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Crystals of Fock spaces and cyclotomic rational double affine Hecke algebras
CRYS TALS OF FO CK SPACES AND CYCLO TOMIC RATIONAL DOUBLE AFFINE HECKE ALGEBRAS

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ABSTRACT. – We define the \( i \)-restriction and \( i \)-induction functors on the category \( \hat{\Theta} \) of the cyclotomic rational double affine Hecke algebras. This yields a crystal on the set of isomorphism classes of simple modules, which is isomorphic to the crystal of a Fock space.

RÉSUMÉ. – On définit les foncteurs de \( i \)-restriction et \( i \)-induction sur la catégorie \( \hat{\Theta} \) des algèbres de Hecke doublem ent affines rationnelles cyclotomiques. Ceci donne lieu à un cristal sur l’ensemble des classes d’isomorphismes de modules simples, qui est isomorphe au cristal d’un espace de Fock.

Introduction

In [1], S. Ariki defined the \( i \)-restriction and \( i \)-induction functors for cyclotomic Hecke algebras. He showed that the Grothendieck group of the category of finitely generated projective modules of these algebras admits a module structure over the affine Lie algebra of type \( A^{(1)} \), with the action of Chevalley generators given by the \( i \)-restriction and \( i \)-induction functors.

The restriction and induction functors for rational DAHA’s (= double affine Hecke algebras) were recently defined by R. Bezrukavnikov and P. Etingof. With these functors, we give an analogue of Ariki’s construction for the category \( \hat{\Theta} \) of cyclotomic rational DAHA’s: we show that as a module over the type \( A^{(1)} \) affine Lie algebra, the Grothendieck group of this category is isomorphic to a Fock space. We also construct a crystal on the set of isomorphism classes of simple modules in the category \( \hat{\Theta} \). It is isomorphic to the crystal of the Fock space. Recall that this Fock space also enters in some conjectural description of the decomposition numbers for the category \( \hat{\Theta} \) considered here. See [16], [17], [14] for related works.
Notation

For $A$ an algebra, we will write $A$-$mod$ for the category of finitely generated $A$-modules. For $f : A \to B$ an algebra homomorphism from $A$ to another algebra $B$ such that $B$ is finitely generated over $A$, we will write

$$f_* : B$-mod \to A$-mod$$

for the restriction functor and we write

$$f^* : A$-mod \to B$-mod, \quad M \mapsto B \otimes_A M.$$

A $\mathbb{C}$-linear category $\mathcal{C}$ is called artinian if the Hom sets are finite dimensional $\mathbb{C}$-vector spaces and every object has a finite length. Given an object $M$ in $\mathcal{C}$, we denote by $\text{soc}(M)$ (resp. $\text{head}(M)$) the socle (resp. the head) of $M$, which is the largest semi-simple subobject (quotient) of $M$.

Let $\mathcal{C}$ be an abelian category. The Grothendieck group of $\mathcal{C}$ is the quotient of the free abelian group generated by objects in $\mathcal{C}$ modulo the relations $M = M' + M''$ for all objects $M, M', M''$ in $\mathcal{C}$ such that there is an exact sequence $0 \to M' \to M \to M'' \to 0$. Let $K(\mathcal{C})$ denote the complexified Grothendieck group, a $\mathbb{C}$-vector space. For each object $M$ in $\mathcal{C}$, let $[M]$ be its class in $K(\mathcal{C})$. Any exact functor $F : \mathcal{C} \to \mathcal{C}'$ between two abelian categories induces a vector space homomorphism $K(\mathcal{C}) \to K(\mathcal{C}')$, which we will denote by $F$ again. Given an algebra $A$ we will abbreviate $K(A) = K(A$-$mod$).

Denote by $\text{Fct}((\mathcal{C}, \mathcal{C}')$ the category of functors from a category $\mathcal{C}$ to a category $\mathcal{C}'$. For $F \in \text{Fct}((\mathcal{C}, \mathcal{C}')$ write $\text{End}(F)$ for the ring of endomorphisms of the functor $F$. We denote by $1_F : F \to F$ the identity element in $\text{End}(F)$. Let $G \in \text{Fct}((\mathcal{C}', \mathcal{C}''))$ be a functor from $\mathcal{C}'$ to another category $\mathcal{C}''$. For any $X \in \text{End}(F)$ and any $X' \in \text{End}(G)$ we write $X'X : G \circ F \to G \circ F$ for the morphism of functors given by $X'X(M) = X'(F(M)) \circ G(X(M))$ for any $M \in \mathcal{C}'$.

Let $e \geq 2$ be an integer and $z$ be a formal parameter. Denote by $\mathfrak{sl}_e$ the Lie algebra of traceless $e \times e$ complex matrices. The type $A^{(1)}$ affine Lie algebra is

$$\tilde{\mathfrak{sl}}_e = \mathfrak{sl}_e \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}e \oplus \mathbb{C}\partial$$

equipped with the Lie bracket

$$[\xi \otimes z^m + ac+b\partial, \xi' \otimes z^n + a'c+b'\partial] = [\xi, \xi'] \otimes z^{m+n} + m\delta_{m,-n} \text{tr}(\xi\xi') c + nb\xi' \otimes z^n - m\xi \otimes z^m,$$

for $\xi, \xi' \in \mathfrak{sl}_e, a, a', b, b' \in \mathbb{C}$. Here $\text{tr} : \mathfrak{sl}_e \to \mathbb{C}$ is the trace map. Let

$$\tilde{\mathfrak{sl}}_e = \mathfrak{sl}_e \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}e.$$

It is the Lie subalgebra of $\tilde{\mathfrak{sl}}_e$ generated by the Chevalley generators

$$e_i = E_{i+1,i} \otimes 1, \quad f_i = E_{i+1,i} \otimes 1, \quad 1 \leq i \leq e - 1$$

$$e_0 = E_{e1} \otimes z, \quad f_0 = E_{1e} \otimes z^{-1}.$$

Here $E_{ij}$ is the elementary matrix with 1 in the position $(i, j)$ and 0 elsewhere. Let $h_i = [e_i, f_i]$ for $0 \leq i \leq e - 1$. We consider the Cartan subalgebra

$$t = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} \mathbb{C}h_i \oplus \mathbb{C}\partial.$$

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and its dual $t^\ast$. For $i \in \mathbb{Z}/e\mathbb{Z}$ let $\alpha_i \in t^\ast$ (resp. $\alpha_i^\vee \in t$) be the simple root (resp. coroot) corresponding to $e_i$. The fundamental weights are \( \{ \Lambda_i \in t^\ast : i \in \mathbb{Z}/e\mathbb{Z} \} \) such that \( \Lambda_i(\alpha_i) = \delta_{ij} \) and \( \Lambda_i(\partial) = 0 \) for any \( i, j \in \mathbb{Z}/e\mathbb{Z} \). Let $\delta \in t^\ast$ be the element given by $\delta(h_i) = 0$ for all $i$ and $\delta(\partial) = 1$. We will write $P$ for the weight lattice of $\tilde{A}_n$. It is the free abelian group generated by the fundamental weights and $\delta$.

1. Reminders on Hecke algebras, rational DAHA’s and restriction functors

1.1. Hecke algebras

Let $\mathfrak{h}$ be a finite dimensional vector space over $\mathbb{C}$. Recall that a pseudo-reflection is a non trivial element $s$ of $GL(\mathfrak{h})$ which acts trivially on a hyperplane, called the reflecting hyperplane of $s$. Let $W \subset GL(\mathfrak{h})$ be a finite subgroup generated by pseudo-reflections. Let $\mathfrak{s}$ be the set of pseudo-reflections in $\mathfrak{h}$.

Here is the functor \((1.1)\) for a precise definition. When the subspace $\mathfrak{T}$ is the functor \((1.1)\) we abbreviate $\mathfrak{t}$.

For any $H \in \mathfrak{g}$, let $W_H$ be the pointwise stabilizer of $H$. This is a cyclic group. Write $e_H$ for the order of $W_H$. Let $s_H$ be the unique element in $W_H$ whose determinant is $\exp(\frac{2\pi i}{e_H})$. Let $q$ be a map from $\mathfrak{g}$ to $\mathbb{C}^\ast$ that is constant on the $W$-conjugacy classes. Following [6, Definition 4.21] the Hecke algebra $\mathcal{H}_q(W, \mathfrak{h})$ attached to $(W, \mathfrak{h})$ with parameter $q$ is the quotient of the group algebra $\mathbb{C}B(W, \mathfrak{h})$ by the relations:

\[ (T_{s_H} - 1) \prod_{t \in W_H \cap s_H} (T_t - q(t)) = 0, \quad H \in \mathfrak{g}. \]

Here $T_{s_H}$ is a generator of the monodromy around $H$ in $\mathfrak{h}_{\text{reg}}/W$ such that the lift of $T_{s_H}$ in $\pi_1(W, \mathfrak{h}_{\text{reg}})$ via the map $\mathfrak{h}_{\text{reg}} \to \mathfrak{h}_{\text{reg}}/W$ is represented by a path from $x_0$ to $s_H(x_0)$. See [6, Section 2B] for a precise definition. When the subspace $\mathfrak{h}_W$ of fixed points of $W$ in $\mathfrak{h}$ is trivial, we abbreviate $B_W = B(W, \mathfrak{h})$, $\mathcal{H}_q(W) = \mathcal{H}_q(W, \mathfrak{h})$.

1.2. Parabolic restriction and induction for Hecke algebras

In this section we will assume that $\mathfrak{h}_W = 1$. A parabolic subgroup $W'$ of $W$ is by definition the stabilizer of a point $b \in \mathfrak{h}$. By a theorem of Steinberg, the group $W'$ is also generated by pseudo-reflections. Let $q'$ be the restriction of $q$ to $\mathfrak{g}' = W' \cap \mathfrak{g}$. There is an explicit inclusion $i_q : \mathcal{H}_q(W') \hookrightarrow \mathcal{H}_q(W)$ given by [6, Section 2D]. The restriction functor $\mathfrak{H}_q(W') : \mathcal{H}_q(W') -\text{mod} \to \mathcal{H}_q(W) -\text{mod}$ is the functor $(i_q)_\ast$. The induction functor $\mathfrak{H}_q(W') : \mathcal{H}_q(W) -\text{mod} \to \mathcal{H}_q(W') -\text{mod}$ is left adjoint to $\mathfrak{H}_q(W')$. The coinduction functor $\mathfrak{H}_q(W') : \mathbf{Hom}_{\mathcal{H}_q(W')} (\mathcal{H}_q(W), -)$ is right adjoint to $\mathfrak{H}_q(W')$. The three functors above are all exact.
Let us recall the definition of $\iota_q$. It is induced from an inclusion $\iota : B_{W'} \hookrightarrow B_W$, which is in turn the composition of three morphisms $\ell, \kappa, j$ defined as follows. First, let $\mathcal{C}' \subset \mathcal{C}$ be the set of reflecting hyperplanes of $W'$. Write

$$\mathfrak{h} = \mathfrak{h}/\mathfrak{h}'W', \quad \mathcal{C} = \{\mathcal{H} = H/\mathfrak{h}'W' : H \in \mathcal{C}'\}, \quad \mathfrak{h}_{\text{reg}} = \mathfrak{h} - \bigcup_{\mathcal{H} \in \mathcal{C}} \mathcal{H}, \quad \mathfrak{h}'_{\text{reg}} = \mathfrak{h} - \bigcup_{H \in \mathcal{C}'} H.$$

The canonical epimorphism $p : \mathfrak{h} \to \overline{\mathfrak{h}}$ induces a trivial $W'$-equivariant fibration $p : \mathfrak{h}'_{\text{reg}} \to \overline{\mathfrak{h}}_{\text{reg}}$, which yields an isomorphism

$$\ell : B_{W'} = \pi_1(\overline{\mathfrak{h}}_{\text{reg}}/W', p(x_0)) \sim \pi_1(\mathfrak{h}'_{\text{reg}}/W', x_0).$$

Endow $\mathfrak{h}$ with a $W$-invariant hermitian scalar product. Let $|| \cdot ||$ be the associated norm. Set

$$\Omega = \{x \in \mathfrak{h} : ||x - b|| < \varepsilon\},$$

where $\varepsilon$ is a positive real number such that the closure of $\Omega$ does not intersect any hyperplane that is in the complement of $\mathcal{C}'$ in $\mathcal{C}$. Let $\gamma : [0, 1] \to \mathfrak{h}$ be a path such that $\gamma(0) = x_0$, $\gamma(1) = b$ and $\gamma(t) \in \mathfrak{h}_{\text{reg}}$ for $0 < t < 1$. Let $u \in [0, 1]$ such that $x_1 = \gamma(u)$ belongs to $\Omega$, write $\gamma_u$ for the restriction of $\gamma$ to $[0, u]$. Consider the homomorphism

$$\sigma : \pi_1(\Omega \cap \mathfrak{h}_{\text{reg}}, x_1) \to \pi_1(\mathfrak{h}_{\text{reg}}, x_0), \quad \lambda \mapsto \gamma^{-1}_u \cdot \lambda \cdot \gamma_u.$$ 

The canonical inclusion $\mathfrak{h}_{\text{reg}} \hookrightarrow \mathfrak{h}'_{\text{reg}}$ induces a homomorphism $\pi_1(\mathfrak{h}_{\text{reg}}, x_0) \to \pi_1(\mathfrak{h}'_{\text{reg}}, x_0)$. Composing it with $\sigma$ gives an invertible homomorphism

$$\pi_1(\Omega \cap \mathfrak{h}_{\text{reg}}, x_1) \to \pi_1(\mathfrak{h}'_{\text{reg}}, x_0).$$

Since $\Omega$ is $W'$-invariant, its inverse gives an isomorphism

$$\kappa : \pi_1(\mathfrak{h}'_{\text{reg}}/W', x_0) \sim \pi_1((\Omega \cap \mathfrak{h}_{\text{reg}})/W', x_1).$$

Finally, we see from above that $\sigma$ is injective. So it induces an inclusion

$$\pi_1((\Omega \cap \mathfrak{h}_{\text{reg}})/W', x_1) \hookrightarrow \pi_1(\mathfrak{h}_{\text{reg}}/W', x_0).$$

Composing it with the canonical inclusion $\pi_1(\mathfrak{h}_{\text{reg}}/W', x_0) \hookrightarrow \pi_1(\mathfrak{h}_{\text{reg}}/W, x_0)$ gives an injective homomorphism

$$j : \pi_1((\Omega \cap \mathfrak{h}_{\text{reg}})/W', x_1) \to \pi_1(\mathfrak{h}_{\text{reg}}/W, x_0) = B_W.$$

By composing $\ell, \kappa, j$ we get the inclusion

$$\iota = j \circ \kappa \circ \ell : B_{W'} \hookrightarrow B_W.$$

It is proved in [6, Section 4C] that $\iota$ preserves the relations in (1.1). So it induces an inclusion of Hecke algebras which is the desired inclusion

$$\iota_q : \mathcal{H}_q(W') \hookrightarrow \mathcal{H}_q(W).$$

For $\iota, \iota' : B_W \hookrightarrow B_W$ two inclusions defined as above via different choices of the path $\gamma$, there exists an element $\rho \in B_W = \pi_1(\mathfrak{h}_{\text{reg}}, x_0)$ such that for any $a \in B_{W'}$ we have $\iota(a) = \rho'(a)\rho^{-1}$. In particular, the functors $\iota_*$ and $(\iota')_*$ from $B_W$-mod to $B_{W'}$-mod are isomorphic. Also, we have $(\iota_q)_* \cong (\iota'_q)_*$. So there is a unique restriction functor $\mathcal{H}_q(W')$ up to isomorphisms.
1.3. Rational DAHA’s

Let $c$ be a map from $\mathcal{O}$ to $\mathbb{C}$ that is constant on the $W$-conjugacy classes. The rational DAHA attached to $W$ with parameter $c$ is the quotient $H_c(W, \mathfrak{h})$ of the smash product of $CW$ and the tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^*$ by the relations

$$[x, x'] = 0, \quad [y, y'] = 0, \quad [y, x] = \langle x, y \rangle - \sum_{s \in \Phi} c_s \langle \alpha_s, y \rangle \langle x, \alpha_s^\vee \rangle,$$

for all $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$. Here $\langle \cdot, \cdot \rangle$ is the canonical pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$, the element $\alpha_s$ is a generator of $\text{Im}(s|_{\mathfrak{h}} - 1)$ and $\alpha_s^\vee$ is the generator of $\text{Im}(s|_{\mathfrak{h}} - 1)$ such that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$.

For $s \in \Phi$ write $\lambda_s$ for the non trivial eigenvalue of $s$ in $\mathfrak{h}^*$. Let $\{ x_i \}$ be a basis of $\mathfrak{h}^*$ and let $\{ y_i \}$ be the dual basis. Let

$$(1.7) \quad \text{eu} = \sum_i x_i y_i + \frac{\text{dim}(\mathfrak{h})}{2} - \sum_{s \in \Phi} \frac{2c_s}{1 - \lambda_s} s$$

be the Euler element in $H_c(W, \mathfrak{h})$. Its definition is independent of the choice of the basis $\{ x_i \}$. We have

$$(1.8) \quad [\text{eu}, x_i] = x_i, \quad [\text{eu}, y_i] = -y_i, \quad [\text{eu}, s] = 0.$$  

1.4. The category $\mathfrak{O}$

The category $\mathfrak{O}$ of $H_c(W, \mathfrak{h})$ is the full subcategory $\mathfrak{O}(W, \mathfrak{h})$ of the category of $H_c(W, \mathfrak{h})$-modules consisting of objects that are finitely generated as $\mathbb{C}[\mathfrak{h}]$-modules and $\mathfrak{h}$-locally nilpotent. We recall from [10, Section 3] the following properties of $\mathfrak{O}(W, \mathfrak{h})$.

