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<http://www.numdam.org/item?id=ASNSP_1960_3_14_1_1_0>
MULTIPLE INTEGRAL PROBLEMS IN THE CALCULUS OF VARIATIONS AND RELATED TOPICS

by

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Introduction.

In this series of lectures, I shall present a greatly simplified account of some of the research concerning multiple integral problems in the calculus of variations which has been reported in detail in the papers [39], [40], [41], [42], [44], [46], and [47]. I shall speak only of problems in non-parametric form and shall therefore not describe the excellent result concerning double integrals in parametric form obtained almost concurrently by Sigalov, Danskin, and Cesari [62], [9], [5]) nor the work of L. C. Young and others on generalized surfaces. Some of my results have been extended in various ways by Cinquini [6], De Giorgi [10], Fichera [17], Nöbêling [51], Sigalov [58], [59], [60], [61], Silova [63], and Stampacchia [67], [68], [69], [70]. However, it is hoped that the results presented here will serve as an introduction to the subject.

The first part of this research reported in these lectures is an extension of Tonelli's work on single and double integral problems in which he employed the so-called direct methods of the calculus of variations ([71] through [78]). His work was stimulated, no doubt, by the succes of Hilbert, Lebesgue [31] and others in the rigorous establishment of Dirichlet's principle in certain important cases. The principle idea of these direct methods is to establish the existence of a function $z$ minimizing an integral by showing (i) that the integral $I(z)$ is lower semicontinuous with respect to some

(*) Presented at the international conference organized by C.I.M.E in Pisa, september 1-10-1958.

1. Annali della Scuola Norm. Sup. 7; Pisa.
kind of convergence, (ii) that $I(z) \geq d$ for the $z$ considered and (iii) that there is a "minimizing sequence" $z_n$ such that $I(z_n) \rightarrow d$ and $z_n \rightarrow z_0$ in the sense required.

In the case of single integral problems, where

$$I(z) = \int_a^b f[x, z(x), z'(x)] \, dx$$

Tonelli (see, for instance [76]) was able to carry through this program for the case that only absolutely continuous functions are admitted, the convergence is uniform, and (essentially) $f(x, z, p)$ is convex in $p$ (if $f(x, z, p) \geq f_0(p)$ where $f_0(p)/\|p\| \rightarrow \infty$, it is seen from the proof of Theorem 2.4 below, that the functions in any minimizing sequence would be uniformly absolutely continuous so that a subsequence would converge uniformly to an absolutely continuous function $z_0$ which would thus minimize $I(z)$. Tonelli was also able to carry through the entire program for certain double integral problems using functions absolutely continuous in his sense (ACT) and uniform convergence [77], [78]. However, in general he had to assume that the integrand $f(x, y, z, p, q)$ satisfied a condition like

$$f(x, y, z, p, q) \geq m \left( p^2 + q^2 \right) - k, \quad |z| > 2, \quad m > 0.$$ 

If $f$ satisfies this condition, Tonelli showed that the functions in any minimizing sequence are equicontinuous, and uniformly bounded on interior domains at least (see Lemma 4.1) and so a subsequence converges uniformly on such domains to a function still in his class. He was also able to handle the case where

$$f(x, y, z, p, q) \geq m \left( p^2 + q^2 \right) - k \quad \text{if} \quad f(x, y, z, 0, 0) = 0,$$

for instance by showing that any minimizing sequence can be replaced by one in which each $z_n$ is monotone in the sense of Lebesgue (see [31] and [37], for instance) and hence equicontinuous on interior domains, etc.

However, Tonelli was not able to get a general theorem to cover the case where $f$ satisfies (0.2) only with $1 < \alpha < 2$. Moreover, if one considers problems involving $\nu > 2$ independent variables, one soon finds that one would have to require $\alpha$ to be $> \nu$ in (0.2) in order to ensure that the functions in any minimizing sequence would be equicontinuous on interior domains. To see this, one needs only to notice that the functions

$$\log \log \left( 1 + \frac{1}{r} \right), \quad 1/r^k, \quad 0 < r \leq 1 \left( r^2 = \sum_{\alpha=1}^\nu (q^\alpha)^2 \right),$$
are limits of ACT functions in which
respectively, are uniformly bounded (see below for notation).

In order to carry through the program, for these more general problems, then, the writer found it expedient to allow functions which are still more general than Tonelli's ACT functions. One obtains these more general functions by merely replacing the requirement of $v$-dimensional continuity in Tonelli's definition by summability, but retaining Tonelli's requirements of absolute continuity along lines parallel to the axes, summable partial derivatives, etc. But then, two such functions may differ on a set of measure zero in such a way that their partial derivatives also differ only on a set of measure zero. It is clear that such functions should be identified and this is done in forming the «spaces $L_{2}$» discussed in Chapter I.

These more general functions have been defined in various ways and studied by various authors in various connections. Beppo Levi [32] was probably the first to use functions of this type in the special case that the function and its first derivatives are in $L_{2}$; any function equivalent to such a function has been called strongly differentiable by Friedrichs and these functions and those of corresponding type involving higher derivatives have been used extensively in the study of partial differential equations (see [2], [3], [11], [18], [19], [20], [21], [24], [28], [30], [42], [45], [46], [47], [50], [57], [61], [66]), G. C. Evans also made use at an early date [14], [15], [16] of essentially these same functions in connection with his work on potential theory. J. W. Calkin needed them in order to apply Hilbert space theory to the study of boundary value problems for elliptic partial differential equations and collaborated with the author in setting down a number of useful theorems about these functions (see [4] and [40]). The functions have been studied in more detail since the war by some of the writers mentioned above and by Aronszajn and Smith who showed that any function in the space $H_{mo}$ (see Professor Nirenberg's lectures) can be represented as a Riesz potential of order $m$ [1]. The writer is sure that many others have also discussed these functions and certainly does not claim that the bibliography is complete.

In Chapter I, the writer presents some of the known results concerning these more general functions. In Chapter II, these are applied to obtain theorems concerning the lower-semicontinuity and existence of minima.
of multiple integrals of the form

\[ I(x, G) = \int_{G} f(x, z(x), Vz(x)) \, dx \]

where the function \( f \) is assumed to be continuous in \((x, z, p)\) for \((x, z, p)\) and convex in \( p \in [p_i^j] \) for each \((x, z)\). In Chapter III, the most general type of function \( f(x, z, p) \) for which the integral \( I(x, G) \) in (0.5) is lower-semicontinuous is discussed. In Chapter IV, the writer discusses his results concerning the differentiability of the solutions of minimum problems. In Chapter V, the writer discusses the recent application by Eells and himself of a variational method in the theory of harmonic integrals.

We consistently use the notations of (0.5). If \( \varphi \) is a vector, \( |\varphi| \) denotes the square root of the sum of the squares of the components. Our functions are all real-valued unless otherwise noted. If \( x \) is a vector or tensor \( z_a, z_{ab}, \) etc., will denote the partial derivatives \( \partial x / \partial x^a, \partial^2 x / \partial x^a \partial x^b, \) etc., or their corresponding generalized derivatives. Repeated indices are summed unless otherwise noted. If \( G \) is a domain \( \partial G \) denotes its boundary and \( G = G \cup \partial G \). \( B(x_0, R) \) denotes the solid sphere with center at \( x_0 \) and radius \( R \); we sometimes abbreviate \( B(0, R) \) to \( BR \). \([a, b]\) denotes the closed cell \( a^x \leq x^a \leq b^x \). All integral are Lebesgue integrals. It is sometimes desirable to consider the behavior of a function (or vector) \( z(x) \) with respect to a particular variable \( x^a \); when this is done, we write \( x = (x^a, x'_a) \) and \( z(x) = z(x^a, x'_a) \) where \( x'_a \) stands the remaining variables; sometimes \((\nu - 1)\) dimensional integrals

\[ \int_{a_n}^{b_n} f(x^a, x'_a) \, dx'_a \]

appear in which case they have their obvious significance. We say that a (vector) function \( z(x) \) satisfies a uniform Lipschitz condition on a set \( S \) if and only if there is a constant \( M \) such that

\[ |z(x_1) - z(x_2)| \leq M \cdot |x_1 - x_2| \quad \text{for } x_1 \text{ and } x_2 \text{ on } S; \]

\( z \) is said to satisfy a uniform Hölder condition on \( S \) with exponent \( \mu, 0 < \mu < 1 \), if and only if there is an \( M \) such that

\[ |z(x_1) - z(x_2)| \leq M \cdot |x_1 - x_2|^\mu \quad \text{for } x_1 \text{ and } x_2 \text{ on } S. \]
A (vector) function $z$ is of class $C^n$ on a domain $G$ if and only if $z$ and its partial derivatives of order $\leq n$ are continuous on $G$; $z$ is said to be of class $C^{n+\mu}$ or $C^\mu$ on $G$ if and only if $z$ is of class $C^n$ on $G$ and all of its partial derivatives of order $\leq n$ satisfy uniform Hölder conditions with exponent $\mu, 0 < \mu < 1$, on $G$; the second notation $C^\mu$ is used when $\mu = 1$ (see Chapter V).
CHAPTER I

Function of class $\mathcal{B}_\lambda$, $\mathcal{B}_\lambda'$, $\mathcal{B}_\lambda'' (\lambda \geq 1)$ and functions which are ACT.

We begin with the definitions of these classes:

**Definition:** A function $z(x) (x = (x^1, \ldots, x^n))$ is of class $\mathcal{B}_\lambda$ on a domain $G$ if and only if $z$ is of class $\mathcal{L}_\lambda$ on $G$ and there are functions $p_\alpha$, $\alpha = 1, \ldots, \nu$, of class $\mathcal{L}_\lambda$ on $G$ with the following property; if $R$ is any cell with closure in $G$, there is a sequence $z_nR$ of functions of class $\mathcal{G}'$ on $R \cup \partial R$ such that $z_n \to z$ and $z_n \to p_\alpha$ strongly in $\mathcal{L}_\lambda$ on $R$.

**Definition:** A function $z$ is of class $\mathcal{B}_\lambda'$ on $G$ if and only if

(i) $z$ is of class $\mathcal{L}_\lambda$ on $G$;

(ii) if $[a, b]$ is any closed cell in $G$, then $z$ is AC (absolutely continuous) in $x^\alpha$ on $[a^\alpha, b^\alpha]$ for almost all $x^\alpha$ on $[a^\alpha, b^\alpha]$, $\alpha = 1, \ldots, \nu$;

(iii) the partial derivatives $z_\alpha$, which exist almost everywhere and are measurable on account of (ii), are of class $\mathcal{L}_\lambda$ on $G$.

**Definition:** A function $z$ is of class $\mathcal{B}_\lambda''$ on $G$ if and only if $z$ is of class $\mathcal{B}_\lambda'$ on $G$ and is continuous there.

**Definition:** A function $z$ is absolutely continuous in the sense of Tonelli (ACT) on $G$ if and only if $z$ is of class $\mathcal{B}_0'$ and is continuous on $G$.

**Definition:** Suppose $z$ is of class $\mathcal{L}_1$ on $G$. We define its $h$-average function on the set $G_h$ by

$$z_h(x) = (2h)^{-n} \int_{x-h}^{x+h} z(\xi) \, d\xi,$$

$G_h$ being the set of all $x$ in $G$ such that the cell $[x-h, x+h] \subset G$.

**Lemma 1.1:** If $z$ is of class $\mathcal{L}_1$ on a domain $G$ and $z_h$ is its $h$-average function defined on $G_h$, then $z_h \to z$ in $\mathcal{L}_1$ as $h \to 0$ on each closed cell $[a, b]$ in $G$ and $z_h$ is continuous on $G_h$.

**Proof:** That $z_h$ is continuous follows from the absolute continuity of the Lebesgue integral. Next, it is well known that $z_h(x) \to z(x)$ as $h \to 0$ for almost all $x$. Finally, choose $h_0 > 0$ so that $[a-h_0, b+h_0] \subset G$, keep $0 < h < h_0$, and let $\varphi(\varepsilon)$ be a function $\to 0$ as $\varepsilon \to 0$ such that $\|z\|_e \leq \varphi[m(e)]$ for $e \subset [a-h_0, b+h_0]$, where

$$\|z\|_e = \left[ \int_e z(x)^{1/\lambda} \, dx \right]^{1/\lambda}.$$
Then the lemma follows, since
\[ \| \mathbf{z}_h - \mathbf{z} \|_e \leq \| \mathbf{z}_h \|_e + \| \mathbf{z} \|_e \leq 2 \varphi \{ m(e) \} \quad \text{for } e \subset [a, b] \]
since
\[ \int_{\varepsilon} |\mathbf{z}_h(x)|^k \, dx \leq (2h)^{k-1} \int_{\varepsilon(\xi)} \left( \int_{\varepsilon(\xi)} |\mathbf{z}(x + \xi)|^k \, dx \right) \, d\xi = \]
\[ = (2h)^{k-1} \int_{\varepsilon(\xi)} \left( \int_{\varepsilon(\xi)} |\mathbf{z}(y)|^k \, dy \right) \, d\xi \leq \left\{ \varphi \{ m(e) \} \right\}^k. \]

where $e(\xi)$ is the set obtained by translating $e$ along the vector $\xi$.

**Theorem 1.1**: If $\mathbf{z}$ is of class $\mathcal{C}_h$ on $G$, the functions $p_a$ are uniquely determined up to null functions. If $\mathbf{z}_h$ is the $h$-average of $\mathbf{z}$ and $p_{ah}$ is that of $p_a$, then $\mathbf{z}_h$ is of class $C'$ on $G_h$ and

\[ z_{h,a}(x) = p_{ah}(x), \quad h > 0. \]

**Proof**: Let $[a, b] \subset G$, choose $h_0$ so $[a - h_0, b + h_0] \subset G$, and keep $0 < h < h_0$. Approximate to $\mathbf{z}$ and $p_a$ by $z_n$ and $z_{nh,a}$ in $L_1$ on $[a - h_0, b + h_0]$. Then for each $h$, we see that $z_{nh,a} = (z_{n,a})_h$ and we may obtain (1.2) by letting $n \to \infty$ on $[a, b]$. The first statement is now obvious.

**Definition**: If $\mathbf{z}$ is of class $\mathcal{C}_h$ on a domain $G$, we define its generalized derivative $D_a \mathbf{z}(x)$ as the Lebesgue derivative at $x$ of the set function

\[ \int_{\varepsilon} p_a(x) \, dx. \]

**Theorem 1.2**: If $\mathbf{z}$ is of class $\mathcal{C}_h$ on $G$, $\mathbf{z}_h$ is its $h$-average function, and $p_{ah}$ is that of its partial derivative $\partial \mathbf{z}/\partial x^a$, then $\mathbf{z}_h$ is of class $C'$ and (1.2) holds. Moreover $\mathbf{z}$ is of class $\mathcal{C}_h$ and its corresponding partial and generalized derivatives coincide almost everywhere.

**Proof**: Let $[a, b] \subset G$, choose $h_0$ so $[a - h_0, b + h_0] \subset G$, and keep $0 < h < h_0$. If $x'_a$ is not in a set of measure 0 on $[a'_a - h_0, b'_a + h_0]$, then $\partial \mathbf{z}/\partial x^a = p_a$ is summable in $x^a$ over $[a - h_0, b + h_0]$ and

\[ p_a(x^a, x'_a) \, dx^a = z(x^a, x'_a) = z(x^a, x'_a). \]

By integrating (1.3), we see that it holds for all $x'_a$ on $[a'_a, b'_a]$ and all $x^a, x'_a$ on $[a^a, b^a]$ if $z$ and $p_a$ are replaced by their $h$-averages. Then (1.2) and the last statement follow.
THEOREM 1.3: (a) If $z_1$ and $z_2$ are equivalent and one is of class $\mathcal{B}_3$ on $G$, then both are and their generalized derivatives coincide.

(b) If $z_1$ and $z_2$ are of class $\mathcal{B}_3$ on a domain $G$ and $z_{1,a}(x) = z_{2,a}(x)$ almost everywhere on $G$, then $z_1$ and $z_2$ differ by a constant and a null function.

These are easily proved using the $h$-average functions.

THEOREM 1.4: (a) Any function $z$ of class $\mathcal{B}_3$ on $G$ is equivalent to a function $z_0$ of class $\mathcal{B}_3$ on $G$.

(b) $z$ is ACT on $G$ if and only if $z$ is of class $\mathcal{B}_3$ there.

Proof: To prove (a), let $R = [a, b]$ be any rational cell in $G$ and approximate to $z$ there by functions $z_n$ of class $C'$ on $[a, b]$. A subsequence, still called $z_n$, converges to $z$ almost everywhere and is such that

$$\lim_{n \to \infty} \int_a^b \left| z_{n,a}^a (x^a, x^\alpha) - z_{a,a}^a (x^a, x^\alpha) \right|^2 \, dx^a = 0$$

for all $x^\alpha$ not in a set $Z_{Ra}$ of $(\nu - 1)$-dimensional measure zero, $\nu = 1, \ldots, \nu$. From (1.4), we see that the $z_n(x^a, x^\alpha)$ are equicontinuous in $x^a$ and converge uniformly on $[a^n, b^n]$ to a function $z_{0,R}^a (x^a, x^\alpha)$ which is $AC$ in $x^a$ if $x^\alpha$ is not in $Z_{Ra}$, $\nu = 1, \ldots, \nu$. Obviously $z_{0,R} = z$ almost everywhere on $R$. Since the union of the $Z_{Ra}$ for $a$ fixed and $R$ running over all rational cells is still of measure zero, we see that the $z_{0,R}$ join up to form a function $z_0$ of class $\mathcal{B}_3$ on $G$.

To prove (b), we note first that if $z$ is ACT on $G$, it is of class $\mathcal{B}_3$ on $G$. Conversely, if $z$ is of class $\mathcal{B}_3$, we may repeat the first part of the proof taking $z_n$ as the $h_n$-average of $z$ and conclude that we may take $z_{0,R}$ always $= z$ since then $z_n$ converges uniformly to $z$ on $R$.

The following theorems are easily proved by approximations:

THEOREM 1.5: The space $\mathcal{B}_h$ of equivalence classes of functions of class $\mathcal{B}_3$ is a Banach space if we define the norm by

$$\| z \|_\lambda = \left\{ \int_{[a,b]} \left[ |z|^2 + \sum_{\alpha=1}^\nu \left| z_{a,\alpha} \right|^2 \right] \, dx \right\}^{1/2}.$$

If $\lambda = 2$, $\mathcal{B}_3$ is a real Hilbert space if we define

$$(z, w) = \int_{[a,b]} \left( zw + \sum_{\alpha=1}^\nu z_{a,\alpha} w_{a,\alpha} \right) \, dx.$$
THEOREM 1.6: If \( u \in B_k \) and \( h \) is of class \( C' \) and satisfies a uniform Lipschitz condition on the bounded domain \( G \), then \( hu \in B_k \) on \( G \) and the generalized derivatives \( (hu)_\alpha \) all exist at any point \( x_0 \) where all the \( u_\alpha(x_0) \) exist.

