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ARITHMETIC FUJITA APPROXIMATION

BY HUAYI CHEN

ABSTRACT. – We prove an arithmetic analogue of Fujita's approximation theorem in Arakelov geometry, conjectured by Moriwaki, by using measures associated to $\mathbb{R}$-filtrations.

RÉSUMÉ. – On démontre un analogue arithmétique du théorème d'approximation de Fujita en géométrie d'Arakelov — conjecturé par Moriwaki — par les mesures associées aux $\mathbb{R}$-filtrations.

1. Introduction

Fujita approximation is an approximative version of Zariski decomposition of pseudo-effective divisors [20]. Let X be a projective variety defined over a field K and L be a *big* line bundle on X, i.e., the *volume* of L, defined as

$$\mathrm{vol}(L) := \limsup_{n\to\infty} \frac{\mathrm{rk}_K H^0(X, L^{\otimes n})}{n^{\dim X}/(\dim X)!},$$

is strictly positive. Fujita's approximation theorem asserts that, for any $\varepsilon > 0$, there exist a projective birational morphism $\nu : X' \to X$, an integer $p > 0$, together with a decomposition $\nu^*(L^{\otimes p}) \cong A \otimes E$, where A is an ample line bundle, E is effective, such that $p^{-\dim X}\mathrm{vol}(A) \geqslant \mathrm{vol}(L) - \varepsilon$. This theorem had been proved by Fujita himself [7] in characteristic 0 case (see also [4]) before its generalization to any characteristic case by Takagi [17]. It is the source of many important results concerning big divisors and the volume function in algebraic geometry, such as the volume function as a limit, its log-concavity and differentiability, etc. We refer readers to [10, 11.4] for a survey, see also [1, 5, 6, 11].

The arithmetic analogue of the volume function and the arithmetic bigness in Arakelov geometry have been introduced by Moriwaki [12, 13]. Let K be a number field and $\mathcal{O}_K$ be its integer ring. Let $\mathscr{X}$ be a projective arithmetic variety of total dimension d over $\mathrm{Spec}\,\mathcal{O}_K$. For any Hermitian line bundle $\overline{\mathscr{L}}$ on $\mathscr{X}$, the *arithmetic volume* of $\overline{\mathscr{L}}$ is defined as

$$\widehat{\mathrm{vol}}(\overline{\mathscr{L}}) := \limsup_{n\to\infty} \frac{\widehat{h}^0(\mathscr{X}, \overline{\mathscr{L}}^{\otimes n})}{n^d/d!}, \tag{1}$$

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where

$$\widehat{h}^0(\mathcal{X},\overline{\mathcal{L}}^{\otimes n}):=\log\big|\{s\in H^0(\mathcal{X},\mathcal{L}^{\otimes n})\,|\,\|s\|_{\sup}\leqslant 1\}\big|.$$

Similarly, $\overline{\mathcal{L}}$ is said to be (arithmetically) *big* if $\widehat{\mathrm{vol}}(\overline{\mathcal{L}})>0$. In [13, 14], Moriwaki has proved that the arithmetic volume function is continuous with respect to $\overline{\mathcal{L}}$, and admits a unique continuous extension to $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{R}}$. In [13], he asked the following question (Remark 5.9 *loc. cit.*): *does the Fujita approximation hold in the arithmetic case?* A consequence of this conjecture is that the right-hand side of (1) is actually a limit (see [13, Remark 4.1]), which is similar to a result of Rumely, Lau and Varley [16] on the existence of the sectional capacity of Hermitian line bundles.

Recall that in [3], the author has proved that, by slope method, one can associate naturally a sequence of Borel probability measures $(\eta_n)_{n\geqslant 1}$ on $\mathbb{R}$ to the Hermitian line bundle $\overline{\mathcal{L}}$ such that

$$\widehat{\deg}(\pi_*(\overline{\mathcal{L}}^{\otimes n}))=[K:\mathbb{Q}]\,\mathrm{rk}(\pi_*(\overline{\mathcal{L}}^{\otimes n}))\int_{\mathbb{R}}x\,\eta_n(\mathrm{d}x).$$

In this probabilistic framework, the existence of sectional capacity is interpreted as the convergence of the sequence of expectations $(\int_{\mathbb{R}}x\,\eta_n(\mathrm{d}x))_{n\geqslant 1}$. The author has actually proved the vague convergence of the sequence $(\eta_n)_{n\geqslant 1}$ to a Borel probability measure, under the ampleness hypothesis of $\mathcal{L}_K$.

In this article, we consider another sequence $(\nu_n)_{n\geqslant 1}$ of Borel probability measures defined by the successive minima of $\pi_*(\overline{\mathcal{L}}^{\otimes n})$ and establish its vague convergence under the bigness hypothesis of $\mathcal{L}_K$. By the arithmetic Riemann-Roch theorem of Gillet and Soulé [9], $\widehat{h}^0(\pi_*(\overline{\mathcal{L}})^{\otimes n})$ is compared to $[K:\mathbb{Q}]\,\mathrm{rk}(\pi_*(\overline{\mathcal{L}}^{\otimes n}))\int_0^\infty x\,\nu_n(\mathrm{d}x)$ and it follows that

$$\widehat{\mathrm{vol}}(\overline{\mathcal{L}})=\lim_{n\to\infty}\frac{\widehat{h}^0(\mathcal{X},\overline{\mathcal{L}}^{\otimes n})}{n^d/d!}=[K:\mathbb{Q}]d\mathrm{vol}(\mathcal{L}_K)\int_0^\infty x\,\nu(\mathrm{d}x),$$

where ν denotes the vague limit of ν_n.

By developing a variant of the convergence result, we prove the arithmetic Fujita approximation. One difficulty is that, if $\overline{\mathcal{A}}$ is an ample Hermitian line subbundle of $\overline{\mathcal{L}}$ which approximates well $\overline{\mathcal{L}}$, then in general the section algebra of $\mathcal{A}_K$ does not approximate that of $\mathcal{L}_K$ at all. In fact, it approximates only the graded linear series of $\mathcal{L}_K$ generated by small sections (see §4.3). To overcome this difficulty, we need a recent result of Lazarsfeld and Mustaţă [11] on a very general approximation theorem for graded linear series of a big line bundle on a projective variety. It permits to approximate the graded linear series of the generic fiber generated by small sections.

Shortly after the first version of this article had been written, X. Yuan told me that he was working on the same subject and had obtained (see [19]) the arithmetic Fujita approximation independently by using a different method inspired by [11].

The organization of this article is as follows. In the second section, we introduce the notion of approximable graded algebras and study their asymptotic properties. We then recall the notion of Borel measures associated to filtered vector spaces. At the end of the section, we establish a convergence result for filtered approximable algebras. In the third section, we recall the theorem of Lazarsfeld and Mustaţă on the approximability of certain graded linear series. We then describe some approximable graded linear series which come from the arithmetic of a Hermitian line bundle on an arithmetic projective variety. The main theorem of the article is established in the fourth section. We prove that the arithmetic volume of a big Hermitian line bundle can be approximated by the arithmetic volume of its graded linear series of finite type, which implies the Moriwaki's conjecture. We also prove that, if a graded linear series generated by small sections approximates well a big Hermitian line bundle $\overline{\mathscr{L}}$, then it also approximates well the asymptotic measure of $\overline{\mathscr{L}}$ truncated at 0.

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2. Approximable algebras and asymptotic measures

In [3], the author has used the measures associated to filtered vector spaces to study asymptotic invariants of Hermitian line bundles. Several convergence results have been established for graded algebras equipped with $\mathbb{R}$-filtrations under the finite generation condition on the underlying graded algebra [3, Theorem 3.4.3]. However, some graded algebras coming naturally from the arithmetic do not satisfy this condition. In this section, we generalize the convergence result to a so-called *approximable graded algebra* case.

2.1. Approximable graded algebras

In the study of projective varieties, graded algebras are natural objects which often appear as graded linear series of a line bundle. In general, such graded algebras are not always finitely generated. However, according to approximation theorems due to Fujita [7], Takagi [17], Lazarsfeld and Mustaţă [11] etc., they can often be approximated arbitrarily closely by their graded subalgebras of finite type. Inspired by [11], we formalize this observation as a notion. In this section, K denotes an arbitrary field. All algebras and all vector spaces are supposed to be over K.

Definition 2.1. – Let $B = \bigoplus_{n\geqslant 0} B_n$ be an integral graded K-algebra. We say that B is *approximable* if the following conditions are verified:

(a) all vector spaces B_n are finite dimensional and $B_n \neq 0$ for sufficiently large n;

(b) for any $\varepsilon \in (0, 1)$, there exists an integer $p_0 \geqslant 1$ such that, for any integer $p \geqslant p_0$, one has
$$\liminf_{n\to\infty} \frac{\mathrm{rk}(\mathrm{Im}(S^n B_p \to B_{np}))}{\mathrm{rk}(B_{np})} > 1 - \varepsilon,$$
where $S^n B_p \to B_{np}$ is the canonical homomorphism defined by the algebra structure on B.

REMARK 2.2. – The condition (a) serves to exclude the degenerate case so that the presentation becomes simpler. In fact, if an integral graded algebra B is not concentrated on B_0, then by choosing an integer $q \geqslant 1$ such that $B_q \neq 0$, we obtain a new graded algebra $\bigoplus_{n \geqslant 0} B_{nq}$ which verifies (a). This new algebra often contains the information about B in which we are interested.

EXAMPLE 2.3. – The following are some examples of approximable graded algebras.

1) If B is an integral graded algebra of finite type such that $B_n \neq 0$ for sufficiently large n, then it is clearly approximable.
2) Let X be a projective variety over $\operatorname{Spec} K$ and L be a big line bundle on X. Then by Fujita's approximation theorem, the total graded linear series $\bigoplus_{n \geqslant 0} H^0(X, L^{\otimes n})$ of L is approximable.
3) More generally, Lazarsfeld and Mustaţă have shown that, with the notation of 2), any graded subalgebra of $\bigoplus_{n \geqslant 0} H^0(X, L^{\otimes n})$ containing an ample divisor (see Definition 3.1) and verifying the condition (a) above is approximable.

We shall revisit the examples 2) and 3) in §3.1.

The following properties of approximable graded algebras are quite similar to some classical results on big line bundles.

PROPOSITION 2.4. – *Let $B = \bigoplus_{n \geqslant 0} B_n$ be an integral graded algebra which is approximable. Then there exists a constant $a \in \mathbb{N} \setminus \{0\}$ such that, for any sufficiently large integer p, the algebra $\bigoplus_{n \geqslant 0} \operatorname{Im}(S^n B_p \to B_{np})$ has Krull dimension a. Furthermore, set $d(B) := a - 1$. The sequence*

$$\left(\frac{\operatorname{rk}(B_n)}{n^{d(B)}/d(B)!}\right)_{n \geqslant 1} \tag{2}$$

converges in $\mathbb{R}_+$.

Proof. – Assume that $B_m \neq 0$ for all $m \geqslant m_0$, where $m_0 \in \mathbb{N}$. Since B is integral, for any integer $n \geqslant 1$ and any integer $m \geqslant m_0$, one has

$$\operatorname{rk}(B_{n+m}) \geqslant \operatorname{rk}(B_n). \tag{3}$$

For any integer $p \geqslant m_0$, denote by $a(p)$ the Krull dimension of $\bigoplus_{n \geqslant 0} \operatorname{Im}(S^{np} B_p \to B_{np})$, and define

$$f(p) := \liminf_{n \to \infty} \frac{\operatorname{rk}(\operatorname{Im}(S^n B_p \to B_{np}))}{\operatorname{rk}(B_{np})}. \tag{4}$$

The approximable condition shows that $\lim\limits_{p \to \infty} f(p) = 1$. Recall that the classical result on Hilbert polynomials implies

$$\operatorname{rk}(\operatorname{Im}(S^n B_p \to B_{np})) \asymp n^{a(p)-1} \qquad (n \to \infty).$$

Thus, if $f(p) > 0$, then $\operatorname{rk}(B_{np}) \asymp n^{a(p)-1}$, and hence by (3), one has $\operatorname{rk}(B_n) \asymp n^{a(p)-1}$ $(n \to \infty)$. So $a(p)$ is constant if $f(p) > 0$. In particular, $a(p)$ is constant when p is sufficiently large. Denote by $d(B)$ this constant minus 1.

