Ann. Scient. Éc. Norm. Sup.,
4e série, t. 35, 2002, p. 489 à 535.

COUPLED MAPS AND ANALYTIC FUNCTION SPACES

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ABSTRACT. – We consider ergodic properties of weakly coupled analytic and expanding circle maps. For weak enough coupling a natural ergodic measure exists and exhibits exponent decay of time correlations. The marginal densities of the natural measure are analytic. A spatial decay of correlations (e.g. polynomial) in these densities may arise from a similar spatial decay of the couplings. The space of couplings and observables is a Banach algebra of analytic functions of infinitely many variables. This algebra acts upon a Banach module of complex measures with analytic marginal densities. A Perron–Frobenius type operator acts on the Banach module and we apply a re-summation technique to derive uniform bounds for this operator. Explicit bounds are calculated for some examples.

RÉSUMÉ. – Nous étudions des propriétés ergodiques des applications analytiques et dilatantes du cercle, faiblement couplées. Lorsque le couplage est assez petit, il y a une mesure naturelle ergodique pour laquelle les corrélations en temps décroissent exponentiellement vite. Les densités marginales de la mesure naturelle sont analytiques. Une décroissance spatiale des corrélations (par exemple polynomiale) de ces densités peut être la manifestation d'une décroissance spatiale similaire pour le couplage. L'espace de couplages et d'observables est une algèbre de Banach de fonctions analytiques d'un nombre infini de variables. Cette algèbre agit sur un module de Banach de mesures complexes qui possèdent des densités marginales analytiques. Un opérateur de type Perron–Frobenius agit sur le module de Banach et nous appliquerons une technique de resommation pour en déduire des bornes uniformes pour cet opérateur. Nous calculons des bornes explicites pour quelques exemples.

1. Introduction

Consider a product of circles, $S_\Omega = \prod_{p\in\Omega} S^1$, over an index set Ω, and a map, $f_\Omega : S_\Omega \to S_\Omega$, which is a direct product of 'uniformly' real-analytic and expanding circle maps. Applying a small real-analytic deformation to this product-map we obtain a *coupled map*, F_Ω. Our aim is to establish some ergodic properties of the dynamical system (S_Ω, F_Ω). The torus S_Ω carries a natural reference measure, the (Kolmogorov extended) Lebesgue probability measure m_Ω (cf. Section 3.1), and one may ask if our dynamical system admits a 'natural' (which is then *a fortiori* unique and ergodic) measure ν_Ω. By 'natural' we mean that Birkhoff time averages converge Lebesgue-almost surely to their ν_Ω-averages. For a certain class of coupled maps (Theorem 2.1) indeed such a natural measure exists and furthermore, the measure exhibits exponential decay of time correlations for a suitable class of observables. If, in addition, Ω carries a metric, a spatial decay in the coupling terms is reflected (Theorem 2.5) in spatial decaying correlations in the marginal densities of the invariant measure ν.

The class of deformations we consider here includes for example a sum of pair, or more generally n-point (with n fixed) interactions for which the norms are summable with

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a sufficiently small sum. Our results apply equally well to a time-dependent sequence of coupled maps.

Example 1.1. – As a model example of a coupled map consider a family of real constants, $c_{p,q}$, $p, q \in \Omega$, for which $\sum_q |c_{p,q}| \leqslant 1$ for all $p \in \Omega$. Also let ε be a real parameter. A coupled map on the Ω-torus is then defined through

$$(1.1)\quad F_p(x) = 2x_p + \frac{\varepsilon}{2\pi} \sum_{q\in\Omega} c_{p,q} \sin(2\pi x_q) \bmod \mathbb{Z}, \quad x = (x_p)_{p\in\Omega} \in (\mathbb{R}/\mathbb{Z})^{\Omega},\ p \in \Omega.$$

For this particular map we prove that when $|\varepsilon| < \varepsilon_0 = 1/31$, the natural measure exists and exhibits exponential decay of time correlations. When Ω is finite the same conclusions hold for all values of $\varepsilon < 1$. It would be interesting to know if this is true in general. We note that when $\varepsilon \geqslant 1$ one may have attracting fixed point(s) of the coupled map.

In the case of a lattice, $\Omega = \mathbb{Z}^d$, $d \geqslant 1$, the above type of dynamical system is in the literature referred to as a ‘coupled map lattice’. In order to extract ergodic properties of this system, one route is, via Markov partitions, to map the dynamics onto a spin system of rapidly decaying interaction. We refer to Bunimovich and Sinai [6], Volevich [19], Jiang [13], Jiang and Pesin [14] and in particular, to Brichmont and Kupiainen [5] for a detailed discussion of the various results obtained.

Another route, close to the one we shall follow, is to look at finite truncations of the lattice, introduce Perron Frobenius type operators and perturbative expansions of such operators. Bricmont and Kupiainen, in their seminal paper [4], obtained the first important results in this direction. Their approach uses so-called ‘cluster-’ or ‘polymer-expansions’, a technique borrowed from statistical mechanics. This technique controls combinatorial aspects of the perturbative expansion but necessitates some additional ‘exponential spatial decay’ of the deformation. A simplified and explicit version of this expansion, inspired by a paper of Maes and van Moffaert [18], was developed in Fischer and Rugh [10] (see also [9]). It was shown in [10] (cf. also [1]) how to define transfer operators acting on (a family of) Banach spaces associated with the full lattice. Ergodic properties are then deduced from spectral properties of these transfer operators.