The action of the Euler element $\text{eu}$ on a module in $\mathfrak{O}(W, \mathfrak{h})$ is locally finite. The category $\mathfrak{O}(W, \mathfrak{h})$ is a highest weight category. In particular, it is artinian. Write $\text{Irr}(W)$ for the set of isomorphism classes of irreducible representations of $W$. The poset of standard modules in $\mathfrak{O}(W, \mathfrak{h})$ is indexed by $\text{Irr}(W)$ with the partial order given by $[10$, Theorem 2.19$]$. More precisely, for $\xi \in \text{Irr}(W)$, equip it with a $CW \ltimes \mathbb{C}[\mathfrak{h}^*]$-module structure by letting the elements in $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ act by zero, the standard module corresponding to $\xi$ is

$$\Delta(\xi) = H_c(W, \mathfrak{h}) \otimes_{CW \ltimes \mathbb{C}[\mathfrak{h}^*]} \xi.$$

It is an indecomposable module with a simple head $L(\xi)$. The set of isomorphism classes of simple modules in $\mathfrak{O}(W, \mathfrak{h})$ is

$$\{ [L(\xi)] : \xi \in \text{Irr}(W) \}. $$

It is a basis of the $\mathbb{C}$-vector space $K(\mathfrak{O}(W, \mathfrak{h}))$. The set $\{ [\Delta(\xi)] : \xi \in \text{Irr}(W) \}$ gives another basis of $K(\mathfrak{O}(W, \mathfrak{h}))$.

We say a module $N$ in $\mathfrak{O}(W, \mathfrak{h})$ has a standard filtration if it admits a filtration

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = N$$

such that each quotient $N_i/N_{i-1}$ is isomorphic to a standard module. We denote by $\mathfrak{O}\Delta(W, \mathfrak{h})$ the full subcategory of $\mathfrak{O}(W, \mathfrak{h})$ consisting of such modules.

**Lemma 1.1.**  
(1) Any projective object in $\mathfrak{O}(W, \mathfrak{h})$ has a standard filtration.

(2) A module in $\mathfrak{O}(W, \mathfrak{h})$ has a standard filtration if and only if it is free as a $\mathbb{C}[\mathfrak{h}]$-module.
Both (1) and (2) are given by [10, Proposition 2.21].

The category $\mathcal{O}_c(W, h)$ has enough projective objects and has finite homological dimension [10, Section 4.3.1]. In particular, any module in $\mathcal{O}_c(W, h)$ has a finite projective resolution. Write $\text{Proj}_{\mathcal{O}}(W, h)$ for the full subcategory of projective modules in $\mathcal{O}_c(W, h)$. Let

$$I : \text{Proj}_{\mathcal{O}}(W, h) \to \mathcal{O}_c(W, h)$$

be the canonical embedding functor. We have the following lemma.

**Lemma 1.2.** For any abelian category $\mathcal{C}$ and any right exact functors $F_1, F_2$ from $\mathcal{O}_c(W, h)$ to $\mathcal{C}$, the homomorphism of vector spaces

$$r_I : \text{Hom}(F_1, F_2) \to \text{Hom}(F_1 \circ I, F_2 \circ I), \quad \gamma \mapsto \gamma 1_I$$

is an isomorphism.

In particular, if the functor $F_1 \circ I$ is isomorphic to $F_2 \circ I$, then we have $F_1 \cong F_2$.

**Proof.** We need to show that for any morphism of functors $\nu : F_1 \circ I \to F_2 \circ I$ there is a unique morphism $\tilde{\nu} : F_1 \to F_2$ such that $\tilde{\nu} 1_I = \nu$. Since $\mathcal{O}_c(W, h)$ has enough projectives, for any $M \in \mathcal{O}_c(W, h)$ there exist $P_0, P_1$ in $\text{Proj}_{\mathcal{O}}(W, h)$ and an exact sequence in $\mathcal{O}_c(W, h)$

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0. \tag{1.9}$$

Applying the right exact functors $F_1, F_2$ to this sequence we get the two exact sequences in the diagram below. The morphism of functors $\nu : F_1 \circ I \to F_2 \circ I$ yields well defined morphisms $\nu(P_1), \nu(P_0)$ such that the square commutes

$$\begin{array}{ccc}
F_1(P_1) & \xrightarrow{F_1(d_1)} & F_1(P_0) \\
\nu(P_1) \downarrow & & \nu(P_0) \downarrow \\
F_2(P_1) & \xrightarrow{F_2(d_1)} & F_2(P_0)
\end{array} \quad \begin{array}{ccc}
F_1(M) & \xrightarrow{F_1(d_0)} & 0 \\
\nu(P_1) \downarrow & & \nu(P_0) \downarrow \\
F_2(M) & \xrightarrow{F_2(d_0)} & 0
\end{array}. $$

Define $\tilde{\nu}(M)$ to be the unique morphism $F_1(M) \to F_2(M)$ that makes the diagram commute. Its definition is independent of the choice of $P_0$, $P_1$, and it is independent of the choice of the exact sequence (1.9). The assignment $M \mapsto \tilde{\nu}(M)$ gives a morphism of functor $\tilde{\nu} : F_1 \to F_2$ such that $\tilde{\nu} 1_I = \nu$. It is unique by the uniqueness of the morphism $\nu(M)$. \hfill \Box

**1.5. The Knizhnik-Zamolodchikov functor**

The Knizhnik-Zamolodchikov functor is an exact functor from the category $\mathcal{O}_c(W, h)$ to the category $\mathcal{M}_q(W, h)\cdot\text{mod}$, where $q$ is a certain parameter associated with $c$. Let us recall its definition from [10, Section 5.3].

Let $\mathcal{D}(\mathfrak{h}_{\text{reg}})$ be the algebra of differential operators on $\mathfrak{h}_{\text{reg}}$. Write

$$H_c(W, \mathfrak{h}_{\text{reg}}) = H_c(W, h) \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}_{\text{reg}}].$$

We consider the Dunkl isomorphism, which is an isomorphism of algebras

$$H_c(W, \mathfrak{h}_{\text{reg}}) \cong \mathcal{D}(\mathfrak{h}_{\text{reg}}) \rtimes \mathbb{C}W.$$
given by \( x \mapsto x, \ w \mapsto w \) for \( x \in \mathfrak{h}^*, \ w \in W, \) and
\[
y \mapsto \partial_y + \sum_{s \in \mathcal{J}} \frac{2c_s}{1 - \lambda_s}(s - 1), \quad \text{for} \ y \in \mathfrak{h}.
\]

For any \( M \in \mathcal{O}_c(W, \mathfrak{h}), \) write
\[
M_{\mathfrak{h}_{\text{reg}}} = M \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}_{\text{reg}}].
\]

It identifies via the Dunkl isomorphism with a \( \mathcal{O}(\mathfrak{h}_{\text{reg}}) \rtimes W \)-module which is finitely generated over \( \mathbb{C}[\mathfrak{h}_{\text{reg}}]. \) Hence \( M_{\mathfrak{h}_{\text{reg}}} \) is a \( W \)-equivariant vector bundle on \( \mathfrak{h}_{\text{reg}} \) with an integrable connection \( \nabla \) given by \( \nabla_y(m) = \partial_y m \) for \( m \in M, \ y \in \mathfrak{h}. \) It is proved in [10, Proposition 5.7] that the connection \( \nabla \) has regular singularities. Now, regard \( \mathfrak{h}_{\text{reg}} \) as a complex manifold endowed with the transcendental topology. Denote by \( \mathcal{O}_{\mathfrak{h}_{\text{reg}}} \) the sheaf of holomorphic functions on \( \mathfrak{h}_{\text{reg}}. \) For any free \( \mathbb{C}[\mathfrak{h}_{\text{reg}}] \)-module \( N \) of finite rank, we consider
\[
N^{\text{an}} = N \otimes_{\mathbb{C}[\mathfrak{h}_{\text{reg}}]} \mathcal{O}_{\mathfrak{h}_{\text{reg}}}^{\text{an}}.
\]

It is an analytic locally free sheaf on \( \mathfrak{h}_{\text{reg}}. \) For \( \nabla \) an integrable connection on \( N, \) the sheaf of holomorphic horizontal sections
\[
N^\nabla = \{ n \in N^{\text{an}} : \nabla_y(n) = 0 \text{ for all } y \in \mathfrak{h} \}
\]
is a \( W \)-equivariant local system on \( \mathfrak{h}_{\text{reg}}. \) Hence it identifies with a local system on \( \mathfrak{h}_{\text{reg}}/W. \)

So it yields a finite dimensional representation of \( CB(W, \mathfrak{h}). \) For \( M \in \mathcal{O}_c(W, \mathfrak{h}) \) it is proved in [10, Theorem 5.13] that the action of \( CB(W, \mathfrak{h}) \) on \( (M_{\mathfrak{h}_{\text{reg}}})^\nabla \) factors through the Hecke algebra \( \mathcal{H}_q(W, \mathfrak{h}). \) The formula for the parameter \( q \) is given in [10, Section 5.2].

The Knizhnik-Zamolodchikov functor is the functor
\[
\text{KZ}(W, \mathfrak{h}) : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{H}_q(W, \mathfrak{h}) \text{-mod}, \quad M \mapsto (M_{\mathfrak{h}_{\text{reg}}})^\nabla.
\]

By definition it is exact. Let us recall some of its properties following [10]. Assume in the rest of this subsection that the algebras \( \mathcal{H}_q(W, \mathfrak{h}) \) and \( CW \) have the same dimension over \( \mathbb{C}. \)

We abbreviate KZ = KZ(W, \mathfrak{h}). The functor KZ is represented by a projective object \( P_{\text{KZ}} \) in \( \mathcal{O}_c(W, \mathfrak{h}). \) More precisely, there is an algebra homomorphism
\[
\rho : \mathcal{H}_q(W, \mathfrak{h}) \to \text{End}_{\mathcal{O}_c(W, \mathfrak{h})}(P_{\text{KZ}})^\text{op}
\]
such that KZ is isomorphic to the functor \( \text{Hom}_{\mathcal{O}_c(W, \mathfrak{h})}(P_{\text{KZ}}, -). \) By [10, Theorem 5.15] the homomorphism \( \rho \) is an isomorphism. In particular KZ(P_{\text{KZ}}) is isomorphic to \( \mathcal{H}_q(W, \mathfrak{h}) \) as \( \mathcal{H}_q(W, \mathfrak{h}) \)-modules.

Now, recall that the center of a category \( \mathcal{C} \) is the algebra \( Z(\mathcal{C}) \) of endomorphisms of the identity functor \( \text{Id}_\mathcal{C}. \) So there is a canonical map
\[
Z(\mathcal{O}_c(W, \mathfrak{h})) \to \text{End}_{\mathcal{O}_c(W, \mathfrak{h})}(P_{\text{KZ}}).
\]
The composition of this map with \( \rho^{-1} \) yields an algebra homomorphism
\[
\gamma : Z(\mathcal{O}_c(W, \mathfrak{h})) \to Z(\mathcal{H}_q(W, \mathfrak{h})),
\]
where \( Z(\mathcal{H}_q(W, \mathfrak{h})) \) denotes the center of \( \mathcal{H}_q(W, \mathfrak{h}). \)
Lemma 1.3. – (1) The homomorphism $\gamma$ is an isomorphism.

(2) For a module $M$ in $\Theta_c(W, h)$ and an element $f$ in $Z(\Theta_c(W, h))$ the morphism

$$KZ(f(M)) : KZ(M) \rightarrow KZ(M)$$

is the multiplication by $\gamma(f)$. See [10, Corollary 5.18] for (1). Part (2) follows from the construction of $\gamma$.

The functor $KZ$ is a quotient functor, see [10, Theorem 5.14]. Therefore it has a right adjoint $S : \mathbb{M}_q(W, h) \rightarrow \Theta_c(W, h)$ such that the canonical adjunction map $KZ \circ S \rightarrow Id_{\mathbb{M}_q(W, h)}$ is an isomorphism of functors. We have the following proposition.

Proposition 1.4. – Let $Q$ be a projective object in $\Theta_c(W, h)$.

(1) For any object $M \in \Theta_c(W, h)$, the following morphism of $\mathbb{C}$-vector spaces is an isomorphism

$$\text{Hom}_{\Theta_c(W, h)}(M, Q) \cong \text{Hom}_{\mathbb{M}_q(W)}(KZ(M), KZ(Q)), \quad f \mapsto KZ(f).$$

In particular, the functor $KZ$ is fully faithful over $\text{Proj}_c(W, h)$.

(2) The canonical adjunction map gives an isomorphism $Q \cong S \circ KZ(Q)$.

See [10, Theorems 5.3, 5.16].

1.6. Parabolic restriction and induction for rational DAHA’s

From now on we will always assume that $h^W = 1$. Recall from Section 1.2 that $W' \subset W$ is the stabilizer of a point $b \in \mathfrak{h}$ and that $\mathfrak{h} = h/h^W$. Let us recall from [4] the definition of the parabolic restriction and induction functors

$$\text{Res}_b : \Theta_c(W, h) \rightarrow \Theta_c(W', \mathfrak{g}), \quad \text{Ind}_b : \Theta_c(W', \mathfrak{g}) \rightarrow \Theta_c(W, h).$$

First we need some notation. For any point $p \in \mathfrak{h}$ we write $\mathbb{C}[\mathfrak{h}]/p$ for the completion of $\mathbb{C}[\mathfrak{h}]$ at $p$, and we write $\widehat{\mathbb{C}[\mathfrak{h}]/p}$ for the completion of $\mathbb{C}[\mathfrak{h}]$ at the $W$-orbit of $p$ in $\mathfrak{h}$. Note that we have $\mathbb{C}[\mathfrak{h}]/0 = \widehat{\mathbb{C}[\mathfrak{h}]/0}$. For any $\mathbb{C}[\mathfrak{h}]$-module $M$ let

$$\widehat{M}_p = \mathbb{C}[\mathfrak{h}]/p \otimes_{\mathbb{C}[\mathfrak{h}]} M.$$

The completions $\widehat{\Theta}_c(W, h)_b$, $\widehat{\Theta}_c(W', h)_b$ are well defined algebras. We denote by $\widehat{\Theta}_c(W, h)_b$ the category of $\widehat{\Theta}_c(W, h)_b$-modules that are finitely generated over $\widehat{\mathbb{C}[\mathfrak{h}]/0}$, and we denote by $\widehat{\Theta}_c(W', h)_b$ the category of $\widehat{\Theta}_c(W', h)_b$-modules that are finitely generated over $\widehat{\mathbb{C}[\mathfrak{h}]/0}$. Let $P = \text{Fun}_{W'}(W, \widehat{\Theta}_c(W', h)_b)$ be the set of $W'$-invariant maps from $W$ to $\widehat{\Theta}_c(W', h)_b$. Let $Z(W, W', \widehat{\Theta}_c(W', h)_b)$ be the ring of endomorphisms of the right $\widehat{\Theta}_c(W', h)_b$-module $P$. We have the following proposition given by [4, Theorem 3.2].

Proposition 1.5. – There is an isomorphism of algebras

$$\Theta : \widehat{\Theta}_c(W, h)_b \rightarrow Z(W, W', \widehat{\Theta}_c(W', h)_b)$$
defined as follows: for \( f \in P, \alpha \in \mathfrak{h}^*, a \in \mathfrak{h}, w \in W, \)
\[
(\Theta(u)f)(w) = f(wu),
\]
\[
(\Theta(x_a)f)(w) = (x_a^{(b)} + \alpha(w^{-1}b))f(w),
\]
\[
(\Theta(y_a)f)(w) = y_a^{(b)}f(w) + \sum_{x \in \beta \in W} \frac{2x_s}{1 - \lambda_s} \frac{\alpha_s(wa)}{x_{\alpha_s}} (f(sw) - f(w)),
\]
where \( x_\alpha \in \mathfrak{h}^* \subset H_c(W, \mathfrak{h}), \quad x_a^{(b)} \in \mathfrak{h}^* \subset H_c(W', \mathfrak{h}), \quad y_a \in \mathfrak{h} \subset H_c(W, \mathfrak{h}), \quad y_a^{(b)} \in \mathfrak{h} \subset H_c(W', \mathfrak{h}). \)

Using \( \Theta \) we will identify \( \widehat{H}_c(W, \mathfrak{h})_b \)-modules with \( Z(W, W', \widehat{H}_c(W', \mathfrak{h})_0) \)-modules. So the module \( P = \text{Fun}_{W',c}(W, \widehat{H}_c(W', \mathfrak{h})_0) \) becomes an \( (\widehat{H}_c(W, \mathfrak{h})_b, \widehat{H}_c(W', \mathfrak{h})_0) \)-bimodule. Hence for any \( N \in \Theta_{c'}(W', \mathfrak{h})_0 \) the module \( P \otimes \widehat{H}_c(W', \mathfrak{h})_0 \) \( N \) lives in \( \Theta_{c}(W, \mathfrak{h})_b \). It is naturally identified with \( \text{Fun}_{W',c}(W, N) \), the set of \( W' \)-invariant maps from \( W \) to \( N \). For any \( \mathbb{C}[\mathfrak{h}^*] \)-module \( M \) write \( E(M) \subset M \) for the locally nilpotent part of \( M \) under the action of \( \mathfrak{h} \).