DEFINITION: A transformation \( T: x = x(y) \) from a domain \( \tilde{G} \) onto \( G \) which is of class \( C' \) is said to be regular if and only if \( T \) is \( 1-1 \) and \( T \) and its inverse are of class \( C' \) and satisfy uniform Lipschitz condition 
\[
|T(x_1) - T(x_2)| \leq M \cdot |x_1 - x_2|, \text{ etc.}
\]

THEOREM 1.7: If \( u \) is of class \( B_k(C^{1'}) \) on the bounded domain \( G \), \( x = x(y) \) is a regular transformation of class \( C' \) from the bounded domain \( \tilde{G} \) onto \( G \) and \( \tilde{u}(y) = u[x(y)] \), then \( \tilde{u} \) is of class \( B_k(C^{1'}) \) on \( \tilde{G} \). Moreover, if \( x_0 = x(y_0) \) and all the generalized derivatives \( u_\alpha(x_0) \) exist, then all the generalized derivatives \( \tilde{u}_\alpha(y_0) \) exist and

\[
(1.5) \quad \tilde{u}_\beta(y_0) = u_\alpha[x(y_0)] \cdot x_\alpha^\beta(y_0)
\]

Proof: That \( \tilde{u} \) is of class \( B_k(C^{1'}) \) and that we may choose the right sides of (1.5) as the «derivative functions» \( \tilde{p}_\alpha \) of the definition is easily proved by approximating \( u \) on interior domains by functions of class \( C' \). Since regular families of sets correspond under regular transformations, the last statement follows easily.

Remarks: It is proved in [40] and [47], for instance, that if \( u \) is of class \( B_k \) on \( G \), it is equivalent to a function \( \overline{u} \) (namely the Lebesgue derivative of \( \int u \, dx \)) which is of class \( B_k \) and is such that any transform as in

Theorem 1.7 retains this property. But the last statement of Theorems 1.7 does not hold for the partial derivatives since this would imply that \( z \) had a total differential almost everywhere contrary to an example of Sake [55]. It is clear how to define the generalized derivative in a given direction and that (Theorem 1.7) if all the \( u_\alpha(x_0) \) exist, then a all the generalized directional derivatives exist at \( x_0 \) and are given by their usual formulas there. It is now easy to prove Rademacher's famous theorem [52] that a Lipschitz function has a total differential almost everywhere: For using the result just mentioned together with Theorem 1.2 we see that if \( z \) is Lipschitz and \( x_0 \) is not in a set of measure zero, then the partial and generalized derivatives all exist at \( x_0 \) and the ordinary directional derivatives in a denumerable everywhere dense set of directions (independent of \( x_0 \)) all exist and are given by their usual formulas; at any such point \( z \) is seen to have a total differential. Thus in Theorem 1.6, \( h \) may be Lipschitz and in Theorem 1.7, the transformation and its inverse may be Lipschitz; in this case (1.5) holds whenever all the generalized derivatives involved exist.
THEOREM 1.8: The most general linear functional on the space $B_1$ is of the form

$$f(x) = \int G (A_0 x + \sum_{a=1}^{\infty} A_a x_a) \, dx$$

where the $A_a (a \geq 0) \in L_p$ with $\lambda^{-1} + \mu^{-1} = 1$ if $\lambda > 1$ or are bounded and measurable on $G$ if $\lambda = 1$.

Proof: Let $A_1$ be the space of all vectors $\varphi = (\varphi_0, \ldots, \varphi_r)$ with components in $L_1$ and

$$\| \varphi \| = \left\{ \int G \left( \sum_{a=0}^{r} \varphi_a^2 \right)^{1/2} \, dx \right\}^{1/2}.$$

From Theorem 1.5 it follows that the subspace of all vectors $(z, z_1, \ldots, z_r)$ for which $z \in B_1$ on $G$ is a closed linear manifold $M$ in $B_1$. Hence if $F(z, z_1, \ldots, z_r) = f(z)$ then $F$ can be extended to the whole space $B$ to have same norm as $f$. Then $F$ is given by (1.6).

From Theorem 1.8 we immediately obtain:

THEOREM 1.9: (a) A necessary and sufficient condition that $z_n$ converges weakly to $z (z_n \Rightarrow z)$ in $B_1$ on $G$ is that $z_n \Rightarrow z$ and the $z_n, a \Rightarrow z, a$ in $L_1$ on $G$.

(b) If $z_n \Rightarrow z$ in $B_1$ on $G$, then $z_n \Rightarrow z$ in $B_1$ on any subdomain.

(c) If $z_n \Rightarrow z$ in $B_1$ on $G$ (bounded), $x = x(y)$ is a regular transformation of class $C'$ from $\tilde{G}$ onto $G$, $z_n(y) = z_n[x(y)]$ and $\tilde{z}(y) = z[x(y)]$, then $\tilde{z}_n \Rightarrow \tilde{z}$ in $B_1$ on $\tilde{G}$.

(d) If $z_n \Rightarrow z$ in $B_1$ on $G$ (bounded) and $h$ is Lipschitz on $G$, then $hz_n \Rightarrow hz$ in $B_1$ on $G$.

DEFINITION: A function $z$ is of class $B_{10}$ on $G$ (bounded) if and only if it is of class $B_1$ there and there exists a sequence $(u_n)$, each of class $C'$ and vanishing on and near the boundary $\partial G$ such that $z_n \rightarrow z$ (strong convergence) in $B_1$ on $G$. The subspace $B_{10}$ of $B_1$ is defined correspondingly. If $z$ and $z^* \in B_1$ on $G$, we say that $z = z^*$ on $G$ in the $B_1$ sense if and only if $z - z^* \in B_{10}$ on $G$.

The following is immediate:

THEOREM 1.10: The subspace $B_{10}$ is a closed linear manifold is $B_1$; if $z_n \Rightarrow z$ in $B_1$ on $G$ and each $z_n \in B_{10}$, then $z \in B_{10}$. If $z \in B_{10}$ and $z_1(x) = z(x)$ for $x$ on $G$ and $z_1(x) = 0$ otherwise, then $z_1 \in B_{10}$ on any $D \supset G$ and $z_1, a(x) = 0$ for almost all $x$ not in $G$.

THEOREM 1.11 (Poincaré's inequality): Suppose $z \in B_{10}$ on $G \subset B(x_0, R)$. Then

$$\int G |z|^1 \, dx \leq \lambda^{-1} R^k \int G |\nabla z|^1 \, dx.$$
Proof: It is sufficient to prove this for \( z \) of class \( C' \) and vanishing on \( \partial B(x_0, R) \) with \( G = B(x_0, R) \). Taking spherical coordinates \((r, p)\) with \( r = |x - x_0| \) and \( p \in \Sigma = \partial B(0, 1) \), we obtain

\[
\int \frac{|u(r, p)|^k}{d \Sigma} \, dp = \int \frac{|u(R, p) - u(r, p)|^k}{d \Sigma} \, dp = \int_r^R \frac{u_r(s, p)}{d \Sigma} \, ds \leq (R - r)^{k-1} \int_r^R \frac{u_r(s, p)}{d \Sigma} \, ds \, d\Sigma
\]

where \( u(r, p) = z(x) \). Thus

\[
\int_{B(\delta_0, R)} |z(x)|^k \, dx = \int_0^R \int_{\Sigma} \left( \int u(r, p) \right)^k \, d\Sigma \, dr \leq \int_0^R (R - r)^{k-1} \left( \int_{\Sigma} u_r(s, p) \right)^k \, d\Sigma \, dr
\]

from which the result follows.

**Theorem 1.12**: Suppose \( z \in C_0^\infty \) on \( G \), \( \Lambda \subseteq G \), \( z^* \in C_0^\infty \) on \( \Lambda \) and coincides with \( z \) on \( \partial \Lambda \) in the \( C_0^\infty \) sense. Then the function \( Z \) such that \( Z(x) = z^*(x) \) on \( \Lambda \) and \( Z(x) = z(x) \) on \( G - \Lambda \) is of class \( C_0^\infty \) on \( G \) and \( z_\Lambda(x) = z_\Lambda(x) \) almost everywhere on \( \Lambda \) and \( Z_\Lambda(x) = z_\Lambda(x) \) almost everywhere on \( G - \Lambda \).

**Proof**: For define \( Z_1(x) = z^*(x) - z(x) \) on \( \Lambda \) and 0 elsewhere. Then \( Z(x) = z(x) + Z_1(x) \) on \( G \) and the result follows from Theorem 1.10.

**Lemma 1.2**: Suppose \( z \in C_0^\infty \) on the cell \([a - h_0, b + h_0]\). Then

\[
\int_a^b |z_h(x) - z(x)|^k \, dx \leq C_1(v, \lambda) \cdot h \cdot \int_{a-h}^{b+h} |V z(y)|^k \, dy,
\]

where \( C_1 \) depends only on the arguments indicated.

**Proof**: Since we may approximate to \( z \) strongly in \( C_0^\infty \) on \([a - h, b + h]\) by functions of class \( C' \) on that closed cell, it is sufficient to prove the lemma for such functions. Then if \( x \in [a, b] \) and \( |\xi| < h \), we see that \( x \) and
Then from which the result follows.

**THEOREM 1.13:** If \( z_n \rightarrow z_0 \) in \( B_{\lambda,1} \) on the bounded domain \( G \), then \( z_n - z_0 \) in \( L_1(G) \), and a subsequence converges strongly in \( L_1(G) \) to some function \( z \).

**Proof:** The first statement follows from the second. For, let \( f(z) \) be any subsequence of \( (z_n) \). A subsequence \( (z_{n_k}) \) converges strongly in \( L_1(G) \) to some function \( z \), which must be (equivalent to) \( z_0 \). Hence the whole sequence \( z_n \rightarrow z_0 \) in \( L_1(G) \).

To prove the second statement, suppose \( G \subset [a, b] \) and extend each \( z_n \) to be 0 outside \( G \); then each \( z_n \in B_{\lambda,1} \) on \( [a-1, a+1] \) with uniformly bounded \( C_{\lambda,1} \) norm. For each \( h \) with \( 0 < h < 1 \), we see that the \( z_{nk} \) are uniformly bounded and equicontinuous on \( [a, b] \). So there is a subsequence, called \( (z_{nk}) \), such that \( z_{nk} \) converges uniformly to some function \( z_h \) for each \( h \) of a sequence \( n \rightarrow 0 \). From lemma 1.2, it is easy to see first that the limiting \( z_h \) form a Cauchy sequence in \( L_1(G) \) having some limit \( z \) and then that \( z_h \rightarrow z \) strongly in \( L_1(G) \).

In order to treat variational problems with fixed boundary values, one can, of course, practically always reduce the problem to one where the given boundary values are zero. Although one can formulate theorems about variational problems having variable boundary values on the boundary of an arbitrary bounded domain (see Chapter II), such problems become more
meaningful if we restrict ourselves to domains $G$ which are bounded and of class $C'$ where boundary values can be defined in a more definite way as we now do.

**Definition:** A bounded domain $G$ is of class $C'$ if and only if each point $x_0$ of the boundary $\partial G$ is interior to a neighborhood $N(x_0)$ on $G \cup \partial G$ which is the image, under a regular transformation $x = \varphi(y)$ of class $C'$, of the half-cube $Q^+$: $|x^\alpha| < 1$ for $\alpha < \nu$ and $0 \leq x^\nu < 1$, where $x(0) = x_0$ and $\partial G \cap N(x_0)$ is the image of the part of $Q^+$ where $x^\nu = 0$. Such a neighborhood $N(x_0)$ is called a boundary neighborhood.

**Definition:** Suppose $C^\infty$ is a domain. A finite sequence $[h_1, \ldots, h_N]$ of functions is said to be a partition of unity of class $C'$ on $G \cup \partial G$ if and only if each $h_i$ is of class $C'$ on $G \cup \partial G$, $0 \leq h_i(x) \leq 1$ on $G \cup \partial G$ for each $i$, and

$$\sum_{i=1}^N h_i(x) \equiv 1 \quad \text{for } x \text{ on } G \cup \partial G.$$  

The support of $h_i$ is the closure of the set of all $x$ on $G \cup \partial G$ for which $h_i(x) > 0$.

**Lemma 1.3:** If $G$ is bounded domain of class $C'$, there is a partition of unity $[h_1, \ldots, h_N]$ of class $C'$ on $G \cup \partial G$ such that the support of each $h_i$ is either interior to a cell in $G$ or is interior to a boundary neighborhood of $G \cup \partial G$.

**Proof.** With each interior point $P$ of $G$ we define $R_P$ as the largest hypercube $|x^\alpha - x_P^\alpha| < h_P$ in $G$ and define $r_P$ as the hypercube $|x^\alpha - x_P^\alpha| < h_P/2$. With each $P$ on $\partial G$, associate a boundary neighborhood $R_P = N(P)$ which is the image under $\tau_P$ of $Q^+$ as in the definition; we define $r_P$ as the part of $R_P$ corresponding under $\tau_P$ to the part of $Q^+$ for which $|x^\alpha| < 1/2$, $\alpha = 1, \ldots, \nu$. There are a finite number $r_1, \ldots, r_N$ of the $r_P$ which cover $G \cup \partial G$. Clearly each corresponding $R_i$ is the image under a regular transformation $\tau_i$ of class $C'$ of either the unit cube $Q$ or the half-cube $Q^+$ where $\tau_i$ corresponds under $\tau_i$ to the part where $|x^\alpha| < 1/2$.

Now, let $\varphi(s)$ be a fixed function of class $C^\infty$ for all $s$ with $\varphi(s) = 1$ for $|s| \leq 1/2$, $\varphi(s) = 0$ for $|s| \geq 3/4$, and $0 \leq \varphi(s) \leq 1$ otherwise. For each $i$, define $k_i(x)$ on $R_i$ as the image under $\tau_i$ of the function $\varphi(y^\alpha) \ldots \varphi(y^\nu)$ and define $k_i(x) = 0$ elsewhere on $G \cup \partial G$. Then the support of $k_i$ is interior to $R_i$, $k_i(x) = 1$ for $x$ on $r_i$, and each $k_i$ is of class $C'$ on $G \cup \partial G$. We then define

$$h_i(x) = k_i(x), \quad h_{i+1}(x) = k_{i+1}(x) \prod_{j=1}^i [1 - k_j(x)], \quad i = 1, \ldots, N - 1.$$
Then we see by induction that
\[ \sum_{i=1}^{N} h_i(x) = 1 - \prod_{i=1}^{N} (1 - h_i(x)) \]
so that the sequence \( \{h_1, \ldots, h_N\} \) satisfies the desired conditions.

**Theorem 1.14:** Suppose \( G \) is bounded and of class \( C' \) and \( z \in \mathcal{E}_{\lambda} \) on \( G \). Then

(i) there is a sequence \( \{z_n\} \) of functions of class \( C' \) on \( G \cup \partial G \) which converges strongly in \( \mathcal{E}_{\lambda} \) to \( z \) on \( G \);

(ii) there is a boundary value function \( \varphi \) in \( L_\lambda \) on \( \partial G \) (with respect to hyperarea) to which every sequence \( \{z_n\} \) in (i) converges strongly in \( L_\lambda \) on \( \partial G \);

(iii) if \( T : x = x(y) \) is a regular transformation of class \( C' \) of \( \widetilde{G} \cup \partial \widetilde{G} \)
onto \( G \cup \partial G \), \( \widetilde{z}(y) = z(x(y)) \), and \( \widetilde{\varphi}(y) = \varphi(x(y)) \), then \( \widetilde{\varphi} \) is the boundary value function for \( z \) on \( \partial \widetilde{G} \);

(iv) if \( \varphi(x) = 0 \) for almost all \( x \) on \( \partial G \), then \( z \in \mathcal{E}_{\lambda} \) on \( G \).

**Proof:** Let \( \{h_1, \ldots, h_N\} \) be a partition of unity on \( G \cup \partial G \) of the type described in Lemma 1.3. Clearly each function \( h_i z \in \mathcal{E}_{\lambda} \) on \( G \) and on \( R_i \), and the transform \( \varphi_i(y) \) under \( \tau \in \mathcal{E}_{\lambda} \) on either \( Q \) or \( Q^+ \); in the former case \( \varphi_i \) vanishes on and near \( \partial Q \) and in the latter, \( \varphi_i \) vanishes near \( \partial Q^+ \cap \Omega \). In the latter case, \( \varphi_i \) is equivalent to a function \( w_i \) which is \( AC \) in \( y^a \) for almost all \( y^a \), \( \alpha = 1, \ldots, r \) on any cell where \( h \leq y^r \leq 1 \) (since \( \varphi_i = 0 \) near \( y^r = 1 \)), where \( h > 0 \). But since \( \varphi_{i_0} \in \mathcal{E}_{\lambda} \), we see that \( w_i \) is \( AC \) in \( y^a \) for \( 0 \leq y^r \leq 1 \) for almost all \( y^r \). If we extend \( w_{i_0} \) to the whole of \( Q \) by setting
\[ w_{i_0}(y^r, y^a) = \frac{w_{i_0}(y^r, y^a)}{y^r} \text{ for } -1 \leq y^r \leq 0 \]
we see that \( w_{i_0} \in \mathcal{E}_{\lambda} \) on \( Q \) and vanishes near \( \partial Q \). Clearly we may approximate each \( w_i \) or \( w_{i_0} \) on \( Q \) strongly in \( \mathcal{E}_{\lambda} \) by functions \( w_{ni} \) of class \( C \) on \( Q \) and vanishing near \( \partial Q \). If we define \( z_n \) on \( R_i \) as the transform of \( w_{ni} \) under \( \tau_i \) and then define \( z_n = z_{n1} + \ldots + z_{nN} \), we see that \( z_n \) has the desired properties.

To prove (ii) we choose, in all cases, \( w_{i_0} \) equivalent to \( w_i \) and \( \mathcal{E}_{\lambda} \) on \( Q \). Then, since \( w_{i_0} \) is \( AC \) in \( x^r \), we see that
\[
\int_{-1}^{1} \left| w_{i_0}(y^r, y^a) - w_{i_0}(y^r, y^a) \right|^2 dy^a
\]
Accordingly, we see that $w_{0} (y, y_{0})$ converges strongly in $L_{2}$ in $y_{0}$ to $w_{0} (0, y_{0})$ as $y \to 0^{+}$. If $z_{n} \to z$ in $B_{1}$, $z_{n}$ of class $C'$, and we let $w_{n1}$ be the transform of $h_{i} z_{n}$ under $\tau_{i}$, then we see that (1.7) holds uniformly. Now let $\{ p \}$ be any subsequence of $\{ n \}$. There is a subsequence $\{ q \}$ of $\{ p \}$ such that (for each $i$) $w_{qi} (y, y_{0})$ converges strongly in $L_{1}$ with respect to $y_{0}$ on $[-1, 1]$ for almost all $y_{0}, 0 < y_{0} \leq 1$. But, on account of the uniformity in (1.7), this convergence is uniform for all $y_{0}, 0 \leq y_{0} \leq 1$. Hence the whole sequence $w_{0} (0, y_{0})$ converges strongly to $w_{0} (0, y_{0})$ in $L_{2}$.

(iii) is now evident. To prove (iii), $\{ z_{n} \}$ be of class $C'$ and converge strongly to $z$ in $B_{2}$ on $\mathcal{G}$. Then $\tilde{z}_{n}$ and each $h_{i} \tilde{z}_{n}$ converges strongly to $0$ on $\partial \mathcal{G}$. If we define the $w_{0i}$ as above, then $w_{0i} (0, y_{0}) = 0$ for almost all $y_{0}$ on $[0, 1, 1]$ if $R_{i}$ is a boundary neighborhood. If we extend such $w_{0i}$ to $Q$ by $w_{0i} (y_{0}, y_{0}) = w_{0i} (-y_{0}, y_{0})$, $y_{0} \leq 0$ we see that $w_{0i}$ is of class $B_{2}$ on $Q$ and that it and its $h$-average functions, for sufficiently small $h$ vanish near $\partial Q$ and along $y_{0} = 0$. By modifying the average function slightly for each $h$ in a sequence $0$ we may construct sequences $w_{ni}$ tending strongly in $B_{2}$ to $w_{0i} = 0$ such that each $W_{ni} = 0$ near $y_{0} = 0$ as near $\partial Q$ for those $i$ for which $R_{i}$ is a boundary neighborhood. The desired $z_{n}$, each of class $C'$ and vanishing near $\partial \mathcal{G}$ can be constructed as above.

