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NOTE ON WEBER'S PARABOLIC CYLINDER FUNCTION $D_n(z)$ AND ITS ASSOCIATED EQUATIONS (FUNCTIONAL & DIFFERENTIAL)

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INTRODUCTION

The main object of the present investigation is to study, from the point of view of Calculus of Functions or of Finite Differences, some of the more salient properties of the Weber's Function $D_n(x)$ with special reference to the (associated) pair of functional equations:

(I) $$f_{n+1}(z) - z f_n(z) + n f_{n-1}(z) = 0$$

and

(II) $$f'_n(z) + \frac{1}{2} z f_n(z) - n f_{n-1}(z) = 0$$

and the (associated) differential equation:

(A) $$\frac{d^2 w}{d z^2} + \left(n + \frac{1}{2} - \frac{1}{4} z^2\right) w = 0 .$$

Although the parabolic cylinder function $D_n(z)$ was first conceived as by Weber in connection with Harmonic Analysis, a good deal of its intrinsic properties were subsequently investigated by a host of prominent mathematicians.

For the sake of brevity, (A) will be called Weber's (differential) equation of *rank* n and symbolised as $D^{(n)}$. Unless otherwise stated, the rank n will be restricted to be a positive integer.

As a matter of convenience the paper has been divided into four sections. Sec. I begins with a study of certain characteristic features of an *arbitrary* solution of (I). Sec. II reckons with the connecting link between the three equations (I), (II) and (A). Sec. III deals with the common solutions of (I) and (II) and their relationship with (A). Lastly, Sec. IV disposes of the « generating » function of the sequence $\left\{ \frac{f_n(z)}{\lfloor n} \right\}$, where $f_n(z)$ is an *analytic* solution of (I); there is finally a passing reference to the special sequence $\left\{ \frac{D_n(z)}{\lfloor n} \right\}$.

Although on certain occasions it has been felt necessary to touch on *known* results, the authors believe that this paper embodies some amount of *original* matter.

SECTION I

Certain characteristic features of arbitrary solutions of the equation I, where n is an integer ≥ 1

§ 1. — The functional equation (I) being of the *second* order, its *complete* solution must insolve *two* arbitrary functions. If $\alpha_n(z)$ and $\beta_n(z)$ be two linearly independent particular solutions of (I) and the parameter n be not restricted to be a positive integer, it is easy to see that the *general* solution of (I) can be thrown into the form:

$$f_n(z) = \alpha_n(z)\, g_n(z) + \beta_n(z)\, h_n(z)\,, \tag{1}$$

where $g_n(z)$ and $h_n(z)$ are *arbitrary* periodic functions of n with *unit* period. When, however, n is restricted to be a positive integer (as will generally be assumed throughout this paper), the functions $g_n(z)$ and $h_n(z)$ are, to all intents and purposes, *independent of n* and so may be regarded simply as *arbitrary* functions of z and symbolised respectively as $g(z)$ and $h(z)$. In other words, *where n is an integer* ≥ 1, *the complete solution of* (I) *admits of the simbolic form:*

$$f_n(z) = \alpha_n(z)\, g(z) + \beta_n(z)\, h(z)\,, \tag{2}$$

where $\alpha_n(z)$ *and* $\beta_n(z)$ *are two linearly independent particular solutions of* (1) *and* $g(z)$ *and* $h(z)$ *are two arbitrary functions of* z.

If, as before, $\alpha_n(z)$ and $\beta_n(z)$ be two particular solutions of (I), we must have:

$$\alpha_{n+1}(z) - z\,\alpha_n(z) + n\,\alpha_{n-1}(z) = 0\,, \tag{3}$$

and

$$\beta_{n+1}(z) - z\,\beta_n(z) + n\,\beta_{n-1}(z) = 0\,. \tag{4}$$

Subtracting (4), multiplied by $\alpha_n(z)$, from (3), multiplied by $\beta_n(z)$ and setting:

$$\gamma_n(z) = \alpha_n(z)\,\beta_{n-1}(z) - \beta_n(z)\,\alpha_{n-1}(z),$$

we obtain:

$$\gamma_{n+1}(z) = n\,\gamma_n(z),$$

i. e.,

(5) $$\gamma_n(z) = (n-1)\,\gamma_{n-1}(z).$$

Putting $n = 1, 2, 3, \ldots,$ successively in (5) and combining the resulting relations, we are squarely led to:

$$\gamma_n(z) = \underline{|n-1}\;\gamma_1(z).$$

We thus arrive at the following proposition:

If $\alpha_n(z)$ and $\beta_n(z)$ be two linearly independent solutions of (I) *and n be restricted to be a positive integer, then the relation:*

(6) $$\alpha_n(z)\,\beta_{n-1}(z) - \beta_n(z)\,\alpha_{n-1}(z) = \underline{|n-1}\,.\{\alpha_1(z)\,\beta_0(z) - \beta_1(z)\,\alpha_0(z)\}$$

must hold identically.

