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ON THE APPROXIMATION OF
INTEGRABLE FUNCTIONS BY FUNCTIONS
OF BOUNDED VARIATION

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1. Introduction.

Let $f(x)$ be a real-valued continuous function of the real variable x defined on $(-\infty, \infty)$, and such that outside a non-degenerate compact interval $[a, b]$: $a \leq x \leq b$ the function $f(x)$ satisfies $f(x) \equiv f(a)$ for $x \leq a$ and $f(x) \equiv f(b)$ for $x \geq b$. Recently Baiada and Cardamone [1] have published the following results:

(i) $f(x)$ has finite total variation V on $[a, b]$ if and only if

$$(1.1) \quad \int_a^b \left| \frac{f(x+h) - f(x)}{h} \right| dx$$

has a finite limit as $h \rightarrow 0$, and in this case the limit of (1.1) is V .

(ii) the curve $C: y = f(x)$ is rectifiable if and only if

$$(1.2) \quad \int_a^b \left(1 + \left[\frac{f(x+h) - f(x)}{h} \right]^2 \right)^{1/2} dx$$

has a finite limit as $h \rightarrow 0$, and in this case the limit of (1.2) is the length of C .

The authors also conjectured that if $f(x)$ is not of bounded variation on $[a, b]$, then (1.1) and (1.2) tend to ∞ as $h \rightarrow 0$.

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The proofs of [1] are long, although quite elementary in nature; moreover, the paper is devoid of references to the extensive literature on curve theory subsequent to the publication of Tonelli's *Fondamenti di Calcolo delle Variazioni*. Actually, conclusion (i) and the corresponding conjecture are contained in the results of Sections III.2.15 and III.2.38-III.2.42 of Radó [7], as independent of the finiteness of the total variation V these results imply that the integral (1.1) tends to V as $h \rightarrow 0^+$, and in turn this result for $f_*(x) = f(-x)$ implies that (1.1) also tends to V as $h \rightarrow 0^-$; correspondingly, an analogous argument yields conclusion (ii) and the associated conjecture.

Now a number of years ago Hardy and Littlewood, (see [4; Theorem 24] and [5; Lemma 9]; also [8; Problem 10, p. 373]), proved that if $f(x)$ is a real-valued function which is (Lebesgue) integrable on $[a, b]$, and $f(x) \equiv 0$ for $x > b$, then a necessary and sufficient condition for there to exist a function of b.v. (bounded variation) on $[a, b]$ which is equal to $f(x)$ a.e. (almost everywhere) on this interval is that the integral (1.1) is $O(1)$ as $h \rightarrow 0^+$.

It is the purpose of this note to present for integrable vector functions a result which embodies the generality of the cited result of Hardy and Littlewood, and also the convergence phenomena of the above conclusions (i), (ii). In particular, the presented theorem contains as simple corollaries the above conclusions (i), (ii), and the corresponding conjectures of Baiada and Cardamone

If $y(x) \equiv (y_\alpha(x))$, $(\alpha = 1, \dots, n)$, is a real-valued vector function of the real variable x on a compact interval $[a, b]$: $a \leq x \leq b$ the symbol $L[a, b; y]$ will denote the supremum of the sums $\lambda[P[a, b]; y] \equiv \sum_{j=1}^m |y(x_j) - y(x_{j-1})|$ on the class of all partitions $P[a, b]: a = x_0 \leq x_1 \leq \dots \leq x_m = b$, $(m = 1, 2, \dots)$, of $[a, b]$, where in general $|\eta|$ signifies the Euclidean norm $(\eta_1^2 + \dots + \eta_n^2)^{1/2}$ of a real vector $\eta \equiv (\eta_\alpha)$, $(\alpha = 1, \dots, n)$. Moreover, for such a $y(x)$ the symbol $\mathcal{E}[y]$ will denote the set of vector functions $z(x)$ equal to $y(x)$ a.e. on $[a, b]$, and $\mathcal{E}_{bv}[y]$ will denote the class of $z \in \mathcal{E}[y]$ which are of b.v. on $[a, b]$. Now if the vector function $y(x)$ is integrable on a non-degenerate compact interval $[a, d]$ and $a < b < d$, then for $0 < h < d - b$ the forward difference operator $\Delta_h y(x) \equiv y(x+h) - y(x)$, and the right-hand integral mean operator $M_h(x; y) \equiv h^{-1} \int_0^h y(x+s) ds$, are well defined for x on $[a, b]$; moreover, the functional

$$(1.3) \quad L_h[a, b; y] \equiv \int_a^b |h^{-1} \Delta_h y(x)| dx$$

is finite for $0 < h < d - b$. Specifically, we shall prove the following theorem.