**Theorem 1.15**: If $G$ is bounded and of class $C'$ and if $z_{n} \to z$ in $B_{1}$ on $G$, then $z_{n} \to z$ in $L_{1}$ on $G$ and $q_{n} \to q$ in $L_{1}$ on $\partial G$. If $\| z_{n} \|$ is uniformly bounded in $B_{1}$, and the set functions $\int_{G} | p_{n} z_{n} | \, dx$ are uniformly absolutely continuous if $\lambda = 1$, there is a subsequence $\{ z_{p} \}$ which converges weakly in $B_{1}$ to some $z$ on $G$.

**Proof**: Let $\{ h_{1}, ..., h_{N} \}$, $w_{ni}$, $w_{1}$, and $w_{0i}$ have meanings as in the proof of Theorem 1.14 and let $w_{0ni}$ be of class $B_{2}$ on $Q$ (or $Q^{+}$) and be equivalent to $w_{ni}$ and extend each $w_{0ni}$ to $Q$ as before. Then (1.7) holds uniformly (in case $\lambda = 1$ this is true on account of the uniform absolute continuity in that case) and $w_{ni} \to w_{i}$ in $L_{1}$ on $Q$ for each $i$. The argument in the proof of (ii) in Theorem 1.14 can be repeated to obtain the desired results. The last statement follows easily.

In the next section, we shall have occasion to discuss vector functions of class $B_{1}$.

**Definition**: A vector function $z = (z_{1}, ..., z_{N})$ is of class $B_{1}$ if and only if each of its components is; in this case

$$\| z \|_{B_{1}} = \left\{ \int_{G} \left[ \sum_{i=1}^{N} (z_{i})^{2} + \sum_{\alpha=1}^{r} (z_{\alpha}^{(i)})^{2} \right]^{1/2} \, dx \right\}^{1/2}.$$
It is clear that all the theorems and lemmas of this section except Theorem 1.11 and lemma 1.2 generalize immediately to vector functions. Those two can be generalized with the help of the following well known lemma:

**Lemma 1.4**: Suppose $f_1, \ldots, f_n$ are summable over the set $S$ with respect to the measure $\mu$. Then $\sqrt{\int f_1^2 + \cdots + f_n^2}$ is also and

$$\left\{ \frac{\sum_{i=1}^{n} \left[ \int f_i(x) \, d\mu \right]}{S} \right\}^{1/2} \leq \int_S \left[ \sum_{i=1}^{n} f_i^2(x) \right]^{1/2} \, dx.$$  

**Proof**: For the left side of (1.8) equals

$$\max_{|a|=1} \sum_{i=1}^{n} a_i f_i(x) \, d\mu \leq \int_S \left[ \sum_{i=1}^{n} f_i^2(x) \right]^{1/2} \, d\mu; \quad \left( a^2 = \sum_{i=1}^{n} a_i^2 \right).$$

In addition, we need the following special case of Rellich's theorem [53]:

**Theorem 1.16**: If the vector $z$ is of class $C^3$ on the hypercube $R$ of side $h$ and $z_R$ is its average over $R$, then

$$\int_R \| z(x) - z_R \|^2 \, dx \leq C_2 (v, k) \cdot h^k \cdot \int_R \| z(x) \|^2 \, dx$$

where $C_2$ depends only on the arguments indicated.

**Proof**: It is sufficient to prove this for vectors of class $C'$ where $R : |x^n| \leq k = h/2$. Then we have

$$\int_R \int_R \left\{ \sum_{i=1}^{N} \left[ \int_0^1 (\xi^n - x^n) \, z_i (x + t (\xi - x)) \, dt \right]^{1/2} \right\} \, dx \, d\xi$$

$$= \int_R \int_R \left\{ \sum_{i=1}^{N} \left[ \int_0^1 \left[ (\xi^n - x^n) \, z_i (x + t (\xi - x)) \right] \, dt \right]^{1/2} \right\} \, dx \, d\xi$$

$$\leq \int_R \int_R \left[ \int_0^1 \| \xi - x \| \cdot \| z (x + t (\xi - x)) \| \, dt \right]^k \, dx \, d\xi$$

$$\leq \int_{-k}^{k} \int_{-k}^{k} \left[ \int_0^1 \| \xi - x \|^4 \cdot \| z (x + t (\xi - x)) \|^4 \, dt \right] \, dx \, d\xi.$$

Setting \( \eta^a = x^a + t(\xi^a - x^a) = (1 - t)x^a + t\xi^a \), the last integral becomes

\[
\int_{-k}^{k} \left\{ \int_{0}^{1} \left[ \int_{(1 - \frac{tk}{v}) \xi - tk}^{(1 - \frac{tk}{v}) \xi + tk} \left| \eta - x \right|^2 \, d\eta \right] \, dt \right\} \, dx
\]

\[
= \int_{-k}^{k} \left\{ \int_{0}^{1} \left[ \int_{R(\eta, t)} \left| \eta - x \right|^2 \, dx \right] \, dt \right\} \, d\eta
\]

where \( R(\eta, t) \) is the intersection of \( R \) with the hypercube \( x^a - \eta^a/(1 - t) \leq tk \). On \( R(\eta, t) \) we see that

\[
\left| \eta - x \right| \leq v^{1/2} \cdot th.
\]

The result follows since \( m[R(h, t)] \leq h^r \) and is \( \leq (2th)^r \) for \( t \leq 1/2 \).
CHAPTER II

Lower-semicontinuity and existence theorems for a class of multiple integral problems.

In this chapter, we consider variational problems for integrals of the form (0,5) in which \( f(x, z, p) \) is continuous in \((x, z, p)\) for all \((x, z, p)\) and is convex in \(p\) for each \((x, z)\) (cf. [42], Chapter III).

**Definitions:** A set \( S \) in a linear space is said to be convex if and only if the segment \( P_1 P_2 \) belongs to \( S \) whenever the points \( P_1 \) and \( P_2 \) do. A function \( \varphi(z) \) \((z = (\xi_1, \ldots, \xi_k))\) is said to be convex on the convex set \( S \) in the \( \xi \)-space if and only if

\[
\varphi[(1 - \lambda)\xi_1 + \lambda\xi_2] \leq (1 - \lambda)\varphi(\xi_1) + \lambda\varphi(\xi_2), \quad 0 \leq \lambda \leq 1,
\]

whenever \( \xi_1 \) and \( \xi_2 \in S \).

The following theorems concerning convex functions are well known and are stated without proof:

**Lemma 2.1:** Suppose \( \varphi(z) \) is convex on the open convex set \( S \) with \( |\varphi(z)| \leq M \) there. Then \( \varphi \) satisfies

\[
|\varphi(\xi_2) - \varphi(\xi_1)| \leq 2M \cdot |\xi_2 - \xi_1|/\delta
\]
on any compact subset of \( S \) at a distance \( \geq \delta \) from \( \partial S \).

**Lemma 2.2:** Suppose \( \varphi \) and each \( \varphi_n \) are convex on the open convex set \( S \) and suppose \( \varphi_n(\xi) \rightarrow \varphi(\xi) \) for each \( \xi \) on \( S \). Then the convergence is uniform on any compact subset of \( S \).

**Lemma 2.3:** A necessary and sufficient condition that \( \varphi \) be convex on the open convex set \( S \) is that for each \( \xi \) in \( S \) there exists a linear function \( a_p \cdot \xi + b \) such that

\[
(2.1) \quad \varphi(\xi) = a_p \xi + b, \quad \varphi(\xi) \geq a_p \xi + b \quad \text{for all} \quad \xi \in S.
\]

If \( \varphi \) is of class \( C^1 \) on \( S \), this condition is equivalent to

\[
E(\xi, \bar{\xi}) = \varphi(\xi) - \varphi(\bar{\xi}) - (\xi - \bar{\xi}) \varphi_p(\xi, \bar{\xi}) \geq 0; \quad \xi, \bar{\xi} \in S.
\]
If \( \varphi \) is of class \( C^n \) on \( S \), this condition is equivalent to

\[
\varphi_{,\alpha} (\xi) \eta^\alpha \eta^\beta \geq 0
\]

for all \( \xi \) on \( S \) and all \( \eta \).

**Definition:** A linear function \( a_p \xi^p + b \) which satisfies (2.1) for some \( \xi \) is said to be supporting to \( \varphi \) at \( \xi \).

**Lemma 2.4:** Suppose \( \varphi \) is convex for all \( \xi \) and satisfies

\[
\lim_{|\xi| \to +\infty} \varphi (\xi)/|\xi| = +\infty.
\]

Then \( \varphi \) takes on its minimum. Also, if \( a_1, \ldots, a_p \) are any numbers, there is a unique \( b \) such that \( a_p \xi^p + b \) is supporting to \( \varphi \) for some \( \xi \). If \( \varphi \) is convex and satisfies (2.2), if \( \varphi (\xi) \geq \varphi (\xi) \) for each \( \xi \), and if \( a_p \xi^p + c \) is supporting to \( \varphi \), then \( c > b \).

**Lemma 2.5:** Suppose that \( \varphi_n \) and \( \varphi \) are everywhere convex and satisfy (2.2) and suppose that \( \varphi_n (\xi) \to \varphi (\xi) \) for each \( \xi \). Suppose \( a_1, \ldots, a_p \) are any numbers and \( b_n \) and \( b \) are chosen so that \( a_p \xi^p + b_n \) and \( a_p \xi^p + b \) are supporting to \( \varphi_n \) and \( \varphi \), respectively. Then \( b_n \to b \). Likewise, if \( a_{np} = a_p \) for each \( p \) and \( b_n \) and \( b \) are chosen so that \( a_{np} \xi^p + b_n \) and \( a_p \xi^p + b \) are all supporting to \( f \), then \( b_n \to b \).

In order to consider variational problems on arbitrary bounded domains, it is convenient to introduce the following type of weaker than weak convergence in \( \mathcal{B}_1 \) on such a domain.

**Definition:** We say that \( z_n \to z_0 \) in \( \mathcal{B}_1 \) on the bounded domain \( G \) if and only if \( z_n \) and \( z_0 \) all \( \in \mathcal{B}_1 \) on \( G \), \( z_n \to z_0 \) in \( \mathcal{B}_1 \) on each cell interior to \( G \) and each \( z_{n,a} \to z_{0,a} \) in \( \mathcal{L}_1 \) on the whole of \( G \).

**Theorem 2.1:** If \( G \) is bounded and of class \( C' \) or if all the \( z_n \in \mathcal{B}_{10} \) on \( G \) and if \( z_n \to z_0 \) in \( \mathcal{B}_1 \) on \( G \), then \( z_n \to z_0 \) in \( \mathcal{B}_1 \) on \( G \).

Proof: The second case can be reduced to the first by extending each \( z_n \) to be zero outside \( G \) and choosing a domain \( G' \) of class \( C' \) such that \( G' \supset G \). Thus we suppose \( G \) of class \( C' \). If we use the notation in the proof of Theorem 1.14, we see that (1.7) holds uniformly for the \( w_{n,i} \) so that an argument similar to those in the proofs of Theorems 1.14 and 1.15 and 1.13 shows that \( w_{n,i} \) converge strongly in \( \mathcal{L}_1 \) on \( Q \) or \( Q^+ \) to something for each \( i \). Thus \( z_n \) converges strongly in \( \mathcal{L}_1 \) on \( G \) to something which must be \( z_0 \).

**Remark:** If \( G \) is not of class \( C' \) and the \( z_n \) are not all in \( \mathcal{B}_{10} \) on \( G \), then an example in [41] shows that \( z_n \nrightarrow z_0 \) in \( \mathcal{B}_1 \) on \( G \) with
the $C_1$ norms of the $z_n$ being uniformly bounded. If for some $\lambda > 1$,
\[
\int_G |\nabla z_n|^\lambda \, dx = \int_G \left[ \sum_{a=1}^{\infty} (z_{n,a}^2)^{\lambda/2} \right] dy \quad (G \text{ bounded})
\]
are uniformly bounded, then a subsequence $\{p\}$ of $\{n\}$ exists such that the $z_{p,a} \to$ something in $L_1$ on the whole of $G$.

**Theorem 2.2**: Suppose that $f(p)$ is defined of all $p = (p_i^n) (i = 1, \ldots, N, a = 1, \ldots, \nu)$ and $f$ is convex. If $z_n \to z_0$ on $G$ and
\[
I(z_0, G) = \int_G f(\nabla z_0) \, dx, \quad I(z_n, G) = \int_G f(\nabla z_n) \, dx,
\]
then $I(z_0, G)$ and $I(z_n, G)$ are each finite or $+\infty$ and
\[
I(z_0, G) \leq \liminf_{n \to \infty} I(z_n, G).
\]

**Proof**: Since $f$ is convex, there are constants $a_i^n$ such that
\[
f(p) \geq f(0) + a_i^n p_i^n \quad \text{for all } p.
\]
Hence
\[
I(x, G) \geq f(0) m(G) + a_i^n \int_G z_i^n(x) \, dx
\]
with a similar inequality for $I(z_n)$. Thus the first statement follows.

If $D \subset G$, we see as above that
\[
I(z_n, G) - I(z_n, D) = I(z_n, G - D) \geq f(0) [m(G - D)] + a_i^n \int_{G-D} z_i^n(x) \, dx \geq \epsilon [m(G - D)]; \quad \lim \epsilon(\epsilon) = 0
\]
by virtue of the uniform absolute continuity of the set functions $\int_G z_i^n(x) \, dx$.

Clearly also $I(x, D) - I(x, G)$ as $D$ runs through an expanding sequence of domains exhausting $G$. Thus it is sufficient to prove the lower semi-continuity for $G$ a hypercube of side $h$, say.

To do this, we define a sequence of summable functions $\phi_q(x)$ as follows: For each $q$ divide $G$ into $2^q$ hypercubes of side $h \cdot 2^{-q}$. On each
of these hypercubes \( R \), define

\[
q_q(x) = f(p_R) + a_i^*(R, q)[z^i_{0,a}(x) - p^i_{R,a}], \quad x \text{ to interior to } R,
\]

where \( p^i_{R,a} \) is the average of \( z^i_{0,a} \) over \( R \) and the \( a_i^*(R, q) \) are chosen so that
\[
f(p_R) + a_i^*(p^i_{R,a} - p^i_{R,a})
\]
is supporting to \( f \) at \( p_R \). We define the \( q_{nq} \) similarly from \( z_n \). Then it follows that

\[
q_q(x) \leq f[Vz_0(x)], \quad q_{nq}(x) \leq f[Vz_n(x)]
\]

(almost everywhere). On the other hand, suppose all the generalized derivatives exist at some \( x_0 \) which is not on \( \partial R \) for any hypercube \( R \) as above for any \( q \). Let \( R \) denote the hypercube containing \( x_0 \). Then as \( q \to 0 \)
\[
p^i_{R,a} - z^i_{0,a}(x_0)
\]
so that \( q_q(x_0) \to f[Vz(x_0)] \) since the \( a_i^*(R, q) \) remain bounded (Lemma 2.5). Hence

\[
(2.3) \quad I(z, G) = \lim_{q \to 0} \int_{\partial} q_q(x) \, dx.
\]

Moreover, for each fixed \( q \), \( p_{R,n} \to p_R \) from the weak convergence so

\[
\int_{\partial} q_q(x) \, dx = \lim_{n \to \infty} \int_{\partial} q_{nq}(x) \, dx \leq \liminf_{n \to \infty} I(z_n, G).
\]

The result follows from (2.3) and (2.4).

**Lemma 2.6:** Suppose \( f(x, z, p) \) is defined and satisfies a uniform Lipschitz condition with constant \( K \) for all \((x, z, p)\), suppose \( f(x, z, p) \) is convex in \( p \) for each \((x, z)\) and suppose \( f(x, z, p) \geq f_0(p) \) for all \((x, z, p)\), where \( f_0(p) \) is convex. Then, if \( z_n \to z_0 \) in \( \mathcal{B}_1 \) on \( G \),

\[
I(z_0, G) \leq \liminf_{n \to \infty} I(z_n, G).
\]

**Proof:** As in the proof of Theorem 2.2, it is sufficient to prove this for a hypercube \( D \) of side \( d \) interior to \( G \). Then \( z_n \to z_0 \) in \( \mathcal{L}_1 \) on \( D \). From the Lipschitz condition, \( f(x, z, p) \leq f(0, 0, 0) + K \cdot |x| + K \cdot |z| + K \cdot |p| \) so that \( I(z, D) \) and \( I(z_n, D) \) are finite.
For each $q$, divide $D$ into $2^{-q}$ hypercubes $R$ of side $2^{-q} \cdot d$. Then, using Theorem 1.16, it follows that

$$\int_{R} \left| f[x, z(x), p(x)] - f[x_R, z_R, p(x)] \right| \, dx \leq K \int_{R} \left| |x - x_R| + |z(x) - z_R| \right| \, dx \leq$$

$$\leq K \cdot 2^{-q} \left( 2^{-1} \cdot \frac{1}{\sqrt{d}} \cdot h^r + \int_{R} |Vz(x)| \, dx \right) (h = 2^{-q})$$

and a similar inequality holds for each $z_n$ with $\epsilon_q$ independent of $n$ on account of the weak convergence. Also

$$\sum_{R} \int_{R} \left| f[x_R, z_R, p_n(x)] - f[x_R, z_R, p(x)] \right| \, dx \leq K \int_{D} \left| z_n(x) - z(x) \right| \, dx.$$

The lemma follows easily from Theorem 2.2 and the inequalities above.

**Theorem 2.3**: Suppose $f(x, z, p)$ is defined and continuous for all $(x, z, p)$, is convex in $p$ for each $(x, z)$ and $f(x, z, p) \geq f_0(p)$ for all $(x, z, p)$ where $f_0(p)$ is convex and $\frac{f_0(p)}{|p|} \to +\infty$ as $p \to \infty$. Then $I(z, G)$ is lower semicontinuous with respect to the convergence $\Rightarrow$.

**Proof**: In order to prove this, it is sufficient to show that $f(x, z, p)$ is the limit of a non-decreasing sequence $f_n(x, z, p)$ each of which has the properties required in Lemma 2.6. In order to do this, let $b(x, z; a)$ ($a \equiv [a^i]$) be chosen so that the function $\varphi(x, z; p; a) = a^i p^i + b(x, z; a)$ is the unique supporting plane (in $p$) to $f$ determined by $a$. By Lemmas 2.4 and 2.5 $b(x, z; a)$ is continuous in $(x, z; a)$ and $b(x, z; a) \geq b_0(a)$, the corresponding function for $f_0$. For each $a$, choose a non-decreasing sequence $b_n(x, z; a)$ of functions, each $\geq b_0(a) - 1$, each satisfying a uniform Lipschitz condition for all $(x, z)$, which converges to $b(x, z; a)$. We then define $\varphi_n(x, z; p; a) = a_i p^i + b_n(x, z; a)$ and we see that $\varphi_n$ is a non-decreasing sequence tending to $\varphi$ for each $a$, each $\varphi_n$ satisfying a uniform Lipschitz condition everywhere.

