ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 10, nº 3-4 (1956), p. 135-145

<http://www.numdam.org/item?id=ASNSP_1956_3_10_3-4_135_0>

MULTIVALUED FUNCTIONS IN GENERALIZED AXIALLY SYMMETRIC POTENTIAL THEORY (*)

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Introduction: In this paper we investigate certain multivalued functions connected with solutions of the equations of generalized axially symmetric potential theory (GASPT). In the first section we obtain a representation for the many valued function conjugate to the Neumann's function for a half plane. Following the terminology of Weinstein [1] we call this Neumann's function the potential of a source ring. An expression representing the values assumed by the various branches of the multivalued stream function was given by Weinstein [1], [2] in terms of integrals of products of Bessel functions. His work corrected and error which had existed since the time of Beltrami, who failed to recognize that the Stokes stream function was multivalued. Subsequently Van Tuyl [3] and Sadowsky and Sternberg [4] showed that the Stokes stream function can be expressed in terms of elliptic integrals, a form which displays the analytic character of the stream function more clearly than the Bessel function representation. In this paper we derive a new expression for the stream function for a source ring in GASPT. This expression exhibits clearly its analytic character. By introducing toroidal coordinates we obtain a representation of the stream function as a sum of two terms. The first is an arc cotangent, a multi-valued quantity which displays the cyclic nature of the stream function. The second term is single-valued and vanishes at the branch point. This latter quantity is written as an integral or a sum of Legendre functions.

(*) This research was supported in part by the United States Air Force under Contract No. AF 18 (600)-573 - monitored by the Office of Scientific Research, Air Research and Development Command.

In the last section we obtain (in the terminology of Weinstein [5]) the stream function for the source disk. Again the solution is represented as the sum of a single-valued portion which vanishes at the branch point, and a multi-valued term which exhibits its cyclic nature. New representations for the stream function and potential of a vortex ring are also given, and an identity is established which relates the stream function for a vortex ring to the potential of a source ring. We show, likewise, that the stream function for a source disk can be represented as the sum of the stream function for a source ring and a quantity which is related to the potential of a vortex ring. The analytical character of all of these functions is clearly displayed.

The stream function for a source ring and toroidal coordinates:

The equations of generalized axially symmetric potential theory, with which we are concerned, are the following:

$$\frac{\partial}{\partial x}\left(y^p \frac{\partial \varphi_p}{\partial x}\right) + \frac{\partial}{\partial y}\left(y^p \frac{\partial \varphi_p}{\partial y}\right) = 0, \tag{1}$$

where φ_p is defined in the half plane $y > 0$ and p is any positive real number. In the special case $p = 1$ (3-dimensions) equation (1) is the equation satisfied by an axially symmetric potential. The Neumann's function for the half plane $y > 0$ is then readily recognized as the potential of a source ring about the axis of symmetry. We, therefore, call the Neumann's function for general p the potential of a source ring.

The stream function ψ_p corresponding to any solution of (1) is given by the Stokes-Beltrami relations

$$y^p \frac{\partial \varphi_p}{\partial x} = \frac{\partial \psi_p}{\partial y}, \quad y^p \frac{\partial \varphi_p}{\partial y} = -\frac{\partial \psi_p}{\partial x}. \tag{2}$$

It follows from (2) that ψ_p is a solution of the equation

$$\frac{\partial}{\partial x}\left(y^{-p} \frac{\partial \psi_p}{\partial x}\right) + \frac{\partial}{\partial y}\left(y^{-p} \frac{\partial \psi_p}{\partial y}\right) = 0. \tag{3}$$

We seek then a function ψ_p which corresponds by (2) to the potential φ_p of the source ring (We suppose the image of the source ring in the xy plane to be at the point (o, b)). It is obvious from (2) that ψ_p is defined up to an additive constant. It is also easily seen from knowledge of the fundamental solution of (1) and the relation (2) that ψ_p assumes constant values on the y-axis but that it is not a single-valued function. These facts have been pointed out by Weinstein [1].

