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ON APPROXIMATION OF SMOOTH SUBMANIFOLDS BY NONSINGULAR REAL ALGEBRAIC SUBVARIETIES ☆

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ABSTRACT. – Let M be a smooth submanifold of dimension m of a nonsingular real algebraic set X. If M can be approximated by nonsingular algebraic subsets of X, then the homology class in $H_m(X,\mathbb{Z}/2)$ represented by M is algebraic. The converse, investigated in this paper, is true only in some exceptional cases.
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RÉSUMÉ. – Soit M une sous-variété différentiable de dimension m d'un ensemble algébrique réel non singulier X. Si M peut être approchée par des sous-ensembles algébriques non singuliers de X, alors la classe d'homologie représentée par M dans $H_m(X,\mathbb{Z}/2)$ est algébrique. La réciproque, étudiée dans cet article, est vraie seulement dans quelques cas exceptionnels.
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1. Introduction

Throughout this paper the term *real algebraic variety* designates a locally ringed space isomorphic to a Zariski closed subset of $\mathbb{R}^n$, for some n, endowed with the Zariski topology and the sheaf of $\mathbb{R}$-valued regular functions; thus we work with affine real algebraic varieties. Morphisms between real algebraic varieties will be called *regular maps*. Basic facts on real algebraic varieties and regular maps can be found in [5]. Every real algebraic variety carries also the Euclidean topology, which is determined by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the Euclidean topology.

Given a compact real algebraic variety X, we denote by $H_d^{\mathrm{alg}}(X,\mathbb{Z}/2)$ the subgroup of $H_d(X,\mathbb{Z}/2)$ of homology classes represented by d-dimensional Zariski closed subsets of X [5,7,8]. If Z is either a Zariski closed subset of X or a compact smooth (of class $\mathcal{C}^\infty$) submanifold of X, $\dim Z = d$, we denote by $[Z]_X$ the homology class in $H_d(X,\mathbb{Z}/2)$ represented by Z.

Recall that a compact smooth submanifold M of a nonsingular real algebraic variety X is said to *admit an algebraic approximation* in X if for each neighborhood $\mathcal{U}$ of the inclusion map $M \hookrightarrow X$ (in the $\mathcal{C}^\infty$ topology on the set $\mathcal{C}^\infty(M,X)$ of all smooth maps from M into X), there exists a smooth embedding $e: M \to X$ such that e is in $\mathcal{U}$ and $e(M)$ is a nonsingular Zariski closed subset of X. Clearly, if M admits an algebraic approximation in X, then $[M]_X$ is in $H_d^{\mathrm{alg}}(X,\mathbb{Z}/2)$, $d = \dim M$. It is natural to ask whether the converse holds true. More precisely, it would be interesting to describe explicitly the set Ω for all pairs of positive integers (n,d), $n > d$,

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with the property that for any compact nonsingular n-dimensional real algebraic variety X, every compact smooth d-dimensional submanifold M of X, with $[M]_X$ in $H_d^{\mathrm{alg}}(X,\mathbb{Z}/2)$, admits an algebraic approximation in X. By [5, Theorem 12.4.11], the pair $(n,n-1)$ is in Ω for all $n \geqslant 2$. The only other pair known to be in Ω is $(3,1)$. Indeed, the following result will be proved in Section 2.

THEOREM 1.1. – *Let X be a compact nonsingular real algebraic variety of dimension 3 and let C be a compact smooth curve in X. Then C admits an algebraic approximation in X if and only if $[C]_X$ is in $H_1^{\mathrm{alg}}(X,\mathbb{Z}/2)$.*

A special case of Theorem 1.1, in which C is assumed to be connected and homologous to the union of finitely many nonsingular algebraic curves in X, is proved in [2].

On the other hand, we have the following negative result.

