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GRADED LIE ALGEBRAS WITH FINITE POLYDEPTH

BY YVES FELIX, STEPHEN HALPERIN AND JEAN-CLAUDE THOMAS

ABSTRACT. – If A is a graded connected algebra then we define a new invariant, $\operatorname{polydepth} A$, which is finite if $\operatorname{Ext}_A^*(M,A) \neq 0$ for some A-module M of at most polynomial growth. THEOREM 1: If $f: X \to Y$ is a continuous map of finite category, and if the orbits of $H_*(\Omega Y)$ acting in the homology of the homotopy fibre grow at most polynomially, then $H_*(\Omega Y)$ has finite polydepth. THEOREM 5: If L is a graded Lie algebra and $\operatorname{polydepth} UL$ is finite then either L is solvable and UL grows at most polynomially or else for some integer d and all r, $\sum_{i=k+1}^{k+d} \dim L_i \geqslant k^r$, $k \geqslant$ some $k(r)$.

RÉSUMÉ. – Si A est une algèbre graduée connexe nous définissons un nouvel invariant, appelé la profondeur polynomiale noté $\operatorname{polydepth} A$, qui est fini s'il existe un A-module gradué M ayant une croissance au plus polynomiale tel que $\operatorname{Ext}_A^*(M,A) \neq 0$. THÉORÈME 1 : si $f: X \to Y$ est une application continue de LS-catégorie finie et si les orbites de l'action de $H_*(\Omega Y)$ sur l'homologie de la fibre homotopique de f possèdent une croissance au plus polynomiale alors $\operatorname{polydepth} H_*(\Omega Y)$ est finie. THÉORÈME 5 : si L est une algèbre de Lie graduée et si $\operatorname{polydepth} UL$ est fini alors, soit L est résoluble et UL possède une croissance au plus polynomiale, soit il existe un entier d tel que pour tout entier r on ait $\sum_{i=k+1}^{k+d} \dim L_i \geqslant k^r$ pour tous les k plus grands qu'un certain $k(r)$.

We work over a field $\Bbbk$ of characteristic different from 2. If $V = \{V_k\}$ is a graded vector space we denote by $V^\# = \{\operatorname{Hom}_{\Bbbk}(V_k, \Bbbk)\}$ the dual graded vector space. A graded Lie algebra is a graded vector space L, equipped with a bilinear map $[\,,\,]: L_i \times L_j \to L_{i+j}$ satisfying

$$[x,y] + (-1)^{ij}[y,x] = 0$$

and

$$\big[x,[y,z]\big] = \big[[x,y],z\big] + (-1)^{ij}\big[y,[x,z]\big]$$

for $x \in L_i$, $y \in L_j$, $z \in L$. It follows that $3[x,[x,x]] = 0$ for x of odd degree, and so if $\operatorname{char} k = 3$ we further require that $[x,[x,x]] = 0$. Finally we consider only graded Lie algebras satisfying $L = \{L_i\}_{i \geqslant 1}$ and each L_i is finite dimensional. (Any graded vector space V with each V_i finite dimensional is said to have *finite type*.)

The universal enveloping algebra of L is denoted by UL and it satisfies the classical Poincaré–Birkhoff–Witt Theorem (in characteristic 3 this uses the $[x,[x,x]] = 0$ requirement).

Important examples appear in topology. Let X be a simply connected topological space with rational homology of finite type. Then the rational homotopy Lie algebra L_X of X is defined by

$$L_X = \pi_*(\Omega X) \otimes \mathbb{Q}; \qquad [\,,\,] = \text{Samelson product},$$

and the Hurewicz map extends to an isomorphism, [10],

$$UL_X \xrightarrow{\cong} H_*(\Omega X;\mathbb{Q}).$$

Analogously, if X is a finite n-dimensional and r-connected CW complex, then for primes $p > n/r$, $H_*(\Omega X;\mathbb{F}_p) = UE$ for some graded Lie algebra E [8].

If M is a module over a (graded) algebra A then the *grade* of M, $\operatorname{grade} M$, is the least integer q (or ∞) such that $\operatorname{Ext}_A^q(M,A) \neq 0$. And if $V = \{V_i\}_{i\geqslant 0}$ is a graded vector space then V has *at most polynomial growth* if for some constant C, and some non-negative integer, d, $\sum_{i\leqslant n} \dim V_i \leqslant Cn^d$, $n \geqslant 1$. In this case the least such d is called the *polynomial bound* for the growth of V and is denoted by $\operatorname{polybd}(V)$. If V does not have at most polynomial growth we put $\operatorname{polybd} V = \infty$ and we say that V grows faster than any polynomial.

In this paper we combine these two notions in the

DEFINITION. – The *polygrade* of an A-module, M, is the sum, $\operatorname{grade} M + \operatorname{polybd} M$, and the *polydepth of* A is the least integer (or ∞) occurring as the polygrade of an A-module.