The ingredients for defining the functors \( \text{Res}_b \) and \( \text{Ind}_b \) consist of:

- the adjoint pair of functors \( (\sim_b, E^b) \) with

\[
\sim_b : \Theta_{c}(W, \mathfrak{h}) \rightarrow \Theta_{c}(W, \mathfrak{h})_b, \quad M \mapsto \widehat{M}_b,
\]
\[
E^b : \Theta_{c}(W, \mathfrak{h})_b \rightarrow \Theta_{c}(W, \mathfrak{h}), \quad N \mapsto E(N),
\]

- the Morita equivalence

\[
J : \Theta_{c'}(W', \mathfrak{h})_0 \rightarrow \Theta_{c}(W, \mathfrak{h})_b, \quad N \mapsto \text{Fun}_{W',c}(W, N),
\]

and its quasi-inverse \( R \) given in Section 1.7 below,

- the equivalence of categories

\[
E : \Theta_{c'}(W', \mathfrak{h})_0 \rightarrow \Theta_{c'}(W', \mathfrak{h}), \quad M \mapsto E(M)
\]

and its quasi-inverse given by \( N \mapsto \widehat{N}_0, \)

- the equivalence of categories

\[
(1.10) \quad \zeta : \Theta_{c'}(W', \mathfrak{h}) \rightarrow \Theta_{c'}(W', \mathfrak{h}), \quad M \mapsto \{ v \in M : yv = 0, \text{ for all } y \in \mathfrak{h}^{W'} \}
\]

and its quasi-inverse \( \zeta^{-1} \) given in Section 1.8 below.

For \( M \in \Theta_{c}(W, \mathfrak{h}) \) and \( N \in \Theta_{c'}(W', \mathfrak{h}) \) the functors \( \text{Res}_b \) and \( \text{Ind}_b \) are defined by

\[
\text{Res}_b(M) = \zeta \circ E \circ R(\widehat{M}_b),
\]
\[
\text{Ind}_b(N) = E^b \circ J(\zeta^{-1}(N)_b).
\]

We refer to [4, Section 2.3] for details.
1.7. The idempotent $x_{pr}$ and the functor $R$

We give some details on the isomorphism $\Theta$ for a future use. Fix elements $1 = u_1, u_2, \ldots, u_r$ in $W$ such that $W \cong \bigoplus_{i=1}^r W'u_i$. Let $\text{Mat}_r(\hat{H}_c(W', h)_{0})$ be the algebra of $r \times r$ matrices with coefficients in $\hat{H}_c(W', h)_{0}$. We have an algebra isomorphism

\begin{equation}
(1.12) \quad \Phi : Z(W, W', \hat{H}_c(W', h)_{0}) \to \text{Mat}_r(\hat{H}_c(W', h)_{0}),
\end{equation}

such that

\[ (Af)(u_i) = \sum_{j=1}^{r} \Phi(A)_{ij}f(u_j), \quad \text{for all } f \in P, 1 \leq i \leq r. \]

Denote by $E_{ij}$, $1 \leq i, j \leq r$, the elementary matrix in $\text{Mat}_r(\hat{H}_c(W', h)_{0})$ with coefficient 1 in the position $(i, j)$ and zero elsewhere. Note that the algebra isomorphism $\Phi \circ \Theta : \hat{H}_c(W, h)_{b} \sim \to \text{Mat}_r(\hat{H}_c(W', h)_{0})$ restricts to an isomorphism of subalgebras

\begin{equation}
(1.13) \quad \mathbb{C}[h]_{b} \cong \bigoplus_{i=1}^{r} \mathbb{C}[h]_{0, E_{ii}}.
\end{equation}

Indeed, there is a unique isomorphism of algebras

\begin{equation}
(1.14) \quad \varpi : \mathbb{C}[h]_{b} \cong \bigoplus_{i=1}^{r} \mathbb{C}[h]_{u_i^{-1}b},
\end{equation}

extending the algebra homomorphism

\[ \mathbb{C}[h] \to \bigoplus_{i=1}^{r} \mathbb{C}[h], \quad x \mapsto (x, x, \ldots, x), \quad \forall x \in \mathfrak{h}^*. \]

For each $i$ consider the isomorphism of algebras

\[ \phi_i : \mathbb{C}[h]_{u_i^{-1}b} \to \mathbb{C}[h]_{0}, \quad x \mapsto u_i x + x(u_i^{-1}b), \quad \forall x \in \mathfrak{h}^*. \]

The isomorphism (1.13) is exactly the composition of $\varpi$ with the direct sum $\bigoplus_{i=1}^{r} \phi_i$. Here $E_{ii}$ is the image of the idempotent in $\mathbb{C}[h]_{b}$ corresponding to the component $\mathbb{C}[h]_{u_i^{-1}b}$. We will denote by $x_{pr}$, the idempotent in $\mathbb{C}[h]_b$ corresponding to $\mathbb{C}[h]_0$, i.e., $\Phi \circ \Theta(x_{pr}) = E_{11}$.

Then the functor

\[ R : \hat{\theta}_c(W, h)_{b} \to \hat{\theta}_c(W', h)_{0}, \quad M \mapsto x_{pr}M \]

is a quasi-inverse of $J$. Here, the action of $\hat{H}_c(W', h)_{0}$ on $R(M) = x_{pr}M$ is given by the following formulas deduced from Proposition 1.5. For any $\alpha \in \mathfrak{h}^*$, $w \in W'$, $a \in \mathfrak{h}^*$, $m \in M$ we have

\begin{align}
(1.15) \quad & x_{\alpha}(b)x_{pr}(m) = x_{pr}((x_{\alpha} - \alpha(b))m), \\
(1.16) \quad & wx_{pr}(m) = x_{pr}(wm), \\
(1.17) \quad & y_{\alpha}(b)x_{pr}(m) = x_{pr}\left(\left(y_{\alpha} + \sum_{s \in \Psi', s \in W'} \frac{2e_s}{1 - \lambda_s} \alpha_s(a)\right)m\right).
\end{align}
In particular, we have
\[ R(M) = \phi'_1(x_{pr}(M)) \]
as \( \mathbb{C}[[h]]_0 \rtimes W' \)-modules. Finally, note that the following equality holds in \( \tilde{H}_c(W, h)_b \)
\[ x_{pr}ux_{pr} = 0, \quad \forall u \in W - W'. \]

1.8. A quasi-inverse of \( \zeta \)

Let us recall from [4, Section 2.3] the following facts. Let \( h^*W' \) be the subspace of \( h^* \)
consisting of fixed points of \( W' \). Set
\[ (h^*W')^\perp = \{ v \in h : f(v) = 0 \text{ for all } f \in h^*W' \}. \]
We have a \( W' \)-invariant decomposition
\[ h = (h^*W')^\perp \oplus h^{W'}. \]
The \( W' \)-space \( (h^*W')^\perp \) is canonically identified with \( \bar{h} \). Since the action of \( W' \) on \( h^{W'} \) is trivial, we have an obvious algebra isomorphism
\[ H_c(W', h) \cong H_c(W', \bar{h}) \otimes \mathcal{D}(h^{W'}). \]
It maps an element \( y \) in the subset \( h^{W'} \) of \( H_c(W', h) \) to the operator \( \partial_y \) in \( \mathcal{D}(h^{W'}) \). Write \( \Theta(1, h^{W'}) \) for the category of finitely generated \( \mathcal{D}(h^{W'}) \)-modules that are \( \partial_y \)-locally nilpotent for all \( y \in h^{W'} \). The algebra isomorphism above yields an equivalence of categories
\[ \Theta_c(W', h) \cong \Theta_c(W', \bar{h}) \otimes \Theta(1, h^{W'}). \]
The functor \( \zeta \) in (1.10) is an equivalence, because it is induced by the functor
\[ \Theta(1, h^{W'}) \cong \mathcal{C} \text{-mod}, \quad M \mapsto \{ m \in M, \partial_y(m) = 0 \text{ for all } y \in h^{W'} \}, \]
which is an equivalence by Kashiwara’s lemma upon taking Fourier transforms. In particular, a quasi-inverse of \( \zeta \) is given by
\[ \zeta^{-1} : \Theta_c(W', \bar{h}) \to \Theta_c(W', h), \quad N \mapsto N \otimes \mathbb{C}[h^{W'}], \]
where \( \mathbb{C}[h^{W'}] \subset \Theta(1, h^{W'}) \) is the polynomial representation of \( \mathcal{D}(h^{W'}) \).

Moreover, the functor \( \zeta \) maps a standard module in \( \Theta_c(W', h) \) to a standard module in \( \Theta_c(W', \bar{h}) \). Indeed, for any \( \xi \in \text{Irr}(W') \), we have an isomorphism of \( H_c(W', h) \)-modules
\[ H_c(W', h) \otimes_{\mathbb{C}[h^*] \rtimes W'} \xi = (H_c(W', \bar{h}) \otimes_{\mathbb{C}[\bar{h}^*] \rtimes W'} \xi) \otimes (\mathcal{D}(h^{W'}) \otimes_{\mathbb{C}[\{h^{W'}\}^*]} \mathbb{C}). \]
On the right hand side \( \mathbb{C} \) denotes the trivial module of \( \mathbb{C}[\{h^{W'}\}^*] \), and the latter is identified with the subalgebra of \( \mathcal{D}(h^{W'}) \) generated by \( \partial_y \) for all \( y \in h^{W'} \). We have
\[ \mathcal{D}(h^{W'}) \otimes_{\mathbb{C}[\{h^{W'}\}^*]} \mathbb{C} \cong \mathbb{C}[h^{W'}] \]
as \( \mathcal{D}(h^{W'}) \)-modules. So \( \zeta \) maps the standard module \( \Delta(\xi) \) for \( H_c(W', h) \) to the standard module \( \Delta(\xi) \) for \( H_c(W', \bar{h}) \).
1.9. – Here are some properties of Res₀ and Ind₀.

Proposition 1.6. – (1) Both functors Res₀ and Ind₀ are exact. The functor Res₀ is left adjoint to Ind₀. In particular the functor Res₀ preserves projective objects and Ind₀ preserves injective objects.

(2) Let Res₀′, and Ind₀′, be respectively the restriction and induction functors of groups. We have the following commutative diagram

\[
\begin{array}{ccc}
\text{Ind}_0 & \cong & \text{Ind}_0' \\
\text{Res}_0 & \cong & \text{Res}_0' \\
K(\theta_c(W, h)) & \overset{\omega}{\longrightarrow} & K(CW) \\
\end{array}
\]

Here the isomorphism \( \omega \) (resp. \( \omega' \)) is given by mapping \([\Delta(\xi)]\) to \([\xi]\) for any \( \xi \in \text{Irr}(W) \) (resp. \( \xi \in \text{Irr}(W') \)).

See [4, Proposition 3.9, Theorem 3.10] for (1), [4, Proposition 3.14] for (2).

1.10. Restriction of modules having a standard filtration

In the rest of Section 1, we study the actions of the restriction functors on modules having a standard filtration in \( \theta_c(W, h) \) (Proposition 1.9). We will need the following lemmas.

Lemma 1.7. – Let \( M \) be an object in \( \theta^s_c(W, h) \).

(1) There is a finite dimensional subspace \( V \) of \( M \) such that \( V \) is stable under the action of \( CW \) and the map

\[
\mathbb{C}[h] \otimes V \to M, \quad p \otimes v \mapsto pv
\]

is an isomorphism of \( \mathbb{C}[h] \times W \)-modules.

(2) The map \( \omega : K(\theta_c(W, h)) \to K(CW) \) in Proposition 1.6(2) satisfies

\[
\omega([M]) = [V].
\]

Proof. – Let

\[
0 = M_0 \subset M_1 \subset \cdots \subset M_l = M
\]

be a filtration of \( M \) such that for any \( 1 \leq i \leq l \) we have \( M_i/M_{i-1} \cong \Delta(\xi_i) \) for some \( \xi_i \in \text{Irr}(W) \). We prove (1) and (2) by recurrence on \( l \). If \( l = 1 \), then \( M \) is a standard module. Both (1) and (2) hold by definition. For \( l > 1 \), by induction we may suppose that there is a subspace \( V' \) of \( M_{l-1} \) such that the properties in (1) and (2) are satisfied for \( M_{l-1} \) and \( V' \).

Now, consider the exact sequence

\[
0 \to M_{l-1} \to M \to \Delta(\xi_l) \to 0.
\]

From the isomorphism of \( \mathbb{C}[h] \times W \)-modules \( \Delta(\xi) \cong \mathbb{C}[h] \otimes \xi \) we see that \( \Delta(\xi_l) \) is a projective \( \mathbb{C}[h] \times W \)-module. Hence there exists a morphism of \( \mathbb{C}[h] \times W \)-modules \( s : \Delta(\xi_l) \to M \) that provides a section of \( j \). Let \( V = V' \oplus s(\xi_l) \subset M \). It is stable under the action of \( CW \). The map \( \mathbb{C}[h] \otimes V \to M \) in (1) is an injective morphism of \( \mathbb{C}[h] \times W \)-modules. Its image is \( M_{l-1} \oplus s(\Delta(\xi_l)) \), which is equal to \( M \). So it is an isomorphism. We have

\[
\omega([M]) = \omega([M_{l-1}]) + \omega([\Delta(\xi_l)]),
\]

by assumption \( \omega([M_{l-1}]) = [V'] \), so \( \omega([M]) = [V'] + [\xi_l] = [V] \).
Lemma 1.8. – (1) Let $M$ be an $\widehat{\mathcal{H}}_c(W, \mathfrak{h})_0$-module free over $\mathbb{C}[[\mathfrak{h}]]_0$. If there exist generalized eigenvectors $v_1, \ldots, v_n$ of $\mathfrak{e}U$ which form a basis of $M$ over $\mathbb{C}[[\mathfrak{h}]]_0$, then for $f_1, \ldots, f_n \in \mathbb{C}[[\mathfrak{h}]]_0$ the element $m = \sum_{i=1}^n f_i v_i$ is $\mathfrak{e}U$-finite if and only if $f_1, \ldots, f_n$ all belong to $\mathbb{C}[\mathfrak{h}]$.

(2) Let $N$ be an object in $\mathcal{O}_c(W, \mathfrak{h})$. If $\widehat{N}_0$ is a free $\mathbb{C}[[\mathfrak{h}]]_0$-module, then $N$ is a free $\mathbb{C}[\mathfrak{h}]_0$-module. It admits a basis consisting of generalized eigenvectors $v_1, \ldots, v_n$ of $\mathfrak{e}U$.

Proof. – (1) It follows from the proof of [4, Theorem 2.3].

(2) Since $N$ belongs to $\mathcal{O}_c(W, \mathfrak{h})$, it is finitely generated over $\mathbb{C}[\mathfrak{h}]$. Denote by $m$ the maximal ideal of $\mathbb{C}[[\mathfrak{h}]]_0$. The canonical map $N' \to \widehat{N}_0/m\widehat{N}_0$ is surjective. So there exist $v_1, \ldots, v_n$ in $N$ such that their images form a basis of $\widehat{N}_0/m\widehat{N}_0$ over $\mathbb{C}$. Moreover, we may choose $v_1, \ldots, v_n$ to be generalized eigenvectors of $\mathfrak{e}U$, because the $\mathfrak{e}U$-action on $N$ is locally finite. Since $\widehat{N}_0$ is free over $\mathbb{C}[[\mathfrak{h}]]_0$, Nakayama’s lemma yields that $v_1, \ldots, v_n$ form a basis of $\widehat{N}_0$ over $\mathbb{C}[[\mathfrak{h}]]_0$. By part (1) the set $N'$ of $\mathfrak{e}U$-finite elements in $\widehat{N}_0$ is the free $\mathbb{C}[\mathfrak{h}]$-submodule generated by $v_1, \ldots, v_n$. On the other hand, since $\widehat{N}_0$ belongs to $\widehat{\mathcal{O}}_c(W, \mathfrak{h})$, by [4, Proposition 2.4] an element in $\widehat{N}_0$ is $\mathfrak{h}$-nilpotent if and only if it is $\mathfrak{e}U$-finite. So $N' = E(\widehat{N}_0)$. On the other hand, the canonical inclusion $N \subset E(\widehat{N}_0)$ is an equality by [4, Theorem 3.2]. Hence $N = N'$. This implies that $N$ is free over $\mathbb{C}[\mathfrak{h}]$, with a basis given by $v_1, \ldots, v_n$, which are generalized eigenvectors of $\mathfrak{e}U$.

Proposition 1.9. – Let $M$ be an object in $\mathcal{O}_c(W, \mathfrak{h})$. 

(1) The object $\text{Res}_0(M)$ has a standard filtration.

(2) Let $V$ be a subspace of $M$ that has the properties of Lemma 1.7(1). Then there is an isomorphism of $\mathbb{C}[\mathfrak{h}] \otimes W'$-modules

$$\text{Res}_0(M) \cong \mathbb{C}[\mathfrak{h}] \otimes \text{Res}^W_{W'}(V).$$

Proof. – (1) By the end of Section 1.8 the equivalence $\zeta$ maps a standard module in $\mathcal{O}_c(W', \mathfrak{h})$ to a standard one in $\mathcal{O}_c(W', \mathfrak{h})$. Hence to prove that $\text{Res}_0(M) = \zeta \circ R(M)$ has a standard filtration, it is enough to show that $N = E \circ R(M)$ has one. We claim that the module $N$ is free over $\mathbb{C}[\mathfrak{h}]$. So the result follows from Lemma 1.1(2).