PROPOSITION 1.2. – *Let N be a compact smooth submanifold of the unit m-sphere S^m, $0 < \dim N < m$. Assume that N is not the boundary of a compact smooth manifold with boundary. Then for any positive integer k, there exist a nonsingular real algebraic variety X and a smooth submanifold M of X such that X is diffeomorphic to $S^m \times S^k$, M is diffeomorphic to $N \times S^k$, the class $[M]_X$ in $H_d(X,\mathbb{Z}/2)$, $d = \dim M$, is null (thus algebraic) and M does not admit an algebraic approximation in X.*

COROLLARY 1.3. – *For n and d with $n-d \geqslant 2$ and $d \geqslant 3$, the pair (n,d) is not in Ω; in other words, there exist a compact nonsingular n-dimensional real algebraic variety X and a compact smooth d-dimensional submanifold M of X with $[M]_X$ in $H_d^{\mathrm{alg}}(X,\mathbb{Z}/2)$ such that M does not admit an algebraic approximation in X.*

To prove Corollary 1.3 we apply Proposition 1.2 with $k = d-2$, $m = n-d+2$, and N diffeomorphic to the real projective plane $\mathbb{R}P^2$.

The first example of a pair (X,M) with M not admitting an algebraic approximation in X, but $[M]_X$ in $H_d^{\mathrm{alg}}(X,\mathbb{Z}/2)$, $d = \dim M$, was given by S. Akbulut and H. King [4]. They showed that if Σ is a nonsingular irreducible real algebraic curve with two connected components Σ_0 and Σ_1 (each necessarily diffeomorphic to S^1), and N is a smooth submanifold of S^4 diffeomorphic to $\mathbb{R}P^2$, then $M = N \times \Sigma_0$ does not admit an algebraic approximation in $X = S^4 \times \Sigma$, despite the fact that the class $[M]_X$ in $H_3(X,\mathbb{Z}/2)$ is null. In this example X is not connected. Corollary 1.3 also follows from a modification of the example of Akbulut and King.

Concerning the remaining series of pairs, namely $(n,1)$ and $(n,2)$, with $n \geqslant 4$, it seems probable that $(n,1)$ belongs to Ω, whereas $(n,2)$ does not. In other words, conjecturally, (n,d) is in Ω if and only if $n-d=1$ or $d=1$.

In the proof of Proposition 1.2 we shall make use of the following, interesting in its own right, result.

THEOREM 1.4. – *Let $f: X \to Y$ be a regular map between nonsingular real algebraic varieties. Assume that X is compact and Y is irreducible. Given two regular values y_1 and y_2 of f, the smooth manifolds $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are cobordant.*

Of course, the case of interest is when y_1 and y_2 belong to distinct connected components of Y.

Proofs of Theorem 1.4 and Proposition 1.2 are given in Section 3.

2. Proof of Theorem 1.1

The reader may refer to [5, Chapter 12] for basic facts concerning algebraic vector bundles on real algebraic varieties. The kth Stiefel–Whitney class of a topological $\mathbb{R}$-vector bundle ξ will be denoted by $w_k(\xi)$.

Proof of Theorem 1.1. – Denote by τ_X the tangent bundle to X. Then $\lambda = \Lambda^3\tau_X$ is an algebraic $\mathbb{R}$-line bundle on X with $w_1(\lambda) = w_1(\tau_X)$ in $H^1_{\mathrm{alg}}(X,\mathbb{Z}/2)$, cf. [8, p. 498]. We shall now construct a smooth $\mathbb{R}$-vector bundle ξ on X of rank 2 and a smooth section $s: X \to \xi$ such that $w_1(\xi) = w_1(\lambda)$, s is transverse to the zero section, and $s^{-1}(0) = C$, where $s^{-1}(0) = \{x \in X \mid s(x) = 0\}$.

