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ON THE PROBLEM OF GEÖCZE

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1. – Introduction.

The comments presented in this note occurred to the writer while studying a recent remarkable paper of MAMBRIANI on the GEÖCZE problem in surface area theory (MAMBRIANI [4]; numbers in square brackets refer to the Bibliography at the end of this note). The results of MAMBRIANI are based on delicate approximations to Lebesgue integrals which he presents in two separate papers [2], [3]. While these approximation theorems are of definite independent interest also, it seemed to the writer that in view of the importance of the result of MAMBRIANI relative to the GEÖCZE problem it may be worthwhile to discuss an alternative approach. The method used in this note is based on an altogether elementary identity involving areas of polyhedra (see the formula (9) in section 2. 2). This identity is an extension, to the parametric case, of a similar identity which the writer developed in the non-parametric case, and is closely related to an ingenious method used previously by HUSKEY in his work on the GEÖCZE problem in the non-parametric case (see [1], [5]). As a matter of fact, our approach yields a certain refinement of the result of MAMBRIANI. To explain this point, let us introduce the following notations. Let

$$(1) \qquad T: \quad x=x(u, v), \quad y=y(u, v), \quad z=z(u, v), \qquad (u, v) \varepsilon Q,$$

be a continuous mapping from the unit square

$$Q: \quad 0 \leqq u \leqq 1, \qquad 0 \leqq v \leqq 1,$$

into Euclidean xyz-space. Then T may be thought of as a representation of a surface S (see the writer's book on *Length and Area*, Bibliography [6]; this book will be referred to by LA). We shall denote by $A(T)$ the area of this surface S in the LEBESGUE sense (see LA, V. 2. 3). We shall consider polyhedra inscribed in S, obtained as follows. We take two sets of numbers

$$(2) \qquad 0 \leqq u_0 < u_1 < \dots < u_n \leqq 1,$$

$$(3) \qquad 0 \leqq v_0 < v_1 < \dots < v_m \leqq 1.$$

Let R be the rectangle $u_0 \leqq u \leqq u_n$, $v_0 \leqq v \leqq v_m$. The lines $u=u_i$, $i=1,\dots n-1$, and $v=v_j$, $j=1,\dots, m-1$, subdivide R into smaller rectangles r. In each r, we draw the diagonal from the upper left to the lower right corner, obtaining a triangulation τ. Let $\xi(u, v)$ be the (uniquely determined) continuous function in R which is linear in each triangle of τ and agrees with $x(u, v)$ at the vertices of τ. Let $\eta(u, v)$, $\zeta(u, v)$ be defined similarly in terms of $y(u, v)$ and $z(u, v)$ respectively. Then the equations $x=\xi(u, v)$ $y=\eta(u, v)$, $z=\zeta(u, v)$, $(u, v) \,\varepsilon\, R$, determine a polyhedron P that Mambriani terms a *Tonelli polyhedron* relative to the representation (1). Let P_k, $k=1, 2,\dots$, be a sequence of TONELLI polyhedra, relative to the fixed representation (1), such that (i) the corresponding rectangles R tend to Q and (ii) the maximum side-length of the triangles in the corresponding triangulation τ approaches zero. We shall then say that the sequence P_k is *admissible* relative to the representation (1). An admissible sequence P_k will be termed *regular* if for each P_k the corresponding numbers (2), (3) satisfy the relations

$$u_n - u_0 = v_m - v_0, \quad u_1 - u_0 = u_2 - u_1 = \dots = u_n - u_{n-1},$$

$$v_1 - v_0 = v_2 - v_1 = \dots = v_m - v_{m-1}, \quad m = n.$$

Let us put

$$A'(T) = \text{gr. l. b.} \liminf E(P_k),$$

where the greatest lower bound is taken with respect to all admissible sequences P_k relative to T, and $E(P_k)$ is the elementary area of P_k. Let $A''(T)$ be defined in the same manner except that only admissible *regular* sequences (relative to T) are used. Clearly

$$A(T) \leqq A'(T) \leqq A''(T). \tag{4}$$

The primary result of MAMBRIANI states that $A(T)=A'(T)$ in a certain important case. Our method yields, for the same case, the relation

$$A(T) = A'(T) = A''(T).$$

In the concluding part 5 of this note, we shall suggest certain further applications of our method, and we shall also call attention to a further problem that may be of interest in connection with the problem of GEÖCZE.