In the case $A = UL$ the unique augmentation $UL \to \Bbbk$ makes $\Bbbk$ into a UL-module, and by definition, the grade of $\Bbbk$ is the *depth* of UL. Since $\operatorname{polybd} \Bbbk = 0$ it follows that:

(1) $$\operatorname{polydepth} UL \leqslant \operatorname{depth} UL.$$

Moreover (cf. Proposition 1.6) if $\dim L < \infty$ then equality holds. We shall abuse notation and refer to these invariants respectively as $\operatorname{polydepth} L$ and $\operatorname{depth} L$.

Note that $\operatorname{Ext}^0_{UL}(UL, UL)$ contains the identity map and so

(2) $$\operatorname{polydepth} L \leqslant \operatorname{polybd} UL.$$

Observe as well that for any graded vector space M, $\operatorname{polybd} M = 0$ if and only if $\dim M$ is finite. Thus $\operatorname{polydepth} L = 0$ if and only if $\operatorname{depth} L = 0$, which happens if and only if L is finite dimensional and concentrated in odd degrees.

Depth has been a useful concept in topology because, on the one hand, Lusternik–Schnirelmann category satisfies [1]

$$\operatorname{depth} L_X \leqslant \operatorname{cat} X$$

and, on the other hand [3–5], finite depth has important implications for the structure of a graded Lie algebra.

The purpose of this paper is to show that essentially the same implications follow from the weaker hypothesis that $\operatorname{polydepth} L$ is finite, while simultaneously identifying a larger class of topological spaces and Lie algebras for which the weaker hypothesis holds.

Indeed, we have

THEOREM 1. – *If $F \to X \xrightarrow{f} Y$ is a fibration of path-connected spaces, then*

$$\operatorname{polydepth} H_*(\Omega Y) \leqslant \operatorname{polybd} H_*(F) + \operatorname{cat} f.$$

Proof. – The fibration determines an action up to homotopy of ΩY on F, which makes $H_*(F)$ into an $H_*(\Omega Y)$-module. According to [6], $\operatorname{grade} H_*(F) \leqslant \operatorname{cat} f$. □

Our main structural theorems read:

THEOREM 2. – *Let* $E(L)$ *denote the linear span of elements* $x \in L_{\text{even}}$ *such that* $\operatorname{ad} x$ *acts nilpotently on each* $y \in L$*. Then*

$$\dim E(L) \leqslant \operatorname{polydepth} L.$$

THEOREM 3. – *The following conditions on a graded Lie algebra* L *are equivalent*

(i) L *is the union of solvable ideals and* $\operatorname{polydepth} L$ *is finite*;

(ii) UL *grows at most polynomially* ($\operatorname{polybd} UL$ *is finite*);

(iii) L_{even} *is finite dimensional, and for some constant* C

$$\sum_{i \leqslant n} \dim L_i \leqslant C \log_2 n, \quad n \geqslant 1.$$

In this case L *is solvable.*

THEOREM 4. – *If* L *is a graded Lie algebra of finite polydepth then the union of the solvable ideals of* L *is a solvable ideal of finite polydepth.*

THEOREM 5. – *Suppose* $\operatorname{polydepth} L$ *is finite and* L *is not solvable. Then there is an integer* d *such that for all* $r \geqslant 1$:

$$\sum_{i=k+1}^{k+d} \dim L_i \geqslant k^r, \quad k \geqslant \textit{ some } k(r).$$

Remark. – In [7] it is shown that if $L = L_X$ where X is a finite 1-connected CW complex, then we may take $d = \dim X$ in Theorem 5.

1. Properties of polydepth

LEMMA 1.1. – *If* M *is a module for some graded algebra* A *of finite type and if* $\operatorname{Ext}_A^q(M, A) \neq 0$ *then* $\operatorname{Ext}_A^q(A \cdot x, A) \neq 0$ *for some* x *in a subquotient module of* M.

Proof. – Recall that $A^{\#} = \operatorname{Hom}_{\Bbbk}(A, \Bbbk)$. Then $\operatorname{Ext}_A^q(M, A)$ is the dual of $\operatorname{Tor}_q^A(M, A^{\#})$ and a direct limit argument shows that for some $x_1, \ldots, x_n \in M$,

$$\operatorname{Tor}_q^A(A \cdot x_1 + \cdots + A \cdot x_n, A^{\#}) \neq 0.$$

Now use the exact sequence associated to the inclusion

$$A \cdot x_1 + \cdots + A \cdot x_{n-1} \in A \cdot x_1 + \cdots + A \cdot x_n. \qquad \square$$

COROLLARY 1.2. – $\operatorname{Polydepth} A$ *is the least* m *such that* $\operatorname{polygrade} N = m$ *for some monogenic* A*-module* N.

Remark. – It follows from the Corollary that we may improve Theorem 1 to the inequality

(3) $$\operatorname{polydepth} H_*(\Omega Y) \leqslant \operatorname{polybd}\big(H_*(\Omega Y) \cdot \alpha\big) + \operatorname{cat} f, \quad \text{some } \alpha \in H_*(F).$$

PROPOSITION 1.3. – *Let* L *be a graded Lie algebra.*

(i) *Each ideal satisfies* $\operatorname{polydepth} I \leqslant \operatorname{polydepth} L$.