In the following, we shall establish the convergence of the sequence (2). It suffices to establish

$$\liminf_{n\to\infty}\frac{\mathrm{rk}(B_n)}{n^{d(B)}}\geqslant\limsup_{n\to\infty}\frac{\mathrm{rk}(B_n)}{n^{d(B)}}. \tag{5}$$

By (3), for any integer $p\geqslant 1$, one has

$$\limsup_{n\to\infty}\frac{\mathrm{rk}(B_n)}{n^{d(B)}}=\limsup_{n\to\infty}\frac{\mathrm{rk}(B_{np})}{(np)^{d(B)}}\quad\text{and}\quad\liminf_{n\to\infty}\frac{\mathrm{rk}(B_n)}{n^{d(B)}}=\liminf_{n\to\infty}\frac{\mathrm{rk}(B_{np})}{(np)^{d(B)}}. \tag{6}$$

Suppose that $f(p)>0$. Then one has

$$\begin{aligned}\limsup_{n\to\infty}\frac{\mathrm{rk}(B_{np})}{(np)^{d(B)}}&=\Big(\limsup_{n\to\infty}\frac{\mathrm{rk}(B_{np})}{\mathrm{rk}(\mathrm{Im}(S^nB_p\to B_{np}))}\Big)\cdot\Big(\lim_{n\to\infty}\frac{\mathrm{rk}(\mathrm{Im}(S^nB_p\to B_{np}))}{(np)^{d(B)}}\Big)\\&=f(p)^{-1}\lim_{n\to\infty}\frac{\mathrm{rk}(\mathrm{Im}(S^nB_p\to B_{np}))}{(np)^{d(B)}}\leqslant f(p)^{-1}\liminf_{n\to\infty}\frac{\mathrm{rk}(B_{np})}{(np)^{d(B)}}.\end{aligned}$$

Combining with (6) and the approximability hypothesis, we obtain (5). □

COROLLARY 2.5. – *Assume that B is approximable. For any $r\in\mathbb{N}$, one has*

$$\lim_{n\to\infty}\frac{\mathrm{rk}(B_{n+r})}{\mathrm{rk}(B_n)}=1.$$

DEFINITION 2.6. – Let B be an integral graded K-algebra which is approximable. Denote by $\mathrm{vol}(B)$ the limit

$$\mathrm{vol}(B):=\lim_{n\to\infty}\frac{\mathrm{rk}(B_n)}{n^{d(B)}/d(B)!}.$$

Note that, if B is the total graded linear series of a big line bundle L, then $\mathrm{vol}(B)$ is just the volume of the line bundle L.

REMARK 2.7. – It might be interesting to know whether any approximable graded algebra can always be realized as a graded linear series of a big line bundle.

2.2. Reminder on $\mathbb{R}$-filtrations

Let K be a field and W be a vector space of finite rank over K. By a *filtration* on W we mean a sequence $\mathcal{F}=(\mathcal{F}_tW)_{t\in\mathbb{R}}$ of vector subspaces of W, satisfying the following conditions:

1) if $t\leqslant s$, then $\mathcal{F}_sW\subset\mathcal{F}_tW$;
2) $\mathcal{F}_tW=0$ for sufficiently positive t, $\mathcal{F}_tW=W$ for sufficiently negative t;
3) the function $t\mapsto\mathrm{rk}(\mathcal{F}_tW)$ is left continuous.

The couple $(W,\mathcal{F})$ is called a *filtered vector space*.

If $W\neq 0$, we denote by $\nu_{(W,\mathcal{F})}$ (or simply ν_W if this does not lead to any ambiguity) the Borel probability measure obtained by taking the derivative (in the sense of distribution) of the function $t\mapsto-\mathrm{rk}(\mathcal{F}_tW)/\mathrm{rk}(W)$. If $W=0$, then there is a unique filtration on W and we define ν_0 to be the zero measure by convention. Note that the measure ν_W is actually a linear combination of Dirac measures. In fact, the filtration $\mathcal{F}$ corresponds to a flag

$W = W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_n = 0$ together with a sequence of real numbers $\lambda_1 < \cdots < \lambda_n$ indicating the jumps. Then one has

$$\nu_{(W,\mathcal{F})} = \sum_{i=1}^{n} \frac{\operatorname{rk}(W_{i-1}/W_i)}{\operatorname{rk} W} \delta_{\lambda_i}.$$

All filtered vector spaces and linear maps preserving filtrations form an exact category. The following proposition shows that mapping $(W, \mathcal{F}) \mapsto \nu_{(W,\mathcal{F})}$ behaves well with respect to exact sequences.

Proposition 2.8. – *Assume that*

$$0 \longrightarrow (W', \mathcal{F}') \longrightarrow (W, \mathcal{F}) \longrightarrow (W'', \mathcal{F}'') \longrightarrow 0$$

is an exact sequence of filtered vector spaces. Then

$$\nu_W = \frac{\operatorname{rk}(W')}{\operatorname{rk}(W)} \nu_{W'} + \frac{\operatorname{rk}(W'')}{\operatorname{rk}(W)} \nu_{W''}.$$

Proof. – For any $t \in \mathbb{R}$, one has

$$\operatorname{rk}(\mathcal{F}_t W) = \operatorname{rk}(\mathcal{F}'_t W') + \operatorname{rk}(\mathcal{F}''_t W''),$$

which implies the proposition by taking the derivative in the sense of distribution. □

Corollary 2.9. – *Let $(W, \mathcal{F})$ be a non-zero filtered vector space, $V \subset W$ be a non-zero subspace, equipped with the induced filtration, and $\varepsilon = 1 - \operatorname{rk}(V)/\operatorname{rk}(W)$.*

1) *For any bounded Borel function h on $\mathbb{R}$, one has*

$$\left| \int h \, \mathrm{d}\nu_W - \int h \, \mathrm{d}\nu_V \right| \leqslant 2\varepsilon \|h\|_{\sup}.$$

2) *One has* $\displaystyle\int_0^{+\infty} x \, \nu_W(\mathrm{d}x) \geqslant (1 - \varepsilon) \int_0^{+\infty} x \, \nu_V(\mathrm{d}x).$

Proof. – The case where $W = V$ is trivial. In the following, we assume that $U := W/V$ is non-zero, and is equipped with the quotient filtration. By Proposition 2.8, one has

$$\nu_W = (1 - \varepsilon)\nu_V + \varepsilon \nu_U = \nu_V + \varepsilon(\nu_U - \nu_V).$$

Therefore

$$\left| \int h \, \mathrm{d}\nu_W - \int h \, \mathrm{d}\nu_V \right| = \varepsilon \left| \int h \, \mathrm{d}\nu_U - \int h \, \mathrm{d}\nu_V \right| \leqslant 2\varepsilon \|h\|_{\sup}.$$

$$\int_0^{+\infty} x \, \nu_W(\mathrm{d}x) = (1 - \varepsilon) \int_0^{+\infty} x \, \nu_V(\mathrm{d}x) + \varepsilon \int_0^{+\infty} x \, \nu_U(\mathrm{d}x) \geqslant (1 - \varepsilon) \int_0^{+\infty} x \, \nu_V(\mathrm{d}x).$$

□

Let $(W, \mathcal{F})$ be a filtered vector space. We denote by $\lambda : W \to \mathbb{R} \cup \{+\infty\}$ the mapping which sends $x \in W$ to

(7) $$\lambda(x) := \sup\{a \in \mathbb{R} \mid x \in \mathcal{F}_a W\}.$$

The function λ takes values in $\operatorname{supp}(\nu_W) \cup \{+\infty\}$, and is finite on $W \setminus \{0\}$. We define

(8) $$\lambda_+(W) := \int_0^{+\infty} x \, \nu_W(\mathrm{d}x), \ \lambda_{\max}(W) := \max_{x \in W \setminus \{0\}} \lambda(x) \quad \text{and} \quad \lambda_{\min}(W) := \min_{x \in W} \lambda(x).$$

By definition, $\lambda_{\min}(W)$ (resp. $\lambda_{\max}(W)$) is the infimum (resp. supremum) of the support of ν_W. Note that $\lambda_{\min}(0) = +\infty$ and $\lambda_{\max}(0) = -\infty$. When $\lambda_{\max}(W) \geqslant 0$, one has $0 \leqslant \lambda_+(W) \leqslant \lambda_{\max}(W)$.

We introduce an order "$\prec$" on the space $\mathcal{M}$ of all Borel probability measures on $\mathbb{R}$. Denote by $\nu_1 \prec \nu_2$, or by $\nu_2 \succ \nu_1$ the relation:

for any bounded increasing function h on $\mathbb{R}$, $\int h \,\mathrm{d}\nu_1 \leqslant \int h \,\mathrm{d}\nu_2$.

This relation can also be described by $\forall x \in \mathbb{R},\ \nu_1([x, +\infty[) \leqslant \nu_2([x, +\infty[)$.

For any $x \in \mathbb{R}$, denote by δ_x the Dirac measure concentrated at x. For any $a \in \mathbb{R}$, let τ_a be the operator acting on the space $\mathcal{M}$ which sends a measure ν to its direct image by the map $x \mapsto x + a$.

PROPOSITION 2.10. – *Let $(V, \mathcal{F})$ and $(W, \mathcal{G})$ be non-zero filtered vector spaces. Assume that $\phi : V \to W$ is an isomorphism of vector spaces and a is a real number such that $\phi(\mathcal{F}_t V) \subset \mathcal{G}_{t+a} W$ holds for all $t \in \mathbb{R}$, or equivalently, $\forall x \in V$, $\lambda(\phi(x)) \geqslant \lambda(x) + a$, then $\nu_W \succ \tau_a \nu_V$.*

See [3, Lemma 1.2.6] for a proof.

2.3. Convergence of measures of an approximable algebra

Let B be an integral graded K-algebra, assumed to be approximable. Assume that, for each integer $n \geqslant 0$, the vector space B_n is equipped with an $\mathbb{R}$-filtration $\mathcal{F}$ such that B is filtered, that is, for all homogeneous elements x_1, x_2 in B of degrees n_1, n_2 in $\mathbb{N}$, respectively, one has

$$\lambda(x_1 x_2) \geqslant \lambda(x_1) + \lambda(x_2),$$

where λ is the function defined in (7).

For any $\varepsilon > 0$, let T_ε be the operator acting on the space $\mathcal{M}$ of all Borel probability measures which sends $\nu \in \mathcal{M}$ to its direct image by the mapping $x \mapsto \varepsilon x$.

The purpose of this subsection is to establish the following convergence result.

THEOREM 2.11. – *Let B be an approximable graded algebra equipped with filtrations as above such that B is filtered. Assume in addition that*

$$\sup_{n \geqslant 1} \lambda_{\max}(B_n)/n < +\infty.$$

Then the sequence $(\lambda_{\max}(B_n)/n)_{n \geqslant 1}$ converges in $\mathbb{R}$, and the measure sequence $(T_{\frac{1}{n}} \nu_{B_n})_{n \geqslant 1}$ converges vaguely to a Borel probability measure on $\mathbb{R}$.

REMARK 2.12. – We say that a sequence $(\nu_n)_{n \geqslant 1}$ of Borel measures on $\mathbb{R}$ converges vaguely to a Borel measure ν if, for any continuous function h on $\mathbb{R}$ whose support is compact, one has

$$\lim_{n \to +\infty} \int h \,\mathrm{d}\nu_n = \int h \,\mathrm{d}\nu. \tag{9}$$

In the case where ν and all ν_n are probability measures, the vague convergence of ν_n to ν implies that the condition (9) holds for all bounded continuous function h.

Proof. – The first assertion has been established in [3, Proposition 3.2.4] in a more general setting without the approximability condition on B. Here we only prove the second assertion.

