Some formulas in a riemannian space


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SOME FORMULAS IN A RIEMANNIAN SPACE

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In this paper a number of theorems and formulas involving two arbitrary affine connections in a Riemannian space \( V^n \) have been established by imposing certain conditions on the affine connections. In section 1 it has been assumed that the covariant derivatives of the metric tensor of the \( V^n \) with respect to the affine connections are the same while in section 2 the torsions of the affine connections have been taken to be the same.

1. Let \( I^i_{jk} \) and \( L^i_{jk} \) be the coefficients of two arbitrary affine connections in a Riemannian space \( V^n \) with metric tensor \( g_{ij} \) and let a comma and a semicolon denote the covariant derivatives of the \( g_{ij} \)'s with respect to the two connections. Then

\[
g_{ij,k} - g_{ij;k} = - g_{is} (I^s_{jk} - L^s_{jk}) - g_{js} (I^s_{ik} - L^s_{ik})
\]

Putting \( T^i_{jk} = I^i_{jk} - L^i_{jk} \), it follows that

\[
g_{ij,k} = g_{ij;k}
\]

if and only if

(I. 1) \[ g_{is} T^s_{jk} + g_{js} T^s_{ik} = 0 \]

Hence we have the following theorem:

**Theorem 1.** The covariant derivatives of the \( g_{ij} \)'s with respect to two affine connections with coefficients \( I^i_{jk} \) and \( I^i_{jk} + T^i_{jk} \) are the same if and only if the tensor \( T^i_{jk} \) satisfies (I. 1).

As an example it is easy to verify that the above result holds with respect to the coefficients of affine connections

\[
I^i_{jk} \quad \text{and} \quad I^i_{jk} \pm g^{is} (g_{jk,s} - g_{ks,j})
\]
This example is an application of Sen's sequence (3) which is defined as follows:

Put $a = \Gamma^t_j$, $a^* = \Gamma^t_j + g^{mt}g_{im,j}$, $a' = \Gamma^t_{ji}$

Then it is known that for every affine connection $a$ there exist uniquely two others $a^*$ and $a'$ which are respectively called the associate and the conjugate of $a$ having the property

$$a^{**} = a' = a$$

In particular, $a$ is self-associate if $a = a^*$ and self-conjugate if $a = a'$. Now if we construct the sequence

$$(1.2) \quad a_1 = a, \quad a_2 = a^*, \quad a_3 = a^*, \quad a_4 = a^*, \quad a_5 = a^*, \quad \ldots$$

then the sequence is a finite cyclic sequence of twelve terms and it is Sen's sequence. In the sequence if we put

$$\alpha = g^{mt}g_{im,j}, \quad \alpha_c = g^{mt}g_{jm,i}, \quad \gamma = g^{mt}g_{ij,m} = \gamma_c$$

$$\beta = g^{mt}g_{ik}(\Gamma^{s}_mj - \Gamma^{s}_jm), \quad \beta_c = g^{mt}g_{ik}(\Gamma^{s}_im - \Gamma^{s}_im)$$

and suppose that $a$ is self-conjugate, then we have

$$a_4 = a_9 = a + \alpha - \gamma, \quad a_5 = a_8 = a + \alpha_c - \gamma, \quad a_6 = a_7 = a + \alpha + \alpha_c - \gamma$$

It follows that

$$(1.3) \quad a_1 - a_5 = a_2 - a_6 = g^{mt}(g_{jk,s} - g_{ks,j})$$

Further, if the covariant derivatives of the $g_{ij}'s$ with respect to two affine connections are the same and if one of them is self-associate then the other is also self-associate, because the covariant derivatives of the $g_{ij}'s$ must vanish. Now when $a_1 = a_2$ is self-associate, then $a_7 = a_8$ is also self-associate. Hence we have the following theorem:

**Theorem 2.** If $a_i$ be coefficients of a self-conjugate affine connection, then the $g_{ij}'s$ have the same covariant derivatives with respect to the pairs of affine connections $(a_1, a_5)$ and $(a_2, a_6)$ of Sen's sequence. And if $a_1$ be self-associate then the $g_{ij}'s$ have the same covariant derivatives with respect to the pair $(a_1, a_7)$. 
Further, let \( \Gamma_{ij}^t \) and \( \Gamma_{ij}^t = T_{ij}^t \) be the coefficients of two affine connections. Then their associates are
\[
\Gamma_{ij}^{*t} \quad \text{and} \quad \Gamma_{ij}^{*t} = g^{mt} g_{ia} T_{mj}^t
\]
and their conjugates are
\[
\Gamma_{ij}^t \quad \text{and} \quad \Gamma_{ij}^t = T_{ij}^t
\]
It follows immediately from Theorem 1 that

**Theorem 3.** If the \( g_{ij}^t \)'s have the same covariant derivatives with respect to two arbitrary affine connections, then the same is true with respect to their associates and conjugates.