2. - An elementary identity.

2. 1. - Let there be given a continuous mapping

$$(1) \qquad T: \quad x=x(u, v), \quad y=y(u, v), \quad z=z(u, v), \qquad (u, v) \,\varepsilon\, Q,$$

from the unit square $Q: 0 \leqq u \leqq 1, 0 \leqq v \leqq 1$ into Euclidean xyz-space. Then T is a representation of an F-surface S (see LA, II. 3. 44). The LEBESGUE area of S will be denoted by $A(T)$, and we assume throughout that

(2) $$A(T) < \infty.$$

It will be convenient to extend the definition of $x(u, v)$, $y(u, v)$, $z(u, v)$ to the whole uv-plane by the following definite process. First, we extend the definition of $x(u, v)$ to the rectangle $0 \leqq u \leqq 2$, $0 \leqq v \leqq 1$ by reflection on the line $u=1$, then we carry out a second extension to the square $0 \leqq u \leqq 2$, $0 \leqq v \leqq 2$ by reflection on the line $v=1$. We have then clearly $x(0, v)=x(2, v)$ for $0 \leqq v \leqq 2$ and $x(u, 0)=x(u, 2)$ for $0 \leqq u \leqq 2$. We can extend therefore $x(u, v)$ in a unique manner to the whole uv-plane subject to the requirement that $x(u+2, v)=x(u, v)$, $x(u, v+2)=x(u, v)$ identically in u, v. The extended function $x(u, v)$ will be denoted by the same symbol $x(u, v)$. The same procedure is applied to the functions $y(u, v)$, $z(u, v)$ appearing in (1).

2. 2. – If (α, β) is any point in the uv-plane, then we shall denote by $Q_{\alpha\beta}$ the square

$$Q_{\alpha\beta}: \quad \alpha \leqq u \leqq \alpha+1, \quad \beta \leqq v \leqq \beta+1.$$

Given a positive integer $n \geqq 2$, we subdivide $Q_{\alpha\beta}$ into n^2 congruent squares by drawing equidistant horizontals and verticals. In each one of the squares of this subdivision, we draw the diagonal from the upper left to the lower right vertex, obtaining a triangulation $\tau_{n\alpha\beta}$ of $Q_{\alpha\beta}$. With this triangulation we associate a polyhedron $\mathfrak{P}_n(\alpha, \beta)$ as follows: $\mathfrak{P}_n(\alpha, \beta)$ is determined by a representation

$$x=\xi(u, v), \quad y=\eta(u, v), \quad z=\zeta(u, v), \qquad (u, v)\,\varepsilon\, Q_{\alpha\beta},$$

where $\xi(u, v)$, $\eta(u, v)$, $\zeta(u, v)$ are linear on each triangle of $\tau_{n\alpha\beta}$ and continuous on $Q_{\alpha\beta}$, and $\xi(u, v)=x(u, v)$, $\eta(u, v)=y(u, v)$, $\zeta(u, v)=z(u, v)$ at every vertex of $\tau_{n\alpha\beta}$. The area of $\mathfrak{P}_n(\alpha, \beta)$ in the elementary sense will be denoted by $\mathfrak{A}_n(\alpha, \beta)$. To obtain a compact formula for this area, we introduce the following notations, where δ is any real number $\neq 0$.

(1) $$x_u^*(u, v, \delta)=\frac{x(u+\delta, v)-x(u, v)}{\delta},$$

(2) $$x_v^*(u, v, \delta)=\frac{x(u, v+\delta)-x(u, v)}{\delta},$$

with similar definitions for the symbols $y_u^*(u, v, \delta)$, $y_v^*(u, v, \delta)$, $z_u^*(u, v, \delta)$, $z_v^*(u, v, \delta)$.

