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Saturnian rings

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SATURNIAN RINGS [1]

by Oliver E. Glenn (Lansdowne, Pennsylvania).

1. - Maxwell contributed extensively to the theory of Saturn's rings, assuming that the potential between a constituent particle and the planet was newtonian. Recently Lichtenstein and Dive have extended this theory. However, opportunities for other points of view have arisen. Einstein's generalized laws of gravitation and also modern theories of radiation and of light pressure, make desirable the study of some new questions relating especially to the motions of small particles within a gravitational field. I have lately proved that, if the force is newtonian, there is a lower limit to the mass of an asteroid in stable motion on a nearly circular orbit, and hence until, by a revision of the hypotheses, new principles of stable motion of small masses upon such an orbit, can be established, the theory of the rings of Saturn necessarily involves obscurities.

Thus the occasion is favorable for a formulation of a new theory of Saturn's rings, the basis for which is a central force function more general than that of Sir Isaac Newton.

2. - **Relations between the planetary mass and the radial distances.** — We assume a special process which, in a region where a stable planet can pursue it's course around a center of potential S, creates a phalanx of small masses which rotate around the center as if to form a narrow strip of a saturnian ring.

If the central force F_1 is arbitrary the orbit of an astral body N is an integral curve of

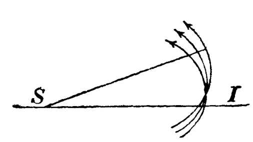

Fig. 1.

$$(1) \qquad d^2u/d\theta^2 + u = F/\gamma^2 u^2,$$

where $F = F_1/m$, $u = 1/r$, and γ is an arbitrary constant. Evidently the mass m is a parameter in the equation of the orbit. This equation can be written,

$$(2) \qquad r = f(\theta, m, s, v, \beta, c, \ldots).$$

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(1) The phenomenon of Saturn's rings was first discovered by Galileo (1564-1642).

The necessary description of the array of constants, $s, \ldots, c, \ldots$, which is to include both initial constants and integration parameters, in as follows: v is the initial velocity at the point I, $s = IS$, and β = the angle between the vector v and s. An instance of (2) is the equation of the orbit of a planet N when the force is given by NEWTON's formula of inverse squares, $F = \mu/r^2$, $(\mu = kmm')$,

$$r = \frac{\mu/s^2v^2 \sin^2 \beta}{1 + [1 + \mu^{-2}(v^2 s - 2\mu) s v^2 \sin^2 \beta]^{\frac{1}{2}} \cos(\theta + c)}. \tag{3}$$

An orbit is here defined to be stable if the planetary body is maintained upon it continually by the potential.

We now assume, in the case of (2), that N disintegrates into unequal parts as it passes I. Each part must follow a separate orbit from I onward, since m is different for each part and the other constants in (2) remain unaltered. Thus the newly-formed masses, while traveling approximately in the same surface that N would have continued in, gradually separate. To avoid local perturbations we could send each mass past I separately. In either case as considered their paths will diverge and give rise to a narrow strip of planetary orbits. This field of orbital curves, which we may take to be suitably restricted in length, will be, as a whole, stable only if each body satisfies conditions relating to it's mass, and other conditions, but the field may be taken to be stable and to be plane, by postulate.

Let θ' be a coordinate angle greater than that of I, the motion being in the direction of the arrows. Suppose the r of each curve, corresponding to θ', to have been observationally determined, giving the n pairs (r_i, θ'), $(i = 1, \ldots, n)$. Under the conditions of the problem, therefore, each m_i is uniquely determined by the corresponding distance r_i. Thus we may say that m is a function of r and determine it by interpolation from the sets (m_i, r_i), $(i = 1, \ldots, n)$, in the form,

$$m = A(s) r^{n-1} + B(s) r^{n-2} + \ldots + M(s) = g(r, s). \tag{4}$$

The general nature of $g(r, s)$ as a function of the parameter s may be seen by imagining (2) to be solved for m. The form (4) however is free from singularities, although valid only over the range $(r_1, \ldots, r_n)$.