THEOREM A. *Suppose that $y(x) \equiv (y_\alpha(x))$, $(\alpha = 1, \dots, n)$, is integrable on $[a, b]$, and is of b.v. on a non-degenerate extension interval $[b, d]$. If L is the infimum of $\Lambda[a, b; z] \equiv L[a, b; z] + |z(b) - y(b^+)|$ for $z \in E[y]$ then:*

(i) $L < \infty$ if and only if $E_{bv}[y]$ is non-empty;

(ii) $L_h[a, b; y] \rightarrow L$ as $h \rightarrow 0^+$;

(iii) if $L < \infty$, then $z_+(x) \equiv \lim_{h \rightarrow 0^+} M_h(x; y)$ exists on $[a, b]$, and: (α) $z_+ \in E_{bv}[y]$; (β) z_+ is right-hand continuous on $[a, b)$ and $z_+(b) = y(b^+)$; (γ) $\Lambda[a, b; z_+] = L[a, b; z_+] = L$; (δ) if $z \in E_{bv}[y]$ and $\Lambda[a, b; z] = L$, then $z(a) = z_+(a)$, and for $a < x \leq b$ there is a $\theta(x)$ such that $0 \leq \theta(x) \leq 1$ and $z(x) = [1 - \theta(x)]z_+(x^-) + \theta(x)z_+(x)$.

For $n = 1$, $y_1(x) \equiv f(x)$, conclusions (i), (ii) imply the Hardy-Littlewood result. Now if $y(x)$ is continuous on $[a, d]$, and of b.v. on $[b, d]$, then $L = \infty$ clearly implies $L[a, b; y] = \infty$, whereas if $L < \infty$ it follows from conclusion (iii) that $z_+(x) \equiv y(x)$ and $L[a, b; y] = L$. Consequently, we have the following corollary.

COROLLARY. *If the vector function $y(x)$ is continuous on $[a, d]$ and of b.v. on $[b, d]$, then $L_h[a, b; y] \rightarrow L[a, b; y]$ as $h \rightarrow 0^+$.*

If $f(x)$ satisfies the conditions of the first paragraph of the Introduction, then for $n = 1$, $y_1(x) \equiv f(x)$ the above Corollary implies conclusion (i) and the corresponding conjecture of [1] for $h \rightarrow 0^+$, and in turn this result for $f_*(x) = f(-x)$ yields corresponding results for $h \rightarrow 0^-$; similarly, for $n = 2$, $y_1(x) \equiv f(x)$, $y_2(x) \equiv x$, the above Corollary implies conclusion (ii) and the corresponding conjecture of [1]:

In § 2 there are listed some well-known properties of the functional $L[a, b; y]$ and the integral mean operator $M_h(x; y)$, which are used in § 3 to establish Theorem A. Finally, in § 4 it is noted that if $y(x) \equiv (y_\alpha(x))$, $(\alpha = 1, \dots, n)$, is a continuous vector function of b.v. on $[a, b]$ then the above Corollary yields a ready proof of the fact that the length of the curve in (x, y_1, \dots, y_n) -space defined by $y_\alpha = y_\alpha(x)$, $(\alpha = 1, \dots, n; a \leq x \leq b)$, is given by

$$\int_a^b [1 + |y'(x)|^2]^{1/2} dx + L[a, b; y] - \int_a^b |y'(x)| dx.$$

For $n = 1$ this result has been presented by Hartman and Kershner [6; § 10] as a corollary of their general results on variational orthogonality.

2. Prefatory results.

For $y(x)$ a real-valued vector function of the real variable x on $[a, b]$ the following properties of $L[a, b; y]$ are well-known. For literature the reader is referred to Cesari [2; Ch. I, § 2], Graves [3; Ch. X, §§ 6, 7, 8], and Radó [7; Chs. III. 2, III. 3]; it is to be commented that although Graves and Radó are concerned specifically with continuous vector functions for the cited properties the modifications of proofs are immediate.

(a) $L[a, b; y] < \infty$ if and only if $y(x)$ is of b.v. on $[a, b]$.

(b) If $a \leq c \leq b$, then $L[a, b; y] = L[a, c; y] + L[c, b; y]$, with obvious interpretation in the case of infinite terms.