Let $\pi: T \to C$ be an open tubular neighborhood of C in X. We identify (T, π, C) with the normal vector bundle v of C in X. Clearly, there exists a smooth section $\sigma: T \to \pi^* v$ such that σ is transverse to the zero section and $\sigma^{-1}(0) = C$. We have

$$\pi^*\nu \mid T\backslash C = \eta \oplus \varepsilon_\sigma, \tag{1}$$

where ε_σ is the trivial smooth $\mathbb{R}$-line subbundle of $\pi^*\nu \mid T\backslash C$ generated by $\sigma \mid T\backslash C$ and η is a smooth $\mathbb{R}$-line bundle on $T\backslash C$. Note that

$$w_1(\eta) = w_1(\lambda \mid T\backslash C). \tag{2}$$

Indeed, since the tangent bundle to C is trivial, v is stably equivalent to $\tau_X \mid C$ and hence

$$w_1(\nu) = w_1(\tau_X \mid C) = w_1(\lambda \mid C) = i^*\big(w_1(\lambda)\big),$$

where $i: C \hookrightarrow X$ is the inclusion map. The composite map $i \circ \pi: T \to X$ is homotopic to the inclusion map $j: T \hookrightarrow X$, which implies

$$w_1(\pi^*\nu) = \pi^*\big(w_1(v)\big) = \pi^*\big(i^*\big(w_1(\lambda)\big)\big) = j^*\big(w_1(\lambda)\big) = w_1\big(\lambda \mid T\big).$$

Making use of (1), we get $w_1(\eta) = w_1(\pi^*\nu \mid T\backslash C) = w_1(\lambda \mid T\backslash C)$ and therefore (2) is proved.

Let ε be the trivial $\mathbb{R}$-line bundle on X with total space $X \times \mathbb{R}$ and let $\tau: X \to \lambda \oplus \varepsilon$ be the smooth section defined by $\tau(x) = (0, (x, 1))$ for all x in X. It follows from (2) that the $\mathbb{R}$-line bundles η and $\lambda \mid T\backslash C$ are isomorphic and hence (1) implies the existence of a smooth isomorphism

$$\varphi: \pi^*\nu \mid T\backslash C \to (\lambda \oplus \varepsilon) \mid T\backslash C$$

of $\mathbb{R}$-vector bundles on $T\backslash C$ such that $\varphi \circ \sigma = \tau$ on $T\backslash C$. Let ξ be the smooth $\mathbb{R}$-vector bundle on X obtained by gluing $\pi^*\nu$ and $(\lambda \oplus \varepsilon) \mid X\backslash C$ over $T\backslash C$ using φ. Similarly, let $s: X \to \xi$ be the smooth section obtained by gluing σ and $\tau \mid X\backslash C$ over $T\backslash C$ using φ. Then $s^{-1}(0) = C$ and s is transverse to the zero section. Furthermore,

$$w_1(\xi \mid X\backslash C) = w_1\big((\lambda \oplus \varepsilon) \mid X\backslash C\big) = w_1(\lambda \mid X\backslash C)$$

and hence $w_1(\xi) = w_1(\lambda)$ since $\dim C = 1$. Thus ξ and s satisfy the required conditions.

Since $w_1(\xi) = w_1(\lambda)$ is in $H^1_{\mathrm{alg}}(X,\mathbb{Z})/2)$ and $w_2(\xi)$ is in $H^2_{\mathrm{alg}}(X,\mathbb{Z})/2)$ (note that $w_2(\xi)$ is Poincaré dual to the homology class $[C]_X$, which belongs to $H_1^{\mathrm{alg}}(X,\mathbb{Z}/2)$), it follows from [6, Theorem 1.6] that ξ is isomorphic to an algebraic $\mathbb{R}$-vector bundle on X. Thus without loss of generality we may assume that ξ is an algebraic $\mathbb{R}$-vector bundle. Let $v: X \to \xi$ be an algebraic section close to s in the $\mathcal{C}^\infty$ topology, cf. [5, Theorem 12.3.2]. Then there exists a smooth embedding of $C = s^{-1}(0)$ into X, close to the inclusion map $i : C \hookrightarrow X$, which transforms C onto the nonsingular Zariski closed curve $v^{-1}(0)$, cf. [1, Theorem 20.2]. Hence C admits an algebraic approximation in X. The proof is complete. □

3. Proofs of Theorem 1.4 and Proposition 1.2

We begin with some general remarks concerning complexification of real algebraic varieties. Given a complex projective variety V, defined over $\mathbb{R}$, we denote by $V(\mathbb{R})$ its set of real points. If $V(\mathbb{R})$ is Zariski dense in V, then $V(\mathbb{R})$ can be regarded in a canonical way as a real algebraic variety (note that $V(\mathbb{R})$ is contained in an affine Zariski open subset of V, defined over $\mathbb{R}$).