First, consider the family of Banach spaces in [10], which we here denote $\mathcal{M}_\theta$, parametrised by $0 < \theta \leqslant 1$. We observe in Appendix C that these spaces may be interpreted naturally as Banach modules over a corresponding family, $\mathcal{H}_\theta$, of Banach algebras (unfortunately not C^*-algebras). These algebras may subsequently be used to construct both couplings and a space of suitable observables. We may then write a perturbative expansion of a transfer operator for the full system, using the $\mathcal{H}_\theta$-algebra norm as an essential tool controlling operator bounds.

Second, through a ‘decoupling’ of the expansion we replace the operator bounds by sums over trees. These sums are encoded in a renormalization map which is essentially a one dimensional map (though its actual realisation has a more complicated appearance). When iterates of this map remain bounded the operator becomes well-defined and the ‘natural’ ergodic properties may even be deduced from the renormalization bounds. Our formulae permit us in particular to calculate explicit bounds (cf. our examples) which hitherto has been virtually impossible. Combinatorial considerations become obsolete in our approach and no spatial structure of Ω is required, whence we made no reference to ‘lattice’ in the title. On the other hand, when Ω is a metric space, e.g. a lattice, then the natural measure may exhibit some additional decay of spatial correlations for the natural measure. In some cases we may capture this decay as well, as the following example shows (cf. also [2]).

Example 1.2. – Consider $\Omega = \mathbb{Z}^n$ with its standard Euclidean metric. As coefficients in the previous example let us set $c_{p,p} = 0$ and take $c_{p,q} = N_\varrho |p-q|^{-n-\varrho}$, $q \neq p$, where $\varrho > 0$ and the normalization constant $0 < N_\varrho < \infty$ is such that $\sum_q c_{p,q} = 1$. We prove that in this case the marginal densities of the invariant measure exhibits a polynomial spatial decay. For $0 < \delta < \varrho$ and $|\varepsilon| < \varepsilon_0 N_{\varrho-\delta}/N_\varrho$ (ε_0 as in the previous example) the exponent of the polynomial decay is at least δ. Proofs of the examples are given in Section 6.

Remarks 1.3. – Why only treat the analytic category? In fact, most of our ingredients carry over almost without changes to less smooth categories, e.g. $C^{k+\alpha}$, $k \geqslant 1$. There remains, however, one obstruction, namely to design a perturbative expansion of the Perron–Frobenius operator. In the analytic category this task is considerably simplified by a multi-dimensional Cauchy integral formula. In a less smooth category the procedure is less clear. One may still attempt to expand operators in a space of tensorial products or rather, some completion thereof. Rigorous bounds are, however, difficult to obtain, notably due to problems with the completion (cf. also Keller and Künzle [16]). Here we play safe and stick to the analytic category. Our treatment is rigorous, complete and fairly self-contained.

Generalizations: The uncoupled maps are chosen among a class of uniformly analytic and expanding maps. Likewise, the interactions are assumed uniformly small. These conditions can be relaxed considerably at the expense of burdening both the notation and the calculations.

The first part of the paper is organized as follows: In Section 1.2 we present a class of uniformly analytic and expanding circle maps. To each site in Ω, we associate a map from this class. The collection of these maps forms an uncoupled dynamical system. We then (Section 1.3) introduce the family of Banach algebras of functions, analytic in infinitely many variables. As already mentioned, these algebras will serve not only to define a class of interactions for the uncoupled maps but also as a space of observables for the correlation functions. In Section 2 we formulate our main Theorems. The remaining sections of the paper are devoted to the proofs, an outline of which is given in the beginning of Section 3.

1.1. Some notation

We consider[1] the circle $S^1 = \mathbb{R}/\mathbb{Z}$ as a subset of the complex cylinder $\mathcal{C} = \mathbb{C}/\mathbb{Z}$ and for $\rho \geqslant 0$ we define the closed annulus on the cylinder,

$$A[\rho] = \{z \in \mathbb{C}/\mathbb{Z}\colon\ |\operatorname{Im} z| \leqslant \rho\}. \tag{1.2}$$

(Thus, S^1 is a closed annulus with $\rho \equiv 0$.) The Ω-torus $S_\Omega = \prod_{p\in\Omega} S^1$ and the Ω-annulus $A_\Omega = \prod_{p\in\Omega} A[\rho]$ are both Haussdorf and, by Tychonoff, also compact for the product topology. For any given topological space X, the space of continuous functions on X is denoted $C(X)$ and for functions on X, $|\cdot|$ (without subscripts) signifies the uniform norm.

Let $\mathcal{F}$ denote the family of all finite subsets, including the empty set, of Ω. For $\Lambda \in \mathcal{F}$ we write $S_\Lambda = \prod_{p\in\Lambda} S^1$ and $A_\Lambda = \prod_{p\in\Lambda} A[\rho] \in \mathcal{C}^\Lambda$ for the Λ-torus and the Λ-annulus, respectively, both viewed as the result of natural projections from the Ω-torus/annulus. $C_\Lambda = C(S_\Lambda)$ denotes the continuous functions over the Λ-torus and $E_\Lambda = E(A_\Lambda) \equiv C^\omega(\operatorname{Int} A_\Lambda) \cap C(A_\Lambda)$ denotes the space of complex valued functions, holomorphic on the interior of A_Λ and continuous on the closed annulus A_Λ. In the case of the empty set $\Lambda = \emptyset$, we let $C_\emptyset = E_\emptyset = \mathbb{C}$, i.e. the field of complex numbers.