Assume that $B_n \neq 0$ holds for any $n \geqslant m_0$, where $m_0 \geqslant 0$ is an integer, and set $\nu_n = T_{\frac{1}{n}}\nu_{B_n}$. The supports of ν_n are uniformly bounded from above since $\sup_{n\geqslant 1}\lambda_{\max}(B_n)/n < +\infty$. Let p be an integer such that $p \geqslant m_0$. Denote by $A^{(p)}$ the graded subalgebra of B generated by B_p. For any integer $n \geqslant 1$, we equipped each vector space $A^{(p)}_{np}$ with the induced filtration, and set $\nu^{(p)}_n := T_{\frac{1}{np}}\nu_{A^{(p)}_{np}}$. Furthermore, we choose, for any $r \in \{p+1,\cdots,2p-1\}$, a non-zero element $e_r \in B_r$, and define

$$M^{(p)}_{n,r} = e_r B_{np} \subset B_{np+r},\quad N^{(p)}_{n,r} = e_{3p-r}M^{(p)}_{n,r} \subset B_{(n+3)p},$$
$$a^{(p)}_{n,r} = \frac{\lambda(e_{3p-r})}{np},\quad b^{(p)}_{n,r} = a^{(p)}_{n,r} + \frac{\lambda(e_r)}{np},$$
$$\nu^{(p)}_{n,r} = T_{\frac{1}{np}}\nu_{M^{(p)}_{n,r}},\quad \eta^{(p)}_{n,r} = T_{\frac{1}{np}}\nu_{N^{(p)}_{n,r}},$$

where $M^{(p)}_{n,r}$ and $N^{(p)}_{n,r}$ are equipped with induced filtrations. Note that, for all $x \in B_{np}$, $y \in M^{(p)}_{n,r}$, one has

$$\lambda(e_r x) \geqslant \lambda(x) + \lambda(e_r),$$
$$\lambda(e_{3p-r}y) \geqslant \lambda(y) + \lambda(e_{3p-r}).$$

By Proposition 2.10, one has $\eta^{(p)}_{n,r} \succ \tau_{a^{(p)}_{n,r}}\nu^{(p)}_{n,r} \succ \tau_{b^{(p)}_{n,r}}\nu_{np}$. Let $h(x)$ be a bounded increasing and continuous function on $\mathbb{R}$ whose support is bounded from below, and which is constant when x is sufficiently positive. One has

$$\int h\,\mathrm{d}\eta^{(p)}_{n,r} \geqslant \int h\,\mathrm{d}\tau_{a^{(p)}_{n,r}}\nu^{(p)}_{n,r} \geqslant \int h\,\mathrm{d}\tau_{b^{(p)}_{n,r}}\nu_{np}. \tag{10}$$

Note that $|h(x+\varepsilon x) - h(x)|$ converges uniformly to zero when $\varepsilon \to 0$. By Corollaries 2.9 1) and 2.5, we obtain

$$\lim_{n\to\infty}\left|\int h\,\mathrm{d}\eta^{(p)}_{n,r} - \int h\,\mathrm{d}\nu_{(n+3)p}\right| = 0, \tag{11}$$

$$\lim_{n\to\infty}\left|\int h\,\mathrm{d}\nu^{(p)}_{n,r} - \int h\,\mathrm{d}\nu_{np+r}\right| = 0. \tag{12}$$

Note that $|h(x+u) - h(x)|$ converges uniformly to zero when $u \to 0$. Combining with the fact that $\lim\limits_{n\to\infty} a^{(p)}_{n,r} = \lim\limits_{n\to\infty} b^{(p)}_{n,r} = 0$, we obtain

$$\lim_{n\to\infty}\left|\int h\,\mathrm{d}\tau_{a^{(p)}_{n,r}}\nu^{(p)}_{n,r} - \int h\,\mathrm{d}\nu^{(p)}_{n,r}\right| = 0, \tag{13}$$

$$\lim_{n\to\infty}\left|\int h\,\mathrm{d}\tau_{b^{(p)}_{n,r}}\nu_{np} - \int h\,\mathrm{d}\nu_{np}\right| = 0. \tag{14}$$

Thus

$$\begin{aligned}&\limsup_{n\to\infty}\left|\int h\,\mathrm{d}\nu_{np+r} - \int h\,\mathrm{d}\nu_{np}\right| = \limsup_{n\to\infty}\left|\int h\,\mathrm{d}\tau_{a^{(p)}_{n,r}}\nu^{(p)}_{n,r} - \int h\,\mathrm{d}\tau_{b^{(p)}_{n,r}}\nu_{np}\right|\\ &\leqslant \limsup_{n\to\infty}\left|\int h\,\mathrm{d}\eta^{(p)}_{n,r} - \int h\,\mathrm{d}\tau_{b^{(p)}_{n,r}}\nu_{np}\right| = \limsup_{n\to\infty}\left|\int h\,\mathrm{d}\nu_{(n+3)p} - \int h\,\mathrm{d}\nu_{np}\right|,\end{aligned} \tag{15}$$

where the first equality comes from (12), (13) and (14). The inequality comes from (10), and the second equality results from (11) and (14).

Let $\varepsilon \in (0,1)$. By the approximability condition on B, there exist two integers $p \geqslant m_0$ and $n_1 \geqslant 1$ such that, for any integer $n \geqslant n_1$, one has

$$\frac{\operatorname{rk} A_{np}^{(p)}}{\operatorname{rk} B_{np}} > 1 - \varepsilon.$$

Therefore, by Corollary 2.9 1), one has

$$\left| \int h \, \mathrm{d}\nu_{np} - \int h \, \mathrm{d}\nu_n^{(p)} \right| \leqslant 2\varepsilon \|h\|_{\sup}. \tag{16}$$

As $A^{(p)}$ is an algebra of finite type, by [3, Theorem 3.4.3], the sequence of measures $(\nu_n^{(p)})_{n\geqslant 1}$ converges vaguely to a Borel probability measure $\nu^{(p)}$. Note that the supports of measures $\nu_n^{(p)}$ are uniformly bounded from above. Hence $(\int h \, \mathrm{d}\nu_n^{(p)})_{n\geqslant 1}$ is a Cauchy sequence. By the relations (15) and (16), we obtain that there exists an integer $n_2 \geqslant 1$ such that, for any integers m and n, $m \geqslant n_2$, $n \geqslant n_2$, one has

$$\left| \int h \, \mathrm{d}\nu_n - \int h \, \mathrm{d}\nu_m \right| \leqslant 8\varepsilon \|h\|_{\sup} + \varepsilon.$$

Since ε is arbitrary, the sequence $(\int h \, \mathrm{d}\nu_n)_{n\geqslant 1}$ converges in $\mathbb{R}$. Denote by $C_0^\infty(\mathbb{R})$ the space of all smooth functions of compact support on $\mathbb{R}$. Since any function in $C_0^\infty(\mathbb{R})$ can be written as the difference of two continuous increasing and bounded functions whose supports are both bounded from below and which are constant on a neighborhood of $+\infty$, we obtain that $h \mapsto \lim\limits_{n\to\infty} \int h \, \mathrm{d}\nu_n$ is a well defined positive continuous linear functional on $(C_0^\infty(\mathbb{R}), \|\cdot\|_{\sup})$. As $C_0^\infty(\mathbb{R})$ is dense in the space $C_c(\mathbb{R})$ of all continuous functions of compact support on $\mathbb{R}$ with respect to the topology induced by $\|\cdot\|_{\sup}$, the linear functional extends continuously to a Borel measure ν on $\mathbb{R}$. Finally, by Corollary 2.9 and by passing to the limit, we obtain that for any $p \geqslant m_0$, one has $|1 - \nu(\mathbb{R})| = |\nu^{(p)}(\mathbb{R}) - \nu(\mathbb{R})| \leqslant 1 - f(p)$, where $f(p)$ was defined in (4). As $\lim\limits_{p\to\infty} f(p) = 1$, ν is a probability measure. □

2.4. A positivity criterion

Let B be an approximable graded K-algebra. Assume that B is equipped with filtrations such that B is filtered. The above theorem shows that the sequence $(\lambda_{\max}(B_n)/n)_{n\geqslant 1}$ converges to a real number which we shall denote by $\lambda_{\max}^{\mathrm{asy}}(B)$. Furthermore, the sequence of measures $(T_{\frac{1}{n}}\nu_{B_n})_{n\geqslant 1}$ converges vaguely to a Borel probability measure on $\mathbb{R}$, denoted by ν_B. Note that the support of ν_B is bounded from above by $\lambda_{\max}^{\mathrm{asy}}(B)$. We define

$$\lambda_+^{\mathrm{asy}}(B) := \int_0^{+\infty} x \, \nu_B(\mathrm{d}x).$$

One has

$$\lambda_+^{\mathrm{asy}}(B) = \lim_{n\to+\infty} \lambda_+(B_n)/n,$$

where $\lambda_+(B_n)$ was defined in (8).

The following theorem shows that $\lambda_+^{\mathrm{asy}}(B)$ is positive if and only if $\lambda_{\max}^{\mathrm{asy}}(B)$ is. This will be useful in the criterion of bigness of Hermitian line bundles (Proposition 3.11).

THEOREM 2.13. – *Let B be a filtered approximable graded K-algebra. Then $\lambda_+^{\mathrm{asy}}(B) > 0$ if and only if $\lambda_{\max}^{\mathrm{asy}}(B) > 0$.*

Proof. – Assume that $\lambda_+^{\mathrm{asy}}(B) > \varepsilon > 0$. Then for sufficiently large n, one has $\lambda_{\max}(B_n) \geqslant \lambda_+(B_n) > \varepsilon n$. Hence $\lambda_{\max}^{\mathrm{asy}}(B) > 0$.

It suffices then to prove the converse implication. Assume that $\alpha > 0$ is a real number such that $\lambda_{\max}^{\mathrm{asy}}(B) > 2\alpha$. Choose sufficiently large $n_0 \in \mathbb{N}$ such that there exists a non-zero $x_0 \in B_{n_0}$ satisfying $\lambda(x_0) \geqslant 2\alpha n_0$. Since the algebra B is filtered, one has $\lambda(x_0^m) \geqslant 2\alpha m n_0$. The algebra B is approximable. Hence there exists an integer p divisible by n_0 such that

$$\liminf_{n\to\infty} \frac{\operatorname{rk}(\operatorname{Im}(S^n B_p \to B_{np}))}{\operatorname{rk} B_{np}} > 0.$$

By Corollary 2.9 2), $\lim\limits_{n\to\infty} \lambda_+(\operatorname{Im}(S^n B_p \to B_{np}))/np > 0$ implies

$$\lim_{n\to\infty} \lambda_+(B_{np})/np = \lim_{n\to\infty} \lambda_+(B_n)/n > 0.$$

Therefore, after replacing B by $\bigoplus_{n\geqslant 0} \operatorname{Im}(S^n B_p \to B_{np})$ we reduce our problem to the case where

1) B is an algebra of finite type generated by B_1,
2) there exists $x_1 \in B_1$, $x_1 \neq 0$ such that $\lambda(x_1) \geqslant 2\alpha$ with $\alpha > 0$,

Furthermore, by Noether's normalization theorem (and by possible extension of the field K), the algebra B is finite over a subalgebra of polynomials $K[x_1, \cdots, x_q]$, where x_1 coincides with the element in the condition 2), $x_2, \cdots, x_q$ are elements in B_1. Note that

$$\lambda(x_1^{a_1} \cdots x_q^{a_q}) \geqslant \sum_{i=1}^q a_i \lambda(x_i) \geqslant 2\alpha a_1 + \sum_{i=2}^q a_i \lambda(x_i). \tag{17}$$

Let $\beta > 0$ such that $-\beta \leqslant \lambda(x_i)$ for any $i \in \{2, \cdots, q\}$. We obtain from (17) that $\lambda(x_1^{a_1} \cdots x_q^{a_q}) \geqslant \alpha a_1$ as soon as $a_1 \geqslant \frac{\beta}{\alpha} \sum_{i=2}^q a_i$. For $n \in \mathbb{N}^*$, let

$$\begin{aligned} u_n &= \#\Big\{(a_1, \cdots, a_q) \in \mathbb{N}^q \,\Big|\, a_1 + \cdots + a_q = n,\ a_1 \geqslant \frac{\beta}{\alpha}(a_2 + \cdots + a_q)\Big\} \\ &= \#\Big\{(a_1, \cdots, a_q) \in \mathbb{N}^q \,\Big|\, a_1 + \cdots + a_q = n,\ a_1 \geqslant \frac{\beta}{\alpha+\beta} n\Big\} \\ &= \binom{n - \lfloor \frac{\beta}{\alpha+\beta} n \rfloor + q - 1}{q-1}, \end{aligned}$$

and $v_n = \#\{(a_1, \cdots, a_q) \in \mathbb{N}^q \mid a_1 + \cdots + a_q = n\} = \binom{n+q-1}{q-1}$. Thus

$$\lim_{n\to\infty} u_n/v_n = \Big(\frac{\alpha}{\alpha+\beta}\Big)^{q-1} > 0$$

and hence

$$\lim_{n\to+\infty} u_n / \operatorname{rk}(B_n) > 0.$$

For any $n \geqslant 1$, let C_n be the subspace of B_n generated by elements of the form $x_1^{a_1} \cdots x_q^{a_q}$, where $a_1 + \cdots + a_q = n$ and $a_1 \geqslant \beta n/(\alpha+\beta)$. By Corollary 2.9 2), one has

$$\lambda_+(B_n) \geqslant \frac{u_n}{\operatorname{rk}(B_n)} \lambda_+(C_n) \geqslant \frac{u_n}{\operatorname{rk}(B_n)} \cdot \frac{\alpha\beta n}{\alpha+\beta}.$$

Hence $\lambda_+^{\mathrm{asy}}(B) > 0$. □

3. Approximable graded linear series in arithmetic

In this section, we recall a result on Fujita approximation for graded linear series due to Lazarsfeld and Mustaţă [11]. We then give several examples of approximable graded linear series which come naturally from the arithmetic setting.