We are now faced with the problem of representing the multi-valued stream function. To do this we make a cut in the z-plane ($z = x + iy$) joining the branch point (o, b) to a point on the boundary. For simplicity we make the slit along the line $x = o$, $o \leq y \leq b$, and introduce an infinite-sheeted Riemann surface in the z-plane. The stream function will then be a single-valued function on the Riemann surface. This suggests a mapping which will take the upper half of the ζ-plane ($\zeta = \xi + i\eta$) onto the infinite-sheeted Riemann surface. Such a mapping is afforded by the introduction of toroidal coordinates, i. e.

$$x + iy = -b \cot 1/2 (\xi + i\eta), \ \eta > 0, \ -\infty < \xi < \infty . \tag{4}$$

If one restricts attention to a single sheet of the Riemann's surface, then for $y < b$ the line segments $x = 0^+$ and $x = 0^-$ correspond to lines $\xi = (2n - 1)\pi$ and $\xi = (2n + 1)\pi$ respectively. It would seem then that the (ξ, η)-coordinate system is the natural system for treating problems in GASPT that involve a single branch point in the half plane. From (4) we obtain

$$x = -\frac{b \sin \xi}{\cosh \eta - \cos \xi}, \quad y = \frac{b \sinh \eta}{\cosh \eta - \cos \xi}, \tag{5}$$

The rim of disk (the branch point) is given by $\eta \to \infty$.

Equations (1), (2), and (3) become now in (ξ, η) — coordinates

$$\frac{\partial}{\partial \xi}\left(y^p \frac{\partial \varphi_p}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(y^p \frac{\partial \varphi_p}{\partial \eta}\right) = 0, \tag{6}$$

$$y^p \frac{\partial \varphi_p}{\partial \xi} = \frac{\partial \psi_p}{\partial \eta}, \quad y^p \frac{\partial \varphi_p}{\partial \eta} = -\frac{\partial \psi_p}{\partial \xi}, \tag{7}$$

and

$$\frac{\partial}{\partial \xi}\left(y^{-p} \frac{\partial \psi_p}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(y^{-p} \frac{\partial \psi_p}{\partial \eta}\right) = 0. \tag{8}$$

We restrict ourselves in this paper to $p > 0$. The case $p = 0$ requires special treatment. However, the solution in this case can be obtained by elementary means. It is well known that for $p > 0$ the potential φ_p of a source ring may be expressed in terms of the Legendre Q-function (see Weinstein [5]). In toroidal coordinates φ_p is given by

$$\varphi_p = \frac{k}{\pi} b^{-p/2} y^{-p/2} Q_{p-2/2} (\coth \eta), \tag{9}$$

where k is the strength of the ring source (We use Hobson's definition [6] for the Q-function and the P-function (to be employed later). We note that φ_p is a single-valued function of x and y, i. e. it assumes the same values on each sheet of the Riemann surface. The value of the conjugate function ψ_p at any point (ξ_0, η_0) in the ξ, η plane is then obtained from (7) by performing the following integration.

$$\psi_p(\xi_0, \eta_0) = - \lim_{a \to \infty} \left[\int_0^{\xi_0} \left(y^p \frac{\partial \varphi_p}{\partial \eta} \right)_{\eta = a} d\xi + \int_{\eta_0}^{a} \left(y^p \frac{\partial \varphi}{\partial \xi} \right)_{\xi = \xi_0} d\eta \right]. \tag{10}$$

In order to evaluate the first integral in (10) we employ the expression for the Q-function given by Hobson [6. p. 206]. As $a \to \infty$ the first integral clearly approaches the constant value k/π. We obtain then the following expsessions for ψ_p:

$$\psi_p(\xi, \eta) = -\frac{k\xi}{\pi} - \frac{pk}{2\pi} \sin \xi \int_{\cosh \eta}^{\infty} \frac{(s^2 - 1)^{\frac{p-2}{4}}}{(s - \cos \xi)^{\frac{p+2}{2}}} Q_{\frac{p-2}{2}}\left(\frac{s}{\sqrt{s^2 - 1}} \right) ds,$$

$$\psi_p(\xi, \eta) = -\frac{k\xi}{\pi} - \frac{k \Gamma\left(\frac{p+2}{2}\right) \sin \xi}{\sqrt{2\pi}} \int_{\cosh \eta}^{\infty} \frac{(s^2 - 1)^{\frac{p-1}{4}}}{(s - \cos \xi)^{\frac{p+2}{2}}} P_{-\frac{1}{2}}^{-\frac{p-1}{2}}(s) \, ds. \tag{11}$$

In the last integral we have replaced the Q-function by a P-function according to Whipple's relation [6, p. 247]

$$Q_{\frac{p-2}{2}}\left(\frac{s}{\sqrt{s^2 - 1}} \right) = \sqrt{\frac{\pi}{2}} \, \Gamma\left(\frac{p}{2} \right) (s^2 - 1)^{1/4} P_{-\frac{1}{2}}^{-\frac{p-1}{2}}(s). \tag{12}$$

From the asymptotic behavior of the P-function it is observed that the integrand in (11) is $0(s^{-2} \log s)$ as $s \to \infty$. This follows immediately from [6, p. 235 (73)] (Note that the asymptotic expression given in [6, p. 436] is incorrect).