Each E_Λ (similar statements for C_Λ) is a Banach space in the uniform norm and through coordinate projections we obtain natural inclusions: $j_{\Lambda,K} : E_K \hookrightarrow E_\Lambda$ whenever $K \subset \Lambda (\in \mathcal{F})$

[1] For technical reasons, it is less convenient to use $\{|z| = 1\}$ as a model for the circle.

and also $j_\Lambda : E_\Lambda \hookrightarrow C(A_\Omega)$ for $\Lambda \in \mathcal{F}$. We denote by $E(A_\Omega)$ the closure of $\bigcup_\Lambda j_\Lambda E_\Lambda$ in $C(A_\Omega)$. The space $E(A_\Omega)$ coincides with the space of weakly holomorphic continuous functions on A_Ω (cf. Appendix B).

1.2. Real-analytic expanding circle maps

DEFINITION 1.4. – For $\rho > 0$ and $\lambda > 1$ we say that $f : A[\rho] \to \mathcal{C}$ is a real-analytic, (ρ,λ)-expanding map of the circle if

(a) f is holomorphic in $\operatorname{Int} A[\rho]$ and continuous on $A[\rho]$.
(b) $f(S^1) = S^1$. (Real-analyticity.)
(c) The intersection $f\partial A[\rho] \cap A[\lambda\rho]$ is empty. (Expansion.)

We let $\mathcal{E}(\rho,\lambda)$ be the class of all such maps.

Remarks 1.5. – As shown in Appendix A a map in $\mathcal{E}(\rho,\lambda)$ is *a fortiori* expanding on the circle in the standard metric sense and λ gives a lower bound for the expansion constant. This also means that λ gives a lower bound for the topological degree of such a map. There is no upper restriction on the topological degree.

The above type of maps also satisfy uniform distortion estimates and therefore induce single-site Perron–Frobenius operators having a uniform ‘spectral gap’. Another issue is that the above conditions are ideal when a coupling is introduced.

For each $p \in \Omega$, choose an element $f_p \in \mathcal{E}(\rho,\lambda)$. The family $f_\Omega = (f_p)_{p\in\Omega}$ is a direct product of uniformly expanding and real-analytic circle maps. When restricted to the torus the pair (S_Ω, f_Ω) defines a dynamical system which we regard as a collection of uncoupled real-analytic expanding maps.

1.3. Coupling and observables

For $\theta \in (0,1]$ consider the subset of functions $H_\theta \subset E(A_\Omega)$ where $\phi \in H_\theta$ has a decomposition in terms of a uniformly converging sum, $\phi = \sum_{\Lambda\in\mathcal{F}} j_\Lambda \phi_\Lambda$, in which each $\phi_\Lambda \in E_\Lambda$ and where the sum $\sum_{\Lambda\in\mathcal{F}} \theta^{-|\Lambda|}|\phi_\Lambda|$ is finite. Let $|\phi|_\theta$ denote the infimum of the latter sum over all possible decompositions of ϕ. Then $|\cdot|_\theta$ is a norm which turns H_θ into a Banach algebra (cf. Appendix C.1 and also Remark 2.2 below). When $\theta = 1$ the $|\cdot|_{\theta=1}$-norm is the same as the uniform norm and $H_1 = E(A_\Omega)$. When $\theta < 1$, functions are ‘penalized’ for depending on many variables.

The particular case $\rho \equiv 0$ is used when studying decay of correlations. The condition on holomorphicity is then void and what remains is a subspace, denoted $C_\theta(S_\Omega) = H_\theta[\rho \equiv 0]$, of the continuous functions on S_Ω, equipped with a norm which we denote $|\cdot|_{C_\theta}$. Again functions are ‘penalized’ for depending on many variables.

Let $0 < \theta \leqslant 1$ and $0 \leqslant \kappa < \infty$ be constants which we call the *deformation parameter* and the *effective coupling strength*, respectively. We consider a family $g_\Omega = (g_p)_{p\in\Omega}$ where each $g_p \in H_\theta$ is real-analytic and such that $|g_p|_\theta \leqslant \kappa$. The map $F_\Omega = (F_p)_{p\in\Omega} : A_\Omega \to \mathcal{C}^\Omega$ obtained by setting

$$F_p : A_\Omega \to \mathcal{C}, \quad F_p : z \mapsto f_p(z_p) + g_p(z), \qquad p \in \Omega, \tag{1.3}$$

is real-analytic and the restriction to the real torus (S_Ω, F_Ω) defines a dynamical system which we call a *coupled map*. The family g_Ω ‘couples’ the single-site components of the ‘uncoupled’ dynamical system (S_Ω, f_Ω).

We denote by $CM[\rho,\lambda,\theta,\kappa]$ the family of all possible coupled analytic maps, F_Ω, for a given choice of parameters $\rho > 0$ and $\lambda > 1$ for the annuli, $\theta \in (0,1]$ for the deformation parameter and $\kappa \in [0,\infty)$ the effective coupling-strength.