Bode has inferred a law for the planetary distances in the solar system in a form which seems analogous to (4),

$$m_1 = \frac{1}{3} r - \frac{4}{3}, \qquad (m_1 = 0, 2^0, 2^1, \ldots, 2^7).$$

This formula is fairly accurate for the planets, and m_1 ranges over a remarkable sequence of values, but Bode's law is only a fact of observation. It has never been justified on theoretical grounds.

The formula (4) holds for a comparatively narrow field of curves which proceed from I: $(s(= s_1), \theta_1)$. If we wish a broader field field Φ we may assume that a second planet moves according to the same initial conditions as the first except that $s=s_2$, instead of $s=s_1$, with s_2-s_1, positive and small. We then obtain (4) with $s=s_2$. Thus a mass-distance formula, valid across a whole band of rotating masses, like a saturnian ring of limited lenthwise extent, is (4), in which s varies over a point-set, and in which the consecutive numbers of this set do not differ by a number too large. Except for the parameter s the coefficients $A,\ldots, M$ are known constants.

The masses distributed according to the law (4), at the distances r from S, will be of irregular sizes, that is, they are represented by the value of a polynomial, which has moreover $n-1$ zeros. Note that if the masses are asteroids in a solar system and we are studying a region in which lies a zero of $g(r, s)$, there will be a gap in the field, because, as was mentioned in the introductory paragraph, asteroids of mass too near to zero are not stable upon orbits nearly circular.

3. - The gravitational field which acts upon a small planetary particle. — It is proved in a former paper by the author that a typical orbit of the band Φ, if it is stable, that is, due to the potential, has a gyroscopic power to right itself when perturbed by small adventitious attractions, must pass through a field which essentially is Φ in the present problem, in coincidence with a curve whose equation is

$$\int dr/p(r)=\lambda\theta+\mu, \qquad (\lambda, \mu \text{ constant}). \tag{5}$$

Here

$$p(r)=ar^{n-1}+br^{n-2}+\ldots+k,$$

while it is an arbitrary polynomial, was derived by a process analogous to that for (4). By an artifice we can connect the formula (4) with the orbital equation (5). We may substitute $g(r, s)$ for $p(r)$ in (5), and when this is done, the author's formula for the central force of a stable orbit, here an orbit C of Φ, is, when the problem has the generality $n=4$, $(A=a,\ B=b,\ldots)$ [2],

$$G=\gamma^2\lambda^2[-A(Ar+B)+L/r^2+M/r^3+U/r^4+V/r^5]=\gamma^2\lambda^2\Gamma/r^5, \tag{6}$$

$$(L=AD+BC,\ M=C^2+2DB+\lambda^{-2},\ U=3CD,\ V=2D^2), \tag{7}$$

where A, $B,\ldots$, are abbreviations of $A(s)$, $B(s),\ldots$, and s is any definite number from it's point-set.

Now the numerator in (6),

$$\Gamma=-A^2r^6-ABr^5+Lr^3+Mr^2+Ur+V,$$

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[2] GLENN, Annali della R. Scuola Normale Superiore di Pisa (1933-XI), Ser. 2, Vol. 2, pp. 297-308. The general form is $G=\gamma^2\lambda^2[2p^2-rpp'+r^2/\lambda^2]/r^5$.

is identically of the form,

(8) $$g(r, s)(-Ar^3+Cr+2D)+\lambda^{-2}r^2=(-Ar^3+Cr+2D)m+\lambda^{-2}r^2,$$

or,

(9) $$G=\gamma^2\lambda^2[-Am/r^2+\lambda^{-2}/r^3+Cm/r^4+2Dm/r^5].$$

In these formulas m varies with r as in (4).