One can easily verify that the expression (11) for $\psi_p(\xi, \eta)$ satisfies the differential equations (8) in the open region of the upper half plane $(\eta > 0)$; for in this open region one may employ the Fourier Series expansion :

$$(13)\qquad \left(\frac{\sinh v}{\cosh v - \cos \xi}\right)^{p/2} = 2 \sum_{m=0}^{\infty}{}' \frac{\Gamma\left(\frac{p}{2}+m\right)}{\Gamma(p/2)} P_{\frac{p-2}{2}}^{-m}(\coth v) \cos m\,\xi , v > 0 ,$$

(where the prime indicate that the term $m = 0$ is to be multiplied by $\left.\frac{1}{2}\right)$. The coefficient of cos m ξ on the right is $O\,(m^{\frac{p-2}{2}} e^{-mv})$ as $m \to \infty$. Thus for $v > 0$ the series may be differentiated term by term yielding the expansion

$$(14)\qquad \frac{p}{2} \sin \xi \frac{(\sinh v)^{p/2}}{(\cosh v - \cos \xi)^{\frac{p+2}{2}}} = 2 \sum_{m=1}^{\infty} m \frac{\Gamma\left(\frac{p}{2}+m\right)}{\Gamma(p/2)} P_{\frac{p-2}{2}}^{-m}(\coth v) \sin m\,\xi_0$$

If we now insert (14) in (11) and employ (12) we obtain the following expression for $\psi(\xi, \eta)$

$$(15)\qquad \psi(\xi, \eta) = -\frac{k\xi}{\pi} + \frac{2k}{\pi\,\Gamma(p/2)} \sum_{m=1}^{\infty} m\, \Gamma\left(\frac{p}{2}+m\right) \sin m\,\xi \times \\ \times \int_1^{\coth \eta} \frac{P_{\frac{p-2}{2}}^{-m}(u)\, Q_{\frac{p-2}{2}}(u)\, du}{u^2 - 1} .$$

The change of order of summation and integration is again valid for $\eta > 0$.

From the differential equations satisfied by the P and Q function we derive the identity

$$(16)\qquad \frac{d}{du}\left\{(u^2-1)\left[P_a^b(u) \frac{d}{du} Q_a(u) - Q_a(u) \frac{d}{du} P_a^b(u)\right]\right\} = b^2 \frac{Q_a(u)\, P_a^b(u)}{(u^2-1)} .$$

We now integrate this identity and insert the result in (15) to obtain the expression,

$$(17)\qquad \psi(\xi, \eta) = -\frac{k\xi}{\pi} - \frac{2k(\mu^2-1)}{\pi\,\Gamma(p/2)} \times$$

$$\times \left[\sum_{m=1}^{\infty} \frac{\Gamma\left(\frac{p}{2}+m\right)}{m} \left\{\frac{d}{d\mu} P_{\frac{p-2}{2}}^{-m}(\mu)\, Q_{\frac{p-2}{2}}(\mu) - P_{\frac{p-2}{2}}^{-m}(\mu) \frac{d}{d\mu} Q_{\frac{p-2}{2}}(\mu)\right\} \sin m\,\xi\right],$$

where $\mu = \coth \eta$. In this form it is easily checked that in the open region $\eta > 0$, $\psi(\xi, \eta)$ satisfies (8). We merely make use of (13) and the valid interchange of differentiation and summation. In this form one also obtains from the asymptotic expressions for the P — and Q — functions the values of $\psi(\xi, 0)$. As $\eta \to 0$ the bracket term in (17) yields the well known Fourier series expansion for the quantity

$$\frac{1}{2} \frac{[(2n+1)\pi - \xi]}{\mu^2 - 1} \Gamma(p/2), \quad 2n\pi < \xi < (2n+1)\pi \tag{18}$$