1.4. Couplings with a spatial decay

When (Ω, d) is a metric space we associate to $p \in \Omega$ and $\Lambda \in \mathcal{F}$ an 'interaction radius' $\mathrm{rad}(p,\Lambda) = \max\{d(p,q)\colon\ q \in \Lambda \cup \{p\}\}$. In order to describe an exponential spatial decay of the couplings let $0 < \xi \leqslant 1$, $p \in \Omega$ and consider $\phi = \sum_{\Lambda \in \mathcal{F}} j_\Lambda \phi_\Lambda$ for which the sum $\sum_\Lambda \theta^{-|\Lambda|} \xi^{-\mathrm{rad}(p,\Lambda)} |\phi_\Lambda|$ is finite. Like in the previous section we let $|\phi|_{\theta,p,\xi}$ denote the infimum over all possible decompositions of ϕ. The pre-factor is sub-multiplicative in Λ. Thus, using Remark C.2., we know that the space $H_{\theta,p,\xi}$ of such functions is again a Banach algebra. When $\xi = 1$ the space is the same as H_θ but when $\xi < 1$ functions are 'penalized' for 'interacting' with points far away from the distinguished point p.

We consider in this case a family $(g_p)_{p\in\Omega}$ where now each $g_p \in H_{\theta,p,\xi}$ is real-analytic and such that $|g_p|_{\theta,p,\xi} \leqslant \kappa$. In this case we say that κ is the *ξ-spatial effective coupling strength* but otherwise we proceed as above.

2. Results

THEOREM 2.1. – *Choose $\rho > 0$, $\lambda > 1$ and let $\theta_c = \theta_c(\rho, \lambda) \in]0, 1/3]$ be the associated critical θ-value (cf. Remark 4.19). When $\theta < \theta_c$ there is $\kappa = \kappa(\rho, \lambda, \theta) > 0$ for which the following holds*:

For each $k \in \mathbb{Z}$, choose $F^{(k)} \in CM = CM[\rho, \lambda, \theta, \kappa]$, i.e. a coupled map in the class associated with these parameters. The (time-dependent) dynamical system $(S_\Omega, F^{(k)}, m_\Omega)$, $k \in \mathbb{Z}$, then has a unique (time-dependent) natural measure, $d\nu_\Omega^k$, $k \in \mathbb{Z}$, invariant under push-forward of the coupled map, $F_*^{(k)}\ d\nu_\Omega^k = d\nu_\Omega^{k+1}$, and exhibiting exponential decay of time-correlations for a suitable class of observables. More precisely:

(a) *Natural measure*: *Let $b \in C(S_\Omega)$. For every $k \in \mathbb{Z}$ and Lebesgue-almost every point $x \in S_\Omega$ the following Birkhoff-average exists*:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} b \circ F^{(k-1)} \circ \cdots \circ F^{(k-n)}(x) = \nu_\Omega^k(b) \equiv \int_{S_\Omega} b\, d\nu_\Omega^k. \tag{2.4}$$

(b) *Decay of time-correlations*: *There are constants, $\vartheta < 1$, $\gamma > 1$ depending on the parameters of CM only such that for $b \in C_\theta$ and $a \in H_\vartheta$,*

$$\left| \int_{S_\Omega} b \circ F^{(k-1)} \circ \cdots \circ F^{(n)} \cdot a\, d\nu_\Omega^n - \nu_\Omega^k(b)\nu_\Omega^n(a) \right| \leqslant 2|b|_{C_\theta} |a|_\vartheta \gamma^{n-k}, \quad k > n \in \mathbb{Z}. \tag{2.5}$$

In particular, when $F^{(k)} \equiv F$ is time independent, the natural measure is ergodic and time-mixing (exponentially when restricted to observables in H_ϑ) in the usual sense.

Remarks 2.2. –

1. For the example in the introduction the interaction term is the real-analytic family $g_p(z) = \varepsilon/2\pi \sum_{q\in\Omega} c_{p,q} \sin(2\pi z_q)$, $z \in M$, $p \in \Omega$. A straight-forward calculation shows that the effective coupling-strength is bounded by $\kappa = \varepsilon/2\pi \cosh(2\pi\rho)\theta^{-1}$. Note that choosing ρ larger and/or θ smaller increases the effective coupling strength without changing the dynamical system (S_Ω, F_Ω). Other bounds, however, improve with such a change, leaving us with the problem of making a 'best' choice for the parameters which will allow for the largest possible value of ε (cf. Section 6).

2. In the proof of the theorem we need θ to be smaller than θ_c which again is smaller than $1/3$. In particular, large cardinalities in the interaction must be 'exponentially penalized'. This could well reflect a true obstruction and not just our lack of understanding (though the critical θ is probably closer to one). This type of condition is quite similar to the one used in [12] for lattice spin systems.