(ii) *Let E be a Lie subalgebra of L. If L has finite polydepth and if for each $x \in L/E$ the orbit $UE \cdot x$ has at most polynomial growth, then E has finite polydepth.*

(iii) *For n sufficiently large the sub Lie algebra E generated by $L_{\leqslant n}$ satisfies*

$$\operatorname{polydepth} E \leqslant \operatorname{polydepth} L.$$

Proof. – (i) This follows from the Hochschild–Serre spectral sequence, converging from $\operatorname{Ext}^p_{UL/I}(\Bbbk, \operatorname{Ext}^q_{UI}(M, UL))$ to $\operatorname{Ext}^{p+q}_{UL}(M, UL)$. (Note that since UL is UI-free, $\operatorname{grade}_{UI}(M)$ is the least q such that $\operatorname{Ext}^q_{UI}(M, UL) \neq 0$.)

(ii) As in Lemma 1.1, $\operatorname{Ext}^q_{UL}(M, UL)$ is dual to $\operatorname{Tor}^{UL}_q(M, (UL)^{\#})$, and this is the homology of the Cartan–Eilenberg–Chevalley complex $\wedge sL \otimes M \otimes (UL)^{\#}$. Write $L = E \oplus V$ and set $F_p = \wedge sE \otimes \wedge^{\leqslant p} sV \otimes M \otimes (UL)^{\#}$. This filtration determines a convergent spectral sequence, introduced by Koszul in [9], and which is the Hochschild–Serre spectral sequence when E is an ideal. The E^1-term of the spectral sequence is $\operatorname{Tor}^{UE}_q(\wedge^p sL/E \otimes M, (UL)^{\#})$, converging to $\operatorname{Tor}^{UL}_{p+q}(M, (UL)^{\#})$.

Each element $z \in \wedge^p sL/E \otimes M$ is contained in a finite sum of UE-modules of the form $s(UE \cdot x_1) \wedge \cdots \wedge s(UE \cdot x_p) \otimes M$ and it follows that

$$\operatorname{polybd}(UE \cdot z) \leqslant p \cdot \operatorname{polybd}(UE \cdot x) + \operatorname{polybd}(M)$$

for some $x \in L/E$. Choose M so that $\operatorname{polydepth} L = \operatorname{polygrade} M$ and apply Lemma 1.1 with $p + q = \operatorname{grade} M$.

(iii) If $\operatorname{Ext}^p_{UL}(M, UL)$ is non-zero and $\operatorname{polybd}(M) < \infty$ it suffices to choose E so that the restriction $\operatorname{Ext}^p_{UL}(M, UL) \to \operatorname{Ext}^p_{UE}(M, UL)$ is non-zero ([4], Proposition 3.1). □

COROLLARY 1.4 (of the proof of (ii)). – *Suppose for some $k \geqslant 1$ that $\operatorname{polybd}(UE \cdot x) \leqslant k$, $x \in L/E$. Then $\operatorname{polydepth} E \leqslant k \operatorname{polydepth} L$.*

COROLLARY 1.5. – *Let E be a sub-Lie algebra of a graded Lie algebra L. If L has finite polydepth and L/E has at most polynomial growth, then E has finite polydepth.*

Example 1. – Let $\mathbb{L}(V)$ be the free Lie algebra on a graded vector space V. Then for any graded Lie algebra L, $L \coprod \mathbb{L}(V)$ has depth 1. Thus the injection $L \to L \coprod \mathbb{L}(V)$ shows that each graded Lie algebra is a sub-Lie algebra of a Lie algebra of finite polydepth. The previous corollary gives restriction on a Lie algebra L for being a sub-Lie algebra of a Lie algebra of finite polydepth, K, when the quotient has at most polynomial growth.

PROPOSITION 1.6. – *If L is a finite dimensional graded Lie algebra then*

$$\operatorname{polydepth} L = \operatorname{depth} L.$$

Proof. – As observed in the introduction, $\operatorname{polydepth} L \leqslant \operatorname{depth} L$. On the other hand, by Lemma 1.1, $\operatorname{polydepth} L = \operatorname{polygrade} M$ for some monogenic module $M = UL \cdot x$. Now Theorem 3.1 in [3] asserts that $\operatorname{polygrade} M = \operatorname{depth} L$. □

2. Proof of Theorem 2

Suppose $I \subset L$ is an ideal. If $\operatorname{Ext}^m_{UL}(M, UL) \neq 0$, then $\operatorname{Ext}^p_{UL/I}(\operatorname{Tor}^{UI}_q(M, k), UL/I) \neq 0$, some $p + q = m$. (Same proof as in: [3], Lemma 4.3, for the case $M = \Bbbk$.) By Lemma 1.1 there is a monogenic UL/I-module N such that N is a subquotient of $\operatorname{Tor}^{UI}_q(M, \Bbbk)$, and grade $N \leqslant p$.