(c) If $y(x)$ is of b.v. on $[a, b]$, and $a \leq x_1 < x_2 \leq b$, then $L[x_1, x_2; y] = |y(x_1) - y(x_1^+)| + L[x_1^+, x_2; y]$, and $L[x_1, x_2; y] = |y(x_2) - y(x_2^-)| + L[x_1, x_2^-, y]$, where $L[x_1^+, x_2; y] = \lim_{u \rightarrow x_1^+} L[u, x_2; y]$, $L[x_1, x_2^-, y] = \lim_{u \rightarrow x_2^-} L[x_1, u; y]$. In particular, if $a \leq x_0 < b$, $\{a < x_0 \leq b\}$, then $L[a, x; y]$ is right-hand, {left-hand}, continuous at $x = x_0$ if and only if $y(x)$ is right-hand, {left-hand}, continuous at $x = x_0$.

(d) If $L[a, b; y] < \infty$ then the derivative vector $y'(x)$ exists a.e. on $[a, b]$,

$y'(x)$ is (Lebesgue) integrable on this interval, and $L[a, b; y] \geq \int_a^b |y'(x)| dx$, with the sign of equality holding only if $y(x)$ is a.c. (absolutely continuous) on $[a, b]$.

(e) If $y^{(k)}(x)$, ($k = 1, 2, \dots$), are real-valued vector functions defined on $[a, b]$, and there is a vector function $y(x)$ such that $y^{(k)}(x) \rightarrow y(x)$ on $[a, b]$ as $k \rightarrow \infty$, then $L[a, b; y] \leq \liminf_{k \rightarrow \infty} L[a, b; y^{(k)}]$.

Moreover, if the vector function $y(x)$ is integrable on $[a, d]$, and $a < b < d$, then the integral mean function

$$(2.1) \quad \eta_h(x) = M_h(x; y), \quad 0 < h < d - b; \quad a \leq x \leq d - h,$$

has the following well-known properties, (see, for example, Graves [3; Ch. XI, Sec. 13]):

1⁰. $\eta_h(x)$ is a.c. on $[a, b]$, and $\eta_h'(x) = h^{-1} \Delta_h y(x)$ a.e. on this interval.

2⁰. $\eta_h(x) \rightarrow y(x)$ a.e. on $[a, b]$ as $h \rightarrow 0$; in particular, this relation holds for each x at which $y(x)$ is right-hand continuous.

In particular, the above properties (d), 1⁰, 2⁰ imply the following result.

3⁰. If $z(x) = y(x)$ a.e. on $[a, b]$, then for $0 < h < d - b$ we have $M_h(x; y) \equiv M_h(x; z)$, and $L[a, b; \eta_h] = L_h[a, b; y] = L_h[a, b; z]$.

3. Proof of Theorem A.

Conclusion (i) of Theorem A is obvious in view of the definition of L and Property (a) of § 2. Conclusion (ii) is equivalent to the two relations

$$(3.1) \quad \limsup_{h \rightarrow 0^+} L_h[a, b; y] \leq L,$$

$$(3.2) \quad \liminf_{h \rightarrow 0^+} L_h[a, b; y] \geq L.$$

Relation (3.1) is true trivially if $L = \infty$. Now if $L < \infty$, and $z \in E_{bv}[y]$, let the domain of definition of $z(x)$ be extended to $[a, d]$ with $z(x) \equiv y(x)$ on $(b, d]$. In view of Properties (b), (c) of § 2 it follows that $L[a, b+h; z] \rightarrow L[a, b; z]$ as $h \rightarrow 0^+$, and therefore (3.1) is a ready consequence of the fact that for arbitrary $z \in E_{bv}[y]$ we have

$$(3.3) \quad L_h[a, b; y] \leq L[a, b+h; z], \quad \text{for } 0 < h < d - b.$$

In turn, (3.3) may be established by the vector analogue of the argument in Section III.2.40 of Radó [7]. Specifically, if $z \in E_{bv}[y]$, the domain of $z(x)$ is extended to $[a, d]$ as indicated above, and $\eta_h(x) = M_h(x; y) = M_h(x; z)$ on $[a, d-h]$, then for $0 < h < d - b$ and a partition $P[a, b]$ with division points $a = x_0 \leq x_1 \leq \dots \leq x_m = b$ we have

$$(3.4) \quad \begin{aligned} \lambda[P[a, b]; \eta_h] &= \sum_{j=1}^m \left| h^{-1} \int_0^h [z(x_j + s) - z(x_{j-1} + s)] ds \right|, \\ &\leq h^{-1} \int_0^h [\sum_{j=1}^m |z(x_j + s) - z(x_{j-1} + s)|] ds. \end{aligned}$$

For each s on $0 \leq s \leq h$ the values $x_j + s$, ($j = 0, 1, \dots, m$), are the division points of a partition of the interval $[a + s, b + s]$, and hence

$$(3.5) \quad \sum_{j=1}^m |z(x_j + s) - z(x_{j-1} + s)| \leq L[a + s, b + s; z].$$

Moreover, since $[a + s, b + s] \subset [a, b + h]$ for $0 \leq s \leq h < d - b$, it follows that $\lambda[P[a, b]; \eta_h] \leq L[a, b + h; z]$ for arbitrary partitions $P[a, b]$ and hence

$$(3.6) \quad L[a, b; \eta_h] \leq L[a, b + h; z],$$

so that (3.3) is a consequence of Property 3⁰ of § 2.