(3) $$J_1^*(u, v, \delta)=y_u^*(u, v, \delta)\, z_v^*(u, v, \delta)-y_v^*(u, v, \delta)\, z_u^*(u, v, \delta).$$

(4) $$J_2^*(u, v, \delta)=z_u^*(u, v, \delta)\, x_v^*(u, v, \delta)-z_v^*(u, v, \delta)\, x_u^*(u, v, \delta).$$

(5) $$J_3^*(u, v, \delta)=x_u^*(u, v, \delta)\, y_v^*(u, v, \delta)-x_v^*(u, v, \delta)\, y_u^*(u, v, \delta).$$

(6) $$W^*(u, v, \delta)=[J_1^*(u, v, \delta)^2+J_2^*(u, v, \delta)^2+J_3^*(u, v, \delta)^2]^{1/2}.$$

A straightforward elementary calculation yields then the formula

$$(7)\quad \mathfrak{A}_n(\alpha, \beta) = \frac{h^2}{2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} [W^*(\alpha + ih, \beta + jh, h) + W^*(\alpha + \overline{i+1}h, \beta + \overline{j+1}h, -h)],$$

where we have put

$$(8)\quad h = \frac{1}{n}.$$

For fixed n, clearly $\mathfrak{A}_n(\alpha, \beta)$ is a continuous function of α and β. We need the integral mean of this function over the square $0 \leqq \alpha \leqq h$, $0 \leqq \beta \leqq h$. Using (7) and (8), a straightforward elementary calculation yields the fundamental identity

$$(9)\quad \frac{1}{h^2} \int_0^h \int_0^h \mathfrak{A}_n(\alpha, \beta)\, d\alpha\, d\beta = \frac{1}{2} \int_0^1 \int_0^1 W^*(u, v, h)\, du\, dv + \frac{1}{2} \int_h^{1+h} \int_h^{1+h} W^*(u, v, -h)\, du\, dv.$$

As noted in the introduction, this identity is merely an extension, to the parametric case, of an identity previously derived by the writer for the non-parametric case [see *LA*, V. 3. 53, formula (2)] as a refinement of certain approximate formulas used by HUSKEY in his work on the GEÖCZE problem (see Bibliography [1], [5]).

3. – Comments on the elementary identity.

3. 1. - Recalling that $h = 1/n$, we put

$$(1)\quad \int_0^1 \int_0^1 W^*(u, v, h)\, du\, dv = I_n',$$

$$(2)\quad \int_h^{1+h} \int_h^{1+h} W^*(u, v, -h)\, du\, dv = I_n''.$$

The identity 2. 2 (9) appears then in the form

$$(3)\quad \frac{1}{h^2} \int_0^h \int_0^h \mathfrak{A}_n(\alpha, \beta)\, d\alpha\, d\beta = \frac{1}{2}(I_n' + I_n'').$$

Since $\mathfrak{A}_n(\alpha, \beta)$ is a continuous function of α, β for fixed n, we have in the square $0 \leqq \alpha \leqq h$, $0 \leqq \beta \leqq h$, by the mean value theorem in Calculus, a point (α_n, β_n) such that

$$(4)\quad \frac{1}{h^2} \int_0^h \int_0^h \mathfrak{A}_n(\alpha, \beta)\, d\alpha\, d\beta = \mathfrak{A}_n(\alpha_n, \beta_n),$$

$$(5)\quad 0 \leqq \alpha_n \leqq h, \quad 0 \leqq \beta_n \leqq h.$$

Comparison with (3) yields the formula

$$\frac{1}{2}(I_n' + I_n'') = \mathfrak{A}_n(\alpha_n, \beta_n). \tag{6}$$

Now $\mathfrak{A}_n(\alpha_n, \beta_n)$ is the elementary area of a polyhedron $\mathfrak{P}_n(\alpha_n, \beta_n)$ (see 2.2), defined by a representation over the square $\alpha_n \leqq u \leqq \alpha_n + 1$, $\beta_n \leqq v \leqq \beta_n + 1$. Let P_n be that portion of $\mathfrak{P}_n(\alpha_n, \beta_n)$ which corresponds to the range

$$\alpha_n \leqq u \leqq \alpha_n + 1 - \frac{1}{n}, \quad \beta_n \leqq v \leqq \beta_n + 1 - \frac{1}{n},$$

and let A_n be the elementary area of P_n. Then $A_n \leqq \mathfrak{A}_n(\alpha_n, \beta_n)$, and hence by (6) we have