For each $n$, we define $f_n(x, z, p) = \max \varphi_n(x, z, p, a)$ for all $a$ for which all the $a_i^p$ are rational numbers having numerator and denominator
both $u$. Then it is clear that the $f_n$ are non-decreasing and each satisfies a uniform Lipschitz condition. Now, let $(x_0, z_0, p_0)$ and $\varepsilon > 0$ be given. Using Lemma 2.5 and the continuity of $b$, we see that there is a rational $\alpha$ such that $\varphi(x_0, z_0, p_0; \alpha) > f(x_0, z_0, p_0) - \varepsilon/2$. Clearly $\varphi_n(x_0, z_0, p_0; \alpha) > \varphi(x_0, z_0, p_0; \alpha) - \varepsilon/2$ for all sufficiently large $n$, so that $f_n(x_0, z_0, p_0) < f(x_0, z_0, p_0)$.

We now turn to existence theorems on arbitrary domains. We begin with the following theorem (cf. [48] and [40], theorem 8.8 and [41]):

**Theorem 2.4**: Suppose $f_0(p)$ is convex in $p$ and $f_0(p)/|p| \to +\infty$ as $p \to -\infty$. Then there is a function $\varphi(q) \to 0$ as $q \to 0$ which depends only on $f$ and $M$ such that if $I(x, G) \leq M$, then

$$
\int_{G} |V z(x)| \, dx \leq \varphi[m(e)].
$$

**Proof**: For each integer $r \geq 1$, let $E_r$ be the set of $x$ in $G$ where $r - 1 \leq |V z(x)| < r$ and $V z(x)$ exists and let

$$
E_r = \bigcup_{k=r+1}^{\infty} E_k \cup Z, \quad r = 0, 1, 2, ...
$$

where $Z$ is the set of measure 0 where $V z(x)$ does not exist. Clearly $E_0 = G$ and if $r \geq 1$ and $x \in G - E_r$, then $|V z(x)| < r$. Let $\alpha_r$ be the inf. of $f_0(p)/|p|$ for $|p| \geq r - 1$. Then $\alpha_r \to +\infty$ as $r \to \infty$. Also

$$
\sum_{k=r+1}^{\infty} \alpha_k \cdot (k - 1) \cdot m(E_k) \leq \int_{G} f_0(V z) \, dx \leq M.
$$

From this we see that

$$
m(E_r) \leq \frac{M}{r \cdot \alpha_r+1}, \quad \int_{E_r} |V z| \, dx \leq \frac{(r + 1)M}{r \cdot \alpha_r+1}
$$

and both $\to 0$ as $r \to \infty$. So, let $e$ be any subset of $G$. Let $r$ be the smallest integer such that $M/r \cdot \alpha_{r+1} \leq m(e)$. Then

$$
\int_{e} |V z| \, dx \leq \int_{e} |V z| \, dx + \int_{e} |V z| \, dx
$$

$$
\leq \frac{M}{\alpha_{r+1}} + \frac{(r + 1)M}{r \cdot \alpha_{r+1}} = \varphi[m(e)]
$$

and $\varphi$ satisfies the conditions.
THEOREM 2.5: Suppose $f(x, z, p)$ satisfies the hypotheses of Theorem 2.3 and $G$ is a bounded domain. Suppose that $I^*$ is a family of functions $z^*$ in $B_1$ which is compact with respect to the convergence $\Rightarrow$ in $B_1$ on $G$. Suppose $F$ is the family of all $z$ in $B_1$ which coincide on $\partial G$ in the $B_1$ sense with some $z^*$ in $I^*$ and suppose $F$ contains some $z_1$ for which $I(z_1, G) < +\infty$. Then $I(z, G)$ takes on its minimum in $F$.

Proof: Let $\{z_n\}$ be a minimizing sequence (i.e. $I(z_n, G) = \inf \{I(z, G) : z \in F\}$); we may assume that $I(z_n, G) \leq M = I(z_1, G)$. Suppose $z_n = z_n^*$ on $G$ where $z_n^* \in I^*$. A subsequence $z_n^* \Rightarrow z_0^*$ in $B_1$ on $G$ and $z_0^* \in I^*$. By Theorem 2.4, the set functions $\int_{\partial G} |V z_q - z^*_q| \, dx$ are uniformly $AC$; the same is true of the set functions $\int_{\partial G} |V (z_q - z^*_q)| \, dx$. Since $G$ is bounded and each $z_q - z^*_q = 0$ on $\partial G$, we see with the aid of Theorem 1.13 that a subsequence $z_r \Rightarrow z_0 = w_0$ in $B_1$ on $G$ and $w_0 = 0$ on $\partial G$. Accordingly $z_r \Rightarrow z_0 = z_0^* + w_0$ in $B_1$ on $G$ and $z_0 \in F$. The theorem follows from the lower-semicontinuity of $I(z, G)$.

Somewhat more meaningful boundary value problems can be studied if we require $G$ to be of class $C'$ at least. We need the following preliminary lemma:

LEMMA 2.7: Suppose $G$ is bounded of class $C$ and $F$ is a family of functions of $B_1$ on $G$ such that

$$\int_{\partial G} |V z| \, dx \leq M, z \in F.$$ 

Suppose that $F$ satisfies one of the following additional conditions:

(i) there is a number $P$ and an open subset $\tau$ of $G$ such that

$$\int_{\tau} |z| \, dx \leq P \text{ for all } z \in F; \text{ or}$$

(ii) there is a number $P$ and an open set $\sigma$ of $\partial G$ such that

$$\int_{\sigma} |z| \, dS \leq P \text{ for all } z \in F.$$ 

Then the $B_1$ norms of the $z$ in $F$ are uniformly bounded.
Proof: We may cover $G \cup \partial G$ with a finite number of hypercubes or boundary neighborhoods $R_1, \ldots, R_Q$; let $\varphi_i$ map $Q$ or $Q^+$ onto $R_i$ as in the proof of Lemma 1.3. We may assume that one of the $R_i \subset \sigma$ in case (i) or that $R_i \cap \partial G \subset \sigma$ in case (ii). In case (ii), we see using equation (1.7) with $y_i^T = 0$ that case (i) holds with $\tau = R_i$ and $P$ replaced by $P_i$; here we have assumed that $w_0$ is equivalent to the transform under $\varphi_i$ of the restriction of $z$ to $R_i$.

Now, let $R_j \cap R_i$ be an open set $\tau_i$. For a given $z$, let $w_{j0}$ be of class $\mathcal{C}_1$ of $Q$ or $Q^+$ and be equivalent to the transform under $\varphi_j$ of the restriction of $z$ to $R_j$. Thus there is a cell $R_{j0} = [a, b]$ in $Q$ or $Q^+$ such that case (i) holds with $z$ replaced by $w_{j0}$ and $P$ by $P_{j0}$ (independently of $z$ in $F$). By using an equation like (1.7), we see in turn that case (i) holds with $R_{j0}$ replaced $R_{j1}, R_{j2}, \ldots, R_{jv} = Q$ or $Q^+$ with $P$ replaced by $P_{j1}, \ldots, P_{jv} = P_j$ where $R_{j2}$ is the cell $-1 \leq x^1 \leq 1, -1 \leq x^2 \leq 1, a^s \leq x^s \leq b^s$ for $s = 3, \ldots, r$, etc. Thus case (i) holds with $\tau$ replaced by $R_j$ and $P$ by $P_j$. Since any $R_k$ can be joined to the first $R_i$ by a sequence $R_n$, each two adjacent members of which have an open set in common, the lemma follows.

We can now prove our second principal existence theorem:

**Theorem 2.6**: Suppose the domain $G$ and the family $F$ satisfy the conditions of Lemma 2.7 for some $\lambda \geq 1$ and hence for $\lambda = 1$ and suppose $F$ contains some vector $\tilde{z}$ for which $I(\tilde{z}, G)$ is finite and suppose $F$ is closed with respect to weak convergence in $\mathcal{C}_1$. Suppose that $f(x, z, p)$ satisfies the conditions of Theorem 2.3 Then $I(z, G)$ takes on its minimum in $F$.

**Proof**: Let $\{z_n\}$ be a minimizing sequence for which $I(z_n, G) \leq I(\tilde{z}, G)$. Then the set functions $\int z_{n, a} \, dx$ are uniformly absolutely continuous on account of Theorem 2.4. Combining this with Theorem 1.15; we see that a subsequence $\{z_p\}$ can be selected which converges weakly on $G$ in $\mathcal{C}_1$ to some $z_0$ in $\mathcal{C}_1$. Since $F$ is closed with respect to weak convergence in $\mathcal{C}_1$, $z_0 \in F$. The result follows from the lower semicontinuity of $I(z, G)$.

**Theorem 2.7**: Suppose $G$ is of class $C^r$, $f(x, z, p)$ satisfies the hypotheses of Theorem 2.3, and $\Gamma$ is a closed family of functions $\varphi$ in $\mathcal{L}_1$ on $\partial G$ such that case (ii) of Lemma 2.7 holds. Suppose $F$ is the family of all functions $z$ in $\mathcal{B}_1$ on $G$, each of which has boundary values in $\Gamma$ and suppose $F$ contains a function $\tilde{z}$ such that $I(\tilde{z}, G)$ is finite. Then $I(z, G)$ takes on its minimum in $F$.

**Proof**: For the subfamily $\overline{F}$ of $z$ in $F$ for which $I(z, G) \leq I(\tilde{z}, G)$ satisfies the conditions of Theorem 2.6, on account of Theorems 2.1, 2.3; and 1.15.
EXAMPLE: As an example of the use of Theorem 2.7, consider the problem of finding the surface $z = z(x)$ ($z = (z^1, z^2, z^3), x = (x^1, x^2)$) of least area of type of a disc bounded by a simple closed $C$ consisting of a fixed arc $C_1$ which has only its end points on a surface $S$ and a variable arc $C_2$ on $S$. Using theorems about conformal mapping this problem can be reduced to that of minimizing the Dirichlet integral

$$I(z, G) = \int_{\partial G} \sqrt{1 + \sum_{i=1}^{3} (z_i^1)^2 + (z_i^2)^2} \, dx^1 \, dx^2$$

among all vectors $z$ of class $\mathcal{B}_2^G$ on $G$, where $\mathcal{G}$ is the unit circular disc, such that the restrictions of $z$ to $\partial G$ carry the upper semicircle of $\partial G$ in a 1-1 continuous way onto the fixed arc $C_1$ with $(0,1)$ corresponding to some fixed point on $C_1$ and carry the lower part of $\partial G$ in a 1-1 continuous way onto the variable arc $C_2$. In order to apply Theorem 2.7, we let $I^*$ consist of all strong limits in $\mathcal{L}_2$ on $\partial G$ of the restrictions of such $z$ to $\partial G$. Any vector $\varphi$ in $I^*$ is equivalent along the upper part of $\partial G$ to a vector which carries that part of $\partial G$ in a « monotone » way onto $C_1$ in which arcs of $C_1$ may correspond to points on $\partial G$; for almost all $x$ on the lower part of $\partial G$, $\varphi(x) \in S$ at any rate. Since any minimizing vector $z_0$ certainly minimizes $I(z, G)$ among all $z$ in $\mathcal{B}_2$ which coincide with $z_0$ on $\partial G$ in the $\mathcal{B}_2$ sense, we see that $z_0$ is harmonic (see Professor Nirenberg's lectures). By arguments like those in [7] and [43], we conclude that $z_0$ is continuous on the upper half of $\partial G$ and yields a conformal map of $G$ onto the surface represented by $z_0$. However, an example of Courant [8] (p. 220, 221), shows that $z_0$ need not be continuous along the lower half of $\partial G$ and that the limiting « curve » $C_2$ need not be an arc even if the surface $S$ is regular and of class $C^\infty$; Lewy [33] has shown that if $S$ is analytic, the curve $C_2$ is analytic.
CHAPTER III

Quasi-convexity and lower-semicontinuity.

In the preceding chapter, we proved theorems concerning the lower-semicontinuity of multiple integrals $I(x, G)$ in cases where the integral function $f(x, z, p)$ is continuous and convex in $p$ for each $(x, z)$. This restriction on $f$ was a natural extension to the case of several unknown functions of the ordinary requirement when $N = 1$ that the variational problem be regular or at least that Hadamard's condition

\[ f_{\alpha \beta}(x, z, p) \lambda_\alpha \lambda_\beta \geq 0 \quad \text{for all } (x, z, p, \lambda) \]

be satisfied, $f$ being assumed of class $C^n$. But (3.1) holds if and only if $f$ is convex in $(p_1, ..., p_N)$ for each $(x^1, ..., x^N, z)$.

The condition (3.1) is arrived at as follows: Suppose a function $z_0(x)$ of class $C'$ minimizes $I(x, G)$ among all functions of $z$ of class $C'$ which have the same boundary values and which are near $z_0$ in the sense that the maximum of $|z(x) - z_0(x)| + |Vz(x) - Vz_0(x)| \leq \delta$ for some $\delta > 0$. Then it can be shown that (3.1) holds for $x$ on $G$, $z = z_0(x)$, and $p = Vz_0(x)$. However, if this procedure is applied in the case where $N > 1$, we obtain only the condition

\[ f_{\alpha \beta}(x, z, p) \lambda_\alpha \lambda_\beta \xi_1 \xi_2 \geq 0 \]

for all $(x, z, p)$ (along the solution $z = z_0(x)$, etc.) and all $(\lambda_1, ..., \lambda_N)$ and $(\xi_1, ..., \xi_N)$ (see Theorem 3.3 below). This does not imply that $f(x, z, p)$ is convex in $p$. Moreover, it is known that integrals $I(x, G)$ which arise in parametric problems are lower semi-continuous with respect to uniform convergence; for the case of the parametric problem for surfaces in 3-space $(\nu = 2, N = 3)$, these integrands have the form

\[ f(x, z, p) = F(x, z, J_1, J_2, J_3) \]

where

\[ J_1 = p_1^3 p_2^3 - p_1^3 p_2^3, \quad J_2 = p_1^3 p_2^3 - p_1^3 p_2^3, \quad J_3 = p_1^3 p_2^3 - p_1^3 p_2^3 \]

and $F$ is convex in $(J_1, J_2, J_3)$, but not in the six $p_4^i$. 

\[ \]
It turns out to be rather easy to derive (see also [44]) a certain necessary and sufficient condition on \( f \) as a function of \( p \) for the lower semicontinuity of \( I(z, G) \) with respect to a certain type of convergence. This question was considered for \( \nu = N = 1 \) by Tonelli ([72], [73], [74], [75]) and by Cesari and others for the parametric case. We begin by deriving this condition and then discuss the relation of that condition to the condition (3.2).

In order not to get involved with the behavior of \( f \) at infinity we shall use the following convergence which obviously implies weak convergence in each \( \mathcal{C}_k \) but does not necessarily imply strong convergence in any \( \mathcal{C}_k \):

**Definition:** We say that \( z_n \to z \) on \( G \to z_n(x) \) converges uniformly to \( z(x) \) on \( G \) and \( z \) and \( z_n \) each satisfy a uniform Lipschitz condition on \( G \) which is independent of \( n \).

**Theorem 3.1:** Suppose \( I(z, G) \) is lower-semicontinuous with respect to this type of convergence at any \( z \) on any \( G \) and \( f \) is continuous. Then

\[
(3.3) \quad \int_0 f(x_0, z_0, p_0 + V \zeta(x)) \, dx \geq f(x_0, z_0, p_0) \cdot m(G)
\]

for any constant \((x_0, z_0, p_0)\), any bounded domain \( G \), and any Lipschitz vector \( \zeta \) which vanishes on \( \partial G \).

**Proof:** Let \( x_0 \) be any point, \( R \) be the cell \( x_0^a \leq x^a \leq x_0^a + h \), \( z_0 \) be any vector of class \( C' \) on \( R \cup \partial R \), \( Q \) be the cell \( 0 \leq x^a \leq 1 \), and \( \zeta \) be any vector which satisfies a uniform Lipschitz condition over the whole space and is periodic of period 1 in each \( x^a \).

For each \( n \), define \( \zeta_n(x) \) on \( R \) by

\[
\zeta_n(x) = n^{-1} h \zeta \left[ n h^{-1} (x - x_0) \right].
\]

Then the \( \zeta_n \) tend to zero in our sense. Then, for each \( n \), \( I(x_0 + \zeta_n, R) \) can be written as a sum of integrals over the sub-hypercubes of \( R \) of side \( n^{-1} h \).

If \( r \) is one these the integral over it is

\[
n^{-r} h^r \int_Q f(x_1 + n^{-1} h \xi, z_n(x_1 + n^{-1} h \xi), p_0(x_1 + n^{-1} h \xi) + V \zeta(\xi)) \, d\xi,
\]

where

\[
r : x_0^a \leq x^a \leq x_0^a + n^{-1} h, \quad x_1^a = x_0^a + k^a n^{-1} h, \quad 0 \leq k^a \leq n - 1
\]

\[
z_n(x) = z_0(x) + \zeta_n(x), \quad x^a = x_1^a + n^{-1} h \xi^a, \quad 0 \leq \alpha \leq 1.
\]
Thus we see that

\[
\lim_{n \to \infty} I(x_0 + \zeta_n, R) = \int_{Q} \int_{R} f[x, z_0(x), p_0(x) + V \zeta(\xi)] \, d\xi \, dx \geq I(z_0, R).
\]

By letting \(z_0\) and \(p_0\) be arbitrary constant vectors, setting \(z_0(x) = x_0 + p_{0a} \cdot (x^a - x_0^a)\), dividing by \(m(R) = h^r\) and letting \(h \to 0\), we obtain (3.3) for \(G = Q\) and \(\zeta\) periodic of period 1 in each \(x^a\). But if \(G\) is any bounded domain and \(\zeta\) vanishes on \(\partial G\), we may choose a hypercube \(Q'\) containing \(G\) and extend \(\zeta(x)\) to be zero in \(Q' - G\). Then a simple change of variable obtains the result in general.

**Definition:** If \(f\) is continuous in \((x, z, p)\) for all \((x, z, p)\) and satisfies (3.3) for all \((x_0, z_0, p_0)\), we say that \(f\) is quasi-convex in \(p\); if \(f\) depends only on \(p\) and satisfies (3.3), we say simply that \(f\) is quasi-convex.

We now prove that the condition (3.3) is sufficient for lower-semicontinuity.

**Lemma 3.1:** Suppose \(R\) is the hypercube \(|x^a - x_0^a| \leq h\), \(f(p)\) is quasi-convex, suppose \(p_0\) is any constant tensor and suppose \(\zeta_n \to 0\) in our sense or \(R\). Then

\[
\liminf_{n \to \infty} \int_{R} f(p_0 + V \zeta_n(x)) \, dx \geq f(p_0) \cdot m(R).
\]

**Proof:** Suppose the \(\zeta_n\) satisfy a uniform Lipschitz condition with constant \(M\) on \(R\). We may assume that \(|\zeta_n(x)| \leq Mk_n h\) where each \(k_n < 1/2\) and \(\lim k_n = 0\). For each \(n\), we begin by defining \(\eta_n(x) = \zeta_n(x)\) on \(\partial R\) and \(\eta_n(x) = 0\) for \(|x^a - x_0^a| \leq (1 - k_n) h\); we then extend each \(\eta_n\) to the whole of \(R\) to satisfy a Lipschitz condition with constant \(\leq M\). Then \(\eta_n \to 0\), \(\zeta_n - \eta_n \to 0\), \(\zeta_n(x) - \eta_n(x) = 0\) on \(\partial R\), and \(\eta_{n,n}(x) \to 0\) for each \(x\) interior to \(R\). Hence

\[
\lim_{n \to \infty} \int_{R} |f[p_0 + V \zeta_n] - f[p_0 + V(\zeta_n - \eta_n)]| \, dx = 0.
\]

The result follows easily from the quasi-convexity of \(f\).

