Let \mathcal{F}^* be the analytic inverse image on X^* of the coherent analytic sheaf $\mathcal{W} \cdot \mathcal{I}$ (i. e. \mathcal{F}^* is the tensor product of the topological inverse image of $\mathcal{W} \cdot \mathcal{I}$ and \mathcal{O}^* over the topological inverse image of \mathcal{O}). Then \mathcal{F}^* is a coherent \mathcal{O}^* -sheaf [4, § 2, (g)].

Let $\mathcal{F} = \pi_0(\mathcal{F}^*)$. By Theorem c, \mathcal{F} is a coherent \mathcal{O} -sheaf. Moreover, since $\mathcal{W} \cdot \tilde{\mathcal{O}} = \mathcal{W} \cdot \pi_0(\mathcal{O}^*) \subset \mathcal{O}$, it follows that \mathcal{F} is a subsheaf of \mathcal{O} and in fact of \mathcal{I} . Finally we remark that by Theorem c, $\pi_\nu(\mathcal{F}^*) = 0$ for $\nu \geq 1$, so that, by Theorem d, we have

$$H^q(X^*, \mathcal{F}^*) = H^q(X, \mathcal{F}).$$

By Theorem a, we have $H^q(X^*, \mathcal{F}^*) = 0$ for $q \geq 1$, so that we conclude that $H^q(X, \mathcal{F}) = 0$ for $q \geq 1$.

We shall first prove Theorem 1 for spaces of finite dimension. Let n be the complex dimension of X , and suppose inductively that Theorem 1, has been proved for all spaces of dimension $\leq n - 1$. We then assert that any closed nowhere dense analytic set Y of X is a Stein space. This follows from the following lemma, and the inductive hypothesis.

LEMMA 1. *Let (X, \mathcal{O}) be a reduced complex space for which the normalisation (X^*, \mathcal{O}^*) is Stein. Then, for any closed analytic set $Y \subset X$, with the induced reduced structure from X , the normalisation Y^* is Stein.*

The proof will be given later.

We go back to the proof of Theorem 1 in the special case.

Let $\mathcal{G}, \mathcal{W}, \mathcal{F}$ be as above and consider the exact sequence

$$(*) \quad 0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{G}/\mathcal{F} \rightarrow 0$$

Now, since $\pi|_{X^* - \pi^{-1}(A)}$ is an analytic isomorphism and, for $x \notin A$, $\tilde{\mathcal{O}}_x = \mathcal{O}_x$, we see that $\mathcal{W}_x = \mathcal{O}_x$ for $x \notin A$ and $\mathcal{F}_x = \mathcal{G}_x$ for $x \notin A$. Hence the set Y of points $x \in X$ with $\mathcal{W}_x \neq \mathcal{O}_x$ (which contains the set of points where $\mathcal{G}_x \neq \mathcal{F}_x$) is a nowhere dense analytic set in X , and so, with its reduced structure, is a Stein space. Moreover, if S is the restriction of \mathcal{G}/\mathcal{F} to Y , then S is a coherent \mathcal{H} -sheaf, where \mathcal{H} is the restriction of \mathcal{O}/\mathcal{W} to Y [6, § 2, Théorème 3]. Now, by our remark above (inductive assumption and Lemma 1), Y is a Stein space. Hence, by Theorem d, $H^q(Y, S) = 0$ for $q \geq 1$. But since $H^q(Y, S) \approx H^q(X, \mathcal{G}/\mathcal{F})$, we conclude that $H^q(X, \mathcal{G}/\mathcal{F}) = 0$ for $q \geq 1$. Hence, since, $H^q(X, \mathcal{F}) = 0$ for $q \geq 1$, we deduce from the exact cohomology sequence associated to (*), that $H^q(X, \mathcal{G}) = 0$ for $q \geq 1$; because of Theorem a, this concludes modulo Lemma 1 the proof of Theorem 1 in the special case when X has finite dimension.

For the proof of Lemma 1, we require the following result.