The restrictions inherent in the fact that we are studying Saturn's rings and not, for example, a zone of asteroids, must here be made. The number A is small, in fact, within numerical approximations, the linear terms of (6) must disappear when the number n of observed pairs (m_i, r_i), of (4), is reduced to three. Secondly all masses represented by (4) are, by postulate, small, so that we can replace m in (8) by the mean mass m_0 among them without greatly altering the value of G. Doing this we obtain within definable numerical approximations,

(10) $$G=\gamma^2\lambda^2[-Am_0/r^2+\lambda^{-2}/r^3+Cm_0/r^4+2Dm_0/r^5],$$

in which m_0 is sufficiently small, remaining constant while, upon C, r ranges over it's interval $(r_1, \ldots, r_n)$. However, with both A and m_0 small and r large, the significant term in (10) is the second. Hence,

THEOREM. - *If the mass of a rotating planetary body N does not exceed a small maximum m_0, r being sufficiently large, the central force function ceases to be newtonian. It becomes the formula of the inverse cube, the accuracy of this approximation being greater the larger r is.*

When the force F in (1) in of the form ν/r^3 the general integral is

(11) $$r=1/(\xi e^{\alpha\theta}+\eta e^{-\alpha\theta}), \qquad (e=2.71828\ldots),$$

ξ, η being arbitrary constants. The logarithmic spiral is a special form of this orbit ($\xi=0$). A curve perhaps more typical of the orbits which form spiral nebulae is that where ξ and η are of opposite signs. The curve is then asymptotic to the line,

$$\theta=\frac{1}{2\alpha}\log(-\eta/\xi).$$

It enters the finite plane along this line, encircles the origin as a spiral of diminishing radial distance and reaches the origin after making an infinite number of circuits. When, for some positive integer q, the following relation exists in (11),

(12) $$\eta/\xi=e^{(4q-2)\alpha\pi}, \qquad (\pi=3.14159\ldots),$$

the $(q-1)^{st}$ and the q^{th} circuits intersect. That is, we can obtain from the formula (11) a branch approximately circular.

When r is comparatively small as in the case of a particle in Saturn's ring or one in a comet which has developed a tail, all terms of G in (9) may be significant and G may be either positive or negative. When G is negative the particle is repelled.

4. - The constant λ. — The unit of mass has to be chosen prior to the calculation of (4). However an indeterminate change in this unit will only have the effect of multiplying the right hand side of (4) by an indeterminate constant, and in (5) this constant has become identified with λ. Thus the choice of the unit of mass affects the determination of the orbital equation for the planetary particle, which fact conditions the arbitrariness of the choice. We could determine λ from (5) by making this curve pass through three properly chosen points in the ring of Saturn.

5. - Vacant bands in saturnian rings. — From $d(r^5G)/dr=0$, we get,

$$r=[\lambda^{-2}\pm\sqrt{(\lambda^{-4}+3ACm_0^2)}]/3Am_0. \tag{13}$$

These are radial coordinates of extremes of G. They will be without meaning in our problem however, unless they fall within the interval $(r_1,\ldots, r_n)$ of the strip which the current choice of s designates. If we equate r from (13) to $s+\delta$, $(\delta \doteq 0,\ \delta^2=0)$ and allow the equality to determine s we obtain a proper s from it's point-set, to give meaning to (13). The condition on s is,

$$C(s)m_0-3A(s)(s+\delta)^2m_0+2(s+\delta)\lambda^{-2}=0. \tag{14}$$

Since A may be assumed to be positive, the positive root (13) will give a negative result when substituted for r in $d^2(r^5G)/dr^2$. Hence the extreme is a maximum. From (8) it will be a positive maximum if m_0 is small enough. Since (9) involves the denominator r^5, it is a small positive maximum (λ can be so chosen). Thus:

THEOREM. - *The maxima of G are small in comparison with their radial coordinates.*

The function G has, at most, two maxima since, by (7), $dG/dr=0$ leads to

$$A^2r^6+2(AD+BC)r^3+3(C^2+2DB+\lambda^{-2})r^2+12CDr+10D^2=0, \tag{15}$$

and this equation has at most four real roots by Descartes' rule of signs. A convenient choice of the arbitrary constant $\gamma^2\lambda^2$ is now all that is required in order that we may be able to draw an accurate graph of the function $F=G(r)$, (as in (6)). (Cf. Fig. 2). For clearness in graphics the distance Oh was much foreshortened. As previously stated m is small, below a limiting value yet to be considered.