**Theorem 3.2:** Suppose \(f(x, z, p)\) is quasi-convex in \(p\), \(G\) is a bounded domain, and \(z_n \Rightarrow z_0\) on \(G\). Then

\[
I(z_0, G) \leq \liminf_{n \to \infty} I(z_n, G).
\]
Proof: Since all the arguments \([x, z_n(x), v z_n(x)]\) and \([x, z_0(x), v z_0(x)]\) remain in a bounded part \(\mathcal{J}\) of \((x, y, p)\)-space and since \(G\) is the union of \(\mathcal{G}_0\) disjoint hypercubes, it is sufficient to prove this for the case of a hypercube \(R\) of side \(h\). Since \(f\) is uniformly continuous on \(\mathcal{J}\), there is a function \(\varepsilon(q)\) with \(\lim_{q \to 0} \varepsilon(q) = 0\) such that

\[
|f'(x', z', p') - f(x'', z'', p'')| \leq \varepsilon(q) \quad \text{if} \quad |x' - x''|^2 + |z' - z''|^2 + |p' - p''|^2 \leq q^2.
\]

For each \(k\), divide \(R\) up into \(2^k\) hypercubes \(R_{hk}\) of side \(2^{-k}h\). Define the functions \(z_k^L(x), v z_k^L(x), p_k^L(x)\) on \(R\) to be equal on each \(R_{hk}\) to the averages over \(R_{hk}\) of \(x, z_0(x), p_0(x)\) respectively, and define

\[
r_k(x) = |x - x_k^L(x)|^2 + |z_0(x) - z_k^L(x)|^2 + |p_0(x) - p_k^L(x)|^2/2.
\]

\[
\zeta_n(x) = z_n(x) - z_0(x).
\]

Then

\[
f[x, z_n(x), v z_n(x)] - f[x, z_0(x), v z_0(x)] = A_n + B_{nk} + C_k + D_{nk}
\]

where

\[
A_n = f[x, z_n(x), p_n(x)] - f[x, z_0(x), p_0(x)]; \quad (p_n(x) = v z_n(x))
\]

\[
B_{nk} = f[x, z_0(x), p_0(x)] - f[x, z_k^L(x), v z_k^L(x); p_k^L(x) + \zeta_n(x)]
\]

\[
C_k = f[x, z_0(x), p_0(x)] - f[x_k^L(x), z_k^L(x), p_k^L(x) + \zeta_n(x)]
\]

\[
D_{nk} = f[x_k^L(x), z_k^L(x), p_k^L(x) + \zeta_n(x)] - f[x_k^L(x), z_k^L(x), p_k^L(x)].
\]

We see that

\[
|A_n| \leq \varepsilon(|z_n(x) - z_0(x)|)
\]

\[
|B_{nk}|, \quad |C_k| \leq \varepsilon[r_k(x)]
\]

and \(I(z_n, R) - I(z_0, R) = J_n + K_{nk} + L_k + P_{nk}\), where these are the integrals of \(A_n, B_{nk}, C_k\), and \(D_{nk}\), respectively. Now, let \(\varepsilon > 0\). We first choose a fixed \(k\) such that \(K_{nk}\) and \(L_k\) are both \(< \varepsilon/2\). From (3.4), (3.5), (3.6), and Lemma 3.1, we see that

\[
\lim_{n \to \infty} J_n = 0, \quad \liminf_{n \to \infty} P_{nk} \geq 0
\]
since \( x_k(x) \), \( x_k(x) \), and \( p_k(x) \) are each constant on each \( R_{ki} \). Thus
\[
\lim_{n \to \infty} \inf [I(x_n, R) - I(x_0, R)] \geq \varepsilon.
\]

Some of the theory of Chapter 2 can be carried over for the more general functions \( f(x, z, p) \) which are quasi-convex in \( p \) but more has to be assumed about how \( f \) behaves as \( p \to \infty \). These theorems are not of great interest and they can be found in [44].

We now investigate the concept of quasi-convexity in more detail.

**Lemma 3.2** [79], [45]: Suppose \( a_{jk}^\beta \) are constants and
\[
\int_G a_{jk}^\beta \xi_j(x) \xi_k(x) \, dx \geq 0
\]
for all \( \xi \) in \( C^{20} \) on domain \( G \), then
(3.7)
\[
a_{jk}^\beta \lambda_\alpha \lambda_\beta \xi_j \xi_k \geq 0 \quad \text{for all } \lambda \text{ and } \xi.
\]

**Proof:** Let \( \lambda^1 \) be a unit vector with \( \lambda^1 = \lambda_\alpha \) and choose \( \lambda^2, \ldots, \lambda^r \) so \( (\lambda^1, \ldots, \lambda^r) \) form a normal orthogonal set. Suppose \( x_0 \in G \) and let \( y_0 = \lambda_\alpha^r \cdot (x^0 - x_0) \). Choose \( h_0 \) and \( R > 0 \) so that the set of all \( x \) for which \( |y_j| \leq h_0 \) and \( |y_j| \leq R \) is in \( G \). Let \( \xi \) be an arbitrary vector and define
\[
\xi^j(x) = \xi_j(y^j) \cdot \psi(|y_i|),
\]
where
\[
\varphi_h(y^j) = h - |y^j| \quad \text{if } |y^j| \leq h, \quad \psi(r) = R - r \quad \text{if } 0 \leq r \leq R
\]
and \( \varphi_h \) and \( \psi(|y_i|) = 0 \) otherwise. Then it is easy to see that
\[
\lim_{h \to 0} (2h)^{-1} I(\xi^j_h, G) = \Gamma_{i=1}^{r} R^\alpha a_{jk}^\beta \lambda_\alpha \lambda_\beta \xi_j \xi_k \psi(r - 1) \geq 0
\]
which proves the lemma.

We now prove the theorem mentioned in the introduction to this chapter.

**Theorem 3.3:** Suppose \( f(x, z, p) \) is of class \( C^\nu \) for all \( (x, z, p) \) near the locus \( S \) of all points \( (x, z, v z_0(x)) \) for \( x \) in \( G \) and suppose \( z_0(x) \) is of class \( C^\nu \) on \( G \cup \partial G \) and minimizes \( I(x, z, s) \) among all Lipschitz \( z \) which coincide with \( z_0 \) on \( \partial G \) and are such that \( |x(x) - z_0(x)| + |v z(x) - v z_0(x)| \leq \delta \) for some \( \delta > 0 \). Then \( (3.2) \) holds for all \( (x, z, p) \) on \( S \).

**Proof:** For, let \( \xi \) be any Lipschitz function vanishing vanishing on and near \( \partial G \). Then \( z_0 + \lambda \xi \) is sufficiently near \( z_0 \) for all sufficiently small
λ. So if \( \phi(\lambda) = 1(x_0 + \lambda \xi) \), we must have

\[
\phi''(0) = \int \frac{f}{G} d \rho \int_\alpha \int_\beta \frac{d}{d\rho} [x \cdot a_0 (x) \cdot p_0 (x)] \xi' \xi' dx \geq 0.
\]

By selecting any point \( x_0 \) in \( G \) and proceeding as in the proof of Lemma 3.2 and then dividing by \( R^{v-1} \), but letting \( R \) and \( h \) both \( \rightarrow 0 \) so that \( h: R \rightarrow 0 \), we obtain (3.2) at \([x_0, z(x_0), p(x_0)]\).

Using the result of Lemma 3.2 and the method of proof of Theorem 3.3, we conclude that if \( f(p) \) is quasi-convex and of class \( C^\infty \), then (3.2) holds with \( x \) and \( z \) omitted. This result and the analogy with convex functions suggest the following theorem which we now prove.

**Theorem 3.4:** If \( f(p) \) is quasi-convex, then \( f(p_a^j + \lambda_a \xi^i) \) is convex in \( \lambda \) for each \( p \) and \( \xi \) and convex in \( \xi \) for each \( p \) and \( \lambda \).

**Proof:** If \( f \) is quasi-convex, it is easy to see that its twice iterated \( h \)-average function \( f_{hh} \) is also quasi-convex and is of class \( C^\infty \) as well. Then any linear function furnishes an absolute minimum to \( f_{hh}(z, G) \) among all Lipschitz functions with the same boundary values. Accordingly, by Theorem 3.3 we see that \( f_{hh} \) satisfies (3.2). But then \( f_{hh} \) has the convexity properties stated in the theorem. Since \( f_{hh} \) converges uniformly to \( f \) on any bounded part of space, the theorem follows.

**Definition:** A function \( f(p) \) which satisfies the conditions in Theorem 3.4 is said to be weakly quasi-convex.

**Remark:** The principal problem, so far unsolved, is whether or not every weakly quasi-convex function is quasi-convex.

**Theorem 3.5:** If \( f(p) \) is weakly quasi-convex, it satisfies a uniform Lipschitz condition on a bounded part of space. If \( p \) is given, there are constants \( A^j_\lambda \) such that

\[
f(p_a^j + \lambda_a \xi^i) \geq f(p_a^j) + A^j_\lambda \lambda_a \xi^i \quad \text{for all } \lambda, \xi.
\]

If \( f \) is also of class \( C' \), then \( A^j_\lambda = f_{a}^j(p) \). If \( f \) is also of class \( C'' \) then (3.2) holds. If \( f \) is continuous and \( f \) for each \( p \), constants \( A^j_\lambda \) exist such that (3.8) holds, then \( f \) is weakly quasi-convex.

**Proof:** If \( f \) is weakly quasi-convex, it is convex in each \( p_a^j \) separately. Hence, if \( |f(p)| \leq M \) on some hypercube, any difference quotient of the form:

\[
|f(p_a^j) - f(p_{2a}^j)|/(p_{2a}^j - p_{1a}^j) \leq 2M/d, \quad p_{2a}^j < p_{1a}^j
\]

where \( d \) is the smaller of \( h_a^j - p_{2a}^j \) and \( p_{1a}^j - a_{1a}^j \).
Next, \( f_{hh} \) is still weakly quasi-convex and of class \( C'' \) so that (3.2) holds. Then, from the convexity in \( \xi \) for each \( \lambda \), for instance, (3.8) holds with \( A_{hh}^\alpha = f_{hhp}^\alpha (p) \). Since \( f \) satisfies a uniform Lipschitz condition near \( p \), we see that the \( A_{hh}^\alpha \) are uniformly bounded as \( h \to 0 \) so a sequence of \( h \to 0 \) can be chosen so that all the \( A_{hh}^\alpha \) tend to limits. Clearly (3.8) holds in the limit. Since the unit vector in the \( p_\alpha \) direction is of form \( \lambda_\alpha \xi^\alpha \), we see that \( A_{hh}^\alpha = f_{hh} \) if \( f \) is of class \( C' \). The last statement follows from theorems on convex functions.

We now define a sufficient condition for \( f \) to be (strongly) quasi-convex.

**Theorem 3.6**: A sufficient condition for \( f \) to be quasi-convex is that for each \( p \) there exist alternating forms

\[
A_{i_1 \cdots i_\mu}^{\alpha_1 \cdots \alpha_\mu} \pi_{a_1}^{j_1} \cdots \pi_{a_\mu}^{j_\mu}, \quad \mu = 1, \ldots, \nu
\]

(in which the coefficients are 0 unless all the \( \alpha_i \cdots \alpha_\mu \) are distinct and all the \( j_1 \cdots j_\mu \) are distinct and an interchange of two \( \alpha \)'s or two \( j \)'s changes the sign) such that for all \( \pi \) we have

\[
f(p + \pi) \geq f(p) + \sum_{\mu=1}^{\nu} A_{i_1 \cdots i_\mu}^{\alpha_1 \cdots \alpha_\mu} \pi_{a_1}^{j_1} \cdots \pi_{a_\mu}^{j_\mu}.
\]

**Proof**: For suppose \( p \) is any constant tensor, \( G \) is any bounded domain, and \( \zeta \) is any Lipschitz vector which vanishes on \( \partial G \). By extending \( \zeta = 0 \) outside \( G \) and approximating to it on a larger domain \( D \) with smooth boundary with functions of class \( C'' \) which vanish on and near \( \partial D \) and using Stokes' theorem we see that the integral of the sum on the right in (3.9) is zero. We now exhibit two interesting cases where the weak quasi-convexity of \( f \) implies its quasi-convexity.

**Theorem 3.7**: If \( f(p) \) is weakly quasi-convex and

\[
f(p) = a_{j_\bar{k}} p_j^i \bar{p}_\bar{k}^i
\]

then \( f \) is quasi-convex ([79], [45]).

**Proof**: For, if \( \zeta \) is Lipschitz and vanishes on \( \partial G \) (which may as well be assumed smooth), then

\[
\int_G f[p + \nabla \zeta (\xi)] \, dx = f(p) \cdot m(G) + \int_G a_{j_\bar{k}} \xi_j^i \bar{\zeta}_\bar{k}^i \bar{x} \, dx
\]

9. **Annali della Scuola Norm. Sup.** • Pisa.
If we introduce Fourier transforms (see [79])

\[ Z^\iota (y) = (2\pi)^{-n/2} \int e^{iy^\alpha \xi^\iota (x)} \, dx \]

we see that

\[ \int_0^\infty a_{jk}^\alpha \xi^j (x) \xi^k (y) \, dx = \int_{-\infty}^\infty a_{jk}^\alpha y^\alpha y^\beta Z^j (y) \overline{Z}^k (y) \, dy \geq 0 \]

since the integrand is \( \geq 0 \) for each \( y \).

**Theorem 3.8:** If \( N = r + 1 \) and

\[ f (y) = F (X_1, \ldots, X_{r+1}) \]

where \( F \) is continuous and

\[ X_j = -\det M_j (j = 1, \ldots, r), \quad X_{r+1} = \det M_{r+1} \]

\[ M_{r+1} = \| p^1_a, \ldots, p^r_a \|, \quad M_j = \| p^1_a, \ldots, p^{j-1}_a, p^{j+1}_a, p^{j+1}_a, \ldots, p^r_a \| \] .

Then \( f \) is quasi-convex in \( p \) if and only if \( F \) is convex in \( (X_1, \ldots, X_{r+1}) \).

We omit the proof which is found in [44]; \( F \) is there required to be homogeneous of the first degree in \( X \) but this is not necessary in the proof.
CHAPTER IV

The differentiability of the solutions of certain variational problems with $v = 2$.

In this chapter we discuss the differentiability of the solutions of certain problems whose existence was proved in § 2. To save time, we shall not discuss the continuity on the boundary but shall consider only the differentiability on the interior. This work was first presented in [42], chapters 4, 6, and 7 and was the culmination of a series of papers on this subject by Lichtenstein [34], [35], Hopf [27], and the writer [39]. Some of these results have recently been generalized by De Giorgi [10] and Nash [49]. Sigalov [61] announced results similar to those presented here.

We begin with the following lemma which has a proper generalization for all values of $v$ (see [42] and [47]):

**Lemma 4.1**: Suppose a vector $z(x) \in \mathcal{B}_2$ on a domain $G$ and suppose that for every vector $x_0 \in G$. Then

$$
(4.1) \quad \int_{B(x_0, r)} |V z|^2 \, dx \leq L^2 (r/a)^{2v} \quad \text{for } 0 \leq r \leq a,
$$

whenever $B(x_0, a) \subseteq G$. Then

$$
(4.2) \quad |z(x_2) - z(x_1)| \leq C_1(\lambda) \cdot L \cdot (|x_1 - x_2|/a)^v \quad \text{for } 0 \leq |x_1 - x_2| \leq a,
$$

where

$$
C_1(\lambda) = 2^{1-\lambda} \pi^{-1/2} \lambda^{-1}
$$

for every pair of points $(x_1, x_2)$ in $G$ such that every point on the segment joining them is at a distance $\geq a$ from $\partial G$.

*Proof*: We note first that if $\xi$ is on the segment and $s \leq a$,

$$
\int_{B(\xi, s)} |V z(y)| \, dy \leq \pi^{1/2} L a^{-v} s^{v+1},
$$
using the Schwarz inequality. Next we write

$$|z(x) - z(x_1)| \leq |z(x) - z(x_2)| + |z(x) - z(x)|$$

$$|z(x) - z(x_k)| = |(x^a - x^a_k)\int_0^1 z_{x_k} [x_k + t(x - x_k)] dt|$$

$$\leq r \int_0^1 |V z [x_k + t(x - x_k)]| dt, r = |x_2 - x_1|, k = 1, 2,$$

and then average with respect to $x$ over $B(\overline{x}, r/2), \overline{x} = (x_1 + x_2)/2$. If for a given $t, 0 < t < 1$, we set $y = x_k + t(x - x_k)$, then $y$ ranges over $B[(1 - t)x_k + tx, rt/2]$. Then

$$\int_{\partial B(x_0, r)} |z(x) - z(x_k)| dx \leq r \int_0^1 t^{-2} \left[ \int |V z(y)|^2 dy \right] dt$$

from which the result follows.

**NOTATION:** If $z \in \mathcal{C}_2$ on $G$, we define $D(z, G) = \int_G |V z|^2 dx$; this is called the Dirichlet integral.

**Lemma 4.2:** Suppose $z \in \mathcal{C}_2$ on $B(x_0, a)$ and suppose

$$D[z, B(x_0, r)] \leq K \cdot D[Z_r, B(x_0, r)] + \psi(r), 0 < r \leq a$$

where

$$\int_0^a s^{-1} \psi(s) ds$$

converges, for every function $Z_r = z$ on $\partial B(x_0, r)$. Then

$$D[z, B(x_0, r)] \leq D[z, B(x_0, a)] (r/a)^{1/2} + K^{-1} r^{1/2} \int_0^a s^{-1} - \frac{1}{K} \psi(s) ds$$

and the right side tends to zero with $r$. 

36

using the Schwarz inequality. Next we write

$$|z(x) - z(x_1)| \leq |z(x) - z(x_2)| + |z(x) - z(x)|$$

$$|z(x) - z(x_k)| = |(x^a - x^a_k)\int_0^1 z_{x_k} [x_k + t(x - x_k)] dt|$$

$$\leq r \int_0^1 |V z [x_k + t(x - x_k)]| dt, r = |x_2 - x_1|, k = 1, 2,$$
Proof: Let \( \varphi (r) = D \{ z, B (x_0, r) \} \). Then \( \varphi \) is absolutely continuous. For almost all \( r, z (r, \theta) \) is \( AC \) in \( \theta \) with \( |z_\theta (r, \theta)| \) in \( L_2 \). For such \( r \), define

\[
Z_r (\theta) = \bar{z} (r) + (\varphi / r) \{ z (r, \theta) - \bar{z} (r) \}, \quad \bar{z} (r) = \frac{1}{2\pi} \int_0^{2\pi} z (r, \theta) \, d\theta.
\]

Using Fourier series, one easily sees that

\[
\int_0^{2\pi} |z (r, \theta) - \bar{z} (r)|^2 \, d\theta \leq \int_0^{2\pi} |z_\theta (r, \theta)|^2 \, d\theta \leq r \varphi' (r)
\]

By computing \( D_2 \{ Z_r, B (x_0, r) \} \) and using (4.5) we see that

\[
\varphi (r) \leq K r \varphi' (r) + \psi (r)
\]

from which (4.4) follows easily. In order to see that the right side of (4.4) tends to zero with \( r \), we note that

\[
r^{1/2} \int_0^a \varphi (q) \, dq \leq \int_0^{a/2} \varphi (q) \, dq + r^{1/2} \int_0^{a/2} \varphi (q) \, dq.
\]

**Theorem 4.1:** Suppose \( f (x, z, p) \) is continuous for all \((x, z, p)\) and is convex in \( p \) for each \((x, z)\), and suppose there are constants \( m, M, \text{ and } k \) such that

\[
m |p|^2 - k \leq f (x, z, p) \leq M |p|^2 + k, \quad M \geq m \geq 0,
\]

for all \( p \). Suppose \( I (z_0, G) \) is finite, \( G \) is a bounded domain, and \( z_0 \) minimizes \( I (z, G) \) among all \( z \) in \( \mathcal{B}_2 \) coinciding with \( z_0 \) on \( \partial G \). Then \( z_0 \) satisfies (4.1) and (4.2) on \( G \) with

\[
|p|^2 \leq f (x, z, p) \leq |p|^2 + k, \quad M \geq m \geq 0.
\]

Thus \( z_0 \) satisfies a uniform Hölder condition on each compact subset of \( G \).