Let us prove the claim. Recall from (1.18) that we have $R(M_0) = \phi_1^*(x_{pr}(M))$ as $\mathbb{C}[[\mathfrak{h}]]_0 \otimes W'$-modules. Using the isomorphism of $\mathbb{C}[\mathfrak{h}] \otimes W$-modules $M \cong \mathbb{C}[\mathfrak{h}] \otimes V$ given in Lemma 1.7(1), we deduce an isomorphism of $\mathbb{C}[\mathfrak{h}]_0 \otimes W'$-modules

$$R(M_0) \cong \phi_1^*(x_{pr}(\mathbb{C}[\mathfrak{h}] \otimes V)) \cong \mathbb{C}[[\mathfrak{h}]]_0 \otimes V.$$

So the module $R(M_0)$ is free over $\mathbb{C}[[\mathfrak{h}]]_0$. The completion of the module $N$ at 0 is isomorphic to $R(M_0)$. By Lemma 1.8(2) the module $N$ is free over $\mathbb{C}[\mathfrak{h}]$. The claim is proved.

(2) Since $\text{Res}_0(M)$ has a standard filtration, by Lemma 1.7 there exists a finite dimensional vector space $V' \subset \text{Res}_0(M)$ such that $V'$ is stable under the action of $\mathbb{C}W'$ and we have an isomorphism of $\mathbb{C}[\mathfrak{h}] \otimes W'$-modules

$$\text{Res}_0(M) \cong \mathbb{C}[\mathfrak{h}] \otimes V'.$$

Moreover, we have $\omega'([\text{Res}_0(M)]) = [V']$ where $\omega'$ is the map in Proposition 1.6(2). The same proposition yields that $\text{Res}^W_{W'}(\omega[M]) = \omega'([\text{Res}_0(M)])$. Since $\omega([M]) = [V]$. 

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its stabilizer in $C$. Since $C$ we have

$$\text{By Dunkl isomorphisms, the left hand side is a corresponding functors for Hecke algebras via the functor $N$ (2.2).}$$

$\ell$ where $h$

$$\text{The isomorphism (2.2) is compatible with the}$$

$$\text{We have}$

$$\text{Recall the inclusion}$$

$$\text{Step 2. Consider the $W'$-equivariant algebra isomorphism}$

$$\phi : C[h] \rightarrow C[h], \quad x \mapsto x + x(b)\text{ for }x \in h^*.$$

$$\text{It induces an isomorphism} \hat{\phi} : C[[h]]_b \cong C[[h]]_0. \text{ The latter yields an algebra isomorphism}$$

$$C[[h]]_b \otimes C[h] \cong C[[h]]_0 \otimes C[h]_{\text{reg}}.$$
To see this note first that by definition, the left hand side is $\mathbb{C}[[\mathfrak{h}]]_{0}[\alpha_{s}^{-1}, s \in \mathfrak{g}]$. For $s \in \mathfrak{g}$, $s \notin W'$ the element $\alpha_{s}$ is invertible in $\mathbb{C}[[\mathfrak{h}]]_{0}$, so we have

$$\mathbb{C}[[\mathfrak{h}]]_{0} \otimes \mathbb{C}[[\mathfrak{b}]]_{\text{reg}} = \mathbb{C}[[\mathfrak{h}]]_{0}[\alpha_{s}^{-1}, s \in \mathfrak{g} \cap W'].$$

For $s \in \mathfrak{g} \cap W'$ we have $\alpha_{s}(b) = 0$, so $\hat{\phi}(\alpha_{s}) = \alpha_{s}$. Hence

$$\hat{\phi}(\mathbb{C}[[\mathfrak{h}]]_{0})[\hat{\phi}(\alpha_{s})^{-1}, s \in \mathfrak{g} \cap W'] = \mathbb{C}[[\mathfrak{h}]]_{0}[\alpha_{s}^{-1}, s \in \mathfrak{g} \cap W'] = \mathbb{C}[[\mathfrak{h}]]_{0} \otimes \mathbb{C}[[\mathfrak{b}]]_{\text{reg}}.$$

**Step 3.** We will assume in Steps 3, 4, 5 that $M$ is a module in $\mathfrak{g}_{c}^{\Delta}(W, \mathfrak{h})$. In this step we prove that $N$ is isomorphic to $\hat{\phi}^{\ast}(M)$ as $\mathbb{C}[[\mathfrak{h}]] \times W'$-modules. Let $V$ be a subspace of $M$ as in Lemma 1.7(1). So we have an isomorphism of $\mathbb{C}[[\mathfrak{h}]] \times W'$-modules

$$(2.3) \quad M \cong \mathbb{C}[[\mathfrak{h}]] \otimes V.$$  

Also, by Proposition 1.9(2) there is an isomorphism of $\mathbb{C}[[\mathfrak{h}]] \times W'$-modules

$$N \cong \mathbb{C}[[\mathfrak{h}]] \otimes \text{Res}_{W'}^{W}(V).$$

So $N$ is isomorphic to $\hat{\phi}^{\ast}(M)$ as $\mathbb{C}[[\mathfrak{h}]] \times W'$-modules.

**Step 4.** In this step we compare $((\hat{\phi}^{\ast}(M))_{0})_{\mathfrak{h}_{\text{reg}}}$ and $(\mathbb{N}_{0})_{\mathfrak{h}_{\text{reg}}}$ as $\mathfrak{D}(\mathfrak{g}_{\text{reg}}^{\ast})_{0}$-modules. The definition of these $\mathfrak{D}(\mathfrak{g}_{\text{reg}}^{\ast})_{0}$-module structures will be given below in terms of connections. By (1.11) we have $N = E \circ R(M_{\mathfrak{b}})$, so we have $\mathbb{N}_{0} \cong R(M_{\mathfrak{b}})$. Next, by (1.18) we have an isomorphism of $\mathbb{C}[[\mathfrak{h}]]_{0} \times W'$-modules

$$R(M_{\mathfrak{b}}) = \hat{\phi}^{\ast}(x_{\text{pr}}(M_{\mathfrak{b}})) = (\hat{\phi}^{\ast}(M))_{0}.$$  

So we get an isomorphism of $\mathbb{C}[[\mathfrak{h}]]_{0} \times W'$-modules

$$\hat{\Psi} : (\hat{\phi}^{\ast}(M))_{0} \rightarrow \mathbb{N}_{0}.$$  

Now, let us consider connections on these modules. Note that by Step 2 we have

$$(\hat{\phi}^{\ast}(M))_{0} = \hat{\phi}^{\ast}(x_{pr}(M_{\mathfrak{b}}))_{\mathfrak{h}_{\text{reg}}}. \quad \text{Write } \nabla \text{ for the connection on } M_{\mathfrak{h}_{\text{reg}}} \text{ given by the Dunkl isomorphism for } H_{c}(W, \mathfrak{h}_{\text{reg}}). \text{ We equip } ((\hat{\phi}^{\ast}(M))_{0})_{\mathfrak{h}_{\text{reg}}} \text{ with the connection } \nabla \text{ given by}$$

$$(\nabla_{a}(x_{\text{pr}}m) = x_{\text{pr}}(\nabla_{a}(m)), \, \forall \, m \in (M_{\mathfrak{b}})_{\mathfrak{h}_{\text{reg}}}, \, a \in \mathfrak{h}). \quad \text{Let } \nabla^{(b)} \text{ be the connection on } N_{\mathfrak{h}_{\text{reg}}} \text{ given by the Dunkl isomorphism for } H_{c}(W', \mathfrak{h}_{\text{reg}}'). \text{ This restricts to a connection on } (\mathbb{N}_{0})_{\mathfrak{h}_{\text{reg}}}. \text{ We claim that } \Psi \text{ is compatible with this connection, i.e., we have}$$

$$(2.4) \quad \nabla^{(b)}_{a}(x_{\text{pr}}m) = x_{\text{pr}}(\nabla_{a}(m)), \, \forall \, m \in (M_{\mathfrak{b}})_{\mathfrak{h}_{\text{reg}}}.$$  

Recall the subspace $V$ of $M$ from Step 3. By Lemma 1.7(1) the map

$$(\mathbb{C}[[\mathfrak{h}]]_{0} \otimes \mathbb{C}[[\mathfrak{b}]]_{\text{reg}}) \otimes V \rightarrow (M_{\mathfrak{b}})_{\mathfrak{h}_{\text{reg}}}, \, \quad p \otimes v \mapsto pv$$
is a bijection. So it is enough to prove (2.4) for \( m = pv \) with \( p \in \mathbb{C}[\mathfrak{h}]_0 \otimes \mathbb{C}[\mathfrak{h}] \), \( v \in V \). We have

\[
\nabla_a^{(b)}(x_{pr}pv) = (y_{a}^{(b)}) - \sum_{s \in \mathfrak{f} \cap W'} \frac{2c_s}{1 - \lambda_s} \frac{\alpha_s(a)}{x_{\alpha_s}^{(b)}} (s - 1)(x_{pr}pv)
\]

\[
= x_{pr}(y_a) + \sum_{s \in \mathfrak{f} \cap W'} \frac{2c_s}{1 - \lambda_s} \frac{\alpha_s(a)}{x_{\alpha_s}^{(b)}} (s - 1)(x_{pr}pv)
\]

\[
= x_{pr}(\nabla_a) + \sum_{s \in \mathfrak{f} \cap W'} \frac{2c_s}{1 - \lambda_s} \frac{\alpha_s(a)}{x_{\alpha_s}^{(b)}} (s)(x_{pr}pv)
\]

(2.5)

Here the first equality is by the Dunkl isomorphism for \( H_c(W', \mathfrak{h}_{\text{reg}}^{'}) \). The second is by (1.15), (1.16), (1.17) and the fact that \( x_{pr}^2 = x_{pr} \). The third is by the Dunkl isomorphism for \( H_c(W, \mathfrak{h}_{\text{reg}}) \). The last is by (1.19). Next, since \( x_{pr} \) is the idempotent in \( \mathbb{C}[\mathfrak{h}]_0 \) corresponding to the component \( \mathbb{C}[[\mathfrak{h}]]_0 \) in the decomposition (1.14), we have

\[
\nabla_a(x_{pr}pv) = (\partial_a(x_{pr}p))v + x_{pr}p(\nabla_a v)
\]

\[
= x_{pr}(\partial_a(p))v + x_{pr}p(\nabla_a v)
\]

\[
= x_{pr}\nabla_a(pv).
\]

Together with (2.5) this implies that

\[
\nabla_a^{(b)}(x_{pr}pv) = x_{pr}\nabla_a(pv).
\]

So (2.4) is proved.

**Step 5.** In this step we prove the isomorphism (2.1) for \( M \in Z^A_c(W, \mathfrak{h}) \). Here we need some more notation. For \( X = \mathfrak{h} \) or \( \mathfrak{h}_{\text{reg}}^{'}, \) let \( U \) be an open analytic subvariety of \( X \), write \( i : U \hookrightarrow X \) for the canonical embedding. For \( F \) an analytic coherent sheaf on \( X \) we write \( i^*(F) \) for the restriction of \( F \) to \( U \). If \( U \) contains 0, for an analytic locally free sheaf \( E \) over \( U \), we write \( \widetilde{E} \) for the restriction of \( E \) to the formal disc at 0.

Let \( \Omega \subset \mathfrak{h} \) be the open ball defined in (1.3). Let \( f : \mathfrak{h} \to \mathfrak{h} \) be the morphism defined by \( \phi \). The preimage of \( \Omega \) via \( f \) is an open ball \( \Omega_0 \) in \( \mathfrak{h} \) centered at 0. We have

\[
f(\Omega_0 \cap \mathfrak{h}_{\text{reg}}^{'}) = \Omega \cap \mathfrak{h}_{\text{reg}}.
\]

Let \( u : \Omega_0 \cap \mathfrak{h}_{\text{reg}}^{'}, v : \Omega \cap \mathfrak{h}_{\text{reg}} \) be the canonical embeddings. By Step 3 there is an isomorphism of \( \mathfrak{W}^{'}, \mathfrak{h}_{\text{reg}} \)-equivariant analytic locally free sheaves over \( \Omega_0 \cap \mathfrak{h}_{\text{reg}} \)

\[
u^*(\mathcal{N}^{an}) \cong \phi^*(\nu^*(\mathcal{M}^{an})).
\]

By Step 4 there is an isomorphism

\[
u^*(\mathcal{N}^{an}) \cong \phi^*(\nu^*(\mathcal{M}^{an}))
\]

which is compatible with their connections. It follows from Lemma 2.2 below that there is an isomorphism

\[
(u^*(\mathcal{N}^{an}))^{\nabla_{(1)}} \cong \phi^*((u^*(\mathcal{M}^{an}))^{\nabla_{(1)}}).
\]
Since $\Omega_0 \cap h'_{\text{reg}}$ is homotopy equivalent to $h'_{\text{reg}}$ via $u$, the left hand side is isomorphic to $(N_{h'_{\text{reg}}})^{\nabla^2}$. So we have

\[ \kappa \circ j \circ \kappa^*(M) \cong KZ(W',h)(N), \]

where $\kappa$, $j$ are as in (1.4), (1.5). Combined with Step 1 we have the following isomorphisms

\[ KZ' \circ \text{Res}_b(M) \cong \ell_* \circ KZ(W',h)(N) \]

(2.6)

\[ = \iota_* \circ KZ(M). \]

They are functorial on $M$.

**Lemma 2.2.** Let $E$ be an analytic locally free sheaf over the complex manifold $h'_{\text{reg}}$. Let $\nabla_1$, $\nabla_2$ be two integrable connections on $E$ with regular singularities. If there exists an isomorphism $\hat{\psi} : (E, \nabla_1) \to (E, \nabla_2)$, then the local systems $E^{\nabla_1}$ and $E^{\nabla_2}$ are isomorphic.

**Proof.** Write $\text{End}(E)$ for the sheaf of endomorphisms of $E$. Then $\text{End}(E)$ is a locally free sheaf over $h'_{\text{reg}}$. The connections $\nabla_1$, $\nabla_2$ define a connection $\nabla$ on $\text{End}(E)$ as follows,

\[ \nabla : \text{End}(E) \to \text{End}(E), \quad f \mapsto \nabla_2 \circ f - f \circ \nabla_1. \]

So the isomorphism $\hat{\psi}$ is a horizontal section of $(\text{End}(E), \nabla)$. Let $(\text{End}(E)^{\nabla})_0$ be the set of germs of horizontal sections of $(\text{End}(E), \nabla)$ on zero. By the Comparison theorem [12, Theorem 6.3.1] the canonical map $(\text{End}(E)^{\nabla})_0 \to (\text{End}(E))^{\nabla}$ is bijective. Hence there exists a holomorphic isomorphism $\psi : (E, \nabla_1) \to (E, \nabla_2)$ which maps to $\hat{\psi}$. Now, let $U$ be an open ball in $h'_{\text{reg}}$ centered at $0$ with radius $\varepsilon$ small enough such that the holomorphic isomorphism $\psi$ converges in $U$. Write $E_U$ for the restriction of $E$ to $U$. Then $\psi$ induces an isomorphism of local systems $(E_U)^{\nabla_1} \cong (E_U)^{\nabla_2}$. Since $h'_{\text{reg}}$ is homotopy equivalent to $U$, we have

\[ E^{\nabla_1} \cong E^{\nabla_2}. \]

**Step 6.** Finally, write $I$ for the inclusion of $\text{Proj}_c(W,h)$ into $\Theta_c(W,h)$. By Lemma 1.1(1) any projective object in $\Theta_c(W,h)$ has a standard filtration, so (2.6) yields an isomorphism of functors

\[ KZ' \circ \text{Res}_b \circ I \to \iota_* \circ KZ \circ I. \]

Applying Lemma 1.2 to the exact functors $KZ' \circ \text{Res}_b$ and $\iota_* \circ KZ$ yields that there is an isomorphism of functors

\[ KZ' \circ \text{Res}_b \cong \iota_* \circ KZ. \]

**2.2.** We give some corollaries of Theorem 2.1.

**Corollary 2.3.** There is an isomorphism of functors

\[ KZ \circ \text{Ind}_b \cong \text{coInd}^W_{h'}, \circ KZ'. \]
Proof. – To simplify notation let us write 
\[ \mathcal{O} = \mathcal{O}_C(W, \mathfrak{h}), \quad \mathcal{O}' = \mathcal{O}_C(W', \mathfrak{h}), \quad \mathcal{H} = \mathcal{H}_q(W), \quad \mathcal{H}' = \mathcal{H}_q(W'). \]
Recall that the functor \( \text{KZ} \) is represented by a projective object \( P_{\text{KZ}} \) in \( \mathcal{O} \). So for any \( N \in \mathcal{O}' \) we have a morphism of \( \mathcal{H}' \)-modules
\[
\text{KZ} \circ \text{Ind}_b(N) \cong \text{Hom}_Q(P_{\text{KZ}}, \text{Ind}_b(N)) \\
\cong \text{Hom}_Q(\text{Res}_b(P_{\text{KZ}}), N)
\]
(2.7)
\[ \to \text{Hom}_{\mathcal{H}'}(\text{KZ}'(\text{Res}_b(P_{\text{KZ}})), KZ'(N)). \]
By Theorem 2.1 we have
\[
\text{KZ}' \circ \text{Res}_b(P_{\text{KZ}}) \cong \text{Res}^W_{\mathcal{H}'} \circ \text{KZ}(P_{\text{KZ}}).
\]
Recall from Section 1.5 that the \( \mathcal{H} \)-module \( \text{KZ}(P_{\text{KZ}}) \) is isomorphic to \( \mathcal{H} \). So as \( \mathcal{H}' \)-modules \( \text{KZ}'(\text{Res}_b(P_{\text{KZ}})) \) is also isomorphic to \( \mathcal{H} \). Therefore the morphism (2.7) rewrites as
\[
\chi(N) : \text{KZ} \circ \text{Ind}_b(N) \to \text{Hom}_{\mathcal{H}'}(\mathcal{H}, KZ'(N)).
\]
It yields a morphism of functors
\[ \chi : \text{KZ} \circ \text{Ind}_b \to \text{Ind}^W_\mathcal{H} \circ \text{KZ}'. \]
Note that if \( N \) is a projective object in \( \mathcal{O}' \), then \( \chi(N) \) is an isomorphism by Proposition 1.4(1). So Lemma 1.2 implies that \( \chi \) is an isomorphism of functors, because both functors \( \text{KZ} \circ \text{Ind}_b \) and \( \text{Ind}^W_\mathcal{H} \circ \text{KZ}' \) are exact. \( \square \)

2.3. – The following lemma will be useful to us.