Certainly the relation (6) will stand, if either $\alpha_n(z)$ or $\beta_n(z)$ be replaced by the Weber function $D_n(z)$, for this function satisfies (1). Finally if (I) be re-written in the form:

$$\frac{f_{n+1}(z)}{f_n(z)} = z - \frac{n}{\dfrac{f_n(z)}{f_{n-1}(z)}},$$

and n be replaced successively by $n-1,\ n-2, \ldots, 3, 2, 1$, and the resulting relations be combined together, we derive the equality:

(7) $$\frac{f_{n+1}(z)}{f_n(z)} = z - \frac{n}{z-}\;\frac{n-1}{z-}\cdots\cdots\frac{2}{z-}\;\frac{f_0(z)}{f_1(z)}.$$

Recollecting that $D_n(z)$ is but a particular solution of (I), and attending to the relations:

$$D_0(z) = e^{-\frac{z^2}{4}} \qquad \text{and} \qquad D_1(z) = z\,e^{-\frac{z^2}{4}},$$

we immediately deduce the following corollary from (7):

$$\frac{D_{n+1}(z)}{D_n(z)} = z - \frac{n}{z-}\;\frac{n-1}{z-}\cdots\cdots\frac{2}{z-}\;\frac{1}{z}.$$

SECTION II

Inter-relations between the two functional equations I and II

§ 2. — Reserving Sec. III for a discussion of the common solutions of (I) and (II), we shall devote the present section to a consideration of the mutual relations that subsist between (I), (II) and (A). There are three distinct cases to consider.

CASE I). Suppose that a sequence of functions $\{f(z)\}$ satisfies both (1) and (II). Then adding (I) and (II), and finally changing n into $n-1$, we obtain:

$$f_n(z) - \frac{1}{2} z f_{n-1}(z) + f'_{n-1}(z) = 0 . \tag{1}$$

If (II) be now differentiated and the derived result be associated with (I) so as to dispense with $f'_{n-1}(z)$, the eliminant is easily seen to be:

$$f''_n(z) + \left(n + \frac{1}{2}\right) f_n(z) + \frac{1}{2} z \{f'_n(z) - n f_{n-1}(z)\} = 0 . \tag{2}$$

Elimination of the set of terms $[f'_n(z) - n f_{n-1}(z)]$ from (II) and (2) leads to:

$$f''_n(z) + \left(n + \frac{1}{2} - \frac{1}{4} z^2\right) f_n(z) = 0 ,$$

shewing that $\{f_n(z)\}$ satisfies (A).

CASE II). Suppose that $\{f_n(z)\}$ satisfies (II) and (A).

Then if we differentiate (II) and in the derived result we replace $f''_n(z)$ by the equivalent value

$$-\left(n + \frac{1}{2} - \frac{1}{4} z^2\right) f_n(z),$$

obtained from (A), we find after easy reductions:

$$\left(n - \frac{1}{4} z^2\right) f_n(z) - \frac{1}{2} z f'_n(z) + n f'_{n-1}(z) = 0 . \tag{3}$$

If in (3) the term $\frac{1}{2} z f'_n(z)$ be replaced by its equivalent value as provided for by (II), we get:

$$f_n(z) - \frac{1}{2} z f_{n-1}(z) + f'_{n-1}(z) = 0 . \tag{4}$$

If, in (4), n be changed into $(n+1)$ and the resulting relation be coupled with (II), so as to get rid of $f'_n(z)$, we obtain:

$$f_{n+1}(z) - z f_n(z) + n f_{n-1}(z) = 0 ,$$

proving that the combination of (II) and (A) implies (I).

CASE III). Suppose that $\{f_n(z)\}$ satisfies both (I) and (A).