**Proof:** Suppose \( B (x_0, r) \subseteq G \) and let \( Z_r \) be any function in \( \mathcal{B}_2 \) on \( B (x_0, r) \) and coinciding with \( z_0 \) on \( \partial B (x_0, r) \). Then, from (4.7)

\[
m D [z_0, B_r] - k \pi r^2 \leq I (z_0, B_r) \leq I (Z_r, B_r) \leq MD (Z_r, B_r) + k \pi r^2
\]

\[
D (z_0, B_r) \leq \frac{M}{m} D (Z_r, B_r) + \frac{2k \pi}{m} r^2 \quad (B_r = B (x_0, r)).
\]

The result follows from Lemma 4.2.
For the remainder of this section, we shall assume that \( f(x,z,p) \) satisfies the following condition in addition to \((4.7)\):

**General Assumptions:** We assume that \( G \) is a bounded domain, \( f \) satisfies the conditions of Theorem 4.1, and

(i) \( f \) is of class \( C^n \) for all \((x,z,p)\).

(ii) there are functions \( m_1(R), M_1(R), \) and \( M_2(R) \) with \( 0 < m_1(R) \leq M_1(R) \) for all \( R > 0 \) such that

\[
(4.9) \quad m_1(R) |\pi| \leq f_p \sum_{\alpha,\beta} |\pi^\alpha_p| \leq M_1(R) |\pi|^2
\]

\[
(4.10) \quad \sum_{j=1}^N \sum_{k=1}^{N} \left[ f_{x_{j_k}} + \sum_{\alpha=1}^{2} f_{p_{\alpha j_k}} + \sum_{\beta=1}^{2} f_{p_{\beta j_k}} \right] \leq M_2(R) \cdot |p|^2
\]

for all \((x,z,p)\) such that \( |x|^2 + |z|^2 \leq R^2\).

**Theorem 4.2:** Suppose \( f \) and \( G \) satisfy the general assumptions, \( z_0 \) satisfies the continuity conclusions of Theorem 4.1, and \( \zeta \) is any Lipschitz function on \( G \) which vanishes on and near \( \partial G \), and \( \varphi(\lambda) = I(z_0 + \lambda \zeta) \). Then \( \varphi'(0) \) exists and

\[
(4.11) \quad \varphi'(0) = \int_{\partial G} \left[ f_{x_j} [x, z_0(x), p_0(x)] \zeta^j(x) + f_{p_{\alpha}} [x, z_0(x), p_0(x)] \pi^\alpha_p \right] dx
\]

**Proof:** Let \( F \) be the compact support of \( \zeta \). Since \( z_0 \) is continuous on \( F \), \( |x|^2 + |z_0(x)|^2 \leq R^2 \), for some \( R \), for all \( x \) on \( F \). Then, for almost all \( x \) on \( F \),

\[
\begin{align*}
&f[x, z_0(x) + \lambda \zeta(x), p_0(x) + \lambda \pi(x)] = f[x, z_0(x), p_0(x)] + \lambda [f_{x_j} \zeta^j + f_{p_{\alpha}} \pi^\alpha_p] + \lambda^2 [A_{\alpha j_k}(x, \lambda) \pi^\alpha_p \zeta^k + 2B_{\alpha j_k}(x, \lambda) \pi^\alpha_p \zeta^k + C_{\alpha j_k}(x, \lambda) \zeta^j \zeta^k]
\end{align*}
\]

where, for instance,

\[
A_{\alpha j_k}(x, \lambda) = \int_0^1 (1 - t) f_{p_{\alpha j_k}}[x, z_0(x) + t \lambda \zeta(x), p_0(x) + t \lambda \pi(x)] dt.
\]

Clearly all the \( A_{\alpha j_k}, B_{\alpha j_k}, \) and \( C_{\alpha j_k} \) are measurable and we conclude also from the general assumptions and the Lipschitz character of \( \zeta \) that

\[
(4.12) \quad \varphi(\lambda) - \varphi(0) - \lambda \int_{\partial G} \left( f_{x_j} \zeta^j + f_{p_{\alpha}} \pi^\alpha_p \right) dx = \lambda^2 K(\lambda)
\]

where \( K(\lambda) \) is uniformly bounded for \( |\lambda| \leq 1 \). The result follows.
DEFINITION: If \( \varphi(0) = 0 \) for every \( \zeta \) as in Theorem 4.2, we say that \( z_0 \) furnishes a stationary value to the integral \( I(z, G) \).

COROLLARY: If \( f, G \), and \( z_0 \) satisfy the conditions of Theorem 4.2 and if \( z_0 \) minimizes \( I(z, G) \) among all sufficiently near \( z \) (\( C_2 \) sense) having the same boundary values, then \( z_0 \) furnishes a stationary value to \( I(z, G) \).

In order to obtain further differentiability properties of the solutions \( z_0 \), we must consider the solutions \( u \) of equations

\[
(4.13) \quad \int_{\Omega} \left[ v^i (a^{ij}_{jk} u^j + b^{ij}_{jk} u^k + c^{ij}_{jk}) + v^j (b^{ij}_{jk} u^j + c^{ij}_{jk} u^k + f^j) \right] \, dx = 0, \quad v \in C^{1,2}_2
\]

where all the coefficients are measurable and satisfy

\[
(4.14) \quad m_1 |\pi|^2 \leq a^{ij}_{jk}(x) \pi^i \pi^j \leq M_1 |\pi|^2 \quad \text{for all} \quad \pi,
\]

\[
(4.15) \quad \int_{B(x_0, r) \cap \Omega} \left( |b|^2 + |c| + |f| \right) \, dx \leq M_2^2 r^{2i}, \quad c \in L_2, \quad 0 < m_1 \leq M_1.
\]

We begin by considering the case where \( b^{ij}_{jk} = c^{ij}_{jk} = 0 \) and set

\[
I_0(u, v; G) = \int_{\Omega} v^i a^{ij}_{jk} u^j \, dx.
\]

From our general assumptions, we see that

\[
(4.16) \quad m_1 \int_{\Omega} |Vu|^2 \, dx \leq I_0(u, u; G) \leq M_1 \int_{\Omega} |Vu|^2 \, dx.
\]

From this result and the Poincaré inequality (Theorem 1.11), we see that the space \( \mathcal{C}^{1,2}_2 \) is a Hilbert space if we take \( I_0(u, v; G) \) as an inner product and that the resulting norm is topologically equivalent to the original \( \mathcal{C}^{1,2}_2 \) norm on \( \mathcal{C}^{1,2}_2 \).

LEMMA 4.3: If \( S \) is any set of finite measure, then

\[
\int_S |x - x_0|^h \, dx \leq 2\pi \cdot h^{-1} s^h, \quad 0 < h < 2, \quad \pi s^2 = m(S).
\]

Proof: Obviously

\[
\int_S |x - x_0|^h \, dx \leq \int_{B(x_0, r)} |x - x_0|^h \, dx.
\]
LEMMA 4.4: Suppose \( u \in C^{20}_{\mathbb{R}} \) on \( G \), \( f \in \mathcal{L}_{1} \) on \( G \), and
\[
\int_{B(x_0, r) \cap G} |f(x)| \, dx \leq L r^{2\lambda}
\]
for every circle \( B(x_0, r) \). Then \( u \cdot f \in \mathcal{L}_{1} \) on \( G \) and satisfies
\[
\int_{B(x_0, r) \cap G} |f(x) \cdot u(x)| \, dx \leq C_{1}(\lambda, \mu) \cdot L \cdot \|Vu\|_{L_{n}} \cdot g^{n} r^{2\lambda - \mu}, \quad 0 < \mu < \lambda,
\]
where
\[
C_{1}(\lambda, \mu) = 2^{-1} \pi^{1/2} \lambda^{1/2} \mu^{-1/2} (\lambda - \mu)^{-1/2}, \quad \pi g^{2} = \mu(G);
\]
u and \( f \) may be tensors.

Proof: The proof for the general vector \( u \) in \( C^{20}_{\mathbb{R}} \) will follow from the result for class \( C^{0} \) which vanishes near \( \partial G \). Let \( x \in G \) and suppose \( \overline{G} \subset B(x_1, R) \) and extend \( u \) to be zero outside \( G \). Then if we set
\[
v^{i}(r, \theta) = u^{i}(x_1 + r \cos \theta, x_2 + r \sin \theta),
\]
we see that
\[
u^{i}(x_1) = v^{i}(0, \theta) = -\frac{1}{2\pi} \int_{0}^{2\pi} v^{i}(r, \theta) \, dx \, d\theta
\]
(4.17)
\[
= -\frac{1}{2\pi} \int_{B(x_0, r) \cap G} |\xi - x_1|^{-2} \left( \xi^n - x^n \right) u^{i}_{,n}(\xi) \, d\xi.
\]
Hence
\[
\int_{B(x_0, r) \cap G} |f(x) \cdot u(x)| \, dx \leq \frac{1}{2\pi} \int_{B(x_0, r) \cap G} |f(x)| \cdot |\xi - x|^{-1} \cdot |Vu(\xi)| \, d\xi \, dx.
\]
(4.18)
Applying the Schwarz inequality judiciously to (4.18), we obtain
\[
\int_{B(x_0, r) \cap G} |f(x) \cdot u(x)| \, dx \leq \frac{1}{2\pi} \left[ \int_{B(x_0, r) \cap G} \int_{G} |\xi - x|^{2\mu - 2} \cdot |f(x)| \, dx \, d\xi \right]^{1/2}
\]
(4.19)
\[
\left[ \int_{B(x_0, r) \cap G} \int_{G} |f(x)| \cdot |\xi - x|^{-2\mu} \cdot |Vu(\xi)|^{2} \, dx \, d\xi \right]^{1/2}.
\]
Using Lemma 4.3 we see that

\[(4.20) \quad \int_{B(x_0,r) \cap G} |\xi - x|^{\alpha - 2\mu} |f(x)| \, dx \, d\xi \leq \pi \mu^{-1} g^{\alpha} \cdot L r^{2\mu}.
\]

Next, define,

\[q_{\xi}(q) = \int_{B(x_0,r) \cap B(x_0,r) \cap G} |f(x)| \, dx.
\]

From our assumption on \(f\), we see that

\[q_{\xi}(q) \leq L q^{2\mu} \quad \text{and} \quad L r^{2\mu}.
\]

Accordingly

\[
\int_{B(x_0,r) \cap G} |\xi - x|^{-2\mu} |f(x)| \, dx = \int_0^{\infty} q_{\xi}^\mu(q) \, dq =
\]

\[= \int_0^{\infty} 2\mu q^{-2\mu - 1} q_{\xi}^\mu(q) \, dq \leq L (\lambda - \mu)^{-1} r^{2\mu - 2\mu}
\]

\[(4.21) \quad \int_{B(x_0,r) \cap G} \int_{B(x_0,r) \cap G} |f(x)| \cdot |\xi - x|^{-2\mu} |Vu(\xi)|^2 \, dx \, d\xi \leq
\]

\[\leq L (\lambda - \mu)^{-1} r^{2\mu - 2\mu} \cdot \|Vu\|_{L^2}^2.
\]

The result follows from (4.20) and (4.21).

**Lemma 4.5**: Suppose \(u\) and \(f\) satisfy the hypotheses of Lemma 4.4. Then \(fu^2 \in L^1\) on \(G\) and

\[
\int_{B(x_0,r) \cap G} |f(x)| \cdot |u(x)|^2 \, dx \leq C_2(\lambda, \mu) \cdot L \cdot \|Au\|_{L^2}^2 \cdot g^{\alpha} \cdot r^{2\mu - \alpha} ; \quad 0 \leq \mu \leq \lambda.
\]

**Proof**: This follows from two applications of Lemma 4.4.

**Theorem 4.3**: There is an \(a_0 > 0\) and depending only on \(m_1, M_1, M_2\), and \(\lambda\) such that if \(0 < a \leq a_0\) and \(B(x_0, a) \subset G\), then

\[I[u, u; B(x_0, a)] \geq \frac{m_1}{2} D[u, B(x_0, a)] \quad \text{for all} \quad u \in C_{B_{20}} \quad \text{on} \quad B(x_0, a).
\]
Proof: For

\[ I[u, u; B(x_0, a)] = \int_{B(x_0, a)} (2b^k_{jk} u^j u^k + c_{jk} u^j u^k) \, dx \]

\[ \geq D_2[u, B(x_0; a)] \cdot |m_1 - 2c^{1/2} M^{1/2} g^{1/2} a^{1-\mu/2} - C_2 g^\mu a^{2^\lambda - \mu}|, \quad 0 < \mu < \lambda, \]

using Lemma 4.5 and the Schwarz inequality.

**Theorem 4.4**: If $0 < a \leq a_0$, $B(x_0, a) \subset G$, $b^k_{jk}$, $c_{jk}$, and $f$ satisfy (4.15) and $e \in L^2$ on $B(x_0, a)$, there exists a unique $u$ in $\mathscr{B}_{20}$ on $B(x_0, a)$ such that (4.13) holds for all $v \in \mathscr{B}_{20}$ on $B(x_0, a)$. Moreover

\[ (4.22) \quad B[u, B(x_0; a)] \leq 2m_1^{-1} \left( \| e \| L^1 + C_1(\lambda, \mu) \cdot M_2 \cdot a^{2^\lambda - 2} \right), \quad 0 < \mu < \lambda. \]

**Proof**: From Theorem 4.3 and the Poincare inequality (Theorem 1.11), we see that the space $\mathscr{B}_{20}$ is a Hilbert space if we introduce $I(u, v)$ as inner product. Since the equation (4.13) ($G = B(x_0, a)$) can be written

\[ (4.23) \quad I(u, v) = L(v), \quad L(v) = \int_{B(x_0, a)} (e^j v^j, f^j v^j) \, dx \]

and since $L(v)$ is a linear functional, we see from Hilbert space theory that there is a unique $u$ in $\mathscr{B}_{20}$ which satisfies the equation. If, now, we revert to $|D[u, B(x_0, a)]|^{1/2}$ as norm, we see from (4.23) and Lemma 4.4 that the norm of $L(v)$ is given by the bracket on the right in (4.22). The inequality (4.22) follows by comparing the $I$ and $D$ norm.

We can now prove the interior boundedness theorem:

**Theorem 4.5**: Suppose $u \in L^2$ on $B(x_0, a) \subset G$ where $0 < a \leq a_0$, $u \in \mathscr{B}_{20}$ on $B(x_0, r)$ and (4.13) holds for each $v \in \mathscr{B}_{20}$ on $B(x_0, r)$ for each $r$ with $0 < r \leq a$. Then

\[ \{ D[u, B(x_0, r)] \}^{1/2} \leq C_3 (m_1, M_1) \| e \| L^1 + C_1 L a^{2^\lambda} + (a - r)^{-1} \| u \| \]

\[ (C_1 = \min_{0 < a < \lambda} C_1(\lambda, \mu)) \]

the norm being the $L^2$ norms.

**Proof**: Let $h$ be a fixed function of class $C^\infty$ with $h(s) = 1$ for $s \leq 0$ and $h(s) = 0$ for $s \geq 1$ and $0 < h(s) \leq 1$. Choose $R$ so $r < R < a$ and define

\[ \xi(x) = h(|x - x_0| - r)/(R - r), \quad v^j = \xi^2 u^j, \quad U^j = \xi u^j. \]
Then $v$ and $U \in C_{20}$ on $B(x_0, R)$. Substituting in (4.13), we obtain

$$
0 = I(U, U; B(x_0, R)) + \int_{B(x_0, R)} \left( \zeta e \frac{U}{\partial} + \zeta f e U + \zeta \xi \xi e u^i - a_{ij} \xi \xi \xi \xi u^i \right) dx
$$

$$
\geq \frac{m_1}{2} || U ||_2^2 - || U ||_2 || e || + C_1(\lambda, \mu) M_2 \cdot R^{2\mu} - h_k (R - r)^{-1} || e ||_2 || u ||_2 - h_k M_1 (R - r)^{-2} || u ||_2^2
$$

where $|| U ||_2$ is the $C_{20}$ -- D-norm and $|| u ||_2$ is the $L_2$ norm. Since (4.24) holds for all $R < \alpha$, the result follows.

**Lemma 4.6:** If $u \in C_{20}$ on $B(x_0, R)$, there is a $u_1 \in C_{20}$ on $B(x_0, 2R)$ such that $u_1(x) = u(x)$ on $B(x_0, R)$ and

$$
D[u_1, B(x_0, 2R)] \leq C_2 \int_{B(x_0, R)} (| V u |^2 + sR^{-2} | u |^2) dx
$$

where $C_2$ is an absolute constant.

**Proof:** Define $u_2(x) = u(x)$ on $B(x_0, R)$ and extend it by reflection in the circle $B(x_0, R)$. Then $u \in C_{20}$ on $B(x_0, 2R)$ and

$$
\int_{B(x_0, 2R)} | V u_2 |^2 dx \leq \int_{B(x_0, R)} | V u |^2 dx
$$

$$
\int_{B(x_0, 2R)} | u_2 |^2 dx \leq 16 \int_{B(x_0, R)} | u |^2 dx
$$

Then, define

$$
u_1(x) = h \left( | x - x_0 | - R/R \right) \cdot u_2(x),$$

where $h$ is function introduced in the proof of Theorem 4.5. Then $v_1$ is easily seen to have the desired properties.

**Theorem 4.6 (Dirichlet growth theorem):** Suppose $0 < a \leq a_0, B(x_0, a) \subset G, u \in C_{20}$ on $B(x_0, a)$, (4.13) holds for all $v \in C_{20}$ on $B(x_0, a)$, and $e$ satisfies the condition

$$
\int_{B(x_0, r)} | e |^2 dx \leq L^2 (r/\delta)^{2\mu}, \quad 0 \leq r \leq \delta = a - | x_1 - x_0 |,
$$

for some $\mu$ with $0 < \mu < \lambda/2$ and $m_1/2M_1$, and every circle $B(x_1, r) \subset B(x_0, a)$. Then $u$ satisfies the condition (4.1) and (4.2) with $G$ replaced by $B(x_0, a)$. $x_0$...
replaced by \( x^i \), \( a \) by \( \delta = a - |x_1 - x_0| \), \( \lambda \) replaced by \( \mu \), and \( L \) replaced by \( C_{\delta} \), where \( C_{\delta} \) depends only on \( m_1, M_1, M_2, L, \lambda, \mu, a \), and \( \| u \| \) where

\[
\int_{B(\delta, a)} (|V u|^2 + a^{-2} |u|^2) \, dx = \| u \|^2.
\]

Thus \( u \) satisfies a uniform Hölder condition on any \( B(x_0, R) \) with \( R < a \) which depends only on the quantities above and \( a - R \).