3.1. Reminder on geometric Fujita approximation

Let K be a field and X be a projective variety (i.e. integral projective scheme) defined over K. Let L be a big line bundle on X. Denote by $B := \bigoplus_{n\geqslant 0} H^0(X, L^{\otimes n})$ the graded K-algebra of global sections of tensor powers of L. For *graded linear series* of L we mean a graded sub-K-algebra of B. The following definition is borrowed from [11].

Definition 3.1. – We say that a graded linear series $W = \bigoplus_{n\geqslant 0} W_n$ of L *contains an ample divisor* if there exist an integer $p \geqslant 1$, an ample line bundle A and an effective line bundle M on X, together with a non-zero section $s \in H^0(X, M)$, such that $L^{\otimes p} \cong A \otimes M$, and that the homomorphism of graded algebras

$$\bigoplus_{n\geqslant 0} H^0(X, A^{\otimes n}) \longrightarrow \bigoplus_{n\geqslant 0} H^0(X, L^{\otimes np})$$

induced by s factors through $\bigoplus_{n\geqslant 0} W_{np}$.

Remark 3.2. – In [11, Definition 2.9], this condition was called the "condition (C)". As a big divisor is always the sum of an ample divisor and an effective one, the total graded linear series B contains an ample divisor.

Definition 3.3. – Let $W = \bigoplus_{n\geqslant 0} W_n$ be a graded linear series of L. Denote by $\mathrm{vol}(W)$ the number

$$\mathrm{vol}(W) := \limsup_{n\to\infty} \frac{\mathrm{rk}(W_n)}{n^{\dim X}/(\dim X)!}. \tag{18}$$

Note that $\mathrm{vol}(B) = \mathrm{vol}(L)$. For a general linear series W of L, one has $\mathrm{vol}(W) \leqslant \mathrm{vol}(L)$. By using the method of Okounkov bodies introduced in [15], Lazarsfeld and Mustaţă have established the following generalization of Fujita's approximation theorem.

Theorem 3.4 (Lazarsfeld-Mustaţă). – *Assume that $W = \bigoplus_{n\geqslant 0} W_n$ is a graded linear series of L which contains an ample divisor and such that $W_n \neq 0$ for sufficiently large n. Then W is approximable.*

In particular, the total graded linear series B is approximable. In [11, Remark 3.4], the authors have explained why their theorem implies Fujita's approximation theorem in its classical form. We include their explanation as the corollary below.

Corollary 3.5 (Geometric Fujita approximation). – *For any $\varepsilon > 0$, there exist an integer $p \geqslant 1$, a birational projective morphism $\varphi : X' \to X$, an ample line bundle A and an effective line bundle M such that*

1) *one has $\varphi^*(L^{\otimes p}) \cong A \otimes M$;*
2) $\mathrm{vol}(A) \geqslant p^{\dim X}(\mathrm{vol}(L) - \varepsilon)$.

Proof. – For any integer p such that $B_p \neq 0$, let $\varphi_p : X_p \to X$ be the blow-up (twisted by L) of X along the base locus of B_p. That is

$$X_p = \operatorname{Proj}\Big(\operatorname{Im}\Big(\bigoplus_{n\geqslant 0} S^n(\pi^*B_p) \longrightarrow \bigoplus_{n\geqslant 0} L^{\otimes np}\Big)\Big).$$

Denote by E_p the exceptional divisor and by s the global section of $\mathcal{O}(E_p)$ which trivializes $\mathcal{O}(E_p)$ outside the exceptional divisor. By definition, one has $\mathcal{O}_{X_p}(1) \cong \varphi_p^* L^{\otimes p} \otimes \mathcal{O}(-E_p)$. On the other hand, the canonical homomorphism $\varphi_p^*\pi^*B_p \to \mathcal{O}_{X_p}(1)$ is surjective, therefore corresponds to a morphism of schemes $i_p : X_p \to \mathbb{P}(B_p)$ such that $i_p^*\mathcal{O}_{\mathbb{P}(B_p)}(1) = \mathcal{O}_{X_p}(1)$. The restriction of global sections of $\mathcal{O}_{\mathbb{P}(B_p)}(n)$ on X_p gives an injective homomorphism

$$\operatorname{Im}(S^n B_p \to B_{np}) \longrightarrow H^0(X_p, \mathcal{O}_{X_p}(n)),$$

where we have identified $H^0(X_p, \mathcal{O}_{X_p}(n))$ with a subspace of $H^0(X_p, \varphi_p^* L^{\otimes n})$ via s. Since the total graded linear series B is approximable, one has

$$\sup_p \liminf_{n\to\infty} \frac{\operatorname{rk}(\operatorname{Im}(S^n B_p \to B_{np}))}{\operatorname{rk}(B_{np})} = 1,$$

which implies

$$\sup_p \lim_{n\to\infty} \frac{\operatorname{rk} H^0(X_p, \mathcal{O}_{X_p}(n))}{(np)^d/d!} = \operatorname{vol}(L).$$

The line bundle $\mathcal{O}_{X_p}(1)$ constructed above is actually nef and big. However, a slight perturbation of L permits to conclude. □

3.2. Arithmetic volume of approximable graded linear series

Let K be a number field and $\mathcal{O}_K$ be its integer ring. Denote by $\delta_K := [K : \mathbb{Q}]$ the degree of K over $\mathbb{Q}$. By a *metrized vector bundle* on $\operatorname{Spec}\mathcal{O}_K$ we mean a projective $\mathcal{O}_K$-module of finite type E together with a family $(\|\cdot\|_\sigma)_{\sigma:K\to\mathbb{C}}$, where $\|\cdot\|_\sigma$ is a norm on $E_{\sigma,\mathbb{C}}$, assumed to be invariant by the complex conjugation. We often use the expression $\overline{E}$ to denote the couple $(E, (\|\cdot\|_\sigma)_{\sigma:K\to\mathbb{C}})$. A metrized vector bundle of rank one is also called a *metrized line bundle*.

On a metrized vector bundle on $\operatorname{Spec}\mathcal{O}_K$, one has a natural filtration defined by its successive minima. Let $\overline{E}$ be a metrized vector bundle on $\operatorname{Spec}\mathcal{O}_K$. Let r be the rank of E and $i \in \{1, \cdots, r\}$. Recall that the i^{th} (logarithmic) *minimum* of $\overline{E}$ is defined as

$$e_i(\overline{E}) := -\log\inf\{a > 0 \,|\, \operatorname{rk}(\operatorname{Vect}_K\{\mathbb{B}(\overline{E}, a)\}) \geqslant i\},$$

where $\mathbb{B}(\overline{E}, a) = \{s \in E \,|\, \forall\sigma : K \to \mathbb{C}, \|s\|_\sigma \leqslant a\}$. Set $e_{\max}(\overline{E}) = e_1(\overline{E})$ and $e_{\min}(\overline{E}) = e_r(\overline{E})$. Define an $\mathbb{R}$-filtration $\mathcal{F}$ on E_K as

$$\mathcal{F}_t E_K := \operatorname{Vect}_K\{\mathbb{B}(\overline{E}, e^{-t})\},$$

called the *minima filtration* of $\overline{E}$. Note that $\lambda_{\max}(E_K, \mathcal{F}) = e_{\max}(\overline{E})$ and $\lambda_{\min}(E_K, \mathcal{F}) = e_{\min}(\overline{E})$.

Let $\overline{E}$ be a metrized vector bundle on $\operatorname{Spec}\mathcal{O}_K$. Denote by $\nu_{\overline{E}}$ the Borel probability measure on $\mathbb{R}$ associated to the filtered vector space $(E_K, \mathcal{F})$ and set

$$\widehat{h}^0(\overline{E}) := \log\#\mathbb{B}(\overline{E}, 1) = \log\#\{s \in E \,|\, \forall\,\sigma : K \to \mathbb{C},\ \|s\|_\sigma \leqslant 1\}.$$

A classical result of Gillet and Soulé leads to the following estimation.

LEMMA 3.6. – *One has*

$$\left|\delta_K r\int_0^{+\infty} x\,\nu_{\overline{E}}(\mathrm{d}x)-\widehat{h}^0(\overline{E})\right|\ll \mathrm{rk}(E)\log\mathrm{rk}(E),\tag{19}$$

where the implicit constant only depends on K.

Proof. – It is a direct consequence of [9, Proposition 6] (see also [9, Theorem 1]). □

REMARK 3.7. – Let $B=\bigoplus_{n\geqslant 0}B_n$ be a graded K-algebra. Assume that, for each n, $\overline{\mathcal{B}}_n$ is a metrized vector bundle on $\operatorname{Spec}\mathcal{O}_K$ such that $B_n=\mathcal{B}_{n,K}$. As explained above, the successive minima of $\overline{\mathcal{B}}_n$ induces a filtration $\mathcal{F}$ on B_n. The graded algebra B is filtered with respect to these filtrations notably when the following conditions are fulfilled.

(a) the K-algebra structure on B gives rise to an $\mathcal{O}_K$-algebra structure on $\bigoplus_{n\geqslant 0}\mathcal{B}_n$;
(b) for any $(m,n)\in\mathbb{N}^2$, any $\sigma:K\to\mathbb{C}$ and for all $s\in B_{n,\sigma,\mathbb{C}}$, $s'\in B_{m,\sigma,\mathbb{C}}$, one has $\|ss'\|_\sigma\leqslant\|s\|_\sigma\|s'\|_\sigma$;

A typical example of such filtered graded algebra is arithmetic graded linear series. Let $\pi:\mathcal{X}\to\operatorname{Spec}\mathcal{O}_K$ be a projective arithmetic variety of total dimension d and $X=\mathcal{X}_K$. Let $\overline{\mathcal{L}}$ be a Hermitian line bundle on $\mathcal{X}$, $L=\mathcal{L}_K$ and B be a graded linear series of L. For any integer $n\geqslant 0$, denote by $\mathcal{B}_n$ the saturation of B_n in $\pi_*(\mathcal{L}^{\otimes n})$. For any embedding $\sigma:K\to\mathbb{C}$, denote by $\|\cdot\|_{\sigma,\sup}$ the sup-norm on $B_{n,\sigma,\mathbb{C}}$. Thus we obtain a metrized vector bundle $(\mathcal{B}_n,g_n)$ on $\operatorname{Spec}\mathcal{O}_K$ with $g_n=(\|\cdot\|_{\sigma,\sup})_{\sigma:K\to\mathbb{C}}$. Then the graded algebra B is filtered with respect to the minima filtrations $\mathcal{F}$ associated to $\overline{\mathcal{B}}_n$.

Inspired by [12], we define the arithmetic volume function of B as follows:

$$\widehat{\mathrm{vol}}(B):=\limsup_{n\to\infty}\frac{\widehat{h}^0(\mathcal{B}_n,g_n)}{n^d/d!}.$$

In particular, when B is the total graded linear series $\bigoplus_{n\geqslant 0}H^0(X,L^{\otimes n})$, then $\widehat{\mathrm{vol}}(B)$ is just the arithmetic volume of $\overline{\mathcal{L}}$ in the sense of Moriwaki [12], denoted by $\widehat{\mathrm{vol}}(\overline{\mathcal{L}})$. Recall that $\overline{\mathcal{L}}$ is said to be (arithmetically) *big* if $\widehat{\mathrm{vol}}(\overline{\mathcal{L}})>0$. Note that if $\overline{\mathcal{L}}$ is big, then L is a big line bundle on X (see [13, Introduction] and [18, Corollary 2.4]).

THEOREM 3.8. – *Assume that the graded linear series B is approximable. Then the sequence $(\frac{1}{n}e_{\max}(\mathcal{B}_n,g_n))_{n\geqslant 1}$ converges in $\mathbb{R}$. Furthermore, for any integer $n\geqslant 1$, let $\nu_n:=T_{\frac{1}{n}}\nu_{(B_n,\mathcal{F})}$ be the normalized probability measure associated to the minima filtration of $(\mathcal{B}_n,g_n)$, then the sequence of measures $(\nu_n)_{n\geq 1}$ converges vaguely to a Borel probability measure ν_B. Moreover, one has*

$$\lambda_+^{\mathrm{asy}}(B):=\int_0^{+\infty}x\,\nu_B(\mathrm{d}x)=\frac{\widehat{\mathrm{vol}}(B)}{\delta_K d\,\mathrm{vol}(B)}.\tag{20}$$

Proof. – Note that the graded algebra B is filtered with respect to the minima filtrations. By Theorem 2.11, it suffices to prove that $e_{\max}(\mathcal{B}_n,g_n)\ll n$. Let Σ be a Zariski dense family of algebraic points in X. Each point P in Σ extends in a unique way to a $\mathcal{O}_{K(P)}$ point of $\mathcal{X}$, where $K(P)$ is the field of definition of P. Therefore we may consider elements in Σ as points of $\mathcal{X}$ valued in algebraic integer rings. Now consider the evaluation map

$\pi_*(\mathcal{L}^{\otimes n}) \longrightarrow \bigoplus_{P\in\Sigma} P^*\mathcal{L}$. It is generically injective since Σ is dense in X. Therefore, one has

$$e_{\max}(\pi_*(\mathcal{L}^{\otimes n}), g_n) \leqslant n \sup_{P\in\Sigma} h_{\overline{\mathcal{L}}}(P).$$

Since Σ is arbitrary, we obtain that $\frac{1}{n}e_{\max}(\pi_*(\mathcal{L}^{\otimes n}), g_n)$ is bounded from above by the essential minimum of $\overline{\mathcal{L}}$ (see [21, §5] for definition. Attention, in [21], the author denoted it by $e_1(\overline{\mathcal{L}})$).