It follows from Hironaka's resolution of singularities theorem [10] that any compact nonsingular real algebraic variety X has a nonsingular projective complexification (V, j). This means, by definition, that V is a complex nonsingular projective variety defined over $\mathbb{R}$ and $j : X \to V(\mathbb{R})$ is a regular isomorphism of real algebraic varieties. If $f : X \to Y$ is a regular map between compact nonsingular real algebraic varieties, then there exists a commutative diagram

$$(*) \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ V & \xrightarrow{g} & W, \end{array}$$

where (V, j) and (W, k) are nonsingular projective complexifications of X and Y, respectively, and g is a regular map defined over $\mathbb{R}$. Indeed, let (U, i) and (W, k) be arbitrary nonsingular projective complexifications of X and Y. The map $k \circ f \circ i^{-1} : U(\mathbb{R}) \to W$ extends to a regular map $f_0 : U_0 \to W$ defined over $\mathbb{R}$, where U_0 is a Zariski open neighborhood, defined over $\mathbb{R}$, of $U(\mathbb{R})$ in U. The existence of (∗) now follows from Hironaka's theorem on resolution of points of indeterminacy [10].

Proof of Theorem 1.4. – In view of Hironaka's theorem [10], Y can be regarded as a Zariski open subset of a compact nonsingular real algebraic variety. This reduces our considerations to the case where Y is compact.

Making use of diagram (∗), we may assume that $X = V(\mathbb{R})$, $Y = W(\mathbb{R})$, and $f : V(\mathbb{R}) \to W(\mathbb{R})$ is the restriction of g. The set Σ of critical points of g is Zariski closed in V and hence the set $g(\Sigma)$ is Zariski closed in W [14, p. 57, Theorem 2]. Since Y is irreducible, it follows that W is also irreducible, and therefore the set $W \backslash g(\Sigma)$ is nonempty and connected [15, p. 126, Theorem 1]. The restriction

$$h : V \backslash g^{-1}\big(g(\Sigma)\big) \to W \backslash g(\Sigma)$$

of g is a locally trivial smooth fibration [12, p. 23]. The connectedness of $W \backslash g(\Sigma)$ implies that any two fibers of h are diffeomorphic. If b is a point of $(W \backslash g(\Sigma)) \cap W(\mathbb{R})$, then $h^{-1}(b)$ is a nonsingular complex projective variety defined over $\mathbb{R}$, whose set of real points is equal to $f^{-1}(b)$. By [9, Theorem 22.4], the smooth manifolds $h^{-1}(b)$ and $f^{-1}(b) \times f^{-1}(b)$ are cobordant.

Suppose now that the regular values y_1 and y_2 of f belong to $W \backslash g(\Sigma)$. Then the smooth manifolds $h^{-1}(y_1)$ and $h^{-1}(y_2)$, being diffeomorphic, are cobordant. Hence $f^{-1}(y_1) \times f^{-1}(y_1)$ and $f^{-1}(y_2) \times f^{-1}(y_2)$ are also cobordant and as such have the same Stiefel–Whitney numbers [13, Theorem 4.10]. It follows that the smooth manifolds $f^{-1}(y_1)$ and $f^{-1}(y_2)$

have the same Stiefel–Whitney numbers (note that if M is a closed smooth manifold and $i_1,\dots,i_r$ are nonnegative integers satisfying $i_1+\dots+i_r=\dim M$, then $\langle w_{i_1}(M)\cup\dots\cup w_{i_r}(M),[M]\rangle=\langle w_{2i_1}(M\times M)\cup\dots\cup w_{2i_r}(M\times M),\ [M\times M]\rangle$) and therefore are cobordant [13, Theorem 4.10].