**Proof:** Let

\[
E_{ij}^n = b^n_{ij} u^k + c^n_j, \quad F_j = b^n_{kj} u^k + c^n_j u^k + f_j.
\]

From our hypotheses on the \( b' \)s, \( c' \)s, \( e' \)s, and \( f' \)s and from Lemmas 4.4, 4.5, and 4.6, we see that

\[
\int_{B(x, r)} |E|^2 \, dx \leq C_{1/2}(\lambda, \mu') \cdot M_2 \cdot \| u \| \cdot a^{\mu'/2} \cdot e^{\lambda - \mu'/2} + L(r/\delta)^{\mu} \quad 0 < \mu' < \lambda,
\]

and

\[
\int_{B(x, r)} |F| \, dx \leq [M_2 r^k + C_{1}(\lambda, \mu') \cdot M_2 a^{\mu'} e^{2k - \mu'}] \cdot \| u \| + M_2^2 r^{2k}.
\]

Moreover, \( u \) satisfies the equation

(4.25) \( I_0[u, v; B(x_1, r)] = - \int_{B(x_1, r)} \left( E_{ij}^n v^i_j + F_j v^j \right) \, dx \); \( v \in C_0^0 \) on \( B(x_1, r) \)

on any \( B(x_1, r) \subset B(x_0, a) \). As in the proof of Theorem 4.4, there is a unique solution \( U_r \) of (4.25) which is in \( C_0^0 \) on \( B(x_1, r) \) and

\[
D[U_r, B(x_1, r)] \leq m_1^{-1} |Z_1 a^1 \| u \| + L^2 (r/\delta)^{2k}
\]

where \( Z_1 \) depends only on the quantities mentioned.

Now \( V_r = u - U_r \) satisfies the homogeneous equation (4.25) and so clearly minimizes \( I_0[V, V; B(x_1, r)] \) among all \( V = V_r(= u) \) on \( \partial B(x_1, r) \). Since \( U_r \in C_0^0 \) on \( B(x_1, r) \), we see that

\[
I_0(V_r, U_r; B_r) = 0 \quad \text{so} \quad I_0(u, u; B_r) = I_0(V_r, V_r; B_r) + I_0(U_r, U_r; B_r),
\]

where \( B_r = B(x_1, r) \). Using the fact that \( I_0(V_r, V_r; B_r) \leq I_0(u, u; B_r) \) for any \( u_r = u \) on \( \partial B(x_1, r) \) and using (4.16), we see that

\[
D(u, B_r) \leq \frac{M_1}{m_1} D(u_r, B_r) + Z_2 (r/\delta)^{2k}
\]
where $Z_2$ depends only on the quantities indicated. The results follow from
Lemmas 4.2 and 4.1.

We can now resume our discussion of a solution $z_0$ of a variational
problem of the type being discussed here.

**Theorem 4.7**: Suppose $z_0$ gives a stationary value to $I(x, G)$ and sati-
ifies the continuity conclusions of Theorem 4.1. Then $z_0 \in C^{1+\nu}$ on each domain
$\Gamma$ with $\bar{\Gamma} \subset G$, where $0 < \mu < 1$, and the derivatives $\partial_i \partial_j z_0'$ on domains inte-
rior to $G$.

**Proof**: Since $q(0) = 0$, we see that the right side of (4.11) holds for
each Lipschitz $\zeta$ with compact support in $G$. So, suppose $B(x_0, a) \subset G$.
Choose $A > a$ so that $B(x_0, A) \subset G$. Then, from Theorem 4.1, we have
$|x|^2 + |z_0(x)|^2 < R^2$, for some $R$, on $B(x_0, A)$. Let $b = (2a + A)/3$
$c = (a + 2A)/3$, $h_0 = (A - a)/3$, let $e_r$ be the unit vector in the $x^r$ direction
for $\gamma = 1, 2$, let $v$ be an arbitrary Lipschitz function having support in
$B(x_0, c)$ and define

$$
\zeta_h(x) = h^{-1} [v(x - he_r) - v(x)],
\zeta^l_h(x) = h^{-1} [z^l_0(x + he_r) - z^l_0(x)]
$$
for $0 < |h| < h_0$. Then $\zeta_h$ has support in $B(x_0, A)$. Substituting $\zeta_h$ into the
equation $q'(0) = 0$ and using (4.11), we see that $u_h$ satisfies equation (4.13)
on $B(x_0, c)$ with coefficients $a_{h,k}^{ij}$, etc., where

$$
(4.26) \quad a_{h,k}^{ij}(x) = \int_0^1 \int_{x_h + he_r} [x^i + t\theta^i_r, (1 - t) z_0(x) + t\theta^i_r, (1 - t) p_0(x)
+ t\theta^i_r, (1 - t) p_0(x)]
\, dt
$$
for almost all $x$. From the general assumptions on $f$ and from the formulas
(4.26) for the coefficients, we see that the bounds (4.14) and (4.15) hold uni-
formly for $0 < |h| < h_0$ with

$$
m_4 = m_4(K), M_4 = M_4(K), M_2 = KM_2(K), 2\lambda = \mu/M, G = B(x_0, c),
$$
where $K$ is a constant depending on $\lambda$ and the distance of $B(x_0, A)$ from
$\partial G$. Clearly each $u_h \in \mathcal{C}^2_2$ on $B(x_0, c)$ and its $L_2$ norm is uniformly boun-
ded there, and we also have

$$
\int_{B(x_0, c)} |e_h|^2 \, dx \leq M_2^2 \rho^{2\lambda}, \quad 0 < |h| < h_0.
$$
Accordingly, we see first from Theorem 4.5 that the $\mathcal{C}^2_2$ norms of the $u_h$
are uniformly bounded on $B(x_0, b)$ and then from Theorem 4.6 that the $u_h$
satisfy a uniform Hölder condition on $\bar{B}(x_0, a)$ independently of $h$. Thus
we may let $h \to 0$ and we see that the derivatives $z^l_0 \in \mathcal{C}^2_2$ and satisfy this
Hölder condition on $\bar{B}(x_0, a)$.
CHAPTER V

A variational method in the theory of harmonic integrals.

In this section, we apply our variational method to the study of harmonic integrals and, more generally, use it to obtain the Kodaira decomposition theorem [29] (see Theorem 5.10 below). This approach was originally suggested by Hodge in his first paper on the subject [25]. The generality of the manifolds allowed and the methods and results obtained are closely related to those obtained by Friedrichs [20] working independently. Of course corresponding results have been obtained on smoother manifolds by a number of other authors using other methods ([12], [23], [26], [29], [38]). In this section, we shall confine ourselves to compact manifolds without boundary. The variational methods are applied to compact manifolds with boundary in [20] and [46]; boundary value problems for forms have been considered by other writers using other methods in [13], [66].

We adopt the usual definition of a compact Riemannian manifold of dimension n (instead of v) and of class $C^k$ or $C^k_\mu (0 < \mu \leq 1)$ any two admissible coordinate systems are related by a transformation of class $C^k_\mu$; respectively. If $0 < \mu < 1$, the class $C^k_\mu$ is the same as what we have called $O^{k+\mu}$; If $\mu = 1$, a function is of class $O^k_1$ if and only if its derivatives of order $\leq k$ satisfy Lipschitz conditions; transformations of class $C^k_1$ are defined similarly. If a coordinate system is of class $C^\infty$, the induced $g_{ij}$ are of class $O^{k-1}_1$. We shall assume that our manifold is of class at least $O^1$.

We shall be concerned with exterior differential forms of degree r on a manifold $M$; we call these simply r-forms. In the domain of a given coordinate system such a form $\omega$ may be represented by

$$\omega = \sum_{i_1 < \ldots < i_r} \omega_{i_1 \ldots i_r} dx^{i_1} \wedge \ldots \wedge dx^{i_r}$$

where $\omega_{i_1 \ldots i_r}$ are the components of $\omega$ in that coordinate system and $\wedge$ denotes the exterior product. In order to take care of the case of non-orientable manifolds, we allow both even and odd forms. If two coordinate systems $(x)$ and $(x')$ overlap, the components transform according to the law

$$\omega_{i_1 \ldots i_r} (x') = \varepsilon \sum_{j_1 < \ldots < j_r} \omega_{j_1 \ldots j_r} (x(x')) \frac{\partial (x^{i_1} \ldots x^{i_r})}{\partial (x^{j_1} \ldots x^{j_r})},$$

$$\varepsilon = \begin{cases} -1 & \text{for even forms}, \\ J/|J| & \text{form odd forms}, \end{cases} \quad J = \frac{\partial (x^1 \ldots x^n)}{\partial (x^{i_1} \ldots x^{i_r})}. $$
Since the Jacobians involved in (5.2) are at least of class \( C^0 \) (Lipschitz), we may say that a form \( \omega \) is of class \( C^2 \) or \( C^{3/2} \) -- its components in each coordinate system are.

Given an \( r \)-form \( \omega \), we define its dual \( *\omega \) by

\[
(*\omega)_{j_1 \ldots j_{n-r}} = \sum_{i_1 < \ldots < i_{r}} \epsilon_{j_1 \ldots j_{n-r}} \sum_{k_1 \ldots k_r} g^{k_1 i_1} \ldots g^{k_r i_r} \omega_{i_1 \ldots i_r}
\]

(5.3)

where \( \epsilon_{j_1 \ldots j_{n-r}} \) is 0 if two indices \( p_i \) are the same or otherwise is \( \pm 1 \) according as \( p_1 \ldots p_n \) is an even or odd permutation, \( k_1 < \ldots < k_r \) are chosen so that \( k_1 \ldots k_r j_1 \ldots j_{n-r} \) is a permutation, \( \Delta^{(k)} \) is the determinant of the \( g^{k j} \), and \( \Gamma = \pm \sqrt{\det g} \) chosen so that \( \Gamma dx^1 \wedge \ldots \wedge dx^n = dS \), the positive volume element. If two forms \( \omega \) and \( \eta \) of the same kind (both even or both odd) of the same degree are in \( \mathcal{L}_2 \) on \( M \), we define their inner product

\[
(\omega, \eta) = \int_M \omega \wedge * \eta;
\]

we form inner products only under these conditions. If \( \omega \) is an \( r \)-form given in the \( x \)-system by (5.1) and if \( \eta \) is an \( s \)-form of the same kind with a corresponding representation, we define

\[
(\omega, \eta) = \sum_{(i)} \omega_{i_1 \ldots i_r} \eta_{j_1 \ldots j_s} \sum_{k_1 \ldots k_r} g^{k_1 i_1} \ldots g^{k_r j_s} \omega_{i_1 \ldots i_r} \eta_{j_1 \ldots j_s}.
\]

Accordingly the inner product \( (\omega, \eta) \) is also given by

\[
(\omega, \eta) = \int_M (F(P; \omega, \eta) \, dS_P)
\]

(5.6)

where \((i) = i_1 \ldots i_r\), where \( i_1 < \ldots < i_r \), etc. In case \( P \) corresponds to \( x_0 \) in the \( x \) system and \( g_{ij}(x_0) = \delta_{ij} \), we see that

\[
F(P; \omega, \eta) = \sum_{(i)} \omega_{i_1 \ldots i_r}(x_0) \eta_{i_1 \ldots i_r}(x_0) \, dS_P \mid dx \mid.
\]
The following theorem is well known and is evident.

**Theorem 5.1.** For each \( r = 0, 1, \ldots, n \) the totality of \( r \)-forms of a fixed kind \( \mathcal{Q}_r \) on \( M \) (with equivalent forms identified) forms a real Hilbert space \( \mathcal{L}_r^2 \) with inner product given by (5.4).

In order to introduce an inner product in \( \mathcal{E}_2^r \) on \( M \), we proceed as follows:

**Definition:** Let \( \mathcal{U} = (U_1, \ldots, U_q) \) be a finite open covering of \( M \) by coordinate patches \( U_q = Q_q(G_q) \), where each \( G_q \) is a Lipschitz domain in \( \mathcal{E}^n \). If \( \omega \) and \( \eta \) are in \( \mathcal{E}_2^r \) on \( M \) we define

\[
(\omega, \eta)_{\mathcal{E}_2^r} = (\omega, \eta) + \sum_{q=1}^Q \sum_{\alpha=1}^n \alpha \omega_{(\alpha)q}^{(q)} \eta_{(\alpha)q}^{(q)} \int_{G_q} dx,
\]

where \( \omega_{(\alpha)q}^{(q)} \) and \( \eta_{(\alpha)q}^{(q)} \) are the components of \( \omega \) and \( \eta \) in \( Q_q \). Then

\[
||\omega||_{\mathcal{E}_2^r} = ((\omega, \omega))^{1/2}_u
\]

is the expression for the norm in \( \mathcal{E}_2^r \) on \( M \) corresponding to the inner product (5.8). It is clear that convergence of \( \omega_k \) to \( \omega \) according to one of the norms (5.9) is equivalent to the strong convergence in \( \mathcal{E}_2^r \) of the components \( \omega_k \) in any coordinate system to those of \( \omega \). Thus we obtain the theorem:

**Theorem 5.2:** For each coordinate cover \( \mathcal{U} \) and each \( r = 0, \ldots, n \) the space of \( r \)-forms in \( \mathcal{E}_2^r \) of a given kind on \( M \) forms a real Hilbert space \( \mathcal{E}_2^r \) with inner product given by (5.8). Any two such inner products are topologically equivalent.

Now, if \( \omega \) is an \( r \) form in \( \mathcal{E}_2^r \), we define \( d\omega \) and \( \delta \omega \) by

\[
\delta \omega = (-1)^{r+n(r-1)} \ast d \ast \omega, \text{ and}
\]

\[
d\omega = \sum_{(i)} \sum_{q=1}^n \omega_{i_1 \ldots i_r q} dx^q \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_r}.
\]

We note that \( d\omega \) is an \((r+1)\)-form (if \( r < n - 1 \)) and \( \delta \omega \) is an \((r-1)\)-form (if \( r \geq 1 \)). Finally, we define the Dirichlet integral by

\[
D(\omega) = (d\omega, d\omega) + (\delta \omega, \delta \omega).
\]

**Theorem 5.3:** \( d \) is a bounded operator from the whole of \( \mathcal{E}_2^r \) into \( \mathcal{L}_2^{r+1} \), and \( \delta \) is a bounded operator from the while of \( \mathcal{E}_2^r \) into \( \mathcal{L}_2^{r-1} \); each of these operators preserves evenness or oddness. \( D(\omega) \) is a lower semi-continuous function with respect to weak convergence in \( \mathcal{E}_2^r \). If \( \omega_k \) tends weakly to \( \omega_0 \) in \( \mathcal{E}_2^r \) on \( M \), then \( \omega_k \) tends strongly to \( \omega_0 \) in \( \mathcal{L}_2^r \) on \( M \).
Proof. The first statement in clear form (5.8) since the $g_{ij}$ are at least Lipschitz and have bounded first derivatives. Now if $\omega_k$ tends weakly in $\mathcal{B}_2$ to $\omega$, $d\omega_k$ and $d\omega$ tend weakly in $\mathcal{L}_2$ to $d\omega$ and $d\omega$, whence the last statement about $D(\omega)$ follows from the lower- semicontinuity of the norm in $\mathcal{L}_2$ with respect to weak convergence. The last statement is an application of Theorem 1.13.

From (5.6) and (5.7), we see that

$$\omega, \eta = (\eta, \omega).$$

(5.12)

In the coordinate system of (5.7), we see that

$$(*\omega)_{ij...j_{n-r}} = e_{i_1}...e_{i_r}j_{i_1...j_{n-r}} \omega_{i_1...i_r} \quad (i \text{ not summed})$$

(5.13)

where $i_1 < ... < i_r$ and $i_1 ... i_r j_1 ... j_{n-r}$ is a permutation. From the form (5.12), we see that

$$** \omega = (-1)^{r(n-r)} \omega.$$

(5.14)

From (5.6) and (5.10) it is easy to see that

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta$$

(5.15)

where $\eta$ is any $s$-form (and $\omega$ is an $r$-form) in $\mathcal{B}_2$. From the rules of exterior multiplication and (5.5), it is easy to see that

$$\eta \wedge \omega = (-1)^{rs} \omega \wedge \eta.$$

(5.16)

From (5.4), (5.12), (5.14), and (5.16), one derives

$$(* \omega, * \eta) = (\omega, \eta).$$

(5.17)

If $M$, $\omega$, and $\zeta$ are all smooth and $\omega$ and $\zeta$ are of the same kind and degrees $r$ and $r - 1$, respectively, we obtain

$$(d\omega, \zeta) = (-1)^{1+r(n-r-1)} (\star d * \omega, \zeta) = (-1)^r (\star d * \omega, \star \zeta)$$

$$= (-1)^r \int_M d * \omega \wedge ** \zeta = (-1)^{r+(n-r-1)} \int_M d * \omega \wedge \zeta$$

$$= (-1)^{r+(n-r-1)} \int_M (d \wedge \omega \wedge \zeta + (-1)^{n-r} \omega \wedge p \zeta$$

$$+ \int_M d \zeta \wedge * \omega = (d\zeta, \omega) = (\omega, d\zeta).$$
since the first integral vanishes by Stoke's theorem for \((n - 1)\)-forms, the bracket being just \(d [\star \omega \wedge \zeta]\) (see (5.15)). We emphasize the result:

\[(5.18) \quad (\delta \omega, \zeta) = (\omega, \delta \zeta) .\]

In the case of smooth manifolds and forms, we see from (5.10) and (5.14) that

\[(5.19) \quad d (\delta \omega) = \delta (d \omega) = 0 .\]

Combining this with (5.18), we see that

\[(5.20) \quad (\delta \omega, \delta \bar{\rho}) = 0 .\]

The formulas (5.18) and (5.20) can be extended to \(\mathcal{B}_2\) forms on manifolds only of class \(C_1\) by using a proper partition of unity (recall Lemma 1.3), such that if the supports of two of the \(h_i\) intersect then their union lies in one coordinate patch, to represent each form as a sum of forms whose supports have the same property. Then, for instance

\[(\delta \omega, \zeta) = \sum_{r,s} (\delta \omega_r, \zeta_s) \]

and each term may be evaluated using one coordinate patch; in that patch, the \(g_{ij}\) and the forms may be approximated by smooth forms.

In the case of a coordinate system of the type in (5.7) where we also assume that all the \(\partial g_{ij}/\partial x^k = 0\) at \(x_0\), we see from (5.10) and (5.13) that the components of \(d \omega\) at \(x_0\) are

\[(5.21) \quad (d \omega)_{i_1...i_{r+1}} = \sum_{q=1}^{r+1} (-1)^{q-1} \omega_{i_1...i_q...i_{q+1}...i_{r+1}} (x_0) \]

where \(i_1 ... i_{r-1} i_1 ... i_{n-r+1}\) is a permutation. From (5.21), we see that Dirichlet integral \(D(\omega)\) in (5.11) reduces to

\[(5.22) \quad D_0(\omega) = \int_G \sum_{(i,k)} \omega_{i,k}^2 \, dx + 2 \sum_{(i,j) \neq \beta} \int_G [\omega_{i,k,\alpha} \omega_{j,\beta,\alpha} - \omega_{i,k,\beta} \omega_{j,\beta,\alpha}] \, dx \]

for the case that \(\omega\) has support in a coordinate patch having domain \(G\) and the \(g_{ij} = \delta_{ij}\) throughout \(G\); the last integrals all vanish in this case.
We now prove the following important lemma, first proved for forms by Gaffney.