Lemma 2.4. – Let \( K, L \) be two right exact functors from \( \mathcal{O}_1 \) to \( \mathcal{O}_2 \), where \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) can be either \( \mathcal{O}_C(W, \mathfrak{h}) \) or \( \mathcal{O}_C(W', \mathfrak{h}) \). Let \( \text{KZ}_2 \) denote the KZ-functor on \( \mathcal{O}_2 \). Suppose that \( K, L \) map projective objects to projective ones. Then the vector space homomorphism
\[
\text{Hom}(K, L) \to \text{Hom}(\text{KZ}_2 \circ K, \text{KZ}_2 \circ L), \quad f \mapsto 1_{\text{KZ}_2} f,
\]
(2.9)
is an isomorphism.

Notice that if \( K = L \), this is even an isomorphism of rings.

Proof. – Let \( \text{Proj}_1, \text{Proj}_2 \) be respectively the subcategory of projective objects in \( \mathcal{O}_1, \mathcal{O}_2 \). Write \( \tilde{K}, \tilde{L} \) for the functors corresponding to \( \text{Proj}_1 \) to \( \text{Proj}_2 \) given by the restrictions of \( K, L \), respectively. Let \( \mathcal{H}_2 \) be the Hecke algebra corresponding to \( \mathcal{O}_2 \). Since the functor \( \text{KZ}_2 \) is fully faithful over \( \text{Proj}_2 \) by Proposition 1.4(1), the following functor
\[ \text{Fct}(\text{Proj}_1, \text{Proj}_2) \to \text{Fct}(\text{Proj}_1, \mathcal{H}_2-\text{mod}), \quad G \mapsto \text{KZ}_2 \circ G \]
is also fully faithful. This yields an isomorphism
\[ \text{Hom}(\tilde{K}, \tilde{L}) \cong \text{Hom}(\text{KZ}_2 \circ \tilde{K}, \text{KZ}_2 \circ \tilde{L}), \quad f \mapsto 1_{\text{KZ}_2} f. \]
Next, by Lemma 1.2 the canonical morphisms
\[ \text{Hom}(K, L) \to \text{Hom}(\tilde{K}, \tilde{L}), \quad \text{Hom}(\text{KZ}_2 \circ K, \text{KZ}_2 \circ L) \to \text{Hom}(\text{KZ}_2 \circ \tilde{K}, \text{KZ}_2 \circ \tilde{L}) \]
are isomorphisms. So the map (2.9) is also an isomorphism. \( \square \)
Let \( b(W, W'') \) be a point in \( \mathfrak{h} \) whose stabilizer is \( W'' \). Let \( b(W', W'') \) be its image in \( \bar{\mathfrak{h}} = \mathfrak{h} / bW'' \) via the canonical projection. Write \( b(W, W') = b \).

**Corollary 2.5.** – There are isomorphisms of functors

\[
\begin{align*}
\text{Res}_b(W', W'') \circ \text{Res}_b(W, W') & \cong \text{Res}_b(W, W''), \\
\text{Ind}_b(W, W') \circ \text{Ind}_b(W', W'') & \cong \text{Ind}_b(W, W'').
\end{align*}
\]

**Proof.** – Since the restriction functors map projective objects to projective ones by Proposition 1.6(1), Lemma 2.4 applied to the categories \( \mathcal{O}_1 = \mathcal{O}_c(W, \mathfrak{h}), \mathcal{O}_2 = \mathcal{O}_{c''}(W'', \mathfrak{h} / bW'') \) yields an isomorphism

\[
\text{Hom}(\text{Res}_b(W', W'') \circ \text{Res}_b(W, W'), \text{Res}_b(W, W'')) \cong \text{Hom}(\text{KZ}'' \circ \text{Res}_b(W', W'') \circ \text{Res}_b(W, W'), \text{KZ}'' \circ \text{Res}_b(W, W')).
\]

By Theorem 2.1 the set on the second row is

\[
(2.10) \quad \text{Hom}(\mathcal{O} W', \mathcal{O} W'' \circ \mathcal{O} W, \mathcal{O} W, \mathcal{O} \text{KZ}, \mathcal{O} \text{KZ}).
\]

By the presentations of Hecke algebras in [6, Proposition 4.22], there is an isomorphism

\[
\sigma : \mathcal{O} W'' \circ \mathcal{O} W, \mathcal{O} \text{KZ} \cong \mathcal{O} W''.
\]

Hence the element \( \sigma(1) \) in the set (2.10) maps to an isomorphism

\[
\text{Res}_b(W', W'') \circ \text{Res}_b(W, W') \cong \text{Res}_b(W, W'').
\]

This proves the first isomorphism in the corollary. The second one follows from the uniqueness of right adjoint functor. \( \square \)

**2.4. Biadjointness of \( \text{Res}_b \) and \( \text{Ind}_b \)**

Recall that a finite dimensional \( \mathbb{C} \)-algebra \( A \) is symmetric if \( A \) is isomorphic to \( A^* = \text{Hom}_\mathbb{C}(A, \mathbb{C}) \) as \( (A, A) \)-bimodules.

**Lemma 2.6.** – Assume that \( \mathcal{H}_q(W) \) and \( \mathcal{H}_q(W') \) are symmetric algebras. Then the functors \( \text{Ind}_W^\mathcal{H} \) and \( \text{coInd}_W^\mathcal{H} \) are isomorphic, i.e., the functor \( \text{Ind}_W^\mathcal{H} \), is biadjoint to \( \text{coInd}_W^\mathcal{H} \).

**Proof.** – We abbreviate \( \mathcal{H} = \mathcal{H}_q(W) \) and \( \mathcal{H}' = \mathcal{H}_q(W') \). Since \( \mathcal{H} \) is free as a left \( \mathcal{H}' \)-module, for any \( \mathcal{H}' \)-module \( M \) the map

\[
(2.11) \quad \text{Hom}_{\mathcal{H}'}(\mathcal{H}, \mathcal{H}') \otimes_{\mathcal{H}'} M \to \text{Hom}_{\mathcal{H}'}(\mathcal{H}, M)
\]

given by multiplication is an isomorphism of \( \mathcal{H} \)-modules. By assumption \( \mathcal{H}' \) is isomorphic to \( (\mathcal{H}')^* \) as \( (\mathcal{H}', \mathcal{H}') \)-bimodules. Thus we have the following \( (\mathcal{H}', \mathcal{H}') \)-bimodule isomorphisms

\[
\text{Hom}_{\mathcal{H}'}(\mathcal{H}, \mathcal{H}') \cong \text{Hom}_{\mathcal{H}'}(\mathcal{H}, (\mathcal{H}')^*) \\
\cong \text{Hom}_\mathbb{C}(\mathcal{H}' \otimes \mathcal{H}, \mathcal{H}, \mathbb{C}) \\
\cong \mathcal{H}^* \\
\cong \mathcal{H}.
\]

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The last isomorphism follows from the fact that $H$ is symmetric. Thus, by (2.11) the functors $\text{Ind}^W_W$ and $\text{coInd}^W_W$ are isomorphic. \hfill \Box

**Remark 2.7.** – It is proved that $\mathcal{H}_q(W)$ is a symmetric algebra for all irreducible complex reflection group $W$ except for some of the 34 exceptional groups in the Shephard-Todd classification. See [5, Section 2A] for details.

The biadjointness of $\text{Res}_b$ and $\text{Ind}_b$ was conjectured in [4, Remark 3.18] and was announced by I. Gordon and M. Martino. We give a proof in Proposition 2.9 since it seems not yet to be available in the literature. Let us first consider the following lemma.

**Lemma 2.8.** – (1) Let $A$, $B$ be noetherian algebras and $T$ be a functor

$$T : A\text{-mod} \to B\text{-mod}.$$ 

If $T$ is right exact and commutes with direct sums, then it has a right adjoint.

(2) The functor

$$\text{Res}_b : \Theta_c(W, h) \to \Theta_c(W', \overline{h})$$

has a left adjoint.

**Proof.** – (1) Consider the $(B, A)$-bimodule $M = T(A)$. We claim that the functor $T$ is isomorphic to the functor $M \otimes A^\text{op}$. Indeed, by definition we have $T(A) \cong M \otimes A$ as $B$-modules. Now, for any $N \in A\text{-mod}$, since $N$ is finitely generated and $A$ is noetherian there exist $m, n \in \mathbb{N}$ and an exact sequence

$$A \oplus m \to A \oplus n \to N \to 0.$$ 

Since both $T$ and $M \otimes A^\text{op}$ are right exact and they commute with direct sums, the fact that $T(A) \cong M \otimes A$ implies that $T(N) \cong M \otimes A N$ as $B$-modules. This proved the claim. Now, the functor $M \otimes A^\text{op}$ has a right adjoint $\text{Hom}_B(M, -)$, so $T$ also has a right adjoint.

(2) Recall that for any complex reflection group $W$, a contravariant duality functor

$$(-)^\vee : \Theta_e(W, h) \to \Theta_{c^1}(W, h^*)$$

was defined in [10, Section 4.2], here $c^1 : \mathfrak{g} \to \mathbb{C}$ is another parameter explicitly determined by $c$. Consider the functor

$$\text{Res}^\vee_c = (-)^\vee \circ \text{Res}_b \circ (-)^\vee : \Theta_e(W, h^*) \to \Theta_{c^1}(W', \overline{h})^*.$$ 

The category $\Theta_{c^1}(W, h^*)$ has a projective generator $P$. The algebra $\text{End}_{\Theta_{c^1}(W, h^*)}(P)^\text{op}$ is finite dimensional over $\mathbb{C}$ and by Morita theory we have an equivalence of categories

$$\Theta_{c^1}(W, h^*) \cong \text{End}_{\Theta_{c^1}(W, h^*)}(P)^\text{op}-\text{mod}.$$ 

Since the functor $\text{Res}^\vee_c$ is exact and obviously commutes with direct sums, by part (1) it has a right adjoint $\Psi$. Then it follows that $(-)^\vee \circ \Psi \circ (-)^\vee$ is left adjoint to $\text{Res}_b$. The lemma is proved. \hfill \Box

**Proposition 2.9.** – Assume that $\mathcal{H}_q(W)$ and $\mathcal{H}_q(W')$ are symmetric algebras. Then the functor $\text{Ind}_b$ is left adjoint to $\text{Res}_b$. 

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The functor adjunctions first give an isomorphism of functors

The goal of this step is to show that there exists an isomorphism of functors

Lemma 2.8 the functor $H$ has a left adjoint. We denote it by $F : \mathcal{H}' \to \mathcal{H}$.

The goal of this step is to show that there exists an isomorphism of functors

To this end, let $S, S'$ be respectively the right adjoints of $KZ, KZ'$, see Section 1.5. We will first give an isomorphism of functors

Let $M \in \mathcal{H'}$-mod and $N \in \mathcal{H}$-mod. Consider the following equalities given by adjunctions

The functor $KZ'$ yields a map

Since the canonical adjunction maps $KZ' \circ S' \to \text{Id}_{\mathcal{H'}}$, $KZ \circ S \to \text{Id}_{\mathcal{H}}$ are isomorphisms (see Section 1.5) and since we have an isomorphism of functors $KZ' \circ E \cong E^\times \circ KZ$ by Theorem 2.1, we get the following equalities

In the last equality we used that $F^\times$ is left adjoint to $E^\times$. So the map (2.12) can be rewritten into the following form

Now, take $N = \mathcal{H}$. Recall that $\mathcal{H}$ is isomorphic to $KZ(P_{KZ})$ as $\mathcal{H}$-modules. Since $P_{KZ}$ is projective, by Proposition 1.4(2) we have a canonical isomorphism in $\mathcal{H}$

Further $E$ maps projectives to projectives by Proposition 1.6(1), so $E \circ S(\mathcal{H})$ is also projective. Hence Proposition 1.4(1) implies that in this case (2.12) is an isomorphism for any $M$, i.e., we get an isomorphism

Further this is an isomorphism of right $\mathcal{H}$-modules with respect to the $\mathcal{H}$-actions induced by the right action of $\mathcal{H}$ on itself. Now, the fact that $\mathcal{H}$ is a symmetric algebra yields that
for any finite dimensional \( \mathcal{H} \)-module \( N \) we have isomorphisms of right \( \mathcal{H} \)-modules
\[
\text{Hom}_{\mathcal{H}}(N, \mathcal{H}) \cong \text{Hom}_{\mathcal{H}}(N, \text{Hom}_{C}(\mathcal{H}, C)) \\
\cong \text{Hom}_{C}(N, C).
\]
Therefore \( a(M, \mathcal{H}) \) yields an isomorphism of right \( \mathcal{H} \)-modules
\[
\text{Hom}_{C}(\text{KZ} \circ F \circ S'(M), C) \rightarrow \text{Hom}_{C}(F^\times (M), C).
\]
We deduce a natural isomorphism of left \( \mathcal{H} \)-modules
\[
\text{KZ} \circ F \circ S'(M) \cong F^\times (M)
\]
for any \( \mathcal{H}' \)-module \( M \). This gives an isomorphism of functors
\[
\psi : \text{KZ} \circ F \circ S' \sim F^\times.
\]
Finally, consider the canonical adjunction map \( \eta : \text{Id}_{\mathcal{H}'} \rightarrow S' \circ \text{KZ}' \). We have a morphism of functors
\[
\phi = (1_{\text{KZ}'}) \circ (\psi_{\text{KZ}'}) : \text{KZ} \circ F \rightarrow F^\times \circ \text{KZ}'.
\]
Note that \( \psi_{\text{KZ}'} \) is an isomorphism of functors. If \( Q \) is a projective object in \( \mathcal{H}' \), then by Proposition 1.4(2) the morphism \( \eta(Q) : Q \rightarrow S' \circ \text{KZ}'(Q) \) is also an isomorphism, so \( \phi(Q) \) is an isomorphism. This implies that \( \phi \) is an isomorphism of functors by Lemma 1.2, because both \( \text{KZ} \circ F \) and \( F^\times \circ \text{KZ}' \) are right exact functors. Here the right exactness of \( F \) follows from that it is left adjoint to \( E \). So we get the desired isomorphism of functors
\[
\text{KZ} \circ F \cong F^\times \circ \text{KZ}'.
\]