The equality (20) is a consequence of Lemma 3.6. In fact, Lemma 3.6 implies that

$$\left|\widehat{h}^0(\mathcal{B}_n, g_n) - nr_n\delta_K \int_0^{+\infty} x\,\nu_n(\mathrm{d}x)\right| \ll r_n \log r_n, \tag{21}$$

where $r_n = \mathrm{rk}(B_n)$. We have shown that $(\nu_n)_{n\geqslant 1}$ converges vaguely to ν_B. Furthermore, Proposition 2.4 shows that $r_n = \mathrm{vol}(B)n^{d-1}/(d-1)! + o(n^{d-1})$. By passing to the limit, we obtain (20). □

Remark 3.9. – The relations (19) and (20) actually imply that

$$\widehat{\mathrm{vol}}(B) = \lim_{n\to+\infty} \frac{\widehat{h}^0(\mathcal{B}_n, g_n)}{n^d/d!}.$$

3.3. Examples of approximable graded linear series in arithmetic

Let $\pi : \mathcal{X} \to \mathrm{Spec}\, \mathcal{O}_K$ be a projective arithmetic variety, $X = \mathcal{X}_K$. Let $\overline{\mathcal{L}}$ be a Hermitian line bundle on $\mathcal{X}$ such that $L := \mathcal{L}_K$ is big on X. In this subsection, we give some examples of approximable graded linear series of L which come from the arithmetic.

Denote by $B = \bigoplus_{n\geqslant 0} H^0(X, L^{\otimes n})$ the sectional algebra of L. For any real number λ, let $B^{[\lambda]}$ be the graded sub-K-module of B defined as follows:

$$B_0^{[\lambda]} := K, \qquad B_n^{[\lambda]} := \mathrm{Vect}_K\big(\{s \in B_n \,|\, \forall\, \sigma : K \to \mathbb{C},\ \|s\|_{\sigma,\sup} \leqslant e^{-\lambda n}\}\big). \tag{22}$$

The following property is straightforward from the definition.

Proposition 3.10. – *For any $\lambda \in \mathbb{R}$, $B^{[\lambda]}$ is a graded linear series of L.*

Note that $B^{[0]}$ is nothing but the graded linear series generated by effective sections. For any integer $n \geqslant 0$ and any real number λ, set $\mathcal{B}_n = \pi_*(\mathcal{L}^{\otimes n})$ and denote by $\mathcal{B}_n^{[\lambda]}$ the saturation of $B_n^{[\lambda]}$ in $\mathcal{B}_n$. We shall use the symbol g_n to denote the family of sup-norms on $\mathcal{B}_n$ or on $\mathcal{B}_n^{[\lambda]}$. By definition, for any integer $n \geqslant 1$ and any $\lambda \in \mathbb{R}$, one has

$$B_n^{[\lambda]} = \mathcal{F}_{n\lambda} B_n,$$

where $\mathcal{F}$ is the minima filtration of $(\mathcal{B}_n, g_n)$.

By Corollary 3.13, we obtain that the sequence $(\frac{1}{n}e_{\max}(\mathcal{B}_n, g_n))_{n\geqslant 1}$ converges to a real number which we denote by $\widehat{\mu}^{\pi}_{\max}(\overline{\mathcal{L}})$.

Proposition 3.11. – *Assume that $L = \mathcal{L}_K$ is big. Then $\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) > 0$ if and only if $\widehat{\mu}^{\pi}_{\max}(\overline{\mathcal{L}}) > 0$.*

Proof. – Let B be the total linear series of L. Since L is big, B is approximable. We equip B with the minima filtrations so that B is filtered. By definition, one has $\widehat{\mu}^{\pi}_{\max}(\overline{\mathcal{L}}) = \lambda^{\mathrm{asy}}_{\max}(B)$. Furthermore, Theorem 3.8 shows that $\lambda^{\mathrm{asy}}_{+}(B) = \widehat{\mathrm{vol}}(\overline{\mathcal{L}})/\delta_K d\mathrm{vol}(B)$, where d is the dimension of $\mathcal{X}$. Therefore, by Theorem 2.13, we obtain the result. □

For any real number λ, denote by $\overline{\mathcal{O}}_\lambda$ the Hermitian line bundle on Spec $\mathcal{O}_K$ whose underlying $\mathcal{O}_K$-module is trivial, and such that $\|\mathbf{1}\|_\sigma = e^{-\lambda}$ for any σ.

Proposition 3.12. – *Let λ be a real number such that $\lambda < \widehat{\mu}^{\pi}_{\max}(\overline{\mathcal{L}})$. Then the graded linear series $B^{[\lambda]}$ contains an ample divisor, and for sufficiently large n, one has $B^{[\lambda]}_n \neq 0$.*

Proof. – Note that $\widehat{\mu}^{\pi}_{\max}(\overline{\mathcal{L}}\otimes\pi^*\overline{\mathcal{O}}_{-\lambda}) > 0$. Since L is big, by Proposition 3.11, $\overline{\mathcal{L}}\otimes\pi^*\overline{\mathcal{O}}_{-\lambda}$ is arithmetically big. Therefore, for sufficiently large n, $\mathcal{L}^{\otimes n}$ has a non-zero global section s_n such that $\|s_n\|_{\sigma,\sup} \leqslant e^{-\lambda n}$ for any $\sigma : K \to \mathbb{C}$, which proves that $B^{[\lambda]}_n \neq 0$. Furthermore, since $\overline{\mathcal{L}} \otimes \pi^*\overline{\mathcal{O}}_{-\lambda}$ is arithmetically big, by [18, Corollary 2.4], there exist an integer $p \geqslant 1$ and two Hermitian line bundles $\overline{\mathcal{A}}$ and $\overline{\mathcal{M}}$, such that $\overline{\mathcal{A}}$ is ample in the sense of Zhang [21], $\overline{\mathcal{M}}$ has a non-zero effective global section s, and that $(\overline{\mathcal{L}} \otimes \pi^*\overline{\mathcal{O}}_{-\lambda})^{\otimes p} \cong \overline{\mathcal{A}} \otimes \overline{\mathcal{M}}$. By taking p sufficiently divisible, we may assume that the graded K-algebra $\bigoplus_{n\geqslant 0} H^0(X, \mathcal{A}_K^{\otimes n})$ is generated by effective sections of $\mathcal{A}$. These sections, viewed as sections of $\overline{\mathcal{A}}\otimes\pi^*\overline{\mathcal{O}}_{\lambda}^{\otimes p}$, have sup-norms $\leqslant e^{-p\lambda}$. Therefore the homomorphism

$$\bigoplus_{n\geqslant 0} H^0(X, \mathcal{A}_K^{\otimes n}) \longrightarrow \bigoplus_{n\geqslant 0} H^0(X, L^{\otimes np})$$

induced by s factors through $\bigoplus_{n\geqslant 0} B^{[\lambda]}_{np}$. □

Corollary 3.13. – *For any real number λ such that $\lambda < \widehat{\mu}^{\pi}_{\max}(\overline{\mathcal{L}})$, the graded linear series $B^{[\lambda]}$ of L is approximable.*

Proof. – This is a direct consequence of Proposition 3.12 and Theorem 3.4. □

4. Arithmetic Fujita approximation

In this section, we establish the conjecture of Moriwaki on the arithmetic analogue of Fujita's approximation. We firstly present an approximation theorem in a form similar to [11, Theorem 3.3] and then explain how to deduce Moriwaki's conjecture from it.

4.1. An approximation theorem

Let $\pi : \mathcal{X} \to$ Spec $\mathcal{O}_K$ be an arithmetic variety of total dimension d and $\overline{\mathcal{L}}$ be a Hermitian line bundle on $\mathcal{X}$ which is arithmetically big.

Write $L = \mathcal{L}_K$ and denote by $B := \bigoplus_{n\geqslant 0} H^0(X, L^{\otimes n})$ the total graded linear series of L. For any integer $n \geqslant 1$, let $\overline{\mathcal{B}}_n$ be the $\mathcal{O}_K$-module $\pi_*(\mathcal{L}^{\otimes n})$ equipped with sup-norms. Define by convention $\overline{\mathcal{B}}_0$ as the trivial Hermitian line bundle on Spec $\mathcal{O}_K$. Set $\widehat{\mu}^{\pi}_{\max}(\overline{\mathcal{L}}) = \lim\limits_{n\to\infty} \frac{1}{n} e_{\max}(\overline{\mathcal{B}}_n)$.

For any real number λ, let $B^{[\lambda]}$ be the graded linear series of L defined in (22). For any integer $n \geqslant 0$, let $\overline{\mathcal{B}}^{[\lambda]}_n$ be the saturation of $B^{[\lambda]}_n$ in $\mathcal{B}_n$, equipped with induced metrics. For any integer $p \geqslant 1$ such that $B^{[0]}_p \neq 0$, let $B^{(p)}$ be the graded sub-K-algebra of B generated by $B_p^{[0]}$. For any integer $n \geqslant 1$, let $\overline{\mathscr{B}}_{np}^{(p)}$ be the saturated metrized vector subbundle of $\overline{\mathscr{B}}_{np}$ such that $\mathscr{B}_{np,K}^{(p)} = B_{np}^{(p)}$.

THEOREM 4.1. – *The following equality holds:*

$$\widehat{\mathrm{vol}}(\overline{\mathscr{L}}) = \sup_p \widehat{\mathrm{vol}}(B^{(p)}),$$

where $B^{(p)}$ is the graded linear series of L generated by $B_p^{[0]}$ defined above.

Proof. – For any integer $n \geqslant 1$, let $\nu_n = T_{\frac{1}{n}}\nu_{(B_n,\mathscr{F})}$, where $\mathscr{F}$ is the minima filtration of $\overline{\mathscr{B}}_n$. We have shown in Theorem 3.8 that the sequence $(\nu_n)_{n\geqslant 1}$ converges vaguely to a Borel probability measure which we denote by ν. Similarly, for any integer $n \geqslant 1$, let $\nu_n^{(p)} = T_{\frac{1}{np}}\nu_{(B_{np}^{(p)},\mathscr{F})}$. The sequence $(\nu_n^{(p)})_{n\geqslant 1}$ also converges vaguely to a Borel probability measure which we denote by $\nu^{(p)}$. Our strategy is to prove that the restriction of ν on $[0,+\infty[$ can be well approximated by the measures $\nu^{(p)}$ (see §4.3 for further discussions).