(see Churchill [7, p. 61]). Thus for $\eta = 0$

$$\begin{aligned} \psi_p(\xi, 0) &= (2n+1)k, \quad (2n-1)\pi < \xi < 2n\pi, \\ \psi_p(\xi, 0) &= (2n-1)k, \quad 2n\pi < \xi < (2n+1)\pi. \end{aligned} \tag{19}$$

From (4) we obtain the following expression for ξ and η as functions of x and y:

$$\eta = \text{arc coth} \frac{x^2 + y^2 + b^2}{2by}, \quad \xi = \text{arc cot} \frac{x^2 + y^2 - b^2}{2bx}. \tag{20}$$

We now rewrite ψ_p as

$$\psi_p(\xi, \eta) = -\frac{k\xi}{\pi} + F_p(\xi, \eta). \tag{21}$$

It is easily seen that $F_p(\xi, \eta)$ is a single-valued function of x and y, an analytic function of x and y in the upper half plane except at the branch point, and in fact an analytic function of p for $0 < \eta < \infty$, and $p > 0$. This latter statement follows immediately from the integral definitions of the P and Q functions. As indicated in (20), ξ is represented as an arc cotangent and hence is an infinite-valued function of x and y. However, it is single-valued, continuous and analytic except at the branch point on the infinitesheeted Riemann surface. This term exhibits clearly the cyclic behavior of the stream function about the branch point (o, b), while the function $F_p(\xi, \eta)$ vanishes at (o, b) and in fact along the entire y-axis.

If we restrict ourselves to one branch of the stream function we obtain an expression which is discontinuous along the line $x = 0$, $0 \leq y \leq b$. If in particular we choose ξ to lie in the range $-\pi < \xi < \pi$ then for $y < b$, $\xi = -\pi$ corresponds to $x = 0^+$ and $\xi = \pi$ to $x = 0^-$. We have then for $x=0$

$$\psi_p(-\pi, \eta) = k,$$

(22)
$$\psi_p(\pi, \eta) = -k,$$

$$\psi_p(0, \eta) = 0.$$

If p is an even integer the P and Q functions reduce to simple functions, and the expression for $\psi_p(\xi, \eta)$ is considerably simplified. In particular for $p=2$

$$(23) \quad \psi_2(\xi, \eta) = \frac{k}{\pi}\left[2\tan^{-1}\left(\frac{e^\eta - \cos\xi}{\sin\xi}\right) - \xi + \eta\frac{\sin\xi}{\cosh\eta - \cos\xi} + \begin{cases} \pi, \xi<0 \\ -\pi, \xi>0 \end{cases}\right].$$

On the other hand if p is an odd integer the stream function may be expressed in terms of derivatives of elliptic functions as was demonstrated by Van Tuyl [3] and Sadowsky an Sternberg [4] for the case $p=1$. We need merely employ the identities

$$P_{-1/2}(s) = \frac{2}{\pi}\left(\frac{2}{1+s}\right)^{1/2} K\left(\sqrt{\frac{s-1}{s+1}}\right)$$

(24)
$$Q_{-\frac{1}{2}}\left(\frac{s}{\sqrt{s^2-1}}\right) = \frac{\pi}{\sqrt{2}}(s^2-1)^{1/4} P_{-1/2}(s),$$

where the K-function is the complete elliptic integral of the first kind.
Stream function for the source disk and vortex ring:

We now restrict ourselves to one sheet of the Riemann surface in the z plane. In other words ξ is chosen to lie in the range $-\pi < \xi < \pi$. Bessel function representations for all the solutions to be discussed in this section were given by Weinstein [5]. However, the form of solution given here shows more clearly their cyclic nature and analytic character.

We consider first the problem in which $\frac{\partial \varphi_p^*}{\partial n} = (p+1)k$ ($k=$ constant) on the faces of the slit $(x=0, 0 \leq y \leq b)$. In three dimensions $(p=1)$ this corresponds to a uniform distribution of charge (simple layer) over a disk of radius b. We adopt then the terminology of three dimensions and call φ_p^* the potential of a source disk. It is apparent from (2) that for $x = 0$

$$\psi_p^*(-\pi, \eta) = k(y^{p+1} + \text{constant})$$

$$\psi_p^*(\pi, \eta) = -k(y^{p+1} + \text{constant}) \tag{25}$$

$$\psi_p^*(0, \eta) = 0.$$

Since $\dfrac{\partial \varphi}{\partial x}$ must remain finite at the rim of the disk, the constants in (25) must be taken as $-b^{p+1}$.