Then re-writing (I) in the form:

$$f_{n+1}(z) = z f_n(z) - n f_{n-1}(z) , \tag{5}$$

and calculating $f'_{n+1}(z)$ and $f''_{n+1}(z)$, we find, on a re-shuffling of terms:

$$\begin{aligned} f''_{n+1}(z) + \left(\overline{n+1} + \frac{1}{2} - \frac{1}{4} z^2\right) f_{n+1}(z) = \\ = z \left[f''_n(z) + \left(\overline{n+1} + \frac{1}{2} - \frac{1}{4} z^2\right) f_n(z) \right] - n \left[f''_{n-1}(z) + \right. \\ \left. + \left(\overline{n+1} + \frac{1}{2} - \frac{1}{4} z^2\right) f_{n-1}(z) \right] + 2 f'_n(z) = \\ = z f_n(z) - 2 n f_{n-1}(z) + 2 f'_n(z) , \end{aligned} \tag{6}$$

for $f_n(z)$ and $f_{n-1}(z)$ satisfy the respective differential equations $D^{(n)}$ and $D^{(n-1)}$.

Further the expression (6) must vanish identically, seeing that $f_{n+1}(z)$ satisfies $D^{(n+1)}$.

Thus the combination of (I) and (A) gives rise to:

$$z f_n(z) - 2 n f_{n-1}(z) + 2 f'_n(z) = 0 ,$$

proving that (I) and (A), taken jointly, lead to (II).

Amalgamating the results of Cases (I), (II) and (III), we infer that *the two functional equations* (I) *and* (II) *and the differential equation* (A) *are so related to one another that the combination of any two of them at once implies the third.*

SECTION III

Common solutions of the two simultaneous equations (I) and (II) and their connection with the differential equation $D_{(n)}$

§ 3. — Our present task is to solve the two *simultaneous* functional equations (I) and (II). The *modus operandi* is suggested by the proved result of § 2, *viz.* that if a sequence $\{f_n(z)\}$ satisfies both (I) and (II), then $f_n(z)$ must satisfy the differential equation $D^{(n)}$ for *every* positive integer n.

Accordingly we may initially choose (*at random*) a primitive $f_n(z)$ of the differential equation $D^{(n)}$ of *known* rank n, so that:

$$f_n''(z) + \left(n + \frac{1}{2} - \frac{1}{4} z^2\right) f_n(z) = 0 . \tag{1}$$

If we now introduce the « contiguous » functions $f_{n-1}(z)$ and $f_{n+1}(z)$ in accordance with the two equations (I) and (II), we may write:

$$f_{n-1}(z) = \frac{1}{n} \left\{ \frac{1}{2} z f_n(z) + f_n'(z) \right\} , \tag{2}$$

and

$$f_{n-1}(z) = \frac{1}{2} z f_n(z) - f_n'(z) . \tag{3}$$

Differentiating (2) and utilising (1), we get:

$$f_{n-1}'(z) = \frac{1}{n} \cdot \left[\frac{1}{2} z f_n'(z) - \left(n - \frac{1}{4} z^2\right) f_n(z) \right] . \tag{4}$$

Again differentiating (4) and looking to (1), we find after a few steps:

$$f_{n-1}''(z) = -\left(n - \frac{1}{2} - \frac{1}{4} z^2\right) f_{n-1}(z) ,$$

shewing that $f_{n-1}(z)$ satisfies $D^{(n-1)}$.

Dealing similarly with (3), we find, by making use of (1),

$$f_{n-1}'(z) = \frac{1}{2} z f_n'(z) + \left(n + 1 - \frac{1}{4} z^2\right) f_n(z) . \tag{5}$$

Another differentiation and simplification by aid of (1) give

$$f_{n+1}''(z) = -\left(n + \frac{3}{2} - \frac{1}{4} z^2\right) f_{n+1}(z) ,$$

shewing that $f_{n+1}(z)$ satisfies $D^{(n+1)}$.

Thus the two « contiguous » functions of $f_n(z)$, *viz.* $f_{n-1}(z)$ and $f_{n+1}(z)$, as defined by (I) and (II), *i. e.*, by (2) and (3), conform to the respective differential equations $D^{(n-1)}$ and $D^{(n+1)}$.

Further coupling (3) and (5) and then reducing, we get:

$$f_n(z) = \frac{1}{n+1} \cdot \left\{ \frac{1}{2} z f_{n+1}(z) + f_{n+1}'(z) \right\} . \tag{6}$$

Comparison of (6) with (2) reveals the fact that $f_n(z)$ is one of the two functions « contiguous » to $f_{n+1}(z)$. If the second « contiguous » function of $f_{n+1}(z)$, *viz.* $f_{n+2}(z)$ be defined by an equation of the type (1), *viz.*

$$f_{n+2}(z) = z f_{n+1}(z) - (n+1) f_n(z)$$

and the foregoing process be gone through again, one can easrly verify that $f_{n+2}(z)$ tallies with $D^{(n+2)}$. Similarly the second « contiguous » function of $f_{n+2}(z)$, *viz.* $f_{n+3}(z)$, may be formed and characterised. Repetition of the foregoing process places at our disposal an *unending* enumerable set of functions, *viz.*,