$$A_n \leqq \frac{1}{2}(I_n' + I_n''). \tag{7}$$

On the other hand, in view of (5), the polyhedra P_n, $n = 2, 3, \ldots$, form an admissible regular sequence relative to the representation 2.1 (1), and consequently (see Introduction)

$$A''(T) \leqq \liminf_{n \to \infty} A_n. \tag{8}$$

From (6), (7), (8) we obtain the inequalities (cf. *Introduction*)

$$A(T) \leqq A'(T) \leqq A''(T) \leqq \frac{1}{2}(\limsup_{n \to \infty} I_n' + \limsup_{n \to \infty} I_n''). \tag{9}$$

3. 2. - We introduce now the following assumption concerning the representation 2.1 (1): the partial derivatives $x_u, x_v, y_u, y_v, z_u, z_v$ exist a. e. (*almost everywhere*) in the unit square Q. We shall use the customary notations

$$J^1 = y_u z_v - y_v z_u, \quad J^2 = z_u x_v - z_v x_u, \quad J^3 = x_u y_v - x_v y_u,$$
$$W = [(J^1)^2 + (J^2)^2 + (J^3)^2]^{1/2}.$$

By *LA*, V. 2. 14, the assumption 2.1 (2) implies that W is summable in Q and

$$\int_0^1 \int_0^1 W \, du \, dv \leqq A(T). \tag{1}$$

Now from the procedure used in 2. 1 to extend the definition of $x(u, v)$, $y(u, v)$, $z(u, v)$ to the whole uv-plane it is obvious that, as a consequence of the assumptions introduced in the present section, the partial derivatives x_u, x_v, y_u, y_v, z_u, z_v exist a. e. in the uv-plane and that W is summable on every bounded measurable set.

Furthermore (cf. 2.2) obviously for $\delta \to 0$,

$$x_u^*(u, v, \delta) \to x_u(u, v), \ldots, \quad z_v^*(u, v, \delta) \to z_v(u, v)$$

a. e. in the uv-plane, and hence also

$$W^*(u, v, \delta) \to W(u, v) \text{ for } \delta \to 0 \tag{2}$$

almost everywhere in the uv-plane. In view of the definition of I_n', I_n'' (see 3.1, it follows from (2) that under suitable further restrictions upon the representation 2.1 (1) we shall have the relations

$$I_n' \underset{n \to \infty}{\longrightarrow} \int_0^1 \int_0^1 W \, du \, dv, \tag{3}$$

$$I_n'' \underset{n \to \infty}{\longrightarrow} \int_0^1 \int_0^1 W \, du \, dv. \tag{4}$$

We shall consider such further restrictions later on, and at this time we state merely an intermediate lemma that summarizes the conclusions that may be derived directly from the fundamental identity 2.2 (9).

3. 3. - LEMMA. - *Using the assumptions and notations of* 3.2, *suppose that the relations* 3.2 (3), 3.2 (4) *hold. Than we have the following statements*:

$$A(T) = \int_0^1 \int_0^1 W \, du \, dv, \tag{i}$$

$$A(T) = A'(T) = A''(T) \quad \text{(cf. } \textit{Introduction}\text{)}. \tag{ii}$$

Proof. If 3.2 (3), 3.2 (4) hold, then by 3.2 (1) and 3.1 (9) there follow the inequalities

$$\int_0^1 \int_0^1 W \, du \, dv \leqq A(T) \leqq A'(T) \leqq A''(T) \leqq \int_0^1 \int_0^1 W \, du \, dv.$$

Since the same quantity occurs at both ends, the sign of equality must hold throughout, and the lemma is proved.

4. - The theorem of Mambriani.

4. 1. - Throughout the present part 4, we make the following assumptions concerning the representation 2.1 (1).

(*i*) The functions $x(u, v)$, $y(u, v)$, $z(u, v)$ are ACT (absolutely continuous in the TONELLI sense, see LA, III. 2. 64) in the unit square $Q: 0 \leqq u \leqq 1, 0 \leqq v \leqq 1$.

(*ii*) The partial derivatives $x_u, x_v, y_u, y_v, z_u, z_v$ are summable with their squares in Q.