In turn, inequality (3.2) is true trivially if the left-hand member is infinite. If $\liminf_{h \rightarrow 0^+} L_h[a, b; y] = L_1 < \infty$, then there is a sequence $\{h_j\}$ ($0 < h_j < d - b$; $j = 1, 2, \dots$), such that $h_j \rightarrow 0^+$ and $L_{h_j}[a, b; y] \rightarrow L_1$, and by 3^o of § 2 we have that $L[a, b; \eta_{h_j}] \rightarrow L_1$. Consequently there is a finite constant M such that $L[a, b; \eta_{h_j}] \leq M$, ($j = 1, 2, \dots$); moreover, $\{\eta_{h_j}(x_0)\}$ is a bounded sequence for each x_0 such that $\eta_h(x_0) \rightarrow y(x_0)$ as $h \rightarrow 0^+$. In view of the Helly selection theorem, (see, for example, Widder [9; p. 29]), it follows that the sequence $\{h_j\}$ may be chosen such that also the sequence $\{\eta_{h_j}(x)\}$ is convergent throughout $[a, b]$. We shall suppose the sequence $\{h_j\}$ so chosen, and set $z_0(x) = \lim_{j \rightarrow \infty} \eta_{h_j}(x)$ on $[a, b]$. In view of (e) and 2^o of § 2 it follows that $z_0 \in E_{bv}[y]$, $z_0(b) = y(b^+)$, and $\Delta[a, b; z_0] = L[a, b; z_0] \leq \lim_{j \rightarrow \infty} L[a, b; \eta_{h_j}] = L_1$, and hence $L = \inf_{z \in E[y]} \Delta[a, b; z] \leq \Delta[a, b; z_0] \leq L_1$.

If $L < \infty$, and $z(x)$ is any vector function belonging to $E_{bv}[y]$, let $z_+(x) = z(x^+)$ for $a \leq x < b$, and $z_+(b) = y(b^+)$. As $z(x) = y(x)$ a.e. on $[a, b]$ for each $z \in E_{bv}[y]$, the vector function $z_+(x)$ is independent of the particular $z \in E_{bv}[y]$ employed in its definition. It is immediate that the thus defined $z_+(x)$ satisfies (α) and (β) in conclusion (iii) of Theorem A. Moreover, if the domain of z_+ is extended to $[a, d]$ with $z_+(x) \equiv y(x)$ on $(b, d]$ then $z_+(x)$ is right-hand continuous at each x on $[a, b]$, and from 2^o of § 2 it follows that $M_h(x; z_+) \rightarrow z_+(x)$ on $[a, b]$ as $h \rightarrow 0^+$; furthermore, by 3^o of § 2 we have $M_h(x; z_+) \equiv M_h(x; y)$ for $a \leq x \leq d - h$, $0 < h < d - b$, and hence $z_+(x)$ has the definitive property of being $\lim_{h \rightarrow 0^+} M_h(x; y)$ on $[a, b]$. In view of (e) of § 2 and conclusion (ii) we have $L[a, b; z_+] \leq \lim_{h \rightarrow 0^+} L[a, b; \eta_h] = \lim_{h \rightarrow 0^+} L_h[a, b; y] = L$. Since $z_+(b) = y(b^+)$ implies $L[a, b; z_+] = \Delta[a, b; z_+]$, and $L \leq \Delta[a, b; z_+]$ in view of the definition of L , we have (γ) of conclusion (iii). Now suppose that $z \in E_{bv}[y]$ and $\Delta[a, b; z] = L$. In particular, $z = z_+$ a.e. on $[a, b]$ since z and z_+ are individually equal to $y(x)$ a.e. on this interval, and consequently $z(x^+) = z_+(x^+) = z_+(x)$ on $[a, b)$, and $z(x^-) = z_+(x^-)$ on $(a, b]$. If $a < u < b$ and $z(u) = z_+(u)$, in view of (b) of § 2 and the extremum condition (γ) of conclusion (iii) it follows that

$$(3.7) \quad L[a, u; z] = L[a, u; z_+],$$

$$(3.8) \quad L[u, b; z] + |z(b) - y(b^+)| = L[u, b; z_+] + |z_+(b) - y(b^+)| = L[u, b; z_+].$$