We may project the natural measure, ν_Ω^k, to a finite Λ-torus to obtain a marginal measure, ν_Λ^k. It is absolutely continuous with respect to the Lebesgue measure, m_Λ, on the Λ-torus. The corresponding density, $h_\Lambda = d\nu_\Lambda^k/dm_\Lambda$, is real-analytic and satisfy the exponential bound (in sup-norm on the torus and with $\vartheta < 1$ as in part (b) of the theorem above), $|h_\Lambda| \leqslant \vartheta^{-|\Lambda|}$. When Ω is not finite, however, one should not expect the measure, ν_Ω^k, on the Ω-torus to be absolutely continuous with respect to the Lebesgue measure on S_Ω. This is due to the infinite dimensionality of the system rather than the coupling. In another context, the following type of example is well-known:

Example 2.3. – Take Ω countable and consider an uncoupled product of identical single-site maps for which the invariant density, h, on S^1 is not identically 1. The tensorial product of $h\,dm$'s gives the invariant natural probability measure on S_Λ which, as with dm itself, extends to a probability measure, $d\nu_\Omega$, on S_Ω. A large deviation argument shows that there are constants $C > 0$ and $\lambda > 0$ such that for each $\Lambda \in \mathcal{F}$ one may find a measurable set $A_\Lambda \subset S_\Lambda (\hookrightarrow S_\Omega)$ for which

$$\nu_\Omega(A_\Lambda) \geqslant 1 - C\mathrm{e}^{-\lambda|\Lambda|} \quad \text{and} \quad m_\Omega(A_\Lambda) \leqslant C\mathrm{e}^{-\lambda|\Lambda|}. \tag{2.6}$$

Taking an increasing sequence of subsets, Λ_n, with $|\Lambda_n| = n$ yields a set $X_N = \bigcup_{n\geqslant N} A_{\Lambda_n}$ with $\nu_\Omega(X_N) = 1$ and $m_\Omega(X_N) \leqslant C\mathrm{e}^{-\lambda N}/(1 - \mathrm{e}^{-\lambda})$. Finally, taking the intersection $X = \bigcap X_N$ we obtain a set of full ν_Ω-measure and zero Lebesgue measure. It is seen that the measure ν_Ω and the Lebesgue measure are mutually singular.

The natural measure in the theorem need not exhibit any decay of spatial correlations as a simple example shows.

Example 2.4. – Let $f \in \mathcal{E}(\rho, \lambda)$ and choose $\Phi \in E(A[\rho] \times A[\rho])$ real-analytic. Fix a point, $p_0 \in \Omega$. We define a coupled map by setting $F_p(z) := f(z_p) + \Phi(z_{p_0}, z_p)$, $p \in \Omega$. When the norm of Φ is small enough, the above Theorem applies and we have a natural measure, ν_Ω. It is straight-forward to check that for each $p\langle\rangle p_0$, the marginal density, $h_{\{p_0,p\}}$, depends only on the dynamics (a skew-product) restricted to these two coordinates. Furthermore, Φ may be chosen so that the 'correlation integral' does not vanish, i.e. $h_{\{p_0,p\}} - h_{\{p_0\}} \otimes h_{\{p\}} \not\equiv 0$. This difference is independent of the choice of $p \in \Omega$ and hence, spatial correlations between the coordinates at p_0 and p do not exhibit any decay at all as a function of p.

When (Ω, d) is a metric space, e.g. a lattice with its Euclidean metric, one may impose further conditions on the spatial decay of the coupling and from this deduce bounds for the decay of spatial correlations. By renormalizing the metric (the strategy of our examples) an apparent exponential bounds could turn out to be e.g. polynomial! Denote by

$$d(\Lambda, K) = \min\{d(p,q):\ p \in \Lambda, q \in K\}$$

the distance between non-empty sets $\Lambda, K \in \mathcal{F}$.

THEOREM 2.5. – *Let ρ, λ, θ, κ and ϑ be as in the theorem above and denote by $(h_\Lambda)_{\Lambda\in\Omega}$ the projective family of densities (with respect to the Lebesgue measure) of the natural measure.*

Suppose that there is $0<\xi<1$ *such that the* ξ*-spatial effective coupling strength is bounded by* κ. *We then have the bound*:

$$\vartheta^{|\Lambda|+|K|}|h_{\Lambda\cup K}-h_\Lambda h_K|\leqslant 2\xi^{d(\Lambda,K)}, \tag{2.7}$$

for $\Lambda,K\in\mathcal{F}$. [*The norm is the sup-norm over the annulus* $A_{\Lambda\cup K}$ *which in particular is stronger than the sup-norm on the torus* $S_{\Lambda\cup K}$.]

3. Proofs

An outline of the proofs is as follows: To each finite 'box' we define a 'confined' dynamical system by imposing fixed boundary conditions outside. We then associate a local Perron Frobenius to each such confined dynamical system. Theorem 3.2 gives an integral representation for the local operator with a very explicit form for the integral kernel and Lemma 3.4 subsequently provides a perturbative expansion of this kernel with precise estimates.

It turns out that the perturbative expansion obtained for the confined system extends quite naturally to the full system. We use this in Section 4 to construct a global Perron Frobenius operator (no boundary conditions anymore). We describe 'configurations' and associated configurational operators (Section 4.2). Each operator is bounded and linear, but certainly leaves the problem of adding them up:

The first step is to map configurations into tree-structures (Section 4.3) and bound the configurational operators by values assigned to the trees (Sections 4.4 and 4.5). This approach is quite alike to Feynman diagrams, probably reflecting some of the authors predelictions. The idea is a two-line argument plus a few diagrams but, unfortunately, in practice turns out to be somewhat harder to realize (Section 4.6). We construct formal sums of the bounds for trees and we show that these formal bounds verify a renormalization equation (Proposition 4.17). Now, when iterates of this renormalization map remain bounded, the formal sums become real sums and the global operator becomes well-defined (Section 4.7). A straight-forward large deviation argument then provides all the estimates we need for the global operator.