For any subdivision $D : 0 = t_0 < t_1 < \cdots < t_m < \widehat{\mu}_{\max}^{\pi}(\overline{\mathscr{L}})$ of the interval $[0, \widehat{\mu}_{\max}^{\pi}(\overline{\mathscr{L}})[$ such that

$$\nu(\{t_1,\cdots,t_m\}) = 0, \tag{23}$$

denote by $h_D : \mathbb{R} \to \mathbb{R}$ the function such that

$$h_D(x) = \sum_{i=0}^{m-1} t_i\,1\!\!1_{[t_i,t_{i+1}[}(x) + t_m\,1\!\!1_{[t_m,\infty[}(x).$$

By Corollary 3.13, for any $\varepsilon > 0$, there exists a sufficiently large integer $p = p(\varepsilon, D) \geqslant 1$ such that $B^{(p)}$ approximates all algebras $B^{[t_i]}$ ($i \in \{0,\cdots,m\}$) simultaneously. That is, there exists $N_0 \in \mathbb{N}$ such that, for any $n \geqslant N_0$, one has

$$\inf_{0\leqslant i\leqslant m} \frac{\mathrm{rk}\left(\mathrm{Im}(S^n B_p^{[t_i]} \to B_{np}^{[t_i]})\right)}{\mathrm{rk}(B_{np}^{[t_i]})} \geqslant 1-\varepsilon.$$

We then obtain that

$$\mathrm{rk}(\mathscr{F}_{npt_i}B_{np}^{(p)}) \geqslant \mathrm{rk}\left(\mathrm{Im}(S^n B_p^{[t_i]} \to B_{np}^{[t_i]})\right) \geqslant (1-\varepsilon)\,\mathrm{rk}(B_{np}^{[t_i]}). \tag{24}$$

Note that

$$np\,\mathrm{rk}(B_{np}^{(p)})\int_0^{+\infty} t\,\nu_{np}^{(p)}(\mathrm{d}t) = -\int_0^{+\infty} t\,\mathrm{d}\,\mathrm{rk}(\mathscr{F}_t B_{np}^{(p)}) = \int_0^{+\infty}\mathrm{rk}(\mathscr{F}_t B_{np}^{(p)})\,\mathrm{d}t$$
$$\geqslant \sum_{i=1}^m (npt_i - npt_{i-1})\,\mathrm{rk}(\mathscr{F}_{npt_i}B_{np}^{(p)}) \geqslant np(1-\varepsilon)\sum_{i=1}^m (t_i - t_{i-1})\,\mathrm{rk}(B_{np}^{[t_i]}),$$

where in the first inequality we have used the decreasing property of $\mathscr{F}$, and the second inequality comes from (24) and the fact that $B_{np}^{[t_i]} = \mathscr{F}_{npt_i}^M B_{np}$.

From the above inequality, one obtains, by Abel's summation formula,

$$\mathrm{rk}(B_{np}^{(p)})\int_0^{+\infty} t\,\nu_{np}^{(p)}(\mathrm{d}t) \geqslant (1-\varepsilon)\,\mathrm{rk}(B_{np})\int h_D(t)\,\nu_{np}(\mathrm{d}t).$$

By (20), one has

$$\lim_{n\to\infty}\frac{\delta_K d}{(np)^{d-1}/(d-1)!}\,\mathrm{rk}(B^{(p)}_{np})\int_0^{+\infty} t\,\nu^{(p)}_{np}(\mathrm{d}t)$$
$$=\delta_K d\mathrm{vol}(B^{(p)})\int_0^{+\infty} t\,\nu^{(p)}(\mathrm{d}t)=\widehat{\mathrm{vol}}(B^{(p)}).$$

Therefore,

$$\widehat{\mathrm{vol}}(B^{(p)})\geqslant\lim_{n\to\infty}\frac{\delta_K d}{(np)^{d-1}/(d-1)!}(1-\varepsilon)\,\mathrm{rk}(B_{np})\int_{\mathbb{R}} h_D\,\mathrm{d}\nu_{np}$$
$$=\delta_K d(1-\varepsilon)\mathrm{vol}(L)\int_{\mathbb{R}} h_D\,\mathrm{d}\nu,$$

where the equality follows from [2, IV §5 $n^\circ 12$ Proposition 22]. Choose a sequence of subdivisions $(D_j)_{j\in\mathbb{N}}$ verifying the condition (23) and such that $h_{D_j}(t)$ converges uniformly to $\max\{t,0\}-\max\{t-\widehat{\mu}^{\pi}_{\max}(\overline{\mathscr{L}}),0\}$ when $j\to\infty$. Note that the support of ν is bounded from above by $\widehat{\mu}^{\pi}_{\max}(\overline{\mathscr{L}})$. Hence one obtains

$$\sup_p\widehat{\mathrm{vol}}(B^{(p)})\geqslant\delta_K d(1-\varepsilon)\mathrm{vol}(L)\int_0^{+\infty} t\,\nu(\mathrm{d}t)=(1-\varepsilon)\widehat{\mathrm{vol}}(\overline{\mathscr{L}}),$$

thanks to (20). The theorem is thus proved. □

4.2. Arithmetic Fujita approximation

In the following, we explain why Theorem 4.1 implies Fujita's arithmetic approximation theorem in the form conjectured by Moriwaki. Our strategy is quite similar to Corollary 3.5, except that the choice of metrics on the approximating invertible sheaf requires rather subtle analysis (because the arithmetic amplitude needs smooth metrics) on the superadditivity of probability measures associated to a filtered graded algebra (see Appendix).

The following is a reminder on some positivity conditions for Hermitian line bundles. Let $g:\mathscr{Y}\to\mathrm{Spec}\,\mathcal{O}_K$ be a projective arithmetic variety such that $\mathscr{Y}_{\mathbb{Q}}$ is smooth. We say that a Hermitian line bundle $\overline{\mathscr{L}}$ (with smooth metrics) is *ample* if the following conditions are fulfilled:

1) $\mathscr{L}$ is relatively ample,
2) the metrics of $\overline{\mathscr{L}}$ are positive, that is, $c_1(\overline{\mathscr{L}})$ is a positive $(1,1)$-form on $\mathscr{Y}(\mathbb{C})$ (the space of $\mathbb{C}$-points of $\mathscr{Y}\otimes_{\mathbb{Z}}\mathbb{C}$, equipped with the analytic topology),
3) for any irreducible closed subscheme Z of $\mathscr{Y}$ which is flat over $\mathrm{Spec}\,\mathcal{O}_K$, the height $h_{\overline{\mathscr{L}}}(Z):=(\widehat{c}_1(\overline{\mathscr{L}})^{\dim Z}\cdot[Z])$ is strictly positive.

By [21, Corollary 5.7], under the conditions 1) and 2), the condition 3) is actually equivalent to

3)′ For any $z\in\mathscr{Y}(\overline{\mathbb{Q}})$, $h_{\overline{\mathscr{L}}}(z)>0$.

Let $\overline{\mathscr{L}}$ be a Hermitian line bundle on $\mathscr{Y}$ with smooth metrics. We say that $\overline{\mathscr{L}}$ is *nef* (cf. [12, §2]) if the following conditions are fulfilled:

1) $\mathscr{L}$ is relatively nef, that is, all fibres of $\mathscr{L}$ are nef,
2) $c_1(\overline{\mathscr{L}})$ is semipositive on $\mathscr{Y}(\mathbb{C})$,
3) for any $z\in\mathscr{Y}(\overline{\mathbb{Q}})$, one has $h_{\overline{\mathscr{L}}}(z)\geqslant 0$.

REMARK 4.2. – It follows from the definition that

(1) the tensor product of an ample Hermitian line bundle with a nef one is ample;
(2) a Hermitian line bundle $\overline{\mathscr{L}}$ with smooth metrics is ample (resp. nef) if and only if, for some integer $n > 0$, $\overline{\mathscr{L}}^{\otimes n}$ is ample (resp. nef);
(3) the pull-back of a nef Hermitian line bundle by a proper morphism is still nef.

Moreover, assume that $\overline{E}$ is a Hermitian line bundle on Spec $\mathcal{O}_K$ and that $\overline{\mathscr{L}}$ is a quotient bundle of $g^*\overline{E}$, then $\mathscr{L}$ is relatively nef and the metrics of $\overline{\mathscr{L}}$ are semipositive. In fact, the surjective map $g^*E \to \mathscr{L}$ corresponds to an embedding $i : \mathscr{Y} \to \mathbb{P}(E)$ with $\overline{\mathscr{L}} \cong i^*\overline{\mathcal{O}_{\mathbb{P}(E)}(1)}$, where the metrics of $\overline{\mathcal{O}_{\mathbb{P}(E)}(1)}$ are Fubini-Study metrics. Therefore, if $e_{\min}(\overline{E}) \geqslant \frac{1}{2}\log(\operatorname{rk} E)$, then $\overline{\mathscr{L}}$ is nef.

THEOREM 4.3 (Arithmetic Fujita approximation). – *For any $\varepsilon > 0$, there exist a birational morphism $\nu : \mathscr{X}' \to \mathscr{X}$, an integer $p \geqslant 1$ together with a decomposition $\nu^*\overline{\mathscr{L}}^{\otimes p} \cong \overline{\mathscr{A}} \otimes \overline{\mathscr{M}}$ such that*

1) *$\overline{\mathscr{M}}$ is effective and $\overline{\mathscr{A}}$ is arithmetically ample;*
2) *one has $p^{-d}\widehat{\operatorname{vol}}(\overline{\mathscr{A}}) \geqslant \widehat{\operatorname{vol}}(\overline{\mathscr{L}}) - \varepsilon$.*

REMARK 4.4. – For establishing Theorem 4.3, it suffices to prove the following weaker result for any big Hermitian line bundle $\overline{\mathscr{L}}$:

> *For any $\varepsilon > 0$ and any $\lambda > 0$, there exist a birational morphism $\nu : \mathscr{X}' \to \mathscr{X}$ with $\mathscr{X}'_{\mathbb{Q}}$ smooth, an integer $p \geqslant 1$ and a decomposition $\nu^*\overline{\mathscr{L}}^{\otimes p} \cong \overline{\mathscr{A}} \otimes \overline{\mathscr{M}}$ such that $\overline{\mathscr{M}}$ is effective, $\overline{\mathscr{A}} \otimes \pi^*\overline{\mathcal{O}}_{\lambda p}$ is nef and that $p^{-d}\widehat{\operatorname{vol}}(\overline{\mathscr{A}}) \geqslant \widehat{\operatorname{vol}}(\overline{\mathscr{L}}) - \varepsilon$.*

We show that this assertion implies Theorem 4.3. Let $\overline{\mathscr{S}}$ be a big Hermitian line bundle on $\mathscr{X}$, $\lambda > 0$ and $m \in \mathbb{N}_*$ sufficiently large so that $\overline{\mathscr{L}}^{\otimes m} \otimes \overline{\mathscr{S}}^{\vee} \otimes \pi^*\overline{\mathcal{O}}_{-\lambda}$ is big. The above assertion implies that, for any $\varepsilon > 0$, there exist a birational morphism $\nu : \mathscr{X}' \to \mathscr{X}$ with $\mathscr{X}'_{\mathbb{Q}}$ smooth, an integer $p > 0$ and a decomposition

$$\nu^*(\overline{\mathscr{L}}^{\otimes mp} \otimes \overline{\mathscr{S}}^{\vee\otimes p} \otimes \pi^*\overline{\mathcal{O}}_{-\lambda p}) \cong \overline{\mathscr{A}} \otimes \overline{\mathscr{M}}$$

with $\overline{\mathscr{A}} \otimes (\pi\nu)^*\overline{\mathcal{O}}_{\lambda p}$ nef, $\overline{\mathscr{M}}$ effective, such that

$$p^{-d}\widehat{\operatorname{vol}}(\overline{\mathscr{A}}) \geqslant \widehat{\operatorname{vol}}(\overline{\mathscr{L}}^{\otimes m} \otimes \overline{\mathscr{S}}^{\vee} \otimes \pi^*\overline{\mathcal{O}}_{-\lambda}) - \varepsilon.$$

Let $\overline{\mathscr{S}}'$ be an ample Hermitian line bundle on $\mathscr{X}'$. Since $\nu^*\overline{\mathscr{S}}$ is big (see [13, Theorem 4.3]), there exists an integer $q > 0$ such that $\nu^*\overline{\mathscr{S}}^{\otimes pq} \otimes \overline{\mathscr{S}}'^{\vee}$ is effective. Consider the decomposition

$$\nu^*\overline{\mathscr{L}}^{\otimes mpq} \cong \big(\overline{\mathscr{A}}^{\otimes q} \otimes (\pi\nu)^*\overline{\mathcal{O}}_{\lambda pq} \otimes \overline{\mathscr{S}}'\big) \otimes \big(\overline{\mathscr{M}}^{\otimes q} \otimes \nu^*\overline{\mathscr{S}}^{\otimes pq} \otimes \overline{\mathscr{S}}'^{\vee}\big).$$

Note that $\overline{\mathscr{A}}' := \overline{\mathscr{A}}^{\otimes q} \otimes (\pi\nu)^*\overline{\mathcal{O}}_{\lambda pq} \otimes \overline{\mathscr{S}}'$ is ample (see Remark 4.2 (1)), $\overline{\mathscr{M}}' := \overline{\mathscr{M}}^{\otimes q} \otimes \nu^*\overline{\mathscr{S}}^{\otimes pq} \otimes \overline{\mathscr{S}}'^{\vee}$ is effective, and

$$(mpq)^{-d}\widehat{\operatorname{vol}}(\overline{\mathscr{A}}') \geqslant (mpq)^{-d}\widehat{\operatorname{vol}}(\overline{\mathscr{A}}^{\otimes q}) \geqslant m^{-d}\big(\widehat{\operatorname{vol}}(\overline{\mathscr{L}}^{\otimes m} \otimes \overline{\mathscr{S}}^{\vee} \otimes \overline{\mathcal{O}}_{-\lambda}) - \varepsilon\big).$$

By the continuity of the arithmetic volume function (cf. [13, Theorem 5.4]), we obtain Theorem 4.3.