We now express $\psi_p^*(\xi, \eta)$ as

$$\psi_p^*(\xi, \eta) = k\{\overline{\psi}_p^*(\xi, \eta) - b^{p+1}\psi_p(\xi, \eta)\}, \tag{26}$$

where $\psi_p(\xi, \eta)$ is given by (11). We sec from (25) that $\overline{\psi}_p^*(\xi, \eta)$ must satisfy

$$\overline{\psi}_p^*(-\pi, \eta) = y^{p+1}$$

$$\overline{\psi}_p^*(\pi, \eta) = -y^{p+1} \tag{27}$$

$$\overline{\psi}_p^*(0, \eta) = 0.$$

We introduce a correspondence principle of Weinstein [2]

$$\overline{\psi}_p^*(\xi, \eta) = y^{p+1}\,\overline{\Phi}_{p+2}^*(\xi, \eta), \tag{28}$$

which relates the stream function with index p to a potential function of index $p+2$. This function $\overline{\Phi}_p^*(\xi, \eta)$ takes the values ± 1 on the two faces of the disk an may be interpreted as the potential of a vortex ring (magnetic disk) in a fictitious space of $p+4$ dimensions. Using the results of Van Nostrand [8], one can easily compute this potential. It is determined as

$$\overline{\Phi}_{p+2}^*(\xi\,\eta) = -\frac{(\cosh\eta - \cos\xi)^{\frac{p+2}{2}}}{\sinh^{\frac{p+1}{2}}\eta}\int_0^\infty A(\alpha)\sinh\alpha\,\xi\, P_{i\alpha-1/2}^{-\frac{p+1}{2}}(\cosh\eta)\,d\alpha, \tag{29}$$

where

$$A(\alpha) = (\alpha/\pi)\Gamma\left(i\alpha + \frac{p}{2} + 1\right)\Gamma\left(-i\alpha + \frac{p}{2} + 1\right)\int_1^\infty \frac{P_{i\alpha-1/2}^{-\frac{p+1}{2}}(s)\,(s^2-1)^{\frac{p+1}{4}}}{(s+1)^{\frac{p+2}{2}}}\,ds. \tag{30}$$

In the case $p = -1$ equations (29) and (30) result from application of the expansion theorem due to Mehler [9].

On the other hand one recognizes that the function $\overline{\Phi}^*_{p+2}(\xi, \eta)$ satisfies approximately the same conditions as the stream function of the source ring. It is a multivalued function which assumes different constant values on the slit and vanishes on the y-axis outside the slit. Let us then formally replace $Q_{\frac{p}{2}-1}\left(\frac{s}{\sqrt{s^2-1}}\right)$ by $Q_{-p/2}\left(\frac{s}{\sqrt{s^2-1}}\right)$ in the first of equations (11). The resulting function still satisfies the differential equation for $\psi_p(\xi, \eta)$ in any region where it is defined. We then replace p by $-(p+2)$ and the resulting expression satisfies the differential equation for $\overline{\Phi}^*_{p+2}(\xi, \eta)$ in any region where it is defined. Thus we have

$$\overline{\Phi}^*_{p+2}(\xi, \eta) = -\frac{\xi}{\pi} + \frac{(p+2)}{2\pi} \sin \xi \int_{\cosh \eta}^{\infty} \frac{(s - \cos \xi)^{p/2}}{(s^2-1)^{\frac{p+4}{4}}} Q_{\frac{p+2}{2}}\left(\frac{s}{\sqrt{s^2-1}}\right) ds =$$

$$(31) \qquad = -\frac{\xi}{\pi} + \frac{\Gamma\left(\frac{p+4}{2}\right)}{\sqrt{2\pi}} \sin \xi \int_{\cosh \eta}^{\infty} \frac{(s - \cos \xi)^{p/2}}{(s^2-1)^{\frac{p+2}{4}}} P_{-1/2}^{-\frac{p+3}{2}}(s)\, ds =$$