What remains is to relate the global operator (which is suitable for norm estimates) and the local operator (which describes the dynamical system). This is done in Section 4.8 and Section 5 then winds up the proof for the theorem on decay of time correlations. In Section 5.1 we deal with the spatial decay (which by now is surprisingly easy!). We conclude in Section 6 by proving the statements in the examples above.

Quite some technical support proves necessary. It would certainly spoil the continuity of the arguments to put this into the proper text. Instead we defer self-containted entities (black boxes), not directly connected with the coupled maps, to Appendices (A)–(E).

Throughout the proofs we consider an arbitrary but fixed choice of the parameters $\rho>0$ and $\lambda>1$. The deformation parameter $\theta\in(0,1]$ and the effective coupling-strength κ will then be subject to restrictions which will be formulated along the way (one could also go directly to Definition 4.18 to see sufficient conditions for the choice of these parameters). In order to simplify the notation we will restrict our attention to a coupled map which is time-independent. The general case consists essentially in keeping track of a few more indices (see Remark 5.2).

3.1. Boundary conditions and weak limits

The set $\mathcal{F}$ of finite subsets of Ω is directed under inclusion and it is therefore natural to use the language of nets rather than sequences. Recall, that if B is a Banach space then a net in B over $\mathcal{F}$ is simply a family $a_\Lambda\in B$, $\Lambda\in\mathcal{F}$. We say that this net converges to $a\in B$, and we write

$a_\Lambda \to a$ as $\Lambda \to \Omega$, if to every $\varepsilon > 0$ there is $K = K_\varepsilon \in \mathcal{F}$ such that $|a - a_\Lambda|_B < \varepsilon$ whenever $K \subset \Lambda \in \mathcal{F}$. Our index set Ω need not be countable, unless the reader feels uneasy with the Axiom of choice (needed e.g. for the compactness of S_Ω when Ω is uncountable), but when it is countable, one could instead of nets consider an increasing co-final sequence of 'boxes' and formulate everything in terms of sequential convergence, but honestly, it would be silly to do so.

In the following, let $\xi \in S_\Omega$ denote an arbitrary but fixed reference point. For $\Lambda \in \mathcal{F}$ we define a holomorphic injection, $i_\Lambda : A_\Lambda \to A_\Omega$, $i_\Lambda(z_\Lambda) = (z_\Lambda, \xi_{\Lambda^c})$ and a ditto natural projection, $q_\Lambda : A_\Omega \to A_\Lambda$, $q_\Lambda(z_\Omega) = z_\Lambda$, all expressed in natural coordinates on the annuli. The map q_Λ is the left-inverse of i_Λ while $r_\Lambda = i_\Lambda \circ q_\Lambda : A_\Omega \to A_\Omega$ is a projection which fixes boundary conditions outside the finite 'box' Λ. We will use the same notation when considering the restriction of the above maps to the tori, S_Ω and S_Λ.

LEMMA 3.1. – *In the uniform topology on $C(S_\Omega)$ we have for fixed $K \in \mathcal{F}$,*

$$\lim_{\Lambda \to \Omega} q_K \circ F \circ r_\Lambda = q_K \circ F. \tag{3.8}$$

Also for $b \in C(S_\Omega)$,

$$\lim_{K,\Lambda \to \Omega} b \circ r_K \circ F \circ r_\Lambda = b \circ F. \tag{3.9}$$

Proof. – For $p \in \Omega$ the boundary conditioned coupling term $g_p \circ r_\Lambda$ converges to g_p in the $C(S_\Omega)$-topology (in fact, even for the stronger H_θ-topology). For finite, fixed K the first statement follows. Given $b \in C(S_\Omega)$, compactness of S_Ω for the product topology implies that $b \circ r_K \to b$ uniformly as $K \to \Omega$. Now $(b \circ r_K) \circ (r_K \circ F \circ r_\Lambda) = b \circ r_K \circ F \circ r_\Lambda$ and the limits $K \to \Omega$ and $\Lambda \to \Omega$ may be taken in any order. □

We also see that if $b \in C(S_\Omega)$ and dm_K denotes the Lebesgue probability measure on S_K then $|m_K(b \circ i_K) - m_\Lambda(b \circ i_\Lambda)| \leqslant |b \circ r_K - b \circ r_\Lambda|$ tends to zero as $K, \Lambda \to \Omega$. Therefore, $\lim_{K \to \Omega} m_K(b \circ i_K)$ exists and defines a positive and continuous linear functional on $C(S_\Omega)$ which takes the value 1 on the constant function 1. By Riesz, the limit functional, m_Ω, defines a probability measure dm_Ω on S_Ω. This construction (due to Kolmogorov) will be used again in Section 4.9 to define more general complex measures on S_Ω.

3.2. The local Perron–Frobenius operator

In this section the value of θ plays no role and it suffices that $\kappa < (\lambda - 1)\rho$.

In the following let $\Lambda \in \mathcal{F} \setminus \emptyset$ be a non-empty finite subset of Ω. We define the Λ-confined coupled map, $F_\Lambda : S_\Lambda \to S_\Lambda$ (similarly for $F_\Lambda : A_\Lambda \to \mathcal{C}^\Lambda$) by setting $F_\Lambda = q_\Lambda \circ F \circ i_\Lambda$. When $w_\Lambda \in A_\Lambda$ we denote by $F_\Lambda^{-1}(w_\Lambda)$ the (finite) set of inverse images in A_Λ obtained by solving $w_\Lambda = F_\Lambda(z_\Lambda)$ for $z_\Lambda \in A_\Lambda$. The condition on the coupling-strength, $\kappa < (\lambda - 1)\rho$, implies (Appendix E) that local inverses indeed exist and are real-analytic in w_Λ. In particular, F_Λ is non-singular and therefore has a well-defined orientation on S_Λ. In the expressions below a prefactor $\pm$ should be understood as the sign of this orientation. It equals the product of signs of orientations of f_p, $p \in \Lambda$.