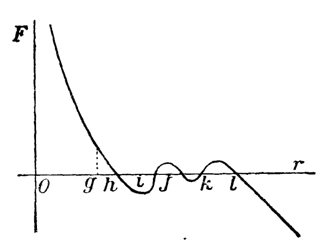

Fig. 2.

The formula (6) was derived, in my paper just quoted, altogether on account of the effect of gravitation. No principle relating to the nature of this force was employed. If radiation (or light pressure) contributes a component to the force, as the latter affects masses of an assigned magnitude (in motion as planets) then this component is already contained in G. In the theory therefore, the reverse action; always away from the SUN, of the particles which form the tail of a comet, is due to a repellant gravitational potential. This repulsion is shown by the curve of G in it's branch below the r axis from l onward.

Throughout the region gl the force G wavers between small attraction and small repulsion.

This circumstance is just what is needed for a simple explanation of the phenomenon of Saturn's rings, these being assumed to be composed of small masses.

At distances less than og the force F is a powerful attraction and a particle within this distance from S will fall to the surface of the planet. Within the interval gh, F is positive but small; hence the centrifugal force, of a small mass within the interval (og, oh), in rotation around S, balances the force of gravity, and the aggregate of the masses within this interval will form a ring. The latter is constrained to lie in the plane of Saturn's inner moons. Perhaps however the plane of the rings might be expected to oscillate a little due to the SUN's potential.

In (oh, oi) F is a repulsion. Particles in the corresponding band are repelled by the force and cannot be in rotation around the planet. This band will be swept clear of particles. This corresponds to the facts of observation. There will be another ring due to the small positive F of the interval (oi, oj), a vacant band jk, a ring kl, while, from l onward, particles are repelled from the planet.

It is obvious that any planet is surrounded by a state of potential proper for the formation of saturnian rings. The zodiacal light thus accords with the theory of these rings. A british astronomer once reported that he had seen a faint but definite ring around Mars. The great observatories did not much encourage this view. (Cf. § 6).

One conjecture relating to the action of comets may be stated. When a comet passes through the zodiacal light, as DONATI's comet (1858), for example, did, we would expect the particles of it's tail to follow the circular motion of the particles of the former luminosity. The tail of DONATI's comet showed this effect clearly by forming a great curved plume. If a comet comes into the region inside of the SUN's saturnian rings, it's appendage should be drawn toward the SUN as it passes perihelion. NEWTON's comet (1680) actually disappeared, for several days, at perihelion.

6. - Arithmetization on the basis of the rings of Saturn. — If a planet actually has all three of it's rings it is clear from fig. 2 that Γ, $[=\Gamma(r)]$, vanishes for five values of r, whence may be obtained five linear equations to determine the ratios, to A^2, of Γ's other five coefficients. If, on the basis of the existing approximative estimates of the values of the zeros of $\Gamma(r)$ for Saturn's rings, we assume

$$Oh = 72000 \text{ mi.}, \qquad Oi = 75000 \text{ mi.}, \qquad Oj = 80000 \text{ mi.},$$
$$Ok = 80500 \text{ mi.}, \qquad Ol = 85500 \text{ mi.},$$

then, choosing 10000 miles as the unit of length, the linear equations are

$$\Gamma(7.2)=0, \qquad \Gamma(7.5)=0,$$
$$\Gamma(8)=0, \qquad \Gamma(8.05)=0, \qquad \Gamma(8.55)=0.$$