Proof. – By [13, Theorem 4.3], we may assume that $\mathscr{X}$ is generically smooth. For any integer $p \geqslant 1$ such that $B_p^{[0]} \neq 0$, let $\phi_p : \mathscr{X}_p \to \mathscr{X}$ be the blow-up (twisted by $\mathscr{L}$) of $\mathscr{X}$ along the base locus of $\mathscr{B}_p^{[0]}$. In other words, $\mathscr{X}_p$ is defined as

$$\mathscr{X}_p = \mathrm{Proj}\Big(\mathrm{Im}\Big(\bigoplus_{n\geqslant 0} \pi^* \mathscr{B}_{np}^{(p)} \longrightarrow \bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes np} \Big)\Big).$$

Let $\mathscr{A}_p = \mathscr{O}_{\mathscr{X}_p}(1)$ and $\mathscr{M}_p$ be the invertible sheaf corresponding to the exceptional divisor. Let s be the global section of $\mathscr{M}_p$ which trivializes $\mathscr{M}_p$ outside the exceptional divisor. By definition, one has $\phi_p^* \mathscr{L}^{\otimes p} \cong \mathscr{A}_p \otimes \mathscr{M}_p$. On the other hand, the canonical homomorphism $\phi_p^* \pi^* \mathscr{B}_p^{[0]} \to \mathscr{A}_p$ induces a morphism $i_p : \mathscr{X}_p \to \mathbb{P}(\mathscr{B}_p^{[0]})$ such that $i_p^*(\mathscr{L}_p) \cong \mathscr{A}_p$, where $\mathscr{L}_p = \mathscr{O}_{\mathbb{P}(\mathscr{B}_p^{[0]})}(1)$. The restriction of global sections of $\mathscr{L}_p^{\otimes n}$ gives an injective homomorphism

$$\mathrm{Im}(S^n \mathscr{B}_p^{[0]} \to \mathscr{B}_{np}) = \mathscr{B}_{np}^{(p)} \longrightarrow H^0(\mathscr{X}_p, \mathscr{A}_p^{\otimes n}), \tag{25}$$

where the last $\mathscr{O}_K$-module is considered as a submodule of $H^0(\mathscr{X}_p, \phi_p^* \mathscr{L}^{\otimes p})$ via s.

For any integer $n \geqslant 1$ and any embedding $\sigma : K \to \mathbb{C}$, denote by $\|\cdot\|_{\sigma,n}$ the quotient Hermitian norm on $\mathscr{A}_{p,\sigma}$ induced by the surjective homomorphism $\phi_p^* \pi^* \mathscr{B}_{np}^{(p)} \to \mathscr{A}_p^{\otimes n}$, where on $\mathscr{B}_{np}^{(n)}$ we have chosen the John norm $\|\cdot\|_{\sigma,\mathrm{John}}$ associated to the sup-norm $\|\cdot\|_{\sigma,\sup}$ (recall that one has $\sqrt{\mathrm{rk}(B_{np}^{(p)})}\|\cdot\|_{\sigma,\sup} \geqslant \|\cdot\|_{\sigma,\mathrm{John}} \geqslant \|\cdot\|_{\sigma,\sup}$, see [8, §4.2] for details), and the corresponding Hermitian vector bundle will be denoted by $\overline{\mathscr{B}}_{np,\mathrm{J}}^{(n)}$. Thus the Hermitian norms on $\mathscr{A}_p$ are positive and smooth. Now let $\sigma : K \to \mathbb{C}$ be an embedding and x be a complex point of $\mathscr{X}_{p,\sigma}$ outside the exceptional divisor. It corresponds to a one-dimensional quotient of $B_{p,\sigma}^{[0]}$, which induces, for any integer $n \geqslant 1$, a one-dimensional quotient $l_{n,x}$ of $B_{np,\sigma}^{(p)}$. By a classical result on convex bodies in Banach space (Hahn-Banach theorem), there exists an affine hyperplane parallel to the $\mathrm{Ker}(B_{np,\sigma}^{(p)} \to l_{n,x})$ and tangent to the closed unit ball of $B_{np,\sigma}^{(p)}$. In other words, there exists $v \in B_{np,\sigma}^{(p)}$ whose image in $\mathscr{A}_{p,\sigma}^{\otimes n}(x)$ has norm $\|v\|_{\sigma,\mathrm{John}}$, which is bounded from below by $\|v\|_{\sigma,\sup}$. Note that, as a section of $L_\sigma^{\otimes n}$ over $X_\sigma(\mathbb{C})$, one has $\|v_x\|_\sigma \leqslant \|v\|_{\sigma,\sup}$, where $\|\cdot\|_\sigma$ denotes the Hermitian metric of index σ of $\overline{\mathscr{L}}$. Hence, for any section u of $\mathscr{A}_{p,\sigma}$ over a neighborhood of x, one has $\|u_x\|_{\sigma,n} \geqslant \|u_x \otimes s_x\|_\sigma$. In fact, by dilation we may assume that $u_x^{\otimes n}$ equals the image of v_x in $\mathscr{A}_{p,\sigma}^{\otimes n}(x)$, and hence

$$\|u_x\|_{\sigma,n} = \|v\|_{\sigma,\mathrm{John}}^{\frac{1}{n}} \geqslant \|v\|_{\sigma,\sup}^{\frac{1}{n}} \geqslant \|v_x\|_\sigma^{\frac{1}{n}} = \|u_x \otimes s_x\|_\sigma.$$

Therefore, if we equip $\mathscr{A}_p$ with metrics $\alpha_n = (\|\cdot\|_{\sigma,n})_{\sigma:K\to\mathbb{C}}$ and define $(\mathscr{M}_p, \beta_n) := \phi_p^* \overline{\mathscr{L}} \otimes (\mathscr{A}_p, \alpha_n)^\vee$. Then the section s of $\mathscr{M}_p$ is an effective section. For any integer $n \geqslant 1$, one has

$$p^{-d}\widehat{\mathrm{vol}}(\mathscr{A}_p, \alpha_n) \geqslant \widehat{\mathrm{vol}}(B^{(p)}, \alpha_n),$$

where

$$\widehat{\mathrm{vol}}(B^{(p)}, \alpha_n) := \lim_{m\to+\infty} \frac{\widehat{h}^0(B_{mp}^{(p)}, \|\cdot\|_{\sigma,n,\sup})}{m^d/d!}.$$

Note that, for any $\sigma : K \to \mathbb{C}$ and any element $v \in B_{np,\sigma}^{(p)}$ considered as a section in $H^0(\mathscr{X}_{p,\sigma}(\mathbb{C}), \mathscr{A}_{p,\sigma}^{\otimes n})$ via (25), the sup-norms of v relatively to the metrics in α_n are bounded from above by the John norms of v considered as a section of L_σ corresponding to the sup-norms induced by the norms of $\overline{\mathcal{L}}$. Note that, for any integer $m \geqslant 1$, one has $e_{\min}(B_{mp}^{(p)}) \geqslant 0$ and $e_{\min}(B_{mp}^{(p)}, \alpha_n) \geqslant -m\lambda_{n,p}$. Thus Corollary A.2 combined with (20) implies that

$$\limsup_{n\to+\infty} \widehat{\mathrm{vol}}(B^{(p)}, \alpha_n) \geqslant \widehat{\mathrm{vol}}(B^{(p)}).$$

Therefore, by Theorem 4.1, for any $\varepsilon > 0$, there exists an integer $p \geqslant 1$ such that $\limsup_{n\to+\infty} p^{-d}\widehat{\mathrm{vol}}(\mathcal{A}_p, \alpha_n) \geqslant \mathrm{vol}(\overline{\mathcal{L}}) - \varepsilon/2$. For any $n, p \in \mathbb{N}_*$, let $\lambda_{n,p} = \frac{1}{2n}\log(\mathrm{rk}\, B_{np}^{(p)})$. Note that $(\mathcal{A}_p, \alpha_n) \otimes \overline{\mathcal{O}}_{2\lambda_{n,p}}$ is nef (see Remark 4.2) since its n^{th} tensor power can be written as a quotient of $\phi_p^*\pi^*(\overline{\mathcal{B}}_{np,\mathrm{J}}^{(p)} \otimes \overline{\mathcal{O}}_{2n\lambda_{n,p}})$ and

$$e_{\min}(\overline{\mathcal{B}}_{np,\mathrm{J}}^{(p)} \otimes \overline{\mathcal{O}}_{2n\lambda_{n,p}}) \geqslant e_{\min}(\overline{\mathcal{B}}_{np,\mathrm{J}}^{(p)}) + 2n\lambda_{n,p} \geqslant \frac{1}{2}\log(\mathrm{rk}\, B_{np}^{(p)}).$$

As $\lim_{n\to+\infty} \lambda_{n,p} = 0$, we have proved the assertion in Remark 4.4, which is actually equivalent to Theorem 4.3. □

4.3. Approximating subalgebras

In this section, we show that if a positive finitely generated subalgebra of $\overline{\mathcal{B}}$ approximates well the arithmetic volume of $\overline{\mathcal{L}}$, then it also approximates well the asymptotic measure of $\overline{\mathcal{L}}$ truncated at 0.

Let $p \geqslant 1$ be an integer. Assume that $\overline{\mathcal{L}}^{\otimes p}$ is decomposed as $\overline{\mathcal{A}} \otimes \overline{\mathcal{M}}$, where $\overline{\mathcal{A}}$ is arithmetically ample and $\overline{\mathcal{M}}$ has a non-zero effective section s. Through the section s we may consider the section algebra $\bigoplus_{n\geqslant 0} H^0(X, \mathcal{A}_K^{\otimes n})$ as a graded sub-K-algebra of B. As $\overline{\mathcal{A}}$ is ample, for sufficiently large n, one has $H^0(X, \mathcal{A}_K^{\otimes n}) \subset \mathcal{F}_0(B_{np})$ (cf. [21, Corollary 4.8]).

Proposition 4.5. – *Let $p \geqslant 1$ be an integer and S be a graded subalgebra of B generated by a subspace of B_p. For any integer $n \geqslant 1$, let $\overline{\mathcal{S}}_n$ be the saturated sub-$\mathcal{O}_K$-module of B_n, equipped with induced metrics, and such that $\mathcal{S}_{n,K} = S_n$; let $\nu_{\overline{\mathcal{S}}_n}$ be the measure associated to the minima filtration of $\overline{\mathcal{S}}_n$. Denote by ν the vague limit of the measure sequence $(T_{\frac{1}{np}}\nu_{\overline{\mathcal{S}}_{np}})_{n\geqslant 1}$. Then for any $x \in \mathbb{R}$, one has*

$$\mathrm{vol}(S)\nu([x, +\infty[) \leqslant \mathrm{vol}(L)\nu_{\overline{\mathcal{L}}}([x, +\infty[), \tag{26}$$

where $\nu_{\overline{\mathcal{L}}}$ is the vague limit of $(T_{\frac{1}{n}}\nu_{\overline{\mathcal{B}}_n})_{n\geqslant 1}$, $\nu_{\overline{\mathcal{B}}_n}$ being the measure associated to the minima filtration of $\overline{\mathcal{B}}_n$. Furthermore, if $e_{\min}(\overline{\mathcal{S}}_{np}) > 0$ holds for sufficiently large n, then

$$\mathrm{vol}(S) \leqslant \mathrm{vol}(L)\nu_{\overline{\mathcal{L}}}([0, +\infty[). \tag{27}$$

Proof. – For any $x \in \mathbb{R}$, one has

$$\mathrm{rk}(S_{np})\nu_{\overline{\mathcal{S}}_{np}}([npx, +\infty[) \leqslant \mathrm{rk}(B_{np})\nu_{\overline{\mathcal{B}}_{np}}([npx, +\infty[),$$

since these two quantities are respectively the ranks of $\mathcal{F}_{npx}S_{np}$ and $\mathcal{F}_{npx}B_{np}$. By letting $n \to +\infty$, one obtains that, for any $x \in \mathbb{R}$,

$$\mathrm{vol}(S)\nu([x, +\infty[) \leqslant \mathrm{vol}(L)\nu_{\overline{\mathcal{L}}}([x, +\infty[).$$

Since the positivity condition on the last minimum implies that $\nu([0, +\infty[) = 1$, one obtains (27). □