Now, let \( \Gamma_{ij}^t \) and \( L_{ij}^t \) be the coefficients of two affine connections and
\[
\Delta_{ij}^t = \frac{1}{2} (\Gamma_{ij}^t + L_{ij}^t), \quad T_{ij}^t = \Gamma_{ij}^t - L_{ij}^t
\]

Also, let \( \Gamma_{ijk}^t, L_{ijk}^t \) and \( \Delta_{ijk}^t \) denote curvature tensors formed with \( \Gamma_{ij}^t, L_{ij}^t \) and \( \Delta_{ij}^t \) respectively.

Then it is known that

(I. 4)
\[
\Delta_{ijk}^t = \frac{1}{2} (\Gamma_{ijk}^t + L_{ijk}^t) = \frac{1}{4} (T_{sk}^t T_{ij}^t - T_{sj}^t T_{ik}^t)
\]

Now if the \( g_{ij}^t \)'s have the same covariant derivatives with respect to \( \Gamma_{ij}^t \) and \( L_{ij}^t \) then by (I. 1)
\[
T_{sk}^t T_{ij}^t = g^{mt} g_{np} T_{mn}^p g^{ms} g_{iq} T_{nj}^q = g^{mt} g_{iq} T_{mk}^q T_{nj}^q
\]
Therefore
\[
g_{ht} (T_{sk}^t T_{ij}^t - T_{sj}^t T_{ik}^t) = g_{ht} g^{mt} g_{iq} (T_{mk}^q T_{nj}^q - T_{mj}^q T_{nk}^q) = g_{ht} (T_{nj}^q T_{hk}^k - T_{nk}^k T_{hn}^k)
\]
Or
\[
\Delta_{hijk}^t = -\frac{1}{2} (\Gamma_{hijk}^t + L_{hijk}^t) = \frac{1}{4} g_{ht} (T_{sk}^t T_{ij}^t - T_{sj}^t T_{ik}^t) = \frac{1}{4} g_{ht} (T_{nj}^q T_{hk}^k - T_{nk}^k T_{hn}^k)
\]
Hence

(I. 5)
\[
\Delta_{hijk}^t = -\frac{1}{2} (\Gamma_{hijk}^t + L_{hijk}^t) = - [\Delta_{hijk}^t - \frac{1}{2} (\Gamma_{hijk}^t + L_{hijk}^t)]
\]
Thus we have the following theorem:

**Theorem 4.** If the $g_{ij}$'s have the same covariant derivatives with respect to $\Gamma_{ij}^t$ and $L_{ij}^t$ and if $A_{ij}^t = \frac{1}{2} (\Gamma_{ij}^t + L_{ij}^t)$, then the curvature tensors formed with them satisfy the relation (I. 5).

This result is easily verified in the case when $\Gamma_{ij}^t$ and therefore $L_{ij}^t$, $A_{ij}^t$ are self-associate. For in this case the curvature tensors are skew in the first two indices (4).

As before, suppose that the covariant derivatives of the $g_{ij}$'s with respect to $\Gamma_{ij}^t$ and $L_{ij}^t$ are the same. Forming the second covariant derivatives it is seen that

\[(I. 6)\]

\[g_{ij,kl} - g_{ij,kl} = - [g_{ij,k} T_{kl}^t + g_{kl,k} T_{ij}^t + g_{ij,s} T_{kl}^t]\]

where

\[\Gamma_{ij}^t - L_{ij}^t = T_{ij}^t = - g^{ml} g_{ij} T_{ml}^n\]

Therefore

by (I. 6) \[g^{ij} (g_{ij,kl} - g_{ij,kl}) = g^{ml} g_{ij,k} T_{ml}^t + g^{ml} g_{kl,k} T_{ml}^t - g^{ij} g_{ij,s} T_{kl}^t =\]

\[= g^{ij} [g_{ij,k} T_{kl}^t + g_{kl,k} T_{ij}^t + g_{ij,s} T_{kl}^t - 2 g_{ij,s} T_{kl}^t] = - g^{ij} (g_{ij,kl} - g_{ij,kl}) - 2 g^{ij} g_{ij,s} T_{kl}^t\]

Therefore

\[g^{ij} (g_{ij,kl} - g_{ij,kl}) = - g^{ij} g_{ij,s} T_{kl}^t\]