Now suppose L/I is finite dimensional. Then Theorem 3.1 in [3] asserts that

$$\operatorname{grade} N + \operatorname{polybd} N = \dim(L/I)_{\mathrm{even}}.$$

On the other hand, write $(L/I)_{\mathrm{even}} = V \oplus W$ where V is the image of $E(L)$. Let $x_i \in L_{\mathrm{even}}$, $y_j \in L_{\mathrm{odd}}$ and $z_k \in E(L)$ represent respectively bases of W, $(L/I)_{\mathrm{odd}}$ and V. Then the elements $x_1^{k_1} \cdots x_s^{k_s} y_1^{\varepsilon_1} \cdots y_t^{\varepsilon_t} z_1^{m_1} \cdots z_u^{m_u}$, where $\varepsilon_i = 0$ or 1, represent a basis for UL/I. Choose the z_k to act locally nilpotently in L. Then this basis applied to any $\omega \in \wedge^q sI$, shows that $\operatorname{polybd}(UL/I) \cdot \omega \leqslant \dim W$. Hence if $u \in \wedge^q sL \otimes M$ represents a generator of N then

$$\operatorname{polybd} N \leqslant \operatorname{polybd}(UL/I \cdot u) \leqslant \operatorname{polybd} M + \dim W.$$

Substitution in the equation above gives

$$\begin{aligned} \dim(L/I)_{\mathrm{even}} &\leqslant \operatorname{grade} N + \operatorname{polybd} M + \dim W \\ &\leqslant \operatorname{grade} M + \operatorname{polybd} M + \dim W. \end{aligned}$$

Choose M so that $\operatorname{grade} M + \operatorname{polybd}(M) = \operatorname{polydepth} L$ and choose $I = L_{>2k}$. Then $V \cong E(L)_{\leqslant 2k}$ and we have

$$\dim E(L)_{\leqslant 2k} \leqslant \operatorname{polydepth} L.$$

Since this holds for all k the theorem is proved.

3. Solvable Lie algebras

LEMMA 3.1. – *Let L be a Lie algebra concentrated in odd degrees. Then*

$$\operatorname{Ext}_{UL}(-, UL) = \operatorname{Hom}_{UL}(-, UL).$$

In particular

$$\operatorname{polydepth} L = \operatorname{polybd} UL.$$

Proof. – Since $L = L_{\mathrm{odd}}$ it is necessarily abelian. Now $\operatorname{Ext}_{UL}(-, UL)$ is the dual of $\operatorname{Tor}^{UL}(-, (UL)^\#)$ and this is the limit of $\operatorname{Tor}^{UL_{\leqslant n}}(-, (UL)^\#)$, which dualizes to

$$\operatorname{Ext}_{UL_{\leqslant n}}(-, UL).$$

Since $UL_{\leqslant n}$ is a finite dimensional exterior algebra and UL is $UL_{\leqslant n}$-free it follows that $\operatorname{Ext}^+_{UL_{\leqslant n}}(-, UL) = 0$, and so $\operatorname{Ext}^+_{UL}(-, UL) = 0$.

Finally, since $\operatorname{Ext}^0_{UL}(UL, UL)$ is non-zero, $\operatorname{polydepth} L \leqslant \operatorname{polybd} UL$. On the other hand if $\operatorname{polydepth} L = m < \infty$, then for some M, we have

$$\operatorname{Ext}^p_{UL}(M, UL) \neq 0 \quad \text{and} \quad \operatorname{polybd} M = m - p.$$

By the above, $p = 0$ and so there is a non-zero UL-linear map $f : M \to UL$. Any $f(m)$ is in some $UL_{<n}$ and if $f(m) \neq 0$ it follows that $UL_{\geqslant n} \xrightarrow{\cong} UL_{\geqslant n} \cdot m$. This implies

$$\operatorname{polybd} M \geqslant \operatorname{polybd} UL \quad \text{and} \quad \operatorname{polydepth} L \geqslant \operatorname{polybd} UL. \qquad \square$$

LEMMA 3.2. – *Let L be a graded Lie algebra of finite polydepth. If I is an ideal in L and* $\operatorname{polybd} I < \infty$ *then* $\operatorname{polydepth} L/I < \infty$.