(7) $$f_n(z), \quad f_{n+1}(z), \quad f_{n+2}(z),$$

which conform to both (I) and (II), and satisfy the respective (differential) equations of the sequence:

$$D^{(n)}, \quad D^{(n+1)}, \quad D^{(n+2)}, \ldots .$$

Plainly it is open to us to proceed in the *rererse* order. In point of fact, the function $f_{n-1}(z)$, « contiguous » to the initial function $f_n(z)$, is already known to satisfy $D^{(n+1)}$. It is clear that, if a method analogous to that adopted as above be employ, we shall be in possession of another *(finite)* sequence of functions which, written in the reverse order, consist of:

(8) $$f_n(z), \quad f_{n+1}(z), \quad f_{n+2}(z), \ldots, f_1(z), \quad f_0(z),$$

which shall comply with (I) and (II) and shall conform respectively to the equations of the sequence:

$$D^{(n)}, \quad D^{(n-1)}, \quad D^{(n-2)}, \ldots, D^{(1)}, \quad D^{(0)}.$$

The initial function $f_n(z)$ is exhibited in both tbe sets (7) and (8) so as to emphasise the ract that it determines, *in its own way*, the whole lot of functions, satisfying (I) and (II).

Taking a broad survey of the results proved in this section, we can affirm that *a particular solution of $f_n(z)$ of a differential equation $D^{(n)}$ of an arbitrarily assigned rank n being selected in the first instance, the entire aggregate of functions:*

$$f_0(z), \quad f_1(z), \quad f_2(z), \ldots, f_n(z), \ldots, f_m(z), \ldots,$$

satisfyng both the functional equations (I) *and* (II) *and therefore satisfying respectively the sequence of differential equations:*

$$D^{(0)}, \quad D^{(1)}, \quad D^{(2)}, \ldots, D^{(n)}, \ldots, D^{(m)}, \ldots,$$

is uniquely determinate. Plainly, the totality of common solutions of the two simultaneous equations (I) and (II) must be counted as ∞^2 in conventional phraseology. By far the most conspicuous among the common solutions of (I) and (II) is, of course, the parabolic cylinder function $D_n(z)$ of Weber.

SECTION IV

On the generating function of the set of functions $\left\{ \frac{f_n(z)}{\lfloor n} \right\}$, were $f_n(z)$ is an analytic solution of (I)

§ 4. — Starting with an arbitrary *analytic* solution $f_n(z)$ of the equation:

$$\text{(I)} \qquad f_{n+1}(z) - z f_n(z) + n f_n(z) = 0 \,, \quad \text{(where } n \text{ is a positive integer)}$$

we propose to sum (when possible) the series:

$$\text{(1)} \qquad V = \sum_{n=0}^{\infty} \frac{h^n f_n(z)}{\lfloor n} \,.$$

Converting (I) into the form:

$$\text{(2)} \qquad (n+1)\, \Phi_{n+1}(z) - z\, \Phi_n(z) + \Phi_{n-1}(z) = 0 \,,$$

by the substitution:

$$\text{(3)} \qquad f_n(z) = \lfloor n \cdot \Phi_n(z) \,,$$

we may write:

$$\text{(4)} \qquad V = \sum_{n=0}^{\infty} h^n \, \Phi_n(z) \,.$$

This being a positive power series in h, term-wise differentiation in the form:

$$\text{(5)} \qquad \frac{\partial V}{\partial h} = \sum_{n=0}^{\infty} (n+1)\, h^n \, \Phi_{n+1}(z)$$

is perfectly legitimate under the condition

$$|h| < R \,,$$

it being postulated that R is the radius of convergence [1] of the series.

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[1] R is presumably a function of z or else a numerical quantity.

If (2) be now multiplied by h^n and the parameter n runs through the series of (integral) values $1, 2, 3, \ldots$, we have, on summation:

$$\sum_{n=1}^{\infty} (n+1) h^n \Phi_{n+1}(z) - z \cdot \sum_{n=1}^{\infty} h^n \Phi_n(z) + \sum_{n=1}^{\infty} h^n \Phi_{n-1}(z) = 0,$$

which, by virtue of (5), can be put in the symbolic form:

$$\frac{\partial V}{\partial h} + P V = Q, \tag{6}$$

where

$$P \equiv h - z \quad \text{and} \quad Q \equiv \Phi_1(z) - z\,\Phi_0(z). \tag{7}$$

When (6) is solved as a *linear* differential equation for V in terms of h, we get:

$$V = e^{-\int P dh} \cdot \left[\int Q\, e^{\int P dh} \cdot dh + C \right], \tag{8}$$

where the constant of integration must needs be independent of h but may involve z.