**Step 2.** Let us now prove that \( F \) is right adjoint to \( E \). By uniqueness of adjoint functors, this will imply that \( F \) is isomorphic to \( \text{Ind}_\mathcal{W} \). First, by Lemma 2.6 the functor \( F^\times \) is isomorphic to \( \text{coInd}_\mathcal{W} \). So \( F^\times \) is right adjoint to \( E^\times \), i.e., we have morphisms of functors
\[
e^\times : E^\times \circ F^\times \rightarrow \text{Id}_{\mathcal{W}}, \quad \eta^\times : \text{Id}_{\mathcal{W}} \rightarrow F^\times \circ E^\times
\]
such that
\[
(e^\times 1_{E^\times}) \circ (1_{F^\times} \eta^\times) = 1_{E^\times}, \quad (1_{F^\times} e^\times) \circ (\eta^\times 1_{E^\times}) = 1_{F^\times}.
\]
Next, both \( F \) and \( E \) have exact right adjoints, given respectively by \( E \) and \( \text{Ind}_\mathcal{W} \). Therefore \( F \) and \( E \) map projective objects to projective ones. Applying Lemma 2.4 to \( \theta_1 = \theta_2 = \theta' \), \( K = E \circ F, L = \text{Id}_{\mathcal{H}'} \) yields that the following map is bijective
\[
(2.13) \quad \text{Hom}(E \circ F, \text{Id}_{\mathcal{H}'}) \rightarrow \text{Hom}(\text{KZ}' \circ E \circ F, \text{KZ}' \circ \text{Id}_{\theta'}), \quad f \mapsto 1_{\text{KZ} \circ F}.
\]
By Theorem 2.1 and Step 1 there exist isomorphisms of functors
\[
\phi_E : E^\times \circ \text{KZ} \sim \text{KZ}' \circ E, \quad \phi_F : F^\times \circ \text{KZ}' \sim \text{KZ} \circ F.
\]
Let
\[
\phi_{EF} = (\phi_E 1_F) \circ (1_{E^\times} \phi_F) : E^\times \circ F^\times \circ \text{KZ} \sim \text{KZ}' \circ E \circ F,
\]
\[
\phi_{FE} = (\phi_F 1_E) \circ (1_{F^\times} \phi_E) : F^\times \circ E^\times \circ \text{KZ} \sim \text{KZ} \circ F \circ E.
\]
Identify
\[
\text{KZ} \circ \text{Id}_{\theta} = \text{Id}_{\mathcal{H}' \circ \text{KZ}}, \quad \text{KZ}' \circ \text{Id}_{\theta'} = \text{Id}_{\mathcal{H}' \circ \text{KZ}}.
\]
We have a bijective map
\[ \text{Hom}(KZ' \circ E \circ F, KZ' \circ \text{Id}_{O'}) \cong \text{Hom}(E \circ F \circ KZ', \text{Id}_{O'} \circ KZ'), \quad g \mapsto g \circ \phi_{EF}. \]
Together with (2.13), it implies that there exists a unique morphism \( \varepsilon : E \circ F \rightarrow \text{Id}_{O'} \) such that
\[ (1_{KZ'} \varepsilon) \circ \phi_{EF} = \varepsilon \circ 1_{KZ}. \]
Similarly, there exists a unique morphism \( \eta : \text{Id}_{O} \rightarrow F \circ E \) such that
\[ (\phi_{FE})^{-1} \circ (1_{KZ} \eta) = \eta \circ 1_{KZ}. \]
Now, we have the following commutative diagram
\[ \begin{array}{ccc}
E \circ KZ & \xrightarrow{\phi_E} & KZ' \circ E \\
\downarrow_{1_{E} \varepsilon \eta} & & \downarrow_{1_{KZ} \varepsilon \eta} \\
E \circ F \circ E \circ KZ & \xrightarrow{1_{E} \varepsilon \phi_{EF}} & E \circ F \circ E \circ KZ' \circ F \circ E \\
\downarrow_{1_{E} \phi_{EF} \varepsilon} & & \downarrow_{1_{KZ} \phi \varepsilon \varepsilon} \\
E \circ F \circ KZ' \circ E & \xrightarrow{1_{KZ} \varepsilon \varepsilon} & KZ' \circ E.
\end{array} \]
It yields that
\[ (1_{KZ} \varepsilon 1_{E}) \circ (1_{KZ'} \varepsilon \eta) = \phi_E \circ (\varepsilon 1_{E} \varepsilon 1_{KZ}) \circ (1_{E} \varepsilon \eta 1_{KZ}) \circ (\phi_E)^{-1}. \]
We deduce that
\[ 1_{KZ'}((\varepsilon 1_{E}) \circ (1_{E} \eta)) = \phi_E \circ (1_{E} \varepsilon 1_{KZ}) \circ (\phi_E)^{-1} = 1_{KZ'} 1_{E}. \]
(2.14)
By applying Lemma 2.4 to \( \theta_1 = \theta, \theta_2 = \theta', K = L = E \), we deduce that the following map is bijective
\[ \text{End}(E) \rightarrow \text{End}(KZ' \circ E), \quad f \mapsto 1_{KZ'} f. \]
Hence (2.14) implies that
\[ (\varepsilon 1_{E}) \circ (1_{E} \eta) = 1_{E}. \]
Similarly, we have \( (1_{E} \varepsilon) \circ (\eta 1_{E}) = 1_{F} \). So \( E \) is left adjoint to \( F \). By uniqueness of adjoint functors this implies that \( F \) is isomorphic to \( \text{Ind}_{b} \). Therefore \( \text{Ind}_{b} \) is biadjoint to \( \text{Res}_{b} \). 

### 3. Reminders on the cyclotomic case

From now on we will concentrate on the cyclotomic rational DAHA's. We fix some notation in this section.

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3.1. – Let $l, n$ be positive integers. Write $\varepsilon = \exp(\frac{2\pi \sqrt{-1}}{e})$. Let $\mathfrak{h} = \mathbb{C}^n$, write $\{y_1, \ldots, y_n\}$ for its standard basis. For $1 \leq i, j, k \leq n$ with $i, j, k$ distinct, let $\varepsilon_k, s_{ij}$ be the following elements of $GL(\mathfrak{h})$:

$$
\varepsilon_k(y_k) = \varepsilon y_k, \quad \varepsilon_k(y_j) = y_j, \quad s_{ij}(y_i) = y_j, \quad s_{ij}(y_k) = y_k.
$$

Let $B_n(l)$ be the subgroup of $GL(\mathfrak{h})$ generated by $\varepsilon_k$ and $s_{ij}$ for $1 \leq k \leq n$ and $1 \leq i < j \leq n$. It is a complex reflection group with the set of reflections

$$
\varphi_n = \{\varepsilon_i^p : 1 \leq i \leq n, 1 \leq p \leq l - 1\} \bigcup \{s_{ij}^{(p)} = s_{ij} \varepsilon_i^p \varepsilon_j^{-p} : 1 \leq i < j \leq n, 1 \leq p \leq l\}.
$$

Note that there is an obvious inclusion $\varphi_{n-1} \hookrightarrow \varphi_n$. It yields an embedding

$$
B_{n-1}(l) \hookrightarrow B_n(l).
$$

This embedding identifies $B_{n-1}(l)$ with the parabolic subgroup of $B_n(l)$ given by the stabilizer of the point $b_n = (0, \ldots, 0, 1) \in \mathbb{C}^n$.

The cyclotomic rational DAHA is the algebra $H_c(B_n(l), \mathfrak{h})$. We will use another presentation in which we replace the parameter $e$ by an $l$-tuple $h = (h, h_1, \ldots, h_{l-1})$ such that

$$
c_{s_{ij}} = -h, \quad c_{\varepsilon_i} = -\frac{1}{2} \sum_{p' = 1}^{l-1} (\varepsilon^{-pp'} - 1) h_{p'}.
$$

We will denote $H_c(B_n(l), \mathfrak{h})$ by $H_{h,n}$. The corresponding category $\mathcal{O}_{\mathfrak{h}}$ will be denoted by $\mathcal{O}_{h,n}$. In the rest of the paper, we will fix the positive integer $l$. We will also fix a positive integer $e \geq 2$ and an $l$-tuple of integers $s = (s_1, \ldots, s_l)$. We will always assume that the parameter $h$ is given by the following formulas,

$$
h = \frac{-1}{e}, \quad h_p = \frac{s_{p+1} - s_p}{e} - \frac{1}{l}, \quad 1 \leq p \leq l - 1.
$$

The functor $KZ(B_n(l), \mathbb{C}^n)$ goes from $\mathcal{O}_{h,n}$ to the category of finite dimensional modules of a certain Hecke algebra $\mathcal{H}_{q,n}$ attached to the group $B_n(l)$. Here the parameter is $q = (q, q_1, \ldots, q_l)$ with

$$
q = \exp(2\pi \sqrt{-1}/e), \quad q_p = q^{s_p}, \quad 1 \leq p \leq l.
$$

The algebra $\mathcal{H}_{q,n}$ has the following presentation:

- Generators: $T_0, T_1, \ldots, T_{n-1},$ 
- Relations:

$$
(T_0 - q_1) \cdots (T_0 - q_l) = (T_i + 1)(T_i - q) = 0, \quad 1 \leq i \leq n - 1,
$$

$$
T_0 T_0 T_1 = T_1 T_0 T_1 T_0, \quad T_i T_j = T_j T_i, \quad \text{if } |i - j| > 1,
$$

$$
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad 1 \leq i \leq n - 2.
$$

The algebra $\mathcal{H}_{q,n}$ satisfies the assumption of Section 2, i.e., it has the same dimension as $\mathbb{C}B_n(l)$. 

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3.2. – For each positive integer \( n \), the embedding \((3.1)\) of \( B_{n-1}(l) \) into \( B_n(l) \) yields an embedding of Hecke algebras

\[
i_q : \mathcal{H}_{q,n-1} \hookrightarrow \mathcal{H}_{q,n},
\]

see Section 1.2. Under the presentation above this embedding is given by

\[
i_q(T_i) = T_i, \quad \forall \ 0 \leq i \leq n - 2,
\]

see [6, Proposition 2.29].

We will consider the following restriction and induction functors:

\[
E(n) = \mathrm{Res}_{b_n}, \quad E(n)^\pi = \pi \mathrm{Res}_{B_{n-1}(l)^{\pi}},
\]

\[
F(n) = \mathrm{Ind}_{b_n}, \quad F(n)^\pi = \pi \mathrm{Ind}_{B_{n-1}(l)^{\pi}}.
\]

The algebra \( \mathcal{H}_{q,n} \) is symmetric (see Remark 2.7). Hence by Lemma 2.6 we have

\[
F(n)^\pi \cong \pi \mathrm{Ind}_{B_{n-1}(l)}.
\]

We will abbreviate

\[
\vartheta_{h,N} = \bigoplus_{n \in \mathbb{N}} \vartheta_{h, n}, \quad \mathrm{KZ} = \bigoplus_{n \in \mathbb{N}} \mathrm{KZ}(B_n(l), \mathbb{C}^n), \quad \mathcal{H}_{q,N} \text{-mod} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_{q,n} \text{-mod}.
\]

So \( \mathrm{KZ} \) is the Knizhnik-Zamolodchikov functor from \( \vartheta_{h,N} \) to \( \mathcal{H}_{q,N} \text{-mod} \). Let

\[
E = \bigoplus_{n \geq 1} E(n), \quad E^\pi = \bigoplus_{n \geq 1} E(n)^\pi,
\]

\[
F = \bigoplus_{n \geq 1} F(n), \quad F^\pi = \bigoplus_{n \geq 1} F(n)^\pi.
\]

So \((E^\pi, F^\pi)\) is a pair of biadjoint endo-functors of \( \mathcal{H}_{q,N} \text{-mod} \), and \((E, F)\) is a pair of biadjoint endo-functors of \( \vartheta_{h,N} \) by Proposition 2.9.

3.3. Fock spaces

Recall that an \( l \)-partition is an \( l \)-tuple \( \lambda = (\lambda^1, \cdots, \lambda^l) \) with each \( \lambda^j \) a partition, that is a sequence of integers \( (\lambda^j)_1 \geq \cdots \geq (\lambda^j)_k > 0 \). To any \( l \)-partition \( \lambda = (\lambda^1, \cdots, \lambda^l) \) we attach the set

\[
\Upsilon_{\lambda} = \{(a, b, j) \in \mathbb{N} \times \mathbb{N} \times (\mathbb{Z}/l\mathbb{Z}) : 0 < b \leq (\lambda^j)_a\}.
\]

Write \( |\lambda| \) for the number of elements in this set, we say that \( \lambda \) is an \( l \)-partition of \( |\lambda| \). For \( n \in \mathbb{N} \) we denote by \( \mathcal{P}_{n,l} \) the set of \( l \)-partitions of \( n \). For any \( l \)-partition \( \mu \) such that \( \Upsilon_{\mu} \) contains \( \Upsilon_{\lambda} \), we write \( \mu/\lambda \) for the complement of \( \Upsilon_{\lambda} \) in \( \Upsilon_{\mu} \). Let \(|\mu/\lambda|\) be the number of elements in this set. To each element \((a, b, j)\) in \( \Upsilon_{\lambda} \) we attach an element

\[
\text{res}((a, b, j)) = b - a + s_j \in \mathbb{Z}/l\mathbb{Z},
\]

called the residue of \((a, b, j)\). Here \( s_j \) is the \( j \)-th component of our fixed \( l \)-tuple \( s \).

The Fock space with multi-charge \( s \) is the \( \mathbb{C} \)-vector space \( \mathcal{F}_s \) spanned by the \( l \)-partitions, i.e.,

\[
\mathcal{F}_s = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\lambda \in \mathcal{P}_{n,l}} \mathbb{C}\lambda.
\]
It admits an integrable \( \tilde{\mathfrak{sl}}_- \)-module structure such that the Chevalley generators act as follows (cf. [11]): for any \( i \in \mathbb{Z}/e \mathbb{Z} \),
\[
e_i(\lambda) = \sum_{|\lambda/\mu|=1, \text{res}(\lambda/\mu)=i} \mu, \quad f_i(\lambda) = \sum_{|\mu/\lambda|=1, \text{res}(\mu/\lambda)=i} \mu.
\]

Let \( n_\lambda \) be the number of elements in the set \( \{(a, b, j) \in \mathbb{T}_\lambda : \text{res}((a, b, j)) = i\} \). The element \( \vartheta \in \tilde{\mathfrak{sl}}_- \) acts on \( \mathcal{S}_\lambda \) by
\[
\vartheta(\lambda) = -n_\lambda \lambda.
\]

For each \( n \in \mathbb{Z} \) set \( \Lambda_n = \Lambda_{\tilde{n}} \), where \( \tilde{n} \) is the image of \( n \) in \( \mathbb{Z}/e \mathbb{Z} \) and \( \Lambda_{\tilde{n}} \) is the corresponding fundamental weight of \( \tilde{\mathfrak{sl}}_- \). Set
\[
\Lambda_\lambda = \Lambda_{n_1} + \cdots + \Lambda_{n_\lambda}.
\]

Each \( l \)-partition \( \lambda \) is a weight vector of \( \mathcal{S}_\lambda \) with weight
\[
\text{wt}(\lambda) = \Lambda_\lambda - \sum_{i \in \mathbb{Z}/e \mathbb{Z}} n_i \alpha_i.
\]

We will call \( \text{wt}(\lambda) \) the weight of \( \lambda \).

In [14, Section 6.1.1] an explicit bijection was given between the sets \( \text{Irr}(B_n(l)) \) and \( \mathcal{P}_{l,n} \). Using this bijection we identify these two sets and index the standard and simple modules in \( \mathcal{S}_{\lambda,\mu} \) by \( l \)-partitions. In particular, we have an isomorphism of \( \mathbb{C} \)-vector spaces
\[
(3.5) \quad \theta : K(\mathcal{S}_{\lambda,\mu}) \sim \mathcal{S}_\lambda, \quad [\Delta(\lambda)] \mapsto \lambda.
\]

3.4. We end this section by the following lemma. Recall that the functor \( \text{KZ} \) gives a map \( K(\mathcal{S}_{\lambda,\mu}) \to K(\mathcal{H}_{\mathbb{Q},n}) \). For any \( l \)-partition \( \lambda \) of \( n \) let \( S_\lambda \) be the corresponding Specht module in \( \mathcal{H}_{\mathbb{Q},n} \)-mod, see [2, Definition 13.22] for its definition.

**Lemma 3.1.** In \( K(\mathcal{H}_{\mathbb{Q},n}) \), we have \( \text{KZ}([\Delta(\lambda)]) = [S_\lambda] \).

**Proof.** Let \( R \) be any commutative ring over \( \mathbb{C} \). For any \( l \)-tuple \( z = (z, z_1, \ldots, z_{l-1}) \) of elements in \( R \) one defines the rational DAHA over \( R \) attached to \( B_n(l) \) with parameter \( z \) in the same way as before. Denote it by \( \mathcal{H}_{R,z,n} \). The standard modules \( \Delta_R(\lambda) \) are also defined as before. For any \( (l+1) \)-tuple \( u = (u, u_1, \ldots, u_l) \) of invertible elements in \( R \) the Hecke algebra \( \mathcal{H}_{R,u,n} \) over \( R \) attached to \( B_n(l) \) with parameter \( u \) is defined by the same presentation as in Section 3.1. The Specht modules \( S_{R,\lambda} \) are also well-defined (see [2]). If \( R \) is a field, we will write \( \text{Irr}(\mathcal{H}_{R,u,n}) \) for the set of isomorphism classes of simple \( \mathcal{H}_{R,u,n} \)-modules.

Now, fix \( R \) to be the ring of holomorphic functions of one variable \( \varpi \). We choose \( z = (z, z_1, \ldots, z_{l-1}) \) to be given by
\[
z = l \varpi, \quad z_p = (s_{p+1} - s_p)l \varpi + e \varpi, \quad 1 \leq p \leq l - 1.
\]

Write \( x = \exp(-2\pi \sqrt{-1} \varpi) \). Let \( u = (u, u_1, \ldots, u_l) \) be given by
\[
u = x, \quad u_p = e^{-1} x^{s_p - s_{p-1}} e, \quad 1 \leq p \leq l.
\]

By [6, Theorem 4.12] the same definition as in Section 1.5 yields a well defined \( \mathcal{H}_{R,u,n} \)-module
\[
T_R(\lambda) = \text{KZ}_R(\Delta_R(\lambda)).
\]
It is a free $R$-module of finite rank and it commutes with the base change functor by the existence and unicity theorem for linear differential equations, i.e., for any ring homomorphism $R \to R'$ over $\mathbb{C}$, we have a canonical isomorphism of $\mathcal{H}_{R',u,n}$-modules
\begin{equation}
T_{R'}(\lambda) = KZ_{R'}(\Delta_{R'}(\lambda)) \cong T_R(\lambda) \otimes_R R'.
\end{equation}

In particular, for any ring homomorphism $a : R \to \mathbb{C}$. Write $\mathbb{C}_a$ for the vector space $\mathbb{C}$ equipped with the $R$-module structure given by $a$. Let $a(z)$, $a(u)$ denote the images of $z$, $u$ by $a$. Note that we have $H_{a(z),n} = H_{R,z,n} \otimes_R \mathbb{C}_a$ and $\mathcal{H}_{a(u),n} = \mathcal{H}_{R,u,n} \otimes_R \mathbb{C}_a$.