The (real) Perron–Frobenius operator associated with the restricted dynamical system (S_Λ, F_Λ) and the Lebesgue measure m_Λ is defined by the identification,

$$\int_{S_\Lambda} dm_\Lambda a \cdot L_\Lambda \phi \equiv \int_{S_\Lambda} dm_\Lambda a \circ F_\Lambda \cdot \phi, \tag{3.10}$$

for $a \in L^\infty(S_\Lambda)$ and $\phi \in L^1(S_\Lambda)$. As F_Λ has no critical points standard arguments show that indeed this defines a bounded linear operator, L_Λ, on $L^1(S_\Lambda)$.

We will derive an integral representation for the Perron–Frobenius operator acting upon the subspace E_Λ of $L^1(S_\Lambda)$. We shall use residue calculus (essentially Cauchy's integral formula) and differential geometry to achieve this. The integral kernel in the Cauchy-formula is not suitable here due to the lack of periodicity (we need to reproduce functions on $\mathcal{C}$) and we replace it by the (still anti-symmetric) kernel,[2] $k(w,z) = 1/2\mathrm{i}\cot(\pi(z-w))$, $w \neq z \in \mathcal{C}$.

The product cylinder $\mathcal{C}^\Lambda$ is an orientable complex manifold. We let $\mu_\Lambda = \bigwedge_{p\in\Lambda} dz_p$ denote the unique holomorphic differential form on $\mathcal{C}^\Lambda$ which extends the Lebesgue volume element m_Λ on $(\mathbb{R}/\mathbb{Z})^\Lambda$. Note that μ_Λ is a holomorphic differential form of maximal degree among holomorphic forms. The product kernel,

$$k_w(z) = \prod_{p\in\Lambda} k(w_p, z_p), \quad w, z \in \mathcal{C}^\Lambda, \tag{3.11}$$

is defined whenever $z_p \neq w_p$ for all $p \in \Lambda$. We write $\Gamma_\Lambda = \Gamma_\Lambda[\rho] = \prod_{p\in\Lambda} \partial A[\rho]$ for the distinguished boundary of the product annulus. By (multi variable) residue calculus it follows that

$$\phi_w = \int_{\Gamma_\Lambda} \mu_\Lambda k_w \phi \tag{3.12}$$

gives an evaluation of $\phi \in E_\Lambda$ at any point in the open product annulus, $w \in \operatorname{Int} A_\Lambda$.

We may define a holomorphic extension of the Perron–Frobenius operator L acting on the space E_Λ: For $\phi \in E_\Lambda$ and any $|\Lambda|$-chain σ in A_Λ we make the following identification (as already mentioned, $\pm$ signifies the orientation of the map F_Λ restricted to the real torus):

$$\int_\sigma \mu_\Lambda \, L_\Lambda \phi \equiv \pm \int_{F_\Lambda^{-1}\sigma} \mu_\Lambda \phi. \tag{3.13}$$

A change of coordinates yields the following formula for the operator:

$$L_\Lambda \phi_w = \pm \sum_{z\,:\,F_\Lambda z = w} \frac{\phi_z}{\det F_{\Lambda *z}}. \tag{3.14}$$

To see that (3.13) is well-defined, note that by Appendix E, F_Λ has local inverses which are holomorphic. Summing the pull-backs of $\mu_\Lambda \phi$ by the inverse branches of F_Λ we therefore obtain a differential form which is *a fortiori* holomorphic on a neighborhood of A_Λ (in general each individual term in the sum is only locally holomorphic!). By maximality of the degree this holomorphic form is proportional to μ_Λ itself. The factor of proportionality is an element of E_Λ and is given by the expression (3.14).

An approximation argument using step functions yields the identity,

$$\int_\sigma \mu_\Lambda a L_\Lambda \phi = \pm \int_{F_\Lambda^{-1}\sigma} \mu_\Lambda a \circ F_\Lambda \phi, \tag{3.15}$$

[2] One may lift from the cylinder to the annulus and note that $1/2\mathrm{i}\cot(\pi(z-w)) = 1/2\pi\mathrm{i}\sum_{n\in\mathbb{Z}} 1/(z-w+n)$ is a sum of translates of the standard Cauchy kernel.

valid for any $|\Lambda|$-chain σ and any continuous function $a \in C(\sigma)$ on the chain. If we here let $\sigma = S_\Lambda = F_\Lambda^{-1} S_\Lambda$ the formula extends to $a \in L^\infty(S_\Lambda)$ and we see that (3.13) extends the real Perron–Frobenius operator when acting upon an element of E_Λ. We shall not make use of the explicit formula (3.14) but rather the following representation theorem (a variation of Proposition 3.3 in [10]):

THEOREM 3.2. – *Under the assumption $\kappa < (\lambda - 1)\rho$, the holomorphic local Perron–Frobenius operator $L_\Lambda : E_\Lambda \to E_\Lambda$, $\Lambda \in \mathcal{F} \setminus \emptyset$ has the following integral representation*:

$$L_\Lambda \phi_w = \pm \int_{\Gamma_\Lambda} \mu_\Lambda k_w \circ F_\Lambda \phi, \quad w \in A_\Lambda,\ \phi \in E_\Lambda. \tag{3.16}$$

Proof. – The integrand on the right hand side is a holomorphic differential form of maximal degree, but with singularities at the pre-images of w under the map F_Λ. The chain Γ_Λ is without boundary and by Stokes' theorem the value of the integral remains unchanged under smooth deformations of the chain as long as we can manage to avoid the singularities. We will consider two such deformations. Let $\rho^+ = \lambda\rho - \kappa > \rho$.