Attending to the relations (7), and noting that V reduces to $\Phi_0[z]$ when h vanishes, we are in a position to re-write (8) in the form:

$$\sum_{n=0}^{\infty} h^n \Phi_n(z) = \{\Phi_1(z) - z\,\Phi_0(z)\}\, e^{zh - \frac{h^2}{2}} \int_0^h e^{\frac{t^2}{z} - zt} \cdot dt + \Phi_0(z) \cdot e^{zh - \frac{h^2}{2}}, \tag{9}$$

it being premised that the path of integration for the line-integral, occurring in the R. S. of (9), is the straight line, joining the origin to the point h in the t plane.

In order to complete the solution, we have to calculate R. To that end we may exhibit the relation (2) in the fractional form:

$$\Phi_{n+1}(z) = \frac{z\,\Phi_n(z) - \Phi_{n-1}(z)}{n+1}. \tag{10}$$

A cursory glance at (10) shews that, the point « z » being supposed to be a *regular* point of the functions of the set $\{\Phi_n(z)\}$, $|\Phi_{n+1}(z)|$ must, for immeasurably large values of n, be indefinitely small in comparison with $|\Phi_n(z)|$ or $|\Phi_{n-1}(z)|$,

i. e.,
$$\lim_{n \to \infty} \left| \frac{\Phi_{n+1}(z)}{\Phi_n(z)} \right| = 0.$$

The radius of convergence R of the series V is (by D'ALEMBERT'S ratio-test), easily seen to be ∞.

As is well-known, (2) being a difference equation of the *second* order, the whole sequence of functions $\{\Phi_n(z)\}$, tallying with (2), becomes perfectly determinate. as soon as the (functional) values $\Phi_0(z)$ and $\Phi_1(z)$ are assigned.

We may then look repon (9) as the *general summation-formula* for $\sum_{n=0}^{\infty} h^n \Phi_n(z)$, *i. e.*, for $\sum_{n=0}^{\infty} \frac{h^n f_n(z)}{\underline{|n}}$, it being tacitly understood that, in any individual case, the *appropriate* (functional) values of $\Phi_0(z)$ and $\Phi_1(z)$ are to be substituted. Needless to add, the formula (9) is valid for *all* positions of h in the *finite* part of the h plane and for *all* positions of z (in the finite part of the z-plane). at which the set of functions $\{\Phi_n(z)\}$ are all analytic, provided, of course, that the line integral, occurring in (9), is taken along the straight line, joining the origen to the point h in the t-plane or h-plane. Should the functions $\{\Phi_n(z)\}$ be all integral (rational or trascendental), the domain for « z » is practically the entire *finite* part of the z-plane, the other conditions remaining intact. An instance in point is afforded by the sequence of Weber functions $\{D_n(z)\}$, which are all integral.

As a matter of fact, on putting

$$\Phi_0(z) = \frac{D_0(z)}{\underline{|0}} = e^{-\frac{z^2}{4}} \qquad \text{and} \qquad \Phi_1(z) = \frac{D_1(z)}{\underline{|1}} = z\, e^{-\frac{z^2}{4}},$$

so that

$$\Phi_1(z) - z\,\Phi_0(z) = 0\,, \qquad \text{(identically)}\,,$$

the formula (9) at once leads to the *known* result [2]:

$$\sum_{n=0}^{\infty} \frac{h^n D_n(z)}{\underline{|n}} = e^{-\frac{z^2}{4} + zh - \frac{h^2}{2}}. \tag{11}$$

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[2] In the above context, the parabolic cylinder-function $D_n(z)$ is being defined as that *particular* solution of the functional equation:

$$(n+1)\,\Phi_{n+1}(z) - z\,\Phi_n(z) + \Phi_{n-1}(z) = 0\,,$$

which conforms to the pair of special conditions:

$$\Phi_0(z) = e^{-\frac{z^2}{4}} \qquad \text{and} \qquad \Phi_1(z) = z\,e^{-\frac{z^2}{4}}.$$

That this definition of $D_n(z)$ is tantamount to the alternative definition that it is the coefficient of $\frac{h^n}{\underline{|n}}$ in the expansion of $e^{-\frac{z^2}{4} + zh - \frac{h^2}{2}}$ (in ascending powers of h) has been in a way corroborated by the concluding portion of the above investigation.