Their solutions are found to be,

$$(16)\quad \begin{cases} \beta/A^2=28.0663, & L/A^2=-7582.8308, \quad M/A^2=99677.8239, \\ U/A^2=-517170.5333, & V/A^2=982860.2947, \quad (\beta=-AB), \end{cases}$$

values which give an arithmetical form to the coefficients of G. The errors in the assumed values of the zeros, however, lead to an impossibility; one very difficult to correct by any revision of these values, based exclusively on the method of trial and error. When equations (7) are solved we get

$$B=-28.07A, \qquad C=245.91A, \qquad D=-701.02A,$$

and the mass equation,

$$(17)\qquad m=A(r^3-28.07r^2+245.91r-701.02),$$

but the identity $L=AD+BC$ lacks 20 of being satisfied and λ turns out to be imaginary; $\lambda^{-2}=-143.98A^2$.

In order to make up the comparatively small deficiencies and secure a real solution for λ, thus proving that G accounts for the ring phenomen, we approach the problem from a different direction. Another expression for G is known in the form,

$$(18)\qquad G(r)=\frac{\gamma^2\lambda^2}{r^5}\left[2(g(r,s))^2-rg(r,s)g'(r,s)+\frac{r^2}{\lambda^2}\right],$$

in which $g'=\partial g/\partial r$. Hence any root z of $g(r,s)$ annuls $\Gamma(r)$ except for A^2z^2/λ^2A^2 which remains as an isolated expression after the substitution. It is convenient to include A^2 with the factor $\gamma^2\lambda^2$ in $G(r)$, thus using Γ in the form in which the coefficient of r^6 is -1.

Let $\varrho<\sigma<a$ be three zeros of $\Gamma(r)$, ϱ and σ being, respectively, the distances from the center of Saturn to the inner edges of the divisions in the rings and a the radial width of the whole ring. It is known that $a=8.6$, approximately, but in what follows these distances are kept literal. There is latitude of choice of their values at least as we pass from the consideration of the rings of Saturn to those of another planet. The constants in $G(r)$ are subject to change in the general problem.

The numbers ϱ, σ, a are roots of a cubic,

$$h(a,r)=A(r-a)(r^2+or+p), \qquad (o^2-4p>0),$$

and if $h(a,r)$ is $g(r,s)$ in (18) we obtain $\Gamma(a)=a^2/\lambda^2A^2$. Thus a would not be a root of $\Gamma(r)$; however if we choose $q<a$ near to a, and adopt $h(q,r)$ as $g(r,s)$

we can determine $1/\lambda^2A^2$ so a will be a root. It is only necessary to solve for $1/\lambda^2A^2$ in

$$\Gamma(a)=\eta(q,a)+a^2/\lambda^2A^2=0,$$

where

$$\eta(q,a)=\{2[h(q,a)]^2-ah(q,a)h'(q,a)\}/A^2.$$

Since $\eta(a,a)=0$, $\eta(q,a)$ is negative and the value of $1/\lambda^2A^2$ determined is a positive number. It will be as small as we please through our choice of q.

We have,

$$m=g(r,s)=A[r^3-(q-o)r^2+(p-oq)r-pq], \tag{19}$$

and, by use of (7),

$$\begin{aligned}\Gamma(r)=&-r^6+(q-o)r^5-[(q-o)(p-oq)+pq]r^3\\&+[(p-oq)^2+2pq(q-o)-a^{-2}\eta(q,a)]r^2-3pq(p-oq)r+2p^2q^2.\end{aligned} \tag{20}$$

The coefficients in $\Gamma(r)$ are now all known numbers, the function being analogous to that determined by computation in (16), except that, in the relation found to connect λ with A, viz., $\lambda^{-2}=-a^{-2}\eta(q,a)A^2$, the number $-\eta(q,a)$ is positive and λ is real. The equation $\Gamma(r)=0$ has three roots which are dimensions of Saturn's rings correct to any desired approximation. These are,

$$\varrho+\varepsilon, \qquad \sigma+\tau, \qquad a, \qquad (|\varepsilon|\doteq 0,\ |\tau|\doteq 0,\ \text{as } a-q\doteq 0). \tag{21}$$