Proof. – Choose M so that $\operatorname{polygrade} M = \operatorname{polydepth} L$. If $m = \operatorname{grade} M$ then it follows (as in [3], proof of Theorem 4.1 for the case $M = \Bbbk$) that for some p,

$$\operatorname{Ext}^p_{UL/I}\big(\operatorname{Tor}^{UI}_{m-p}(M, \Bbbk), UL/I\big) \neq 0.$$

Since $\operatorname{Tor}^{UI}_{m-p}(M, \Bbbk)$ is a subquotient of $\wedge^{m-p} sI \otimes M$ it follows that it has polynomial growth at most equal to $(m-p) \operatorname{polybd} I$. □

Proof of Theorem 3. –

(i) ⇒ (ii). Let I be the sum of the ideals in L concentrated in odd degrees. Then I is an ideal of this form, necessarily abelian, and L/I has no ideals concentrated in odd degrees. Moreover $\operatorname{polybd} UI = \operatorname{polydepth} I \leqslant \operatorname{polydepth} L$ (Lemma 3.1 and Proposition 1.3) and hence $\operatorname{polydepth} L/I < \infty$ (Lemma 3.2).

Next we show that every solvable ideal J in L/I is finite dimensional, by induction on the solvlength. Indeed, if J is abelian then $J_{\text{even}} = E(J)$. Since

$$\operatorname{polydepth} J \leqslant \operatorname{polydepth} L/I \quad \text{(Proposition 1.3)},$$

Theorem 2 asserts that J_{even} is finite dimensional. Thus for some r, $J_{\geqslant r}$ is an ideal concentrated in odd degrees, i.e. $J_{\geqslant r} = 0$.

Now if J has solvlength k then its $(k+1)$st derived algebra is abelian and so finite dimensional. Thus for some r, $J_{\geqslant r}$ has solvlength $k-1$. By induction, J is finite dimensional.

By hypothesis L/I is the sum of its solvable ideals. Since these are finite dimensional, each $x \in (L/I)_{\text{even}}$ acts locally nilpotently. Thus $(L/I)_{\text{even}} = E(L/I)$, and this is finite dimensional by Theorem 2. But $L_{\text{even}} \cong (L/I)_{\text{even}}$ since I is concentrated in odd degrees.

Suppose $L_{\text{even}} \subset L_{\leqslant 2n}$. Since $L_{>2n}$ is an ideal in odd degrees of finite polydepth, $\operatorname{polybd} UL_{>n} < \infty$, while trivially $\operatorname{polybd} UL/L_{>2n} < \infty$. Hence $\operatorname{polybd} UL < \infty$.

(ii) ⇒ (iii). Clearly $\operatorname{polybd} UL \geqslant \dim L_{\text{even}}$, so the latter must be finite. It is trivial from the Poincaré–Birkhoff–Witt theorem that if $\sum_{i\leqslant n} \dim L_i = d(n)$ then

$$\sum_{i\leqslant nd(n)} \dim(UL)_i \geqslant 2^{d(n)}.$$

Thus $2^{d(n)} \leqslant K[nd(n)]^r$ for some constant K and some integer r, $r \geqslant 1$. It follows that

$$d(n) \leqslant \log_2 K + r\log_2 n + r \log_2 d(n) \leqslant r \log_2 n + \frac{1}{2} d(n),$$

n sufficiently large.

(iii) ⇒ (i). Choose N so that $I = L_{\geqslant N}$ is concentrated in odd degrees. Then UI is an exterior algebra and so

$$\sum_{i \leqslant n} \dim(UI)_i \leqslant 2^{\sum_{i\leqslant n} \dim L_i} \leqslant n^C$$

for some constant C. Thus, since L/I is finite dimensional, $\operatorname{polybd} UL$ is finite. The identity of UL is in $\operatorname{Ext}^0_{UL}(UL, UL)$ and so $\operatorname{polydepth} L \leqslant \operatorname{polybd} UL$.

Finally, since I is abelian and L/I is finite dimensional, L itself is solvable. This also proves the last assertion. □

Proof of Theorem 4. – This is immediate from Proposition 1.3(i) and Theorem 3. □

PROPOSITION 3.3. – *Suppose I is a solvable ideal in a Lie algebra L of finite polydepth. Then*
(i) $\operatorname{polydepth} L/I \leqslant \operatorname{polydepth} L$;
(ii) $\dim I_{\text{even}} \leqslant \operatorname{polydepth} I \leqslant \operatorname{polybd} UI$.

Proof. – (i) As noted in the proof of Lemma 3.2,

$$\operatorname{Ext}^p_{UL/I}\big(\operatorname{Tor}^{UI}_{m-p}(M,\Bbbk), UL/I\big) \neq 0,$$

where $m + \operatorname{polybd} M = \operatorname{polydepth} L$. Also the Tor is a subquotient of $\wedge^{m-p} sI \otimes M$. By Theorem 3

$$\sum_{i\leqslant n} \dim(\wedge^{m-p} sI \otimes M)_i \leqslant (C_1 \log_2 n)^{m-p} \cdot C_2 \cdot n^{\operatorname{polybd} M}.$$