Denote the Knizhnik-Zamolodchikov functor of $H_{a(z),n}$ by $KZ_{a(z)}$ and the standard module corresponding to $\lambda$ by $\Delta_{a(z)}(\lambda)$. Then we have an isomorphism of $\mathcal{H}_{a(u),n}$-modules
\[ T_R(\lambda) \otimes_R \mathbb{C}_a \cong KZ_{a(z)}(\Delta_{a(z)}(\lambda)). \]

Let $K$ be the fraction field of $R$. By [10, Theorem 2.19] the category $\mathcal{O}_{K,z,n}$ is split semisimple. In particular, the standard modules are simple. We have
\[ \{ T_K(\lambda), \lambda \in \mathcal{P}_{n,l} \} = \text{Irr}(\mathcal{H}_{K,u,n}). \]

The Hecke algebra $\mathcal{H}_{K,u,n}$ is also split semisimple and we have
\[ \{ S_{K,\lambda}, \lambda \in \mathcal{P}_{n,l} \} = \text{Irr}(\mathcal{H}_{K,u,n}), \]
see for example [2, Corollary 13.9]. Thus there is a bijection $\varphi : \mathcal{P}_{n,l} \to \mathcal{P}_{n,l}$ such that $T_K(\lambda)$ is isomorphic to $S_{K,\varphi(\lambda)}$ for all $\lambda$. We claim that $\varphi$ is identity. To see this, consider the algebra homomorphism $\pi_0 : R \to \mathbb{C}$ given by $\varpi \mapsto 0$. Then $\mathcal{H}_{a(0),n}$ is canonically isomorphic to the group algebra $\mathbb{C}B_0(l)$, thus it is semi-simple. Let $\overline{K}$ be the algebraic closure of $K$. Let $\overline{R}$ be the integral closure of $R$ in $\overline{K}$ and fix an extension $\overline{a}_0$ of $a_0$ to $\overline{R}$. By it’s deformation theorem (see for example [9, Section 68A]), there is a bijection
\[ \psi : \text{Irr}(\mathcal{H}_{K,u,n}) \cong \text{Irr}(\mathcal{H}_{a(0),n}) \]
such that
\[ \psi(T_{\overline{K}}(\lambda)) = T_{\overline{K}}(\lambda) \otimes_{\overline{R}} \mathbb{C}_{\overline{a}_0}, \quad \psi(S_{\overline{R},\lambda}) = S_{\overline{R},\lambda} \otimes_{\overline{R}} \mathbb{C}_{\overline{a}_0}. \]

By the definition of Specht modules we have $S_{\overline{R},\lambda} \otimes_{\overline{R}} \mathbb{C}_{\overline{a}_0} \cong \lambda$ as $\mathbb{C}B_0(l)$-modules. On the other hand, since $a_0(0) = 0$, by (3.6) we have the following isomorphisms
\[ T_{\overline{K}}(\lambda) \otimes_{\overline{R}} \mathbb{C}_{\overline{a}_0} \cong T_R(\lambda) \otimes_R \mathbb{C}_{a_0} \cong KZ_0(\Delta_0(\lambda)) = \lambda. \]

So $\psi(T_{\overline{K}}(\lambda)) = \psi(S_{\overline{R},\lambda})$. Hence we have $T_{\overline{R}}(\lambda) \cong S_{\overline{R},\lambda}$. Since $T_{\overline{K}}(\lambda) = T_K(\lambda) \otimes_K \overline{K}$ is isomorphic to $S_{\overline{R},\varphi(\lambda)} \otimes_K \overline{R}$, we deduce that $\varphi(\lambda) = \lambda$. The claim is proved.

Finally, let $m$ be the maximal ideal of $R$ consisting of the functions vanishing at $\varpi = -1/\epsilon l$. Let $\overline{R}$ be the completion of $R$ at $m$. It is a discrete valuation ring with residue field $\mathbb{C}$. Let $a_1 : \overline{R} \to \overline{R}/m\overline{R} = \mathbb{C}$ be the quotient map. We have $a_1(z) = h$ and $a_1(u) = q$. Let $\overline{K}$ be the fraction field of $\overline{R}$. Recall that the decomposition map is given by
\[ d : K(\mathcal{H}_{K,u,n}) \to K(\mathcal{H}_{q,n}), \quad [M] \mapsto [L \otimes_\overline{R} \mathbb{C}_{a_1}]. \]
Here \( L \) is any free \( \widehat{K} \)-submodule of \( M \) such that \( L \otimes_{\widehat{R}} \widehat{K} = M \). The choice of \( L \) does not affect the class \([L \otimes_{\widehat{R}} \widehat{C}_{a_1}] \) in \( K(\mathcal{H}_{q,n}) \). See [2, Section 13.3] for details on this map. Now, observe that we have

\[
d([S_{\widehat{R}}]) = [S_{\widehat{R}} \otimes_{\widehat{R}} C_{a_1}] = [S_{\Delta}],
\]

\[
d([T_{\widehat{R}}]) = [T_{\widehat{R}} \otimes_{\widehat{R}} C_{a_1}] = [KZ(\Delta(\lambda))].
\]

Since \( \widehat{K} \) is an extension of \( K \), by the last paragraph we have \([S_{\widehat{R}}] = [T_{\widehat{R}}(\lambda)] \). We deduce that \([KZ(\Delta(\lambda))] = [S_{\Delta}] \). \( \square \)

4. \( i \)-restriction and \( i \)-induction

We define in this section the \( i \)-restriction and \( i \)-induction functors for the cyclotomic rational DAHA’s. This is done in parallel with the Hecke algebra case.

4.1. – Let us recall the definition of the \( i \)-restriction and \( i \)-induction functors for \( \mathcal{H}_{q,n} \). First define the Jucy-Murphy elements \( J_0, \ldots, J_{n-1} \) in \( \mathcal{H}_{q,n} \) by

\[
J_0 = T_0, \quad J_i = q^{-1}T_iJ_{i-1}T_i \quad \text{for } 1 \leq i \leq n - 1.
\]

Write \( Z(\mathcal{H}_{q,n}) \) for the center of \( \mathcal{H}_{q,n} \). For any symmetric polynomial \( \sigma \) of \( n \) variables the element \( \sigma(J_0, \ldots, J_{n-1}) \) belongs to \( Z(\mathcal{H}_{q,n}) \) (cf. [2, Section 13.1]). In particular, if \( z \) is a formal variable the polynomial \( C_n(z) = \prod_{i=0}^{n-1}(z - J_i) \) in \( \mathcal{H}_{q,n} [z] \) has coefficients in \( Z(\mathcal{H}_{q,n}) \).

Now, for any \( a(z) \in \mathbb{C}(z) \) let \( P_{n,a(z)} \) be the exact endo-functor of the category \( \mathcal{H}_{q,n} \)-mod that maps an object \( M \) to the generalized eigenspace of \( C_n(z) \) in \( M \) with the eigenvalue \( a(z) \).

For any \( i \in \mathbb{Z}/e\mathbb{Z} \) the \( i \)-restriction functor and \( i \)-induction functor

\[
E_i(n)^{\infty} : \mathcal{H}_{q,n} \text{-mod} \to \mathcal{H}_{q,n-1} \text{-mod}, \quad F_i(n)^{\infty} : \mathcal{H}_{q,n-1} \text{-mod} \to \mathcal{H}_{q,n} \text{-mod}
\]

are defined as follows (cf. [2, Definition 13.33]):

\[
E_i(n)^{\infty} = \bigoplus_{a(z) \in \mathbb{C}(z)} P_{n-1,a(z)/(z-q^i)} \circ E(n)^{\infty} \circ P_{n,a(z)},
\]

\[
F_i(n)^{\infty} = \bigoplus_{a(z) \in \mathbb{C}(z)} P_{n,a(z)/(z-q^i)} \circ F(n)^{\infty} \circ P_{n-1,a(z)}.
\]

We will write

\[
E_i^{\infty} = \bigoplus_{n \geq 1} E_i(n)^{\infty}, \quad F_i^{\infty} = \bigoplus_{n \geq 1} F_i(n)^{\infty}.
\]

They are endo-functors of \( \mathcal{H}_{q,n} \)-mod. For each \( \lambda \in \mathcal{P}_{n,l} \) set

\[
a_\lambda(z) = \prod_{v \in \Upsilon_\lambda} (z - q^{\text{res}(v)}).
\]

We recall some properties of these functors in the following proposition.
Proposition 4.1. – (1) The functors \( E_i(n) \), \( F_i(n) \) are exact. The functor \( E_i(n) \) is biadjoint to \( F_i(n) \).

(2) For any \( \lambda \in \mathcal{P}_{n,i} \), the element \( C_n(z) \) has a unique eigenvalue on the Specht module \( S_\chi \).

(3) We have
\[
E_i(n)([S_\lambda]) = \sum_{\text{res}(\lambda/\mu) = i} [S_\mu], \quad F_i(n)([S_\lambda]) = \sum_{\text{res}(\mu/\lambda) = i} [S_\mu].
\]

(4) We have
\[
E(n)^\neq = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} E_i(n)^\neq, \quad F(n)^\neq = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} F_i(n)^\neq.
\]

Proof. – Part (1) is obvious. See [2, Theorem 13.21(2)] for (2) and [2, Lemma 13.37] for (3). Part (4) follows from (3) and [2, Lemma 13.32].

4.2. – By Lemma 1.3(1) we have an algebra isomorphism
\[
\gamma : Z(\bar{\theta}_{h,n}) \cong Z(\mathcal{H}_{q,n}).
\]
So there are unique elements \( K_1, \ldots, K_n \in Z(\bar{\theta}_{h,n}) \) such that the polynomial
\[
D_n(z) = z^n + K_1 z^{n-1} + \cdots + K_n
\]
maps to \( C_n(z) \) by \( \gamma \). Since the elements \( K_1, \ldots, K_n \) act on simple modules by scalars and the category \( \bar{\theta}_{h,n} \) is artinian, every module \( M \in \bar{\theta}_{h,n} \) is a direct sum of generalized eigenspaces of \( D_n(z) \). For \( a(z) \in \mathbb{C}(z) \), let \( Q_{n,a(z)}^{\text{geo}} \) be the exact endo-functor of \( \bar{\theta}_{h,n} \) which maps an object \( M \) to the generalized eigenspace of \( D_n(z) \) in \( M \) with the eigenvalue \( a(z) \).

Definition 4.2. – The \( i \)-restriction functor and the \( i \)-induction functor
\[
E_i(n) : \bar{\theta}_{h,n} \to \bar{\theta}_{h,n-1}, \quad F_i(n) : \bar{\theta}_{h,n-1} \to \bar{\theta}_{h,n}
\]
are given by
\[
E_i(n) = \bigoplus_{a(z) \in \mathbb{C}(z)} Q_{n-1,a(z)/(z-q^i)} \circ E(n) \circ Q_{n,a(z)},
\]
\[
F_i(n) = \bigoplus_{a(z) \in \mathbb{C}(z)} Q_{n,a(z)-(z-q^i)} \circ F(n) \circ Q_{n-1,a(z)}.
\]

We will write
\[
E_i = \bigoplus_{n \geq 1} E_i(n), \quad F_i = \bigoplus_{n \geq 1} F_i(n).
\]

We have the following proposition.

Proposition 4.3. – For any \( i \in \mathbb{Z}/e\mathbb{Z} \), there are isomorphisms of functors
\[
KZ \circ E_i(n) \cong E_i(n)^\neq \circ KZ, \quad KZ \circ F_i(n) \cong F_i(n)^\neq \circ KZ.
\]

Proof. – Since \( \gamma(D_n(z)) = C_n(z) \), by Lemma 1.3(2) for any \( a(z) \in \mathbb{C}(z) \) we have
\[
KZ \circ Q_{n,a(z)}^{\text{geo}} \cong P_{n,a(z)} \circ KZ.
\]
So the proposition follows from Theorem 2.1 and Corollary 2.3.
The next proposition is the DAHA version of Proposition 4.1.

**Proposition 4.4.** — (1) The functors $E_i(n)$, $F_i(n)$ are exact. The functor $E_i(n)$ is biadjoint to $F_i(n)$.

(2) For any $\lambda \in \mathcal{P}_{n,i}$ the unique eigenvalue of $D_n(z)$ on the standard module $\Delta(\lambda)$ is $\alpha_\lambda(z)$.

(3) We have the following equalities

$$E_i(n)([\Delta(\lambda)]) = \sum_{\text{res}(\lambda/\mu) = i} [\Delta(\mu)], \quad F_i(n)([\Delta(\lambda)]) = \sum_{\text{res}(\mu/\lambda) = i} [\Delta(\mu)].$$

(4) We have

$$E(n) = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} E_i(n), \quad F(n) = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} F_i(n).$$

**Proof.** — (1) This is by construction and by Proposition 2.9.

(2) Since a standard module is indecomposable, the element $D_n(z)$ has a unique eigenvalue on $\Delta(\lambda)$. By Lemma 3.1 this eigenvalue is the same as the eigenvalue of $C_n(z)$ on $S_\lambda$.

(3) Let us prove the equality for $E_i(n)$. The Pieri rule for the group $B_n(1)$ together with Proposition 1.6(2) yields

$$E(n)([\Delta(\lambda)]) = \sum_{|\lambda/\mu| = 1} [\Delta(\mu)], \quad F(n)([\Delta(\lambda)]) = \sum_{|\mu/\lambda| = 1} [\Delta(\mu)].$$

So we have

$$E_i(n)([\Delta(\lambda)]) = \bigoplus_{a \lambda(z) \in \mathbb{C}[z]} Q_{n-1,a \lambda(z)/(z-q^e)}(E(n)(Q_{n,a \lambda(z)}([\Delta(\lambda)])))$$

$$= Q_{n-1,a \lambda(z)/(z-q^e)}(E(n)(Q_{n,a \lambda(z)}([\Delta(\lambda)])))$$

$$= Q_{n-1,a \lambda(z)/(z-q^e)}(E(n)([\Delta(\lambda)]))$$

$$= Q_{n-1,a \lambda(z)/(z-q^e)}(\sum_{|\lambda/\mu| = 1} [\Delta(\mu)])$$

$$= \sum_{\text{res}(\lambda/\mu) = i} [\Delta(\mu)].$$

The last equality follows from the fact that for any $l$-partition $\mu$ such that $|\lambda/\mu| = 1$ we have $a_\lambda(z) = a_\mu(z)(z - q^{\text{res}(\lambda/\mu)})$. The proof for $F_i(n)$ is similar.

(4) It follows from part (3) and (4.6). □

**Corollary 4.5.** — Under the isomorphism $\theta$ in (3.5) the operators $E_i$ and $F_i$ on $K(\vartheta_{h,N})$ go respectively to the operators $e_i$ and $f_i$ on $\mathcal{F}_\lambda$. When $i$ runs over $\mathbb{Z}/e\mathbb{Z}$ they yield an action of $\mathfrak{sl}_e$ on $K(\vartheta_{h,N})$ such that $\theta$ is an isomorphism of $\mathfrak{sl}_e$-modules.

**Proof.** — This is clear from Proposition 4.4(3) and from (3.3). □

5. $\mathfrak{sl}_e$-categorification

In this section, we construct an $\mathfrak{sl}_e$-categorification on the category $\mathfrak{sl}_e$ (Theorem 5.1).
5.1. – Recall that we put \( q = \exp(\frac{2\pi \sqrt{-1}}{\tau}) \) and \( P \) denotes the weight lattice of \( \widehat{\mathfrak{sl}}_e \). Let \( \mathcal{C} \) be a \( \mathbb{C} \)-linear artinian abelian category. For any functor \( F: \mathcal{C} \rightarrow \mathcal{C} \) and any \( X \in \End(F) \), the generalized eigenspace of \( X \) acting on \( F \) with eigenvalue \( a \in \mathbb{C} \) will be called the \( a \)-eigenspace of \( X \) in \( F \). By [15, Definition 5.29] an \( \mathfrak{sl}_e \)-categorification on \( \mathcal{C} \) is the data of

(a) an adjoint pair \((U, V)\) of exact functors \( \mathcal{C} \rightarrow \mathcal{C} \),
(b) \( X \in \End(U) \) and \( T \in \End(U^2) \),
(c) a decomposition \( \mathcal{C} = \bigoplus_{\tau \in \mathbb{P}} \mathcal{C}_\tau \),

such that, set \( U_i \) (resp. \( V_i \)) to be the \( q^i \)-eigenspace of \( X \) in \( U \) (resp. in \( V \))\(^{(1)}\) for \( i \in \mathbb{Z}/e\mathbb{Z} \), we have

1. \( U = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} U_i \),
2. the endomorphisms \( X \) and \( T \) satisfy

\[
(1_U T) \circ (T 1_U) = (1_U T) \circ (1_U T) = T + q 1_U T = 0,
\]

\[
T \circ (1_U X) \circ T = qX 1_U,
\]

3. the action of \( e_i = U_i, f_i = V_i \) on \( K(\mathcal{C}) \) with \( i \) running over \( \mathbb{Z}/e\mathbb{Z} \) gives an integrable representation of \( \widehat{\mathfrak{sl}}_e \).
4. \( U_i(\mathcal{C}_\tau) \subset \mathcal{C}_{\tau + \alpha_i} \) and \( V_i(\mathcal{C}_\tau) \subset \mathcal{C}_{\tau - \alpha_i} \),
5. \( V \) is isomorphic to a left adjoint of \( U \).

5.2. – We construct an \( \mathfrak{sl}_e \)-categorification on \( \mathcal{O}_{h,N} \) in the following way. The adjoint pair will be given by \((E, F)\). To construct the part (b) of the data we need to go back to Hecke algebras. Following [7, Section 7.2.2] let \( X^\times \) be the endomorphism of \( E^\times \) given on \( E(n)^\times \) as the multiplication by the Jucy-Murphy element \( J_{n-1} \). Let \( T^\times \) be the endomorphism of \( (E^\times)^2 \) given on \( E(n)^\times \circ E(n-1)^\times \) as the multiplication by the element \( T_{n-1} \in \mathcal{H}_{q,n} \). The endomorphisms \( X^\times \) and \( T^\times \) satisfy the relations (5.1). Moreover the \( q^i \)-eigenspace of \( X^\times \) in \( E^\times \) and \( F^\times \) gives respectively the \( i \)-restriction functor \( E_i^\times \) and the \( i \)-induction functor \( F_i^\times \) for any \( i \in \mathbb{Z}/e\mathbb{Z} \).