COROLLARY 4.6. – *With the notation of Proposition 4.5, assume that*

$$\widehat{\mathrm{vol}}(S) := \lim_{n\to\infty} \frac{\widehat{h}^0(\overline{\mathscr{S}}_{np})}{(np)^d/d!} \geqslant (1-\varepsilon)\widehat{\mathrm{vol}}(\overline{\mathscr{L}}),$$

where $0<\varepsilon<1$ *is a constant. Then one has*

$$0 \leqslant \delta_K d \int_0^{+\infty} \Big[\mathrm{vol}(L)\nu_{\overline{\mathscr{L}}}([x,+\infty[) - \mathrm{vol}(S)\nu([x,+\infty[)\Big]\,\mathrm{d}x \leqslant \varepsilon\widehat{\mathrm{vol}}(\overline{\mathscr{L}}). \tag{28}$$

Proof. – By (20), one obtains

$$\widehat{\mathrm{vol}}(\overline{\mathscr{L}}) = \delta_K d\,\mathrm{vol}(L)\int_0^{+\infty} t\,\nu_{\overline{\mathscr{L}}}(\mathrm{d}t) = \delta_K d\,\mathrm{vol}(L)\int_0^{+\infty} \nu_{\overline{\mathscr{L}}}([x,+\infty[)\,\mathrm{d}x.$$

Similarly,

$$\widehat{\mathrm{vol}}(S) = \delta_K d\,\mathrm{vol}(S)\int_0^{+\infty}\nu([x,+\infty[)\,\mathrm{d}x.$$

Hence the inequality (28) results from (26). □

Appendix

Comparison of filtered graded algebras

Let $B=\bigoplus_{n\geqslant 0}B_n$ be an integral graded algebra of finite type over an infinite field K. We suppose that $B_1\neq 0$ and that B is generated as a K-algebra by B_1. Assume that each B_n is equipped with an $\mathbb{R}$-filtration $\mathscr{F}$ such that B becomes a filtered graded algebra (see §2.3 for definition). For all integers $m,n\geqslant 0$, let $\mathscr{F}^{(m)}$ be another $\mathbb{R}$-filtration on B_n such that B equipped with $\mathbb{R}$-filtrations $\mathscr{F}^{(m)}$ is filtered. For all integers $m,n\geqslant 1$, let $\nu_n = T_{\frac{1}{n}}\nu_{(B_n,\mathscr{F})}$ and $\nu_n^{(m)} = T_{\frac{1}{n}}\nu_{(B_n,\mathscr{F}^{(m)})}$. Assume in addition that $\lambda_{\max}(B_n,\mathscr{F})\ll n$ and $\lambda_{\max}(B_n,\mathscr{F}^{(m)})\ll_m n$. By Theorem 2.11, the sequence of measures $(\nu_n^{(m)})_{n\geqslant 1}$ (resp. $(\nu_n)_{n\geqslant 1}$) converges vaguely to a Borel probability which we denote by $\nu^{(m)}$ (resp. ν).

The purpose of this section is to establish the following comparison result:

PROPOSITION A.1. – *Let* φ *be an increasing, concave and Lipschitz function on* $\mathbb{R}$. *Assume that, for any* $m\geqslant 1$ *and any* $t\in\mathbb{R}$, *one has* $\mathscr{F}_tB_m\subset\mathscr{F}_t^{(m)}B_m$, *then*

$$\limsup_{m\to+\infty}\int_{\mathbb{R}}\varphi\,\mathrm{d}\nu^{(m)} \geqslant \int_{\mathbb{R}}\varphi\,\mathrm{d}\nu. \tag{29}$$

Proof. – By Noether's normalization theorem, there exists a graded subalgebra A of B such that A is isomorphic to the polynomial algebra generated by A_1. We still use $\mathscr{F}^{(m)}$ (resp. $\mathscr{F}$) to denote the induced filtrations on A. Let $\widetilde{\nu}_n = T_{\frac{1}{n}}\nu_{(A_n,\mathscr{F})}$ and $\widetilde{\nu}_n^{(m)} = T_{\frac{1}{n}}\nu_{(A_n,\mathscr{F}^{(m)})}$. For any integer $m\geqslant 1$ and any $t\in\mathbb{R}$, one still has $\mathscr{F}_tA_m\subset\mathscr{F}_t^{(m)}A_m$. Furthermore, by [3, Proof of Theorem 3.4.3, Step 1], the sequence of measures $(\widetilde{\nu}_n^{(m)})_{n\geqslant 1}$ (resp. $(\widetilde{\nu}_n)_{n\geqslant 1}$) converges vaguely to $\nu^{(m)}$ (resp. ν). Therefore, we may suppose that $B=A$ is a polynomial algebra. In this case, [3, Proposition 3.3.3] implies that

$$nm\int\varphi\,\mathrm{d}\nu_{nm}^{(m)} \geqslant nm\int\varphi\,\mathrm{d}\nu_m^{(m)} \geqslant nm\int\varphi\,\mathrm{d}\nu_m,$$

since $\nu_m^{(m)} \succ \nu_m$. By letting $n \to \infty$, we obtain

$$\int \varphi \,\mathrm{d}\nu^{(m)} \geqslant \int \varphi \,\mathrm{d}\nu_m,$$

which implies (29). □

In the following, we apply Proposition A.1 to study algebras in metrized vector bundles. From now on, K denotes a number field. We assume given an $\mathcal{O}_K$-algebra $\mathcal{B} = \bigoplus_{n\geqslant 0} \mathcal{B}_n$, generated by $\mathcal{B}_1$, and such that

1) each $\mathcal{B}_n$ is a projective $\mathcal{O}_K$-module of finite type;
2) for any integer $n \geqslant 0$, $B_n = \mathcal{B}_{n,K}$;
3) the algebra structure of $\mathcal{B}$ is compatible to that of B.

For each integer $n \geqslant 1$, assume that g is a family of norms on $\mathcal{B}_n$ such that $(\mathcal{B}_n, g)$ becomes a metrized vector bundle on $\operatorname{Spec} \mathcal{O}_K$. For all integers $n \geqslant 1$ and $m \geqslant 1$, let $g^{(m)}$ be another metric structure on $\mathcal{B}_n$ such that $(\mathcal{B}_n, g^{(m)})$ is also a metrized vector bundle on $\operatorname{Spec} \mathcal{O}_K$. Let $\nu_{(\mathcal{B}_n,g)}$ and $\nu_{(\mathcal{B}_n,g^{(m)})}$ be the measures associated to the minima filtration of $(\mathcal{B}_n, g)$ and of $(\mathcal{B}_n, g^{(m)})$, respectively.

Corollary A.2. – *With the notation above, assume in addition that*

1) *$(\mathcal{B}, g)$ and all $(\mathcal{B}, g^{(m)})$ verify the conditions in Remark 3.7;*
2) *$e_{\max}(\mathcal{B}_n, g) \ll n$ and $e_{\max}(\mathcal{B}_n, g^{(m)}) \ll_m n$;*
3) *the identity homomorphism $\mathrm{Id} : (\mathcal{B}_m, g) \to (\mathcal{B}_m, g^{(m)})$ is effective, that is, for any $\sigma : K \to \mathbb{C}$, the norm of $\mathrm{Id}_{\sigma,\mathbb{C}}$ is $\leqslant 1$.*

Let ν and $\nu^{(m)}$ be respectively the limit measures of $(T_{\frac{1}{n}}\nu_{(\mathcal{B}_n,g)})_{n\geqslant 1}$ and $(T_{\frac{1}{n}}\nu_{(\mathcal{B}_n,g^{(m)})})_{n\geqslant 1}$. Then for any increasing, concave and Lipschitz function φ on $\mathbb{R}$, one has

$$\limsup_{m\to\infty} \int_{\mathbb{R}} \varphi \,\mathrm{d}\nu^{(m)} \geqslant \int_{\mathbb{R}} \varphi \,\mathrm{d}\nu.$$

In particular, if $\liminf_{n\to\infty} \frac{1}{n} e_{\min}(\mathcal{B}_n, g) \geqslant 0$ and if $\liminf_{m\to\infty} \liminf_{n\to\infty} \frac{1}{n} e_{\min}(\mathcal{B}_n, g^{(m)}) \geqslant 0$, then

$$\limsup_{m\to\infty} \int_0^{+\infty} x\,\nu^{(m)}(\mathrm{d}x) \geqslant \int_0^{+\infty} x\,\nu(\mathrm{d}x). \tag{30}$$

Proof. – The first assertion is a direct consequence of Proposition A.1. In particular, one has

$$\limsup_{m\to\infty} \int_{\mathbb{R}} x\,\nu^{(m)}(\mathrm{d}x) \geqslant \int_{\mathbb{R}} x\,\nu(\mathrm{d}x).$$

The hypothesis $\liminf_{n\to\infty} \frac{1}{n} e_{\min}(\mathcal{B}_n, g) \geqslant 0$ implies that the support of ν is bounded from below by 0, so

$$\int_0^{+\infty} x\,\nu(\mathrm{d}x) = \int_{\mathbb{R}} x\,\nu(\mathrm{d}x).$$

For any integer $m \geqslant 1$, let $a_m = \liminf_{n\to\infty} \frac{1}{n} e_{\min}(\mathcal{B}_n, g_m)$ and $b_m = \min(a_m, 0)$. One has $\int_{\mathbb{R}} x\,\nu^{(m)}(\mathrm{d}x) = \int_{b_m}^{+\infty} x\,\nu^{(m)}(\mathrm{d}x)$. Note that

$$0 \geqslant \int_{b_m}^{+\infty} x\,\nu^{(m)}(\mathrm{d}x) - \int_0^{+\infty} x\,\nu^{(m)}(\mathrm{d}x) = \int_{b_m}^0 x\,\nu^{(m)}(\mathrm{d}x) \geqslant b_m.$$

As b_m converges to 0 when $m \to \infty$, we obtain (30). □

REFERENCES

[1] S. Boucksom, C. Favre, M. Jonsson, Differentiability of volumes of divisors and a problem of Teissier, *J. Algebraic Geom.* **18** (2009), 279–308.

[2] N. Bourbaki, *Éléments de mathématique. Fasc. XIII. Livre VI: Intégration. Chapitres 1, 2, 3 et 4: Inégalités de convexité, espaces de Riesz, mesures sur les espaces localement compacts, prolongement d'une mesure, espaces L^p*, Hermann, 1965.

[3] H. Chen, Convergence des polygones de Harder-Narasimhan, to appear in *Mémoires de la SMF*.

[4] J.-P. Demailly, L. Ein, R. Lazarsfeld, A subadditivity property of multiplier ideals, *Michigan Math. J.* **48** (2000), 137–156.

[5] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, M. Popa, Asymptotic invariants of line bundles, *Pure Appl. Math. Q.* **1** (2005), 379–403.

[6] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, M. Popa, Restricted volumes and base loci of linear series, *Amer. J. Math.* **131** (2009), 607–651.

[7] T. Fujita, Approximating Zariski decomposition of big line bundles, *Kodai Math. J.* **17** (1994), 1–3.

[8] É. Gaudron, Pentes des fibrés vectoriels adéliques sur un corps global, *Rend. Semin. Mat. Univ. Padova* **119** (2008), 21–95.

[9] H. Gillet, C. Soulé, On the number of lattice points in convex symmetric bodies and their duals, *Israel J. Math.* **74** (1991), 347–357.

[10] R. Lazarsfeld, *Positivity in algebraic geometry. II*, Ergebnisse Math. Grenzg. **49**, Springer, 2004.

[11] R. Lazarsfeld, M. Mustaţă, Convex bodies associated to linear series, *Ann. Sci. Éc. Norm. Supér.* **42** (2009), 783–835.

[12] A. Moriwaki, Arithmetic height functions over finitely generated fields, *Invent. Math.* **140** (2000), 101–142.

[13] A. Moriwaki, Continuity of volumes on arithmetic varieties, *J. Algebraic Geom.* **18** (2009), 407–457.

[14] A. Moriwaki, Continuous extension of arithmetic volumes, *Int. Math. Res. Not.* **2009** (2009), 3598–3638.

[15] A. Okounkov, Brunn-Minkowski inequality for multiplicities, *Invent. Math.* **125** (1996), 405–411.

[16] R. Rumely, C. F. Lau, R. Varley, Existence of the sectional capacity, Mem. Amer. Math. Soc. **145**, 2000.

[17] S. Takagi, Fujita's approximation theorem in positive characteristics, *J. Math. Kyoto Univ.* **47** (2007), 179–202.

[18] X. Yuan, Big line bundles over arithmetic varieties, *Invent. Math.* **173** (2008), 603–649.

[19] X. Yuan, On volumes of arithmetic line bundles, preprint arXiv:0811.0226.

[20] O. Zariski, The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface, *Ann. of Math.* **76** (1962), 560–615.

[21] S. Zhang, Positive line bundles on arithmetic varieties, *J. Amer. Math. Soc.* **8** (1995), 187–221.

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