Hence

$$\operatorname{polybd} \operatorname{Tor}^{UI}_{m-p}(M,\Bbbk) \leqslant \operatorname{polybd} M + 1,$$

and so

$$\operatorname{polydepth} L/I \leqslant \operatorname{polydepth} M + p + 1.$$

If $p < m$ then this gives $\operatorname{polydepth} L/I \leqslant \operatorname{polydepth} L$. If $p = m$ then

$$\operatorname{Tor}^{UI}_{m-p}(M,\Bbbk) = M \otimes_{UI} \Bbbk.$$

Hence in this case $\operatorname{polybd} \operatorname{Tor}^{UI}_{m-p}(M,\Bbbk) \leqslant \operatorname{polybd} M$ and again

$$\operatorname{polydepth} L/I \leqslant \operatorname{polydepth} L.$$

(ii) Since $I_{\text{even}} \subset I_{\leqslant 2n}$, some n (Theorem 3) we may apply the first assertion to obtain

$$\begin{aligned} \dim I_{\text{even}} &= \operatorname{depth} I/I_{>2n} \quad \text{(cf. [2])}\\ &= \operatorname{polydepth} I/I_{>2n} \quad \text{(Proposition 1.6)}\\ &\leqslant \operatorname{polydepth} I. \end{aligned}$$

The second inequality has already been observed:

$$\operatorname{polybd} UI = \operatorname{polygrade} UI \geqslant \operatorname{polydepth} I. \qquad □$$

Example 2. – Consider a Lie algebra L concentrated in odd degrees with a basis $\{x_i,\ i \geqslant 1\}$ satisfying the degree relations

$$\deg x_i > \sum_{j<i} \deg x_j.$$

Then for each n, $\dim(UL)_n \leqslant 1$. The identity on UL shows that $\operatorname{polydepth} L = 1$.

Example 3. – Consider the graded Lie algebra $L=\mathbb{L}(a,x_n)_{n\geqslant 2}/I$, with $\deg a=2$, $\deg x_n=2^n+1$, and where I is generated by the relations

$$\big[(\operatorname{ad}a)^k x_r,(\operatorname{ad}a)^l x_s\big]=0,\quad k,l\geqslant 0,\ r,s\geqslant 2,\quad \text{and}\quad \operatorname{ad}^{n+1}(a)(x_n)=0.$$

Then $\operatorname{polybd} UL=2$, so that L has finite polygrade. On the other hand, L is solvable but not nilpotent, and is the union of the infinite sequence of the finite dimensional Lie algebras I_N generated by $a,x_2,\dots,x_N$.

PROPOSITION 3.4. – *Let L be the direct sum of non-solvable Lie algebras $L(i)$, $i\leqslant n$. If $\operatorname{polydepth} L(i)<\infty$ for $1\leqslant i\leqslant n$, then*

$$n\leqslant \operatorname{polydepth} L\leqslant \sum_i \operatorname{polydepth} L(i).$$

Proof. – We first prove by induction on n that for any UL-module M that has at most polynomial growth, we have

$$\operatorname{Ext}^{<n}_{UL}(M,UL)=0.$$

Consider the Hochschild–Serre spectral sequence

$$\operatorname{Ext}^p_{UL(1)}\big(\operatorname{Tor}_q^{U(L(2)\oplus\cdots\oplus L(n))}\big(M,\big(U\big(L(2)\oplus\cdots\oplus L(n)\big)^{\#}\big),UL(1)\big)\big)$$
$$\Rightarrow \operatorname{Ext}^{p+q}_{UL}(M,UL). \tag{4}$$

Since $L(1)$ commutes with the other $L(i)$ it follows that for each monogenic $UL(1)$-module N that is a subquotient of $\operatorname{Tor}_q^{U(L(2)\oplus\cdots\oplus L(n))}(M,U(L(2)\oplus\cdots\oplus L(n))^{\#})$ we have

$$\operatorname{polybd} N\leqslant \operatorname{polybd} M.$$

Now since $L(1)$ is not solvable, $\operatorname{polybd} UL(1)=\infty$ and the argument in the proof of Lemma 3.1 shows that

$$\operatorname{Ext}^0_{UL(1)}\big(N,UL(1)\big)=0.$$

Thus (Lemma 1.1) the left hand in (4) vanishes for $p=0$. By induction on n it vanishes for $q<n-1$ and so $\operatorname{Ext}^{<n}_{UL}(M,UL)=0$. Thus $\operatorname{polydepth} L\geqslant n$.

On the other hand, there are $UL(i)$-modules $M(i)$ such that $\operatorname{polygrade} M(i)=\operatorname{polydepth} L(i)$. Then $\bigotimes_{i=1}^n M(i)$ is a UL-module that has at most polynomial growth and whose polygrade is the sum of the polygrades of the $M(i)$. □

4. Growth of Lie algebras

PROPOSITION 4.1. – *Let L be a non-solvable graded Lie algebra of finite polydepth. Then for each integer $r\geqslant 1$ there is a positive integer $d(r)$ such that*

$$\sum_{i=k+1}^{k+d(r)} \dim L_i\geqslant k^r,\quad k \textit{ sufficiently large}.$$

Proof. – We distinguish two cases.

Case A: L_{even} contains an infinite dimensional abelian sub Lie algebra E.