By Theorem 2.1 we have an isomorphism \( KZ \circ E \cong E^\times \circ KZ \). This yields an isomorphism

\[
\End(KZ \circ E) \cong \End(E^\times \circ KZ).
\]

By Proposition 1.6(1) the functor \( E \) maps projective objects to projective ones, so Lemma 2.4 applied to \( \theta_1 = \theta_2 = \theta_{h,N} \) and \( K = L = E \) yields an isomorphism

\[
\End(E) \cong \End(KZ \circ E).
\]

Composing it with the isomorphism above gives a ring isomorphism

\[
\sigma_E: \End(E) \xrightarrow{\sim} \End(E^\times \circ KZ).
\]

Replacing \( E \) by \( E^2 \) we get another isomorphism

\[
\sigma_{E^2}: \End(E^2) \xrightarrow{\sim} \End((E^\times)^2 \circ KZ).
\]

\(^{(1)}\) Here \( X \) acts on \( V \) via the isomorphism \( \End(U) \cong \End(V)^{op} \) given by adjunction, see [7, Section 4.1.2] for the precise definition.
The data of $X \in \text{End}(E)$ and $T \in \text{End}(E^2)$ in our $\tilde{\mathfrak{s}_l}$-categorification on $\mathfrak{h}_{h, N}$ will be provided by

$$X = \sigma^{-1}_E(X^{\tau}_1), \quad T = \sigma^{-1}_E(T^{\tau}_1).$$

Finally, the part (c) of the data will be given by the block decomposition of the category $\mathfrak{h}_{h, N}$. Recall from Proposition 4.1(4) and Proposition 4.4(4) that we have

$$H \otimes X \text{ with } F,$$

restricts to an isomorphism and

$$\text{F}(\text{mod}) \text{ where } S \text{ modules }$$

is an

$$(b) \text{ the adjoint pair } (E, F),$$

(c) the decomposition $\mathfrak{h}_{h, N} = \bigoplus_{\tau \in P}(\mathfrak{h}_{h, N})_{\tau},$

where $(\mathfrak{h}_{h, N})_{\tau}$ is the block corresponding to $(\mathfrak{h}_{h, N})_{\tau}$ via KZ.

5.3. – Now we prove the following theorem.

**Theorem 5.1.** – The data of

(a) the adjoint pair $(E, F)$,

(b) the endomorphisms $X \in \text{End}(E), T \in \text{End}(E^2)$,

(c) the decomposition $\mathfrak{h}_{h, N} = \bigoplus_{\tau \in P}(\mathfrak{h}_{h, N})_{\tau};$

is an $\tilde{\mathfrak{s}_l}$-categorification on $\mathfrak{h}_{h, N}$.

**Proof.** – First, we prove that for any $i \in \mathbb{Z}/e\mathbb{Z}$ the $q^i$-generalized eigenspaces of $X$ in $E$ and $F$ are respectively the $i$-restriction functor $E_i$ and the $i$-induction functor $F_i$ as defined in (4.4). Recall from Proposition 4.1(4) and Proposition 4.4(4) that we have

$$E = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} E_i \quad \text{and} \quad E^{\tau} = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} E^{\tau}_i.$$

By the proof of Proposition 4.3 we see that any isomorphism

$$\text{KZ} \circ E \cong E^{\tau} \circ \text{KZ}$$

restricts to an isomorphism $\text{KZ} \circ E_i \cong E_i^{\tau} \circ \text{KZ}$ for each $i \in \mathbb{Z}/e\mathbb{Z}$. So the isomorphism $\sigma_E$ in (5.2) maps $\text{Hom}(E_i, E_j)$ to $\text{Hom}(E_i^{\tau} \circ \text{KZ}, E_j^{\tau} \circ \text{KZ})$. Write

$$X = \sum_{i,j \in \mathbb{Z}/e\mathbb{Z}} X_{ij}, \quad X^{\tau}_1 = \sum_{i,j \in \mathbb{Z}/e\mathbb{Z}} (X^{\tau}_1)_{ij}$$

with $X_{ij} \in \text{Hom}(E_i, E_j)$ and $(X^{\tau}_1)_{ij} \in \text{Hom}(E_i^{\tau} \circ \text{KZ}, E_j^{\tau} \circ \text{KZ})$. We have

$$\sigma_E(X_{ij}) = (X^{\tau}_1)_{ij}.$$

Since $E_i^{\tau}$ is the $q^i$-eigenspace of $X^{\tau}$ in $E^{\tau}$, we have $(X^{\tau}_1)_{ij} = 0$ for $i \neq j$ and $(X^{\tau}_1)_{ii} - q^i$ is nilpotent for any $i \in \mathbb{Z}/e\mathbb{Z}$. Since $\sigma_E$ is an isomorphism of rings, this implies that $X_{ij} = 0$ and $X_{ii} - q^i$ is nilpotent in $\text{End}(E)$. So $E_i$ is the $q^i$-eigenspace of $X$ in $E$. The fact that $F_i$ is the $q^i$-eigenspace of $X$ in $F$ follows from adjunction.
Now, let us check the conditions (1)–(5):

1. It is given by Proposition 4.4(4).

2. Since $X^\infty$ and $T^\infty$ satisfy relations in (5.1), the endomorphisms $X$ and $T$ also satisfy them. Because these relations are preserved by ring homomorphisms.

3. It follows from Corollary 4.5.

4. By the definition of $(\theta_{h, N})_\tau$ and Lemma 3.1, the standard modules in $(\theta_{h, N})_\tau$ are all the $\Delta(\lambda)$ such that $\text{wt}(\lambda) = \tau$. By (3.4) if $\mu$ is an $l$-partition such that $\text{res}(\lambda/\mu) = i$ then $\text{wt}(\mu) = \text{wt}(\lambda) + \alpha_i$. Now, the result follows from (4.5).

5. This is Proposition 2.9.

6. Crystals

Using the $\hat{\mathfrak{sl}}_n$-categorification in Theorem 5.1 we construct a crystal on the classes of simple objects in $\theta_{h, N}$ and prove that it coincides with the crystal of the Fock space $\mathcal{F}_n$ (Theorem 6.3).

6.1. – A crystal (or more precisely, an $\hat{\mathfrak{sl}}_n$-crystal) is a set $B$ together with maps

\[ \text{wt} : B \to P, \quad \tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\}, \quad \epsilon_i, \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\}, \]

such that

- we have $\varphi_i(b) = \epsilon_i(b) + \langle \alpha_i^\vee, \text{wt}(b) \rangle$,
- if $\tilde{e}_ib \in B$, then $\text{wt}(\tilde{e}_ib) = \text{wt}(b) + \alpha_i$, \quad $\epsilon_i(\tilde{e}_ib) = \epsilon_i(b) - 1$, \quad $\varphi_i(\tilde{e}_ib) = \varphi_i(b) + 1$,
- if $\tilde{f}_ib \in B$, then $\text{wt}(\tilde{f}_ib) = \text{wt}(b) - \alpha_i$, \quad $\epsilon_i(\tilde{f}_ib) = \epsilon_i(b) + 1$, \quad $\varphi_i(\tilde{f}_ib) = \varphi_i(b) - 1$,
- let $b, b' \in B$, then $\tilde{e}_ib = b'$ if and only if $\tilde{e}_ib' = b$,
- if $\varphi_i(b) = -\infty$, then $\tilde{e}_ib = 0$ and $\tilde{f}_ib = 0$.

Let $b$ be the Lie subalgebra of $\hat{\mathfrak{sl}}_n$ generated by the elements $e_i$, $i \in \mathbb{Z}/e\mathbb{Z}$ and $t$. We say that an $\hat{\mathfrak{sl}}_n$-module $V$ is $b$-locally finite if

- we have $V = \bigoplus_{\mu \in P} V_\mu$, where $V_\mu = \{v \in V : hv = \mu(h)v, \forall h \in t\}$,
- for any $v \in V$, the $b$-submodule of $V$ generated by $v$ is finite dimensional.

Let $V$ be a $b$-locally finite $\hat{\mathfrak{sl}}_n$-module. For any nonzero vector $v \in V$ and any $i \in \mathbb{Z}/e\mathbb{Z}$ we set

\[ l_i(v) = \text{max}\{l \in \mathbb{N} : e_i^l(v) \neq 0\}. \]

Write $l_i(0) = -\infty$. For $l \geq 0$ let

\[ V_i^{<l} = \{v \in V : l_i(v) < l\}. \]

A weight basis of $V$ is a basis $B$ of $V$ such that each element of $B$ is a weight vector. Following A. Berenstein and D. Kazhdan (cf. [3, Definition 5.30]), a perfect basis of $V$ is a weight basis $B$ together with maps $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$ for $i \in \mathbb{Z}/e\mathbb{Z}$ such that

- for $b, b' \in B$ we have $\tilde{f}_ib = b'$ if and only if $\tilde{e}_ib' = b$,
- we have $\tilde{e}_i(b) \neq 0$ if and only if $e_i(b) \neq 0$,
- if $e_i(b) \neq 0$ then we have

\[ e_i(b) \in \mathbb{C}^*\tilde{e}_i(b) + V_i^{<l_i(b)-1}. \]
We denote it by \((B, \bar{e}_i, \bar{f}_i)\). For such a basis let \(wt(b)\) be the weight of \(b\), let \(e_i(b) = l_i(b)\) and let 
\[
\varphi_i(b) = e_i(b) + \langle \alpha_i^\vee, \text{wt}(b) \rangle
\]
for all \(b \in B\). The data
\[
(B, \text{wt}, \bar{e}_i, \bar{f}_i, e_i, \varphi_i)
\]
is a crystal. We will always attach this crystal structure to \((B, \bar{e}_i, \bar{f}_i)\). We call \(b \in B\) a primitive element if \(e_i(b) = 0\) for all \(i \in \mathbb{Z}/e\mathbb{Z}\). Let \(B^+\) be the set of primitive elements in \(B\). Let \(V^+\) be the vector space spanned by all the primitive vectors in \(V\). The following lemma is \([3, \text{Claim 5.32}]\).

**Lemma 6.1.** – For any perfect basis \((B, \bar{e}_i, \bar{f}_i)\) the set \(B^+\) is a basis of \(V^+\).

**Proof.** – By definition we have \(B^+ \subset V^+\). Given a vector \(v \in V^+\), there exist \(\zeta_1, \ldots, \zeta_r \in C^*\) and distinct elements \(b_1, \ldots, b_r \in B\) such that \(v = \sum_{j=1}^{r} \zeta_j b_j\). For any \(i \in \mathbb{Z}/e\mathbb{Z}\) let \(l_i = \max\{l_i(b_j) : 1 \leq j \leq r\}\) and \(J = \{j : l_i(b_j) = l_i, 1 \leq j \leq r\}\). Then by the third property of perfect basis there exist \(\eta_j \in C^*\) for \(j \in J\) and a vector \(w \in V^{<l_i-1}\) such that \(0 = e_i(v) = \sum_{j \in J} \zeta_j \eta_j \bar{e}_i(b_j) + w\). For distinct \(j, j' \in J\), we have \(b_j \neq b_j'\), so \(\bar{e}_i(b_j)\) and \(\bar{e}_i(b_j')\) are different unless they are zero. Moreover, since \(l_i(\bar{e}_i(b_j)) = l_i - 1\), the equality yields that \(\bar{e}_i(b_j) = 0\) for all \(j \in J\). So \(l_i = 0\). Hence \(b_j \in B^+\) for \(j = 1, \ldots, r\).

**6.2.** – Given an \(\mathfrak{s}_L\)-categorification on a \(\mathbb{C}\)-linear artinian abelian category \(\mathcal{C}\) with the adjoint pair of endo-functors \((U, V)\), \(X \in \text{End}(U)\) and \(T \in \text{End}(U^2)\), assume that the \(\mathfrak{s}_L\)-module \(K(\mathcal{C})\) is \(b\)-locally finite, then one can construct a perfect basis of \(K(\mathcal{C})\) as follows. For \(i \in \mathbb{Z}/e\mathbb{Z}\) let \(U_i, V_i\) be the \(q^i\)-eigenspaces of \(X\) in \(U\) and \(V\). By definition, the action of \(X\) restricts to each \(U_i\). One can prove that \(T\) also restricts to endomorphism of \((U_i)^2\), see for example the beginning of Section 7 in [7]. It follows that the data \((U_i, V_i, X, T)\) gives an \(\mathfrak{s}_L\)-categorification on \(\mathcal{C}\) in the sense of [7, Section 5.21]. By [7, Proposition 5.20] this implies that for any simple object \(L\) in \(\mathcal{C}\), the object \(\text{head}(U_i(L))\) (resp. \(\text{soc}(V_i(L))\)) is simple unless it is zero.

Let \(B_{\mathcal{C}}\) be the set of isomorphism classes of simple objects in \(\mathcal{C}\). As part of the data of the \(\mathfrak{s}_L\)-categorification, we have a decomposition \(\mathcal{C} = \bigoplus_{r \in \mathbb{P}} \mathcal{C}_r\). For a simple module \(L \in \mathcal{C}_r\), the weight of \([L]\) in \(K(\mathcal{C})\) is \(r\). Hence \(B_{\mathcal{C}}\) is a weight basis of \(K(\mathcal{C})\). Now for \(i \in \mathbb{Z}/e\mathbb{Z}\) define the maps
\[
\bar{e}_i : B_{\mathcal{C}} \rightarrow B_{\mathcal{C}} \cup \{0\}, \quad [L] \mapsto [\text{head}(U_i(L))],
\]
\[
\bar{f}_i : B_{\mathcal{C}} \rightarrow B_{\mathcal{C}} \cup \{0\}, \quad [L] \mapsto [\text{soc}(V_i(L))].
\]

**Proposition 6.2.** – The data \((B_{\mathcal{C}}, \bar{e}_i, \bar{f}_i)\) is a perfect basis of \(K(\mathcal{C})\).

**Proof.** – Fix \(i \in \mathbb{Z}/e\mathbb{Z}\). Let us check the conditions in order. First, for two simple modules \(L, L' \in \mathcal{C}\), we have \(\bar{e}_i([L]) = [L']\), if and only if if \(0 \neq \text{Hom}(U_i(L), L') = \text{Hom}(L, V_i(L'))\), and only if \(\bar{f}_i([L']) = [L]\). The second condition follows from the fact that any non trivial module has a non trivial head. The third condition is [7, Proposition 5.20(d)].
6.3. – Let $B_{\mathcal{F}}$ be the set of $l$-partitions. In [11] this set is given a crystal structure. We will call it the crystal of the Fock space $\mathcal{F}$.

**Theorem 6.3.** – (1) The set

$$B_{\mathcal{O}_h,n} = \{ [L(\lambda)] \in K(\mathcal{O}_h,n) : \lambda \in \mathcal{P}_{n,l}, n \in \mathbb{N} \}$$

and the maps

$$\tilde{e}_i : B_{\mathcal{O}_h,n} \to B_{\mathcal{O}_h,n} \cup \{ 0 \}, \quad [L] \mapsto [\text{head}(E_i L)],$$

$$\tilde{f}_i : B_{\mathcal{O}_h,n} \to B_{\mathcal{O}_h,n} \cup \{ 0 \}, \quad [L] \mapsto [\text{soc}(F_i L)].$$

define a crystal structure on $B_{\mathcal{O}_h,n}$.

(2) The crystal $B_{\mathcal{O}_h,n}$ given by (1) is isomorphic to the crystal $B_{\mathcal{F}}$.

**Proof.** – (1) The Fock space $\mathcal{F}$ is a locally finite $b$-module. So applying Proposition 6.2 to the $\tilde{\mathfrak{sl}}_e$-categorification in Theorem 5.1 yields that $(B_{\mathcal{O}_h,n}, \tilde{e}_i, \tilde{f}_i)$ is a perfect basis. Therefore it defines a crystal structure on $B_{\mathcal{O}_h,n}$ by (6.2).

(2) It is known that $B_{\mathcal{F}}$ is a perfect basis of $\mathcal{F}$. Identify the $\tilde{\mathfrak{sl}}_e$-modules $\mathcal{F}$ and $K(\mathcal{O}_h,n)$. By Lemma 6.1 the set $B_{\mathcal{F}}$ and $B_{\mathcal{O}_h,n}$ are two weight bases of $\mathcal{F}$. So there is a bijection $\psi : B_{\mathcal{F}} \to B_{\mathcal{O}_h,n}$ such that $\text{wt}(b) = \text{wt}(\psi(b))$. Since $\mathcal{F}$ is a direct sum of highest weight simple $\tilde{\mathfrak{sl}}_e$-modules, this bijection extends to an automorphism $\psi$ of the $\tilde{\mathfrak{sl}}_e$-module $\mathcal{F}$. By [3, Main Theorem 5.37] any automorphism of $\mathcal{F}$ which maps $B_{\mathcal{F}}$ to $B_{\mathcal{O}_h,n}$ induces an isomorphism of crystals $B_{\mathcal{F}} \cong B_{\mathcal{O}_h,n}$. \hfill $\Box$

**Remark 6.4.** – One can prove that if $n < e$ then a simple module $L \in \mathcal{O}_h,n$ has finite dimension over $\mathbb{C}$ if and only if the class $[L]$ is a primitive element in $B_{\mathcal{O}_h,n}$. In the case $n = 1$, we have $B_{\mathcal{F}}(l) = \mu_1$, the cyclic group, and the primitive elements in the crystal $B_{\mathcal{F}}$ have explicit combinatorial descriptions. This yields another proof of the classification of finite dimensional simple modules of $H_1(\mu_1)$, which was first given by W. Crawley-Boevey and M. P. Holland. See type $A$ case of [8, Theorem 7.4].

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