Since the signs of the numerical coefficients of the powers of r in (20) alternate, the maximum number of real positive roots of $\Gamma(r)$ is five and of real negative roots is one, by DESCARTES' rule of signs. It has four real roots since it was constructed to have three, but it must have five and therefore six real roots if $G(r)$ is to describe the ring phenomenon. In view of the small value of $-a^{-2}\eta(q,a)$ the three roots additional to those of the set (21) are approximately those of the equation,

$$E(r)\equiv A^{-1}\{-2h(q,r)+rh'(q,r)\}=r^3+(oq-p)r+2pq=0. \tag{22}$$

THEOREM. - *With ϱ, σ, a, chosen or determined to be appropriate dimensions, as specified, $G(r)$ accounts for the ring configuration. The Cassini division is bounded by the limits (ϱ,χ) and the Encke division by the limits (σ,t), χ, t, being the real positive roots of $E(r)=0$.*

The equation $E(r)=0$ always has two real positive roots χ, t. In fact,

$$E(\varrho)=\varrho(\varrho-\sigma)(\varrho-q), \qquad E(\sigma)=\sigma(\sigma-\varrho)(\sigma-q), \qquad E(q)=q(q-\varrho)(q-\sigma).$$

Hence $E(\varrho)$ is $+$, $E(\sigma)$ is $-$, $E(q)$ is $+$, and there is a root χ on the interval (ϱ,σ) and one, t, on the interval (σ,q). Since the sum of the roots of $E(r)$ is zero, one, $-(\chi+t)$, will be negative. This proves the theorem.

With $\varrho=7.1$, $\sigma=8.1$, $q=8.6$ we find that CASSINI's division is bounded by the limits ($\varrho, \chi = 7.4664$) and ENCKE's division by the limits ($\sigma, t = 8.3671$). The limits χ, t, are obtained by solving, by HORNER's method, the equation $E(r) = 0$, which here becomes,

$$r^3 - 188.23r + 989.172 = 0.$$

The negative root of $\Gamma(r) = 0$ has no physical interpretation.

It can be emphasized from fig. 2 that the rotating masses are small since their motion is in response to small forces. It is clear from a graph that the mass equation (19), valid in the vicinity of q, and by extension over the whole interval (ϱ, a), is one proper to represent a typical phalanx of the rotating masses.

7. - **The orbits of the constituent particles.** — According to the known theory of formula (6) the orbit of a particle rotating in the ring will coincide, in the sense of analytical approximation, with the curve (5) throughout a segment of their length. When $p(r)$ is $h(q, r)$ the integral of (5) is,

$$(r-\varrho)^u(r-q)^v/(r-\sigma)^w = ke^{\lambda A\theta},$$

where u, v, w are all positive and k is an arbitrary constant. If the negative value of λA is chosen, as θ increases indefinitely, two branches of this transcendental curve approach, in the form of spirals, the respective limiting circles $r-\varrho=0$, $r-q=0$. (Fig. 3). With λA positive a spiral branch approaches the limiting circle $r-\sigma=0$. The limit circles and the spirals close to them show that the orbits of the separate particles are circles. They are thus obtained as circles notwithstanding the comparatively complicated form of the force function $G(r)$.

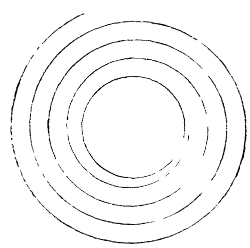

Fig. 3.

At the beginning of section 6 we had reached the conclusion that the gravitational function $G(r)$ accounts for the rings provided $\Gamma(r)$ could be numerically constructed with five real positive roots equal to the radial distances to the five outer edges of the rings and with λ real. These conditions have all been satisfied; also some variations from the respective dimensions of Saturn's rings are shown to be possible in the gravitational field of another planet.