4. 2. - Using the notations $A(T)$, $A'(T)$, $A''(T)$ in the sense explained in the introduction, we shall derive presently the relations

$$(1) \qquad A(T)=A'(T)=A''(T).$$

The relation $A(T)=A'(T)$ is the theorem of MAMBRIANI [4], while the relation $A(T)=A''(T)$ represents a refinement of his result. Our proof of (1) will depend upon the lemma in 3. 3 and upon a classical theorem on term-wise integration which shall state presently.

4. 3. - On a bounded measurable set E in the uv-plane, let $f_n(u, v)$, $n=0, 1, 2,...$, be a sequence of non-negative summable functions such that $f_n \to f_0$ a. e. on E. Then the relation

$$\int_E f_n \underset{n\to\infty}{\longrightarrow} \int_E f_0$$

holds, by a classical theorem of VITALI, if and only if the sequence f_n possesses the following property (V): for every $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that

$$\int_e f_n < \varepsilon, \quad n=1, 2,...,$$

for every measurable set $e \subset E$ for which $|e| < \delta$.

4. 4. - Returning to the situation described in 4. 1, we first make the obvious remark that the functions $x(u, v)$, $y(u, v)$, $z(u, v)$, extended to the whole uv-plane according to the process described in 2. 1, satisfy the conditions 4. 1 (*i*), 4. 1 (*ii*) not merely in the unit square Q, but also in every rectangle

$$(1) \qquad R: \quad a \leqq u \leqq b, \quad c \leqq v \leqq d.$$

4. 5. - We first assert that

$$\iint_R \left| \frac{x(u+h, v)-x(u, v)}{h} \right|^2 du\, dv \to \iint_R |x_u(u, v)|^2 du\, dv$$

if $h=1/n \to 0$, for every rectangle R of the form 4. 4 (1). Indeed, since a.e. in R

$$\frac{x(u+h, v)-x(u, v)}{h} \to x_u(u, v),$$

we have by the lemma of FATOU

$$\liminf \iint_R \left| \frac{x(u+h, v)-x(u, v)}{h} \right|^2 du\, dv \geqq \iint_R |x_u(u, v)|^2 du\, dv,$$

and hence it is sufficient to show that

$$\text{(1)} \qquad \limsup \iint_R \left| \frac{x(u+h, v) - x(u, v)}{h} \right|^2 du\, dv \leqq \iint_R | x_u(u, v) |^2 du\, dv.$$

Now in view of the assumptions made in 4. 1 (cf. 4. 4), we have

$$\iint_R \left| \frac{x(u+h, v) - x(u, v)}{h} \right|^2 du\, dv = \iint_R \left[\frac{1}{h} \int_0^h x_u(u+\xi, v)\, d\xi \right]^2 du\, dv \leqq$$

$$\leqq \frac{1}{h^2} \iint_R \left[\int_0^h x_u(u+\xi, v)^2\, d\xi \cdot \int_0^h d\xi \right] du\, dv = \frac{1}{h} \int_0^h \left[\iint_R x_u(u+\xi, v)^2\, du\, dv \right] d\xi \leqq$$

$$\leqq \frac{1}{h} \int_0^h \left[\int_a^{b+h} \int_c^d x_u(u, v)^2 du\, dv \right] d\xi = \int_a^{b+h} \int_c^d x_u(u, v)^2 du\, dv.$$

Hence (1) follows for $h = 1/n \to 0$. Of course, the relation derived in the present section 4. 5 is classical, and we sketched its proof merely for the convenience of the reader.