LEMMA 2. *Let X, Y be normal complex spaces (reduced) and $\pi: X \rightarrow Y$ a proper holomorphic map with discrete fibres onto Y . Then, X is Stein if and only if Y is Stein.*

PROOF. The fact that if Y is Stein, then so is X follows at once from [2, Satz B]. Conversely, suppose X Stein. We may suppose X and Y connected. Then, there is a nowhere dense analytic set $M \subset Y$ such that $\pi|_{X - \pi^{-1}(M)}$ is an *unramified* covering of $Y - M$ (say with p sheets); we may suppose also that M contains the singular locus of Y . Then, if f is holomorphic on X , and, for $y \in Y - M$, $a_\nu(y)$ is the ν^{th} elementary symmetric function of the values of f at the points of $\pi^{-1}(y)$, then the $a_\nu(y)$ remain bounded as $y \rightarrow y_0 \in M$ and since Y is normal, can be extended to holomorphic functions a_ν on Y . Moreover, we have $f^p(x) + \sum_{\nu \geq 1} f^{p-\nu}(x) a_\nu(\pi(x)) = 0$.

It is now obvious that if $|f|$ is unbounded on a set $D \subset X$, then at least one a_ν is unbounded on $\pi(D)$. Since X is holomorphconvex, so is Y . Now Y can contain no compact analytic set T of positive dimension since $\pi^{-1}(T)$ would then be a compact analytic set of positive dimension in X , and this cannot exist since holomorphic functions on X separate points. If we use the fact that a holomorph-convex reduced complex space which contains no compact analytic sets of positive dimension is Stein (an easy consequence of [2, Satz B]), we see that Y is Stein.

PROOF OF LEMMA 1. Let $\pi: X^* \rightarrow X$ be the natural map, and $Y^1 = \pi^{-1}(Y)$. Since Y^1 is a closed subspace of the Stein space X^* , Y^1 is Stein. Hence, by [2, Satz B], its normalization \tilde{Y} is Stein. Clearly, we have a proper holomorphic map $\varphi: \tilde{Y} \rightarrow Y$ which has discrete fibres. Let Y^* be the normalisation of Y and $\pi^1: Y^* \rightarrow Y$ the natural map. Since \tilde{Y} is normal, there exists a holomorphic map $\varphi^1: \tilde{Y} \rightarrow Y^*$ such that $\pi^1 \circ \varphi^1 = \varphi$. Since, clearly φ^1 must be proper, surjective and have discrete fibres, and since \tilde{Y} is Stein, we see, by Lemma 2, that Y^* is Stein, which is Lemma 1.

To prove Theorem 1 in the general case, we proceed as follows. Let X_k , $k = 1, 2, \dots$ be the union of the irreducible components of dimension $\leq k$ of X . The normalisation of X_k is a union of connected components of X and so is Stein. By the special case of Theorem 1 which is already proved, each X_k is Stein.

Let now D be any discrete subset of X and let $D_k = D \cap X_k$, $E_1 = D_1$ and $E_{k+1} = D_{k+1} - D_k$. Let h be a holomorphic function on D (i. e. assignment of a complex number to each point of D) and, for $k \geq 1$, h_k the restriction of h to E_k . Since X_1 is Stein, there is a holomorphic function f_1 on X_1 , so that $f_1|_{E_1} = h_1$. Clearly $E_2 \cup X_1$ is a closed subspace of X_2 , so that there is, since X_2 is Stein, a holomorphic function f_2 on X_2 such that $f_2|_{X_1} = f_1$, $f_2|_{E_2} = h_2$. Proceeding thus, we construct f_{k+1} holomorphic on X_{k+1} so that $f_{k+1}|_{X_k} = f_k$, $f_{k+1}|_{E_{k+1}} = h_{k+1}$. If $f = \lim f_k$, then f is holomorphic on X and clearly $f|_D = h$. Hence X is itself Stein, and this proves Theorem 1 in the general case.

Using Theorem 1 and Lemma 2, it is possible to prove Lemma 2 without the assumption of normality. We formulate this as a separate Theorem.

THEOREM 2. *Let X, Y be reduced complex spaces, $\pi: X \rightarrow Y$ a proper holomorphic map onto Y . Then, if X is Stein, so is Y .*

PROOF. Since X is Stein, X contains no compact analytic sets of positive dimension. Hence every fibre of π , being a compact analytic set, is a finite set.