Choose n so that $\dim E_{\leqslant n} \geqslant (r+3)\,\mathrm{polydepth}\,L$. Then there is a finite sequence

$$L = I(0) \supset I(1) \supset \cdots \supset I(l)$$

in which $I(j)$ is an ideal in $I(j-1)$ and $I(l)_{\leqslant n} = E_{\leqslant n}$.

By Proposition 1.3, $\mathrm{polydepth}\,I(q) \leqslant \mathrm{polydepth}\,L$. Thus without loss of generality we may suppose that $L = I(l)$, i.e. that $L_{\leqslant n}$ is an abelian sub Lie algebra concentrated in even degrees and that $\dim L_{\leqslant n} \geqslant (r+3)\,\mathrm{polydepth}\,L$.

Let M be a UL-module such that

$$\mathrm{grade}\,M + \mathrm{polybd}\,M = \mathrm{polydepth}\,L$$

and put $m = \mathrm{grade}\,M$. As observed in the proof of Proposition 1(ii),

$$\mathrm{Ext}^q_{UL_{\leqslant n}}(\wedge^p sL_{>n} \otimes M, UL_{\leqslant n}) \neq 0,$$

for some $p + q = m$. It follows that for some $z \in \wedge^p sL_{>n} \otimes M$,

$$\mathrm{polybd}\,UL_{\leqslant n} \cdot z + q \geqslant \dim L_{\leqslant n}$$

(Theorem 3.1 in [3]). Hence for some $x \in L$,

$$p(\mathrm{polybd}\,UL_{\leqslant n} \cdot x) \geqslant \dim L_{\leqslant n} - q - \mathrm{polybd}\,M.$$

Since $p + q + \mathrm{polybd}\,M = \mathrm{polydepth}\,L$ we conclude that

$$(2 + \mathrm{polybd}\,UL_{\leqslant n} \cdot x) \cdot \mathrm{polydepth}\,L \geqslant \dim L_{\leqslant n} \geqslant (r+3)\,\mathrm{polydepth}\,L.$$

As observed in the introduction, since $\dim L = \infty$, $\mathrm{polydepth}\,L > 0$. It follows that

(5) $$\mathrm{polybd}(UL_{\leqslant n} \cdot x) \geqslant r + 1.$$

On the other hand, $UL_{\leqslant n}$ is the polynomial algebra $\Bbbk[y_1, \ldots, y_s]$ on a basis $y_1, \ldots, y_s$ of $L_{\leqslant n}$. Because of (5) it is easy to see (induction on s) that this basis can be chosen so that for some $w \in UL_{\leqslant n} \cdot x$, $\Bbbk[y_1, \ldots, y_{r+1}] \to \Bbbk[y_1, \ldots, y_{r+1}] \cdot w$ is injective. Put $d = \prod_{i=1}^{r+1} \deg y_i$ and note that

$$\sum_{i=k+\deg w+1}^{k+\deg w+d} \dim L_i \geqslant \sum_{i=k+1}^{k+d} \dim \Bbbk[y_1, \ldots, y_{r+1}]_i \geqslant \frac{1}{r!} k^r.$$

It follows that for any r, and k sufficiently large,

$$\sum_{i=k+1}^{k+d} \dim L_i \geqslant k^{r-1}.$$

This proves Proposition 5 in case A.

Case B: Every abelian sub Lie algebra of L_{even} is finite dimensional.

Let I be the sum of the solvable ideals in L. Then I_{even} is finite dimensional and $\mathrm{polydepth}\,L/I$ is finite (Theorem 3 and Proposition 3.3). Thus all abelian sub Lie algebras of L/I are finite dimensional. Thus it is sufficient to prove case B when L has no solvable ideals.

There are now two possibilities: either $L = L_{\text{even}}$, or L has elements of odd degree. In the latter case the sub Lie algebra generated by L_{odd} is an ideal, hence non-solvable and of finite polydepth. Let $L(s)$ denote the sub Lie algebra generated by the first s linearly independent elements $x_1, \ldots, x_s$ of odd degree. For s sufficiently large, $\operatorname{polydepth} L(s) \leqslant \operatorname{polydepth} L$ (Proposition 1) and $\dim L(s)_{\text{even}} > \operatorname{polydepth} L$ (obvious). Thus $L(s)$ cannot be solvable (Theorem 3). In other words, we may assume that either $L = L_{\text{even}}$ or else L is generated by finitely many elements $x_1, \ldots, x_s$ of odd degree. In either case set $E = L_{\text{even}}$, and note that $\dim E$ is infinite.

Define a sequence of elements z_i and sub Lie algebras $E(i)$ by setting $E(1) = E$, z_i is a non-zero element in $E(i)$ and $E(i+1) \subset E(i)$ is the sub Lie algebra of elements on which $\operatorname{ad} z_i$ acts nilpotently.

Since E contains no infinite dimensional abelian Lie algebra some $E(N+1) = 0$ and $E(1)/E(2) \oplus \cdots \oplus E(N)/E(N+1)$ is a graded vector space isomorphic with E.