In order to complete the proof, we still have to consider the case where at least one of the points y_1 and y_2 is not in $W\backslash g(\Sigma)$. Note that $(W\backslash g(\Sigma))\cap W(\mathbb{R})$ is dense in $W(\mathbb{R})$. Thus it suffices to show that for any regular value a of f, there exists an open neighborhood N of a in $W(\mathbb{R})$ such that every point b in N is a regular value of f, and the smooth manifolds $f^{-1}(a)$ and $f^{-1}(b)$ are cobordant. In fact any connected open neighborhood N of a in $W(\mathbb{R})$, which does not contain any critical value of f, has the required properties. Indeed, let $\varphi: W(\mathbb{R})\to W(\mathbb{R})$ be a smooth diffeomorphism homotopic to the identity map and satisfying $\varphi(a)=b$. Then f and $\varphi\circ f$ are smooth homotopic maps for which b is a regular value. Thus $f^{-1}(b)$ and $(\varphi\circ f)^{-1}(b)=f^{-1}(a)$ are cobordant [11, p. 170, Lemma 1.2]. The proof is complete. □

Proof of Proposition 1.2. – Choose a compact nonsingular irreducible real algebraic variety Σ, with two connected components Σ_0 and Σ_1, each diffeomorphic to S^k. For example,

$$\Sigma=\{(x_0,x_1,\dots,x_k)\in\mathbb{R}^{k+1}\mid x_0^4-4x_0^2+1+x_1^2+\dots+x_k^2=0\}.$$

Let $\varphi: S^k\to\Sigma$ be a smooth map, which is a diffeomorphism of S^k onto Σ_0. Denote by B^{m+1} the unit $(m+1)$-ball and let $\pi: B^{m+1}\times S^k\to S^k$ be the canonical projection. Clearly, the unoriented bordism class of the restriction $f: S^m\times S^k\to\Sigma$ of $\varphi\circ\pi: B^{m+1}\times S^k\to\Sigma$ is zero. This fact allows us to make use of [3, Theorem 2.8.4] (we only need absolute case of this theorem, that is, $P=\emptyset,\ L=\emptyset$), and hence there exist a nonnegative integer q, a smooth embedding $e: S^m\times S^k\to\mathbb{R}^{m+1}\times\mathbb{R}^{k+1}\times\mathbb{R}^q$, a nonsingular Zariski closed subset X of $\mathbb{R}^{m+1}\times\mathbb{R}^{k+1}\times\mathbb{R}^q$, and a regular map $g: X\to\Sigma$ such that $X=e(S^m\times S^k)$ and $g\circ e: S^m\times S^k\to\Sigma$ is close to f in the $\mathcal{C}^\infty$ topology.

Note that the restriction $f_0: N\times S^k\to\Sigma$ of f is a submersion with fiber $f_0^{-1}(y)$ diffeomorphic to N for y in Σ_0 and empty for y in Σ_1. Setting $M=e(N\times S^k)$, we deduce that the restriction $g_M: M\to\Sigma$ of g is a submersion with fiber $g_M^{-1}(y)$ diffeomorphic to N for y in Σ_0 and empty for y in Σ_1.

Suppose that M admits an algebraic approximation in X. Choose a smooth embedding $i: M\hookrightarrow X$, close in the $\mathcal{C}^\infty$ topology to the inclusion map $M\hookrightarrow X$, whose image $Y=i(M)$ is a Zariski closed nonsingular subset of X. Then the restriction $g_Y: Y\to\Sigma$ of g is a submersion with fiber $g_Y^{-1}(y)$ diffeomorphic to N for y in Σ_0 and empty for y in Σ_1. In particular, since N is not the boundary of a compact smooth manifold with boundary, g_Y has fibers that are not cobordant. This leads to a contradiction with Theorem 1.4, g_Y being regular and Σ irreducible. Hence M does not admit an algebraic approximation in X and the proof is complete. □

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