**Lemma 5.1:** Given \( \varepsilon > 0 \), \( 0 \leq r \leq n \), and \( P_0 \) on \( M \), there is an admissible coordinate system mapping \( B(0, \varepsilon) \), for some \( \varepsilon > 0 \), onto a neighborhood \( U \) of \( P_0 \), and a constant \( l \) such that

\[
D(\omega) \geq (1 - \varepsilon) \int_{B(0,\varepsilon)} \omega^2 \omega \cdot d\omega - l(\omega, \omega)
\]

for any \( r \)-form \( \epsilon \mathcal{B}_g \) whose support is in \( U \).

**Proof:** We begin by choosing a fixed coordinate system mapping some \( B_R = B(0, R) \) onto a neighborhood \( U_R \) of \( P_0 \), carrying the origin into \( P_0 \), and satisfying \( g_{ij}(0) = \delta_{ij} \). From our formulas for \( d\omega \) and \( \omega \cdot d\omega \), we see that

\[
D(\omega) = \int_{B_R} [d(g_{ij}) \omega_{ij,a} \omega_{ij,b} + 2b(i) \omega_{ij,a} \omega_{ij,b} + c(i) \omega_{ij,a} \omega_{ij,b}] dx
\]

where the \( a \)'s are combination of the \( g_{ij} \) only and so are Lipschitz and the \( b \)'s and \( c \)'s are combinations of the \( g_{ij} \) and their first derivatives and so are bounded and measurable at least. Since the \( a \)'s are Lipschitz and since

\[
|2 \alpha \beta| \leq \eta^2 + \eta^{-1} \beta^2
\]

we see that we may choose \( \eta \) so small that

\[
D(\omega) \geq \left( 1 - \frac{\varepsilon}{2} \right) D_0(\omega) - \frac{\varepsilon}{2} \int_{B_R} \omega^2 \omega \cdot d\omega - l(\omega, \omega)
\]

The result follows from (5.22).

The following important theorem corresponds to Garding's Inequality for differential equations:

**Theorem 5.4:** For each \( r = 0, \ldots, n \) and coordinate covering \( \mathcal{U} \) of \( M \), there exist constants \( K_{2R} > 0 \) and \( L_{2R} \) such that

\[
D(\omega) \geq K_{2R} (\omega, \omega)_{2R} - L_{2R} (\omega, \omega)
\]

for every \( \omega \in \mathcal{B}_g \).

**Proof:** From Theorem 5.2 it is sufficient to prove this for some particular \( \mathcal{U} \). Let \( \mathcal{U} = (U_1, \ldots, U_Q) \) be an open covering of \( M \) by coordinate patches such that each \( x \in M \) is in some \( U_k \) satisfying (5.23) with \( \varepsilon = \frac{1}{2} \).
say. Let $G_1, \ldots, G_Q$ be the domain in $E^n$ such that $U_k = Q_k(G_k)$ for all $k$. There exists a finite sequence $\Phi_1, \ldots, \Phi_s$ of Lipschitz functions on $M$, each of which has support interior to some $U_q$, and such that

$$\sum_{s=1}^{S} \Phi_s(x) = 1$$

for all $x \in M$.

Now if (5.25) were false for the $\mathcal{U}$ just described, there would exist a sequence $(u_p)$ of $r$-forms in $\mathcal{V}_2$ such that $D(u_p)$ and $(u_p, u_p)$ were uniformly bounded but $\|u_p\|_U - \infty$. Then, for some $s$, $q$, and some subsequence, still called $u_p$, we would have

$$\int_{\mathcal{G}_q} \sum_{(i,a)} (\Phi_s u_p)^{q_i} dx - \infty$$

where $\Phi_s$ has support in $U_q$, since

$$\|u_p\|_{\mathcal{G}_q} \leq \sum_{s=1}^{s} \|\Phi_s u_p\|_{\mathcal{G}_q}$$

and

$$\|\Phi_s u_p\|_{\mathcal{G}_q}^2 = (\Phi_s u_p, \Phi_s u_p) + \sum_{(i,a)} \int_{\mathcal{G}_q} (\Phi_s u_p)^{q_i} dx.$$

But it is easy to see that $D(\Phi_s u_p)$ and $(\Phi_s u_p, \Phi_s u_p)$ are uniformly bounded. From our choice of neighborhoods we have reached a contradiction with the fact that

$$D(\Phi_s u_p) \geq \frac{1}{2} \int_{\mathcal{G}_q} \sum_{(i,a)} (\Phi_s u_p)^{q_i} dx.$$

We can now present the variational method. We begin with the following lemma:

**Lemma 5.2**: Let $\mathcal{M}$ be any closed linear manifold in the space $L^r_2$ of $r$-forms on $M$ (of some one kind). Then either there is no form $\omega$ of $\mathcal{M}$ which is in $\mathcal{B}_2$ or there is a form $\omega_0$ in $\mathcal{M} \cap \mathcal{B}_2$ with $(\omega_0, \omega_0) = 1$ which minimizes $D(\omega)$ among all such forms.

**Proof**: If $\mathcal{M}$ contains no form in $\mathcal{B}_2$, there is nothing to prove. Otherwise let $[\omega_k]$ be a minimizing sequence, i.e., one such that $(\omega_k, \omega_k) = 1$ and $\omega_k \in \mathcal{M} \cap \mathcal{B}_2$ for each $k = 1, 2, \ldots$, and such that $D(\omega_k)$ approaches its infimum for all $\omega \in \mathcal{M} \cap \mathcal{B}_2$. From Theorem 5.4 it follows that the
((\omega_k, \omega_k))_{2^r} are uniformly bounded. Accordingly, a subsequence, still called
\{\omega_k\}, exists which converges weakly in \mathcal{B}_2 to some form \omega_0. But from
Theorem 5.3 \omega_k tends strongly in \mathcal{L}^2_{2r} to \omega_0 and \mathcal{D}(\omega) is lower-semiconti-
uous with respect to weak convergence in \mathcal{L}^2_{2r}. The proof of the lemma is
now complete.

**Definition:** A harmonic field \omega on \Omega is a form in \mathcal{B}_2 on \Omega for which
d\omega = \delta \omega = 0 almost everywhere. We will let \mathcal{H}^r denote the linear mani-
fold of harmonic fields on \Omega of degree \nu. (Strictly speaking we have \mathcal{H}_e^r and
\mathcal{H}_o^r for even and odd forms, respectively).

**Theorem 5.5:** For each \nu = 0; \ldots, n (= \dim \Omega) the linear manifold \mathcal{H}^r
is finite dimensional.

**Proof.** The \mathcal{B}_2 forms are dense in \mathcal{L}^2_{2r}, since the Lipschitz forms are.
Let \mathcal{M}_1 = \mathcal{B}_2. There is a form \omega_1 in \mathcal{M}_1 \cap \mathcal{B}_2 which minimizes \mathcal{D}(\omega) among
all such forms with \langle \omega, \omega \rangle = 1. Let \mathcal{M}_k be the closed linear manifold in
\mathcal{L}^2_{2r} orthogonal to \omega_1, and let \omega_2 be the corresponding minimizing form in
\mathcal{M}_2. By continuing this process, we may determine successive minimizing
forms \omega_1, \omega_2, \omega_3, \ldots, each satisfying \langle \omega_k, \omega_k \rangle = 1 and being orthogonal to
all the preceding ones.

Now if \mathcal{D}(\omega_k) > 0, there are no harmonic fields \neq 0 since \mathcal{D}(\omega) \leq
\mathcal{D}(\omega_k) \leq \ldots. On the other hand, suppose \mathcal{D}(\omega_k) = 0 for all values of \nu.
Then by Theorem 5.4, \langle (\omega_k, \omega_k) \rangle_{2^r} is uniformly bounded in \nu, whence a
subsequence \{\omega_p\} converges weakly in \mathcal{B}_2^r and hence strongly in \mathcal{L}^2_{2r}
...
THEOREM 5.7: Suppose \( \omega_0 \) is any form in \( \mathcal{L}_2^r \) and orthogonal to \( \mathcal{H}^r \). Then there is a unique form \( \Omega_0 \) in \( \mathcal{B}_2^r \) and orthogonal to \( \mathcal{H}^r \) such that

\[
(\delta \Omega_0, d\zeta) + (\delta \Omega_0, \delta \zeta) = (\omega_0, \zeta)
\]

for every \( \zeta \) in \( \mathcal{B}_2^r \). Moreover, the transformation from \( \omega_0 \) to \( \Omega_0 \) is a bounded linear transformation from \( \mathcal{L}_2^r \) into \( \mathcal{B}_2^r \).

Proof: From Theorem 5.5, we see that

\[
I(\omega) \equiv D(\omega) - 2\langle \omega, \omega \rangle \geq \lambda_0 \| \omega \|_{2f}^2 - P \| \omega \|_{2f}
\]

since \( \langle \omega, \omega \rangle \) is a bounded linear functional on \( \mathcal{B}_2^r \); here \( \| \omega \|_{2f}^2 = \langle (\omega, \omega) \rangle_{2f} \). Hence \( I(\omega) \) is bounded below and is lower-semicontinuous with respect to weak convergence in \( \mathcal{B}_2^r \) if \( \omega \) is orthogonal \( (L_2 \text{-sense}) \) to \( \mathcal{H}^r \). Accordingly there is a minimizing form \( \Omega_0 \). If \( \zeta \) is any form in \( \mathcal{B}_2^r \) orthogonal to \( \mathcal{H}^r \), we then see that

\[
I(\Omega_0 + \lambda \zeta) = I(\Omega_0) + 2\lambda \langle d\Omega_0, d\zeta \rangle + (\delta \Omega_0, \delta \zeta) - (\omega_0, \zeta) + \lambda^2 D(\zeta)
\]

which shows that (5.28) holds for all such \( \zeta \) and \( \Omega_0 \) is unique. But then (5.28) holds all \( \zeta \) in \( \mathcal{B}_2^r \) since any such \( \zeta \) is uniquely representable in the form \( \zeta = H + \zeta_0 \) where \( dH = \delta H = 0 \) and \( \zeta_0 \) is in \( \mathcal{B}_2^r \) and orthogonal to \( \mathcal{H}^r \). Finally, if we set \( \zeta = \Omega_0 \) in (5.28) and use Theorem 5.7, we see that

\[
\| \Omega_0 \|_{2f} \leq \lambda_0^{-1} \| \Omega_0 \|_{2f} \cdot \| \omega_0 \|_{2f}
\]

from which the last statement follows.

Definition: The form \( \Omega_0 \) of Theorem 5.7 is called the potential of \( \omega_0 \).

We observe that if all forms in (5.28) and the manifold \( M \) were sufficiently smooth, the equation (5.28), together with equation (5.18) would imply that

\[
\Delta \Omega_0 = d \delta \Omega_0 + \delta d \Omega_0 = \omega_0.
\]

In any coordinate system, (5.30) reduces to a system of second order equations in the components of the forms; if \( r \geq 1 \), these equations involve the second derivatives of the \( g_{ij} \) as well as those of the components of \( \Omega_0 \). However, all the results stated so far hold for manifolds of class \( C^1 \) in which case the requisite second derivatives of the \( g_{ij} \) certainly do not exist.
DEFINITION: We say that \( \omega \) is of class \( \mathcal{C}_{21} \), \( 0 < \lambda < \pi/2 \), if for each coordinate system \( \theta \) with domain \( B_{R} \), there is a constant \( L = L(\theta, \omega) \) such that

\[
\int_{B_{r}} \omega^{2}_{\theta} \, dx \leq L^{2} r^{2\lambda}, \quad 0 \leq r \leq R (B_{r} = B(0, r)).
\]

The class \( \mathcal{C}_{21} \) is defined similarly.

The importance of the spaces \( \mathcal{C}_{21} \) arises from the fact that if \( \omega \in \mathcal{C}_{21} \) with \( \lambda = \mu - 1 + \pi/2, 0 < \mu < 1 \), then \( \omega \in \mathcal{C}_{\mu}^{0} \); this follows from the straightforward extension of Lemma 4.1, to \( n \) dimensions. We can now state the following results concerning differentiability.

**Theorem 5.8:** Suppose that \( \omega \in \mathcal{C}_{21} \) is its potential.

(i) If \( M \) is of class \( \mathcal{C}_{1} \), the \( \Omega, d\Omega, \) and \( \delta\Omega \in \mathcal{C}_{21} \).

(ii) If \( M \) is of class \( \mathcal{C}_{1}, \) and \( \omega \in \mathcal{C}_{21} \), then \( \Omega, \delta\Omega, \) and \( \delta\Omega \in \mathcal{C}_{21} \)

hence in \( \mathcal{C}_{\mu}^{0} \) if \( \lambda = n/2 - 1 + \mu, 0 < \mu < 1 \).

(iii) If \( M \) is of class \( \mathcal{C}_{1} \) and \( \omega \in \mathcal{C}_{21} \), then \( \delta\Omega \) and \( \delta\Omega \) are the potentials of \( d\omega \) and \( d\omega \), respectively.

(iv) If \( M \) is of class \( \mathcal{C}_{k}^{\mu} \) and \( \omega \in \mathcal{C}_{k-2}^{\mu}, k \geq 2, 0 < \mu < 1 \), then \( \Omega, \delta\Omega, \) and \( \delta\Omega \in \mathcal{C}_{k-1}^{\mu} \).

(v) If \( M \) and \( \omega \) are of class \( \mathcal{C}^{\pi} \) or analytic, then so is \( \Omega \). In all case, if we set \( \alpha = d\Omega \) and \( \beta = \delta\Omega \) we have

\[
\delta\alpha + d\beta = \delta (d\Omega) + d(\delta\Omega) = \omega, \quad d\alpha = \delta\beta = 0.
\]

**Theorem 5.9:** Suppose that \( H \) is a harmonic field.

(i) If \( M \in \mathcal{C}_{1} \), then \( H \in \mathcal{C}_{21} \) with \( \lambda = n/2 - 1 + \mu \) for any \( \mu, 0 < \mu < 1 \).

(ii) If \( M \in \mathcal{C}_{k}^{\mu}, k \geq 2, 0 < \mu < 1 \), then \( H \in \mathcal{C}_{k-1}^{\mu} \).

(iii) If \( M \in \mathcal{C}^{\pi} \) or is analytic, then so is \( H \).

In both Theorems 5.8 and 5.9, 0-forms have an additional degree of differentiability (except in the second part of Theorem 5.8 (iv)). It should be observed that we can form \( \Lambda \Omega \) as indicated in (5.31) even though the individual components of \( \Omega \) do not have the necessary second derivatives (if \( r > 0 \)).

**Proof:** Obviously \( H \) satisfies (5.28) with \( \omega_{0} = 0 \). Then equations (5.28) are a special case of the more general equations

\[
(\delta\omega - \varphi, d\zeta) + (d\omega - \psi, \delta\zeta) = (\omega_{0}, \zeta)
\]

Using (5.24) and (5.22) we see that equations (5.32) are equivalent to equations of the form (4.13), if \( \zeta \) has support on some one coordinate patch, where the \( a^\prime \)s are Liptschitz, the \( b^\prime \)s and \( c^\prime \)s are bounded and measurable,
and the $e'$ s and $f'$ s $\in \mathcal{L}_2$. Such systems have been studied extensively by the writer in [75] and [47]. Since Professor Nirenberg's lectures are concerned with differentiability problems, the results and their proofs are omitted.

The results concerning $\omega_2$ follow directly from the result just mentioned. To prove the differentiability of $d\Omega$ and $d\Omega$, we select a coordinate patch and find that we can approximate to $\Omega$, $\omega$, and the $g_{ij}$ by smooth functions so that $\Omega$ is a potential of $\omega$ with respect to the altered $g_{ij}$ at each stage. Then, if $\xi$ has support interior to this patch, we see that (5.31), (5.18), and (5.20) imply that $\alpha$ and $\beta$ satisfy

\[
(d\alpha, d\xi) + (d\alpha, \omega, d\xi) = 0
\]

The interior boundedness theorem (like Theorem 4.5) and an approximation theorem for such systems allow us to pass to the limit in (5.33). If $\omega \in \mathcal{B}_2$, we use (5.33) and (5.18) to see that $\alpha$ and $\beta$ are the potentials of $d\omega$ and $d\omega$, respectively.

The following theorem complements the well-known orthogonal decomposition of Kodaira [29].

**THEOREM 5.10:** If $\omega$ is any form in $\mathcal{L}_2$, then there exists a harmonic field $H$ and forms $\alpha$, $\beta$, and $\Omega$ in $\mathcal{B}_2$ such that

\[
\omega = H + d\alpha + d\beta, \quad d\alpha = d\beta = 0,
\]

where $\Omega$ is the potential of $\omega - H$. If the first equation of (5.34) holds for a harmonic field $H_1$ and forms $\alpha_1$ and $\beta_1$ in $\mathcal{B}_2$, then $H_1 = H$, $d\alpha_1 = d\alpha$, and $d\beta_1 = d\beta$.

The sets $\mathcal{C}$ or all forms $d\alpha$ for $\alpha$ in $\mathcal{B}_2^{\nu+1}$ and $\mathcal{D}$ of all forms $d\beta$ for $\beta$ in $\mathcal{B}_2^{\nu-1}$ are closed linear manifolds in $\mathcal{L}_2$ and

\[
\mathcal{L}_2 = \mathcal{H} \oplus \mathcal{C} \oplus \mathcal{D}.
\]

If $M \in C^1$ and $\omega \in \mathcal{L}_{2\nu}$ or $\mathcal{B}_{2\nu}$, $0 \leq \lambda < n/2$, then $d\alpha$ and $d\beta$ have the same properties.

If $M \in C^k_\mu$ and $\omega \in C_\mu$ with $k \geq 2$, $0 < \mu < 1$, $0 < \sigma < 1$, and either $l < k - 1$ or $l = k - 1$ and $\sigma \leq \mu$, then $d\alpha$ and $d\beta$ have the same differentiability properties as $\omega$.

If $M$ and $\omega \in C^\infty$ or are anatitic, so are $d\alpha$ and $d\beta$. 
Proof: The first statement and the differentiability results follow immediately from Theorems 5.8 and 5.9 If $H, \alpha, \text{ and } \beta$ all $\in \mathcal{B}_2$ (and have properly related degrees), formulas (5.18) and (5.20) and the definition of harmonic field imply that $H, \delta \alpha, \text{ and } d\beta$ are orthogonal in $L_2$. To see that the sets $\mathcal{C}^r$ and $\mathcal{D}^r$ are closed we see, by following the construction in the first paragraph of the theorem with $\omega = \delta \alpha$ and $d\beta$ in turn, that if $\alpha$ and $\beta \in \mathcal{B}_2$, there are forms $\alpha_1$ and $\beta_1$ in $\mathcal{B}_2$ and orthogonal to $\mathcal{H}$ such that

$$\delta \alpha_1 = \delta \alpha, \quad d\alpha_1 = 0, \quad \delta \beta_1 = 0, \quad d\beta_1 = d\beta.$$  

Then if $\delta \alpha_n \to \sigma$ in $L_2$, we see that the $\alpha_{1n} \to$ some $\alpha_1$ in $\mathcal{B}_2$ by Theorem 5.6. A corresponding result holds if $d\beta_n \to \tau$ in $L_2$. 

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