If $u \rightarrow a^+$ through values such that $z(u) = z_+(u)$, it follows from (3.7) and (c) of § 2 that $|z(a^+) - z(a)| = |z_+(a^+) - z_+(a)| = 0$, and hence $z(a) = z(a^+) = z_+(a^+) = z_+(a)$. Correspondingly, if $u \rightarrow b^-$ through values such that $z(u) = z_+(u)$ it follows that $|z(b) - z(b^-)| + |z(b) - y(b^+)| = |z_+(b) - z_+(b^-)|$, that is $|z(b) - z_+(b^-)| + |z(b) - z_+(b)| = |z_+(b) - z_+(b^-)|$, and consequently there exists a $\theta(b)$ such that $0 \leq \theta(b) \leq 1$ and $z(b) = [1 - \theta(b)]z_+(b^-) + \theta(b)z_+(b)$.

If $a < x < b$, and $u \rightarrow x^+$ through values such that $z(u) = z_+(u)$, it follows from (3.7) and (c) of § 2 that $L[a, x; z] + |z(x) - z(x^+)| = L[a, x; z_+] + |z_+(x) - z_+(x^+)| = L[a, x; z_+]$, and an argument similar to that employed in the consideration of $x = b$ leads to the existence of a $\theta(x)$ such that $0 \leq \theta(x) \leq 1$ and $z(x) = [1 - \theta(x)]z_+(x^-) + \theta(x)z_+(x)$.

4. Remarks.

Now consider an n -dimensional vector function $y(x) \equiv (y_\alpha(x))$, ($\alpha = 1, \dots, n$), which is measurable on $[a, b]$, and set $y(x) \equiv y(b)$ for $x > b$. If $y(x)$ has a derivative a.e. on $[a, b]$, then for $0 < h < b - a$ the scalar function $g(x, h) = ([1 + |h^{-1} \Delta_h y(x)|^2]^{1/2} + |h^{-1} \Delta_h y(x)|)^{-1} = [1 + |h^{-1} \Delta_h y(x)|^2]^{1/2} - |h^{-1} \Delta_h y(x)|$ is measurable in x on $[a, b]$, and $0 \leq g(x, h) \leq 1$. Moreover, a.e. on $[a, b]$ we have $g(x, h) \rightarrow g(x) \equiv ([1 + |y'(x)|^2]^{1/2} + |y'(x)|)^{-1} = [1 + |y'(x)|^2]^{1/2} - |y'(x)|$ as $h \rightarrow 0^+$, so that by the Lebesgue bounded convergence theorem

$\int_a^b g(x, h) dx \rightarrow \int_a^b g(x) dx$ as $h \rightarrow 0^+$. Now if $Y(x) \equiv (Y_\beta(x))$, ($\beta = 1, \dots, n + 1$),

is the $(n + 1)$ -dimensional vector function with $Y_\alpha(x) \equiv y_\alpha(x)$, ($\alpha = 1, \dots, n$),

$Y_{n+1}(x) \equiv x$, then $L_h[a, b; Y] = \int_a^b [1 + |h^{-1} \Delta_h y(x)|^2]^{1/2} dx$ and $L_h[a, b; y] =$

$= \int_a^b |h^{-1} \Delta_h y(x)| dx$. In particular, if $y(x)$ is continuous and of bounded

variation on $[a, b]$, then the Corollary to Theorem A implies that $L_h[a, b; Y] \rightarrow L[a, b; Y]$ and $L_h[a, b; y] \rightarrow L[a, b; y]$ as $h \rightarrow 0^+$, while $[1 + |y'(x)|^2]^{1/2}$ and $|y'(x)|$ are individually integrable on $[a, b]$. Consequently we have the following result, which for the case of $n = 1$ was presented by Hartman and Kershner [6; § 10] as a corollary of their general results on variational orthogonality.

THEOREM 4.1. *If $y(x) \equiv (y_\alpha(x))$, ($\alpha = 1, \dots, n$), is continuous and of b.v. on $[a, b]$, and $Y(x) \equiv (Y_\beta(x))$, ($\beta = 1, \dots, n + 1$), with $Y_\alpha(x) \equiv y_\alpha(x)$, ($\alpha = 1, \dots, n$), $Y_{n+1}(x) \equiv x$, then*

$$L[a, b; Y] = \int_a^b [1 + |y'(x)|^2]^{1/2} dx + L[a, b; y] - \int_a^b |y'(x)| dx.$$

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