4. 6. - We assert (see 2. 2) that the family of functions $| x_u^*(u, v, h) |^2$, $h = 1/n$, $n = 1, 2, \ldots$, possesses the property (V) in every rectangle R of the form 4. 4 (1). Indeed, by 4. 5 we have the relation

$$\iint_R | x_u^*(u, v, h) |^2 du\, dv \to \iint_R | x_u(u, v) |^2 du\, dv$$

for $h = 1/n \to 0$. Since clearly $| x_u^*(u, v, h) |^2 \to | x_u(u, v) |^2$ a. e. in R, the property (V) follows by 4. 3. In a similar manner, it follows of course that the families $| x_v^*(u, v, h) |^2$, $| y_u^*(u, v, h) |^2$, $| y_v^*(u, v, h) |^2$, $| z_u^*(u, v, h) |^2$, $| z_v^*(u, v, h) |^2$ possess the property (V) in every rectangle R of the form 4. 4 (1). Now since (see 2. 2)

$$| J_1^*(u, v, h) | \leqq \frac{1}{2} [y_u^*(u, v, h)^2 + y_v^*(u, v, h)^2 + z_u^*(u, v, h)^2 + z_v^*(u, v, h)^2],$$

it follows that the family $J_1^*(u, v, h)$, and hence by an analogous argument the families $| J_2^*(u, v, h) |$, $| J_3^*(u, v, h) |$ all possess the property (V) in R. Finally, since (see 2. 2)

$$W^*(u, v, h) \leqq | J_1^*(u, v, h) | + | J_2^*(u, v, h) | + | J_3^*(u, v, h) |,$$

it follows that the family $W^*(u, v, h)$ possesses the property (V) in R. An entirely analogous reasoning applies of course to the family $W^*(u, v, -h)$.

4. 7. - From the preceding results there follow, by 3. 2 (2) and 4. 3, the relations

(1) $$\iint_R W^*(u, v, h)\, du\, dv \to \iint_R W\, du\, dv,$$

(2) $$\iint_R W^*(u, v, -h)\, du\, dv \to \iint_R W\, du\, dv$$

for every rectangle R of the form 4. 4 (1) (and indeed for every bounded measurable set, a fact that we do not have to use). Applying this result with $R=Q$, we obtain (cf. 3. 1) the relation 3. 2 (3). To derive the relation 3. 2 (4), let us choose a rectangle R_0 that contains the unit square Q in its interior. Since on R_0 the family $W^*(u, v, -h)$ possesses the property (V), it is clear that the relation 3. 2 (4) is equivalent to the relation 4. 7 (2) with $R=Q$. Thus 3. 2 (3), 3. 2 (4) are both established, and hence 4. 2 (1) follows by the lemma in 3. 3.

5. - Conclusion.

5. 1. - Inspection reveals that the assumptions made in 4. 1 were used only to show that the families $|J_1^*(u, v, \delta)|$, $|J_2^*(u, v, \delta)|$, $|J_3^*(u, v, \delta)|$, with $\delta = \pm 1/n$, $n=1, 2, \ldots$, possess the property (V) (see 4. 3). Once this fact is established, the rest of the argument depends only upon the lemma in 3. 3, which in turn follows directly from the elementary identity 2. 2 (9), by means of well-known properties of the LEBESGUE area. These remarks suggest further applications which the experienced reader will formulate readily.

5. 2. - Comparison with the best previous result in the non-parametric case, due to HUSKEY [1] (see also the presentation in LA, V. 3.50 — V. 3.57) leads to an interesting question. Let S be a surface which admits of a non-parametric representation

(1) $$T_0: \quad z=f(x, y), \quad 0 \leqq x \leqq 1, \quad 0 \leqq y \leqq 1,$$

where $f(x, y)$ is single-valued and continuous in the unit square, and suppose that the LEBESGUE area $A(S)=A(T_0)$ of S is finite. By a theorem of C. B. MORREY (see LA, V. 2.43) the surface S admits then of a representation

(2) $$T: \quad x=x(u, v), \quad y=y(u, v), \quad z=z(u, v), \qquad (u, v) \varepsilon Q,$$

where Q is the unit square $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$, which satisfies the assumptions of 4. 1. Hence, as pointed out by MAMBRIANI, it follows that

(3) $$A(S)=A(T)=A'(T).$$

In fact, using our refinement of the result of MAMBRIANI, we can even assert (see 4.2) that

$$A(S)=A(T)=A'(T)=A''(T). \tag{4}$$

However, in previous work on the GEÖCZE problem in the non-parametric form, the intention was to obtain this type of result *in terms of the original non-parametric representation* T_0 (see the formulation in LA, V. 3.50). Of course, this point of view may represent an unduly severe statement of the GEÖCZE problem, but it may also represent a fruitful challenge.

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