Let X^*, Y^* be the normalisations of X, Y respectively and $\pi_X: X^* \rightarrow X$, $\pi_Y: Y^* \rightarrow Y$ the corresponding projections. Let $\varphi = \pi \circ \pi_X: X^* \rightarrow Y$. Then φ is a surjective proper holomorphic map of X^* onto Y with discrete fibres. Since X^* is normal, there is a holomorphic map $\varphi^1: X^* \rightarrow Y^*$ which is surjective, so that $\pi_Y \circ \varphi^1 = \varphi$. Since X is Stein, so is X^* ; by Lemma 2, so is Y^* . By Theorem 1, we deduce that Y itself is Stein.

Finally we give a sketch of a direct proof for spaces with isolated singularities in particular, for spaces of one dimension. This proof has the

«merit» of not depending on the heavy machinery of direct and inverse images of analytic sheaves.

Let X be a reduced complex space with isolated singularities, A the set of singular points of X and X^* the normalisation of X . We suppose that X^* is Stein. Let $\{X_k^*\}$ be a sequence of relatively compact open sets in X^* with the following properties.

a) X_k^* is Stein, $X_k^* \subset\subset X_{k+1}^*$ and $\partial X_k^* \cap \pi^{-1}(A) = \emptyset$ [here $\pi: X^* \rightarrow X$ is the natural map].

b) X_k^* is X^* -convex, i. e. if K is a compact subset of X_k^* then $\widehat{K} = \{x \in X_k^* \mid |f(x)| \leq \sup |f(K)| \text{ for all } f \text{ holomorphic in } X^* \text{ is compact.}$

Let $X_k = \pi(X_k^*)$. We assert that (i) X_k is Stein and that (ii) X_k is X_{k+1} -convex. It then follows that X is Stein.

PROOF OF (i). Since $X_k^* \subset\subset X^*$, for any f holomorphic in X_k^* which vanishes on $X_k^* \cap \pi^{-1}(A)$, there exists an integer $\lambda > 0$ so that $f^\lambda = g \circ \pi$ for some g holomorphic on X_k . Clearly we may find, for any $x_0 \in \partial X_k$, an f holomorphic on X_k^* , vanishing on $X_k^* \cap \pi^{-1}(A)$, such that $|f(y)| \rightarrow \infty$ as $y \rightarrow y_0$ if $y_0 \in \pi^{-1}(x_0) \cap \partial X_k^*$.

If λ is such that $f^\lambda = g \circ \pi$, then clearly $|g(x)| \rightarrow \infty$ as $x \rightarrow x_0$. Hence X_k is holomorph-convex. As in the proof of Lemma 2, X_k has no compact analytic sets of positive dimension and so is Stein.

PROOF OF (ii). If K is a compact set of X_k and $x_0 \in \partial X_k$, then, there exists f holomorphic on X_{k+1}^* , vanishing on $\pi^{-1}(A) \cap X_{k+1}^*$ so that, if $y_0 = \pi^{-1}(x_0)$, then $|f(y_0)| > \sup_{y \in K^*} f(y)$ where $K^* = \pi^{-1}(K) \cap X_k^*$ (note that for the existence of f , we need the fact that $\partial X_k^* \cap \pi^{-1}(A) = \emptyset$).

Choose $\lambda > 0$ so that $f^\lambda = g \circ \pi$ where g is holomorphic on X_{k+1} . Then $|g(x_0)| > \sup_{x \in K} |g(x)|$. Hence X_k is X_{k+1} -convex.

REFERENCES

1. H. CARTAN. *Variétés analytiques complexes et cohomologie*, Colloque sur les fonctions de plusieurs variables complexes de Bruxelles, (1953), 41-55.
2. H. GRAUERT. *Charakterisierung der holomorph-vollständigen komplexen Räume*, Math. Ann. 129 (1955), 233-259.
3. H. GRAUERT. *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen*, Publications mathématiques de l'Inst. des Hautes Etudes Scientifiques, Paris, N° 5, 1960.
4. H. GRAUERT und R. REMMERT. *Bilder und Urbilder analytische Garber*, Annals of Math. 68 (1958), 393-443.
5. H. GRAUERT und R. REMMERT. *Komplexe Räume*, Math. Ann. 136 (1958), 245-318.
6. J.-P. SERRE. *Faisceaux algébriques cohérents*, Annals of Math. 61 (1955), 197-278.

*Tata Institute of Fundamental Research,
Bombay, India.*