Interchanging $k$ and $l$ and subtracting

\[g^{ij} (g_{ij,kl} - g_{ij,kl}) = g^{ij} g_{ij,s} (T_{kl}^t - T_{kl}^t)\]

Finally using Ricci's identity

\[(I. 7)\]

\[g^{ij} (g_{ij,kl} - g_{ij,kl}) = g^{ij} g_{ij,s} [(\Gamma_{kl}^t - L_{kl}^t) - (\Gamma_{kl}^t - L_{kl}^t)]\]

Let us further suppose that $\Gamma_{ij}^t$ and $L_{ij}^t$ are both self-associate or both self-conjugate. Then the right hand side of (I. 7) vanishes. We have therefore

\[g^{ij} [(\Gamma_{ij}^t + L_{ij}^t) - (\Gamma_{ij}^t + L_{ij}^t)] = 0\]

whence

\[(I. 8)\]

\[g^{ij} (\Gamma_{ij}^t - L_{ij}^t) = 0\]

As said before, this result is obvious when both the affine connections are self-associate.
Hence we have the following theorem:

**Theorem 5.** If the $g_{ij}$'s have the same covariant derivatives with respect to two self-conjugate affine connections with coefficients $\Gamma^t_{ij}$, $L^t_{ij}$ and if $\Gamma^t_{ijk}$, $L^t_{ijk}$ be the corresponding covariant curvature tensors, then (I. 8) holds.

2. The torsion of an affine connection with coefficients $\Gamma^t_{ij}$ is defined to be the tensor $\frac{1}{2} (\Gamma^t_{ij} - \Gamma^t_{ji})$ (2). It follows that two arbitrary affine connections with coefficients $\Gamma^t_{ij}$ and $\Gamma^t_{ij} + T^t_{ij}$ have the same torsion if and only if $T^t_{ij}$ is symmetric in $i$ and $j$. It is now easy to see that if two affine connections have the same torsion the same is true of their conjugates. E. g., in Sen's sequence each of the pairs $(a_1, a_8)$, $(a_2, a_9)$, $(a_4, a_7)$ and therefore their conjugates $(a_{12}, a_7)$, $(a_3, a_8)$, $(a_{14}, a_6)$ have the same torsion.

Again, let $a = \Gamma^t_{ij}$, $b = L^t_{ij}$ be the coefficients of two affine connections and $a - b = T^t_{ij}$. Their associates $a^*$ and $b^*$ will have the same torsion if the tensor

$$T^t_{ij} + g^{mt}(g_{im,j} - g_{im,j}) = g^{mt} g_{is} T^*_{sj}$$

is symmetric in $i,j$ i. e., if

$$(2.1)\quad g_{is} T^*_{sj} = g_{js} T^*_{is}$$

Putting $g_{is} T^*_{sj} = T^*_{ij}$ we have the following theorem:

**Theorem 6.** Let $a$ and $b$ have the same torsion; then their associates will also have the same torsion if the tensor $T^*_{ijk}$ is symmetric in all the indices.

Let $e_0 = (a, b) = \frac{1}{2} (a + b) + (a - b)$, $\bar{e}_0 = (b, a) = \frac{1}{2} (a + b) + (b - a)$

$$e_1 = (e_0, \bar{e}_0) = \frac{1}{2} (a + b) + 2(a - b), \bar{e}_1 = \frac{1}{2} (a + b) + 2(b - a)$$

Similarly for $e_2$, $\bar{e}_2$ etc.

Then

$$e_r = \frac{1}{2} (a + b) + 2^r (a - b)$$

Therefore

$$e_r - \bar{e}_r = 2^r (a - b)$$

It follows that if $a$ and $b$ have the same torsion then the same is true of $e_r$ and $\bar{e}_r$. 

Further, we have the following theorem:

**Theorem 7.** If the associates of $a$ and $b$ have the same torsion, the same is true of the associates of $c_r$ and $d_r$.

Let $I_{ij}^t$ and $L_{ij}^t$ have the same torsion. Then applying the condition that $T_{ij}^t$ is symmetric in $i$, $j$, we obtain from (I. 4) the cyclical property, namely

\[(2.3) A_{ijk}^t = \frac{1}{2} (I_{ijk}^t + L_{ijk}^t) + A_{ikj}^t - \frac{1}{2} (I_{jki}^t + L_{jki}^t) + A_{kij}^t - \frac{1}{2} (I_{kij}^t + L_{kij}^t) = 0\]

This result is obvious if $I_{ij}^t$ and therefore $L_{ij}^t$, $A_{ij}^t$ are self-conjugate.

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**References**

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