$$= -\frac{\xi}{\pi} - \frac{2}{\pi} \sum_{m=1}^{\infty} (-1)^m \frac{\Gamma\left(\frac{p+4}{2}\right)}{\Gamma\left(\frac{p+4-2m}{2}\right)} (\mu^2 - 1) \sin m\xi \times$$

$$\times \left[\frac{d}{d\mu} P^{-m}_{\frac{p+2}{2}}(\mu)\, Q_{\frac{p+2}{2}}(\mu) - \frac{d}{d\mu} Q_{\frac{p+2}{2}}(\mu)\, P^{-m}_{\frac{p+2}{2}}(\mu)\right],$$

the interchange of summation and integration or differentiation being valid for $\eta > 0$. We can now easily check that this function satisfies the boundary conditions and is regular analytic on an infinite-sheeted Riemann surface except at the branch point. We note that $\overline{\Phi}^*_{p+2}(\xi, \eta)$ as given by (31) is finite at every point in the closed plane region $y \geq 0$ including the branch point. But it can easily be shown that a function $\overline{\Phi}^*_{p+2}(\xi, \eta)$ which remains finite in the half space and satisfies the prescribed boundary conditions is unique. Hence equation (31) represents this unique solution. If p is an even integer the series in the third of expressions (31) terminates. In particular if $p = -2$ only the first term remains giving the well known solution for the potential of two separated vortices in two dimensions. For

$p = -1$ (31) yields the potential of a three dimensional vortex ring. On the x-axis

$$\bar{\Phi}_1^* (\xi , \eta) = - \sin \xi/2 , \tag{32}$$

which agrees with the value obtained by Sadowsky and Sternberg [4]. Note that the second term in (31) is again a single-valued function of x and y. If we rewrite (31) as

$$\bar{\Phi}_{p+2}^* (\xi , \eta) = - \xi/\pi - F_{p+2}^* (\xi , \eta) . \tag{33}$$

then $\psi_p^* (\xi , \eta)$ may be expressed as

$$\psi_p^* (\xi , \eta) = - \frac{k}{\pi} \{y^{p+1} - b^{p+1}\} \xi - k \{y^{p+1} F_{p+2}^* (\xi , \eta) + b^{p+1} F_p (\xi , \eta)\} , \tag{34}$$

where $F_p (\xi , \eta)$ is given by (21). The function $F_{p+2}^* (\xi , \eta)$ and $F_p (\xi , \eta)$ vanish at the branch point $(\eta \to \infty)$. The cyclic behavior of $\psi_p^* (\xi , \eta)$ is clearly exhibited by the term $- k \{y^{p+1} - b^{p+1}\} \xi$. On the x and y axes ψ_p^* takes the values

$$\begin{aligned}
&\psi_p^* (0 , \eta) = 0 \\
&\psi_p^* (- \pi , \eta) = k (y^{p+1} - b^{p+1}) \\
&\psi_p^* (\pi , \eta) = - k (y^{p+1} - b^{p+1}) \\
&\psi_p^* (\xi , 0) = - k\, b^{p+1}, \ \xi < 0 \\
&\psi_p^* (\xi , 0) = k\, b^{p+1}, \ \xi > 0 .
\end{aligned} \tag{35}$$

The stream function $\psi'_p (\xi , \eta)$ for a vortex ring may be obtained, as is well known by differentiation of $\psi_p^* (\xi , \eta)$ with respect to x. On the other hand since the function $\varphi'_p (\xi , \eta)$, conjugate to $\psi'_p (\xi , \eta)$, is precisely the $\bar{\Phi}_p^* (\xi , \eta)$ just treated, one is lead at once to the following expression for $\psi'_p (\xi , \eta)$,

$$\psi'_p (\xi , \eta) = \frac{k}{\pi} b^{p/2} y^{p/2} Q_{p/2} (\coth \eta) . \tag{36}$$

This is obtained by replacing $Q_{\frac{p-2}{2}}$ by $Q_{-p/2}$ in (9) and changing p to $-p$ throughout. It is clear then that

$$\psi'_p (\xi , \eta) = b^{p+1} y^{p+1} \varphi_{p-2} (\xi , \eta) , \tag{37}$$

where $\varphi_{p+2}(\xi, \eta)$ is the potential of the source ring. The potential $\varphi'_p(\xi, \eta)$ is obtained directly from (31) by replacing $p+2$ by p throughont.

The stream functions for doublet rings may be obtained by differentiation of (13) with respect to x or b.

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