Put $d = \prod \deg z_i$ and $d_i = d/\deg z_i$. Then $(\operatorname{ad} z_1)^{d_1} \oplus \cdots \oplus (\operatorname{ad} z_N)^{d_N}$ is an injective transformation of $E(1)/E(2) \oplus \cdots \oplus E(N)/E(N+1)$ of degree d. Since this space is isomorphic with E it follows that

$$(6) \qquad \sum_{i=s}^{t} \dim E_i \leqslant \sum_{i=s+d}^{t+d} \dim E_i \quad \text{and thus} \quad \sum_{i=k+1}^{k+d} \dim E_i \geqslant \frac{d}{k+d} \sum_{i=1}^{k+d} \dim E_i, \quad k \geqslant 1.$$

On the other hand, choose n so that

$$\dim E_{\leqslant n} \geqslant (r+3) \cdot \operatorname{polydepth} L.$$

(This is possible because E is infinite dimensional.) Set $I = L_{>n}$. Let M be an L-module with $\operatorname{polydepth} L = \operatorname{polygrade} M$. As in the proof of Theorem 4.1 in [3], $\operatorname{Ext}^p_{UL/I}(\operatorname{Tor}^{UI}_q(M, \Bbbk), UL/I) \neq 0$ for $p + q = \operatorname{grade} M$. Thus p, q and $\operatorname{polybd} M$ are all bounded above by $\operatorname{polydepth} L$. Now Theorem 3.1 of [3] asserts that for some $\alpha \in \operatorname{Tor}^{UI}_q(M, \Bbbk)$, $UL/I \cdot \alpha$ has polynomial growth at least equal to $\dim(L/I)_{\text{even}} - p$. This means that for some positive C,

$$\sum_{i \leqslant k} \dim(UL/I \cdot \alpha)_i \geqslant C k^{(\dim(L/I)_{\text{even}}) - p}, \quad k \text{ sufficiently large.}$$

Since $\operatorname{Tor}^{UI}_*(M, \Bbbk)$ is the homology of $\wedge^* sI \otimes M$, it follows that

$$\sum_{i \leqslant k} \dim \operatorname{Tor}^{UI}_q(M, \Bbbk)_i \leqslant \left(\sum_{i \leqslant k} \dim L_i \right)^q \sum_{i \leqslant k} \dim M_i.$$

But $(L/I)_{\text{even}} \cong E_{\leqslant n}$ and so a quick calculation gives

$$(7) \qquad \sum_{i \leqslant k} \dim L_i \geqslant K k^{r+1}, \quad k \text{ sufficiently large.}$$

Finally, recall that either $L = L_{\text{even}}$ or else L is generated by the elements of odd degree x_i. In the former case $L = E$ and the proposition follows from (6) and (7). In the second case we have

$$L_{\text{odd}} = [x_1, E] + \cdots + [x_s, E] + \Bbbk x_1 + \cdots + \Bbbk x_s,$$

and hence (7) yields

$$\sum_{i\leqslant k} E_i \geqslant \frac{K}{s+1} k^{r+1} + s, \quad k \text{ sufficiently large.}$$

Combined with (6) this formula gives the proposition. □

Proof of Theorem 5. – Since L is not solvable we may choose n so that

$$\dim(L_{\text{even}})_{\leqslant n} > \operatorname{polydepth} L \quad \text{(Theorem 3)},$$

and so that the sub Lie algebra generated by $L_{\leqslant n}$ satisfies $\operatorname{polydepth} E \leqslant \operatorname{polydepth} L$ (Proposition 1.3). Then E is not solvable (Proposition 3.3).

Let $x_1, \ldots, x_s$ generate E (see beginning of case B) and put $d = \max \deg x_i$. Letting UE act via the adjoint representation on E we have that

$$UE_{[0,q]} \cdot E_{[k+1,k+d]} \supset E_{[k+1,k+q]}.$$

For any $r \geqslant 1$ choose $q = q(r)$ so that

$$\sum_{i=k+1}^{k+q} \dim E_i \geqslant k^{r+1},$$

k sufficiently large (Proposition 4.1). Then

$$\sum_{i=k+1}^{k+d} \dim L_i \geqslant \sum_{i=k+1}^{k+d} \dim E_i \geqslant \frac{1}{\dim UE_{[0,q]}} k^{r+1} \geqslant k^r, \quad k \text{ sufficiently large.}$$

Since d is independent of r, the theorem is proved. □

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Yves FELIX
Institut de Mathématiques,
Université Catholique de Louvain,
1348 Louvain-La-Neuve, Belgique

Stephen HALPERIN
College of Computer,
Mathematical and Physical Sciences
University of Maryland,
College Park, MD 20742-3281, USA

Jean-Claude THOMAS
Faculté des Sciences,
Université d'Angers,
Bd. Lavoisier,
49045 Angers, France
E-mail: jean-claude.thomas@univ-angers.fr