First, a smooth homotopy $\Phi^t = \Phi^t_\Lambda$, $t \in [0,1]$ (with boundary conditions relative to Λ as before) between the uncoupled maps, f_Λ, and the coupled map F_Λ is given by:

$$\Phi^t_p = f_p + t g_p, \quad p \in \Lambda. \tag{3.17}$$

Each g_p has a θ-norm smaller than κ and since the sup-norm does not exceed the θ-norm,

$$\left|\Phi^t_p - F_p\right| \leqslant (1-t)\kappa. \tag{3.18}$$

Let $\rho^t = \lambda\rho - t\kappa$ and consider the family of distinguished boundaries: $\Gamma^t = \Gamma_\Lambda[\rho^t]$, $t \in [0,1]$. For $z \in (\Phi^t)^{-1}\Gamma^t$ we have

$$\left|\operatorname{Im} F_p(z)\right| \geqslant \left|\operatorname{Im} \Phi^t_p(z)\right| - (1-t)\kappa = \rho^+ > \rho. \tag{3.19}$$

We therefore conclude that when $w \in A_\Lambda$ the kernel $k_w \circ F_\Lambda$ remains non-singular along the path $(\Phi^t)^{-1}\Gamma^t$, $t \in [0,1]$.

Next, we note that for each $p \in \Lambda$ one has $U = f_p^{-1} A[\lambda\rho] \subset \operatorname{Int} A[\rho]$. Since f_p has no critical values in $A[\lambda\rho]$ the set U is topologically itself an annulus (with smooth boundaries) and we may find a smooth homotopy from $f_p^{-1}\Gamma[\lambda\rho]$ to $\Gamma[\rho]$ which stays in $A[\rho] - U$ and preserves the orientation. Let us do so for each $p \in \Lambda$. Their direct product then gives a smooth homotopy between $\Gamma_\Lambda[\rho]$ and $f_\Lambda^{-1}\Gamma[\lambda\rho]$ during which $|\operatorname{Im} f_p| \geqslant \lambda\rho$ for each $p \in \Lambda$. Hence, $k_w \circ F_\Lambda$ is non-singular under this homotopy as well.

Composing the above homotopies we obtain a deformation from the chain Γ_Λ to the chain $F_\Lambda^{-1}\Gamma_\Lambda[\rho^+]$ under which the integrand remains non-singular. By Stokes, we can make this replacement of chains without changing the value of the integral. But then the value equals

$$\int_{F_\Lambda^{-1}\Gamma_\Lambda[\rho^+]} \mu_\Lambda k_w \circ F_\Lambda \phi = \pm \int_{\Gamma_\Lambda[\rho^+]} \mu_\Lambda k_w L_\Lambda \phi = \pm L_\Lambda \phi_w \tag{3.20}$$

in which we have simply used the identity (3.15) and the multi-variable Cauchy formula (3.12) for the product-domain $\Gamma_\Lambda[\rho^+]$. $\square$

Some elementary properties of the Perron–Frobenius operator, L_{f_p}, associated with a single-site map, f_p, $p \in \Omega$, will be used. Applying the above theorem we see that

$$L_{f_p}\phi_{w_p} = \pm \int_{\Gamma_p} \mu_p k_{w_p} \circ f_p \phi \tag{3.21}$$

defines a holomorphic Perron–Frobenius operator on $E(A[\rho])$ which extends naturally (Appendix B) to an operator acting upon E_Λ (provided, of course, that $p \in \Lambda$). For a function $\phi \in E_\Lambda$ we will write

$$\ell_p : E_\Lambda \to E_\Lambda, \qquad \ell_p \phi(z_\Lambda) = \oint_{S^1} dz'_p \phi(z'_p, z_{\Lambda-p}) \tag{3.22}$$

for the average in the p'th coordinate (again with $p \in \Lambda$). If ϕ_Λ is considered as the density (with respect to Lebesgue) of a complex measure on S_Λ then $\ell_p \phi_\Lambda$ yields its marginal density on the factor space $S_{\Lambda-p}$.

LEMMA 3.3. – *The family of operators L_{f_p}, $p \in \Lambda$ is commutative. The same is true for the family ℓ_p, $p \in \Lambda$. Also L_{f_p} and ℓ_q commute when $p \neq q$. Finally, when $p = q$ one has*

$$\ell_p L_{f_p} = \ell_p. \tag{3.23}$$

Let us first consider the single site operator L_{f_p} acting upon $\phi_p \in E_p = E(A[\rho])$. Setting $a \equiv 1$ in the defining Eq. (3.10) we get: $\int_{S_p} dm_p L_{f_p} \phi_p = \int_{S_p} dm_p \phi_p$ and this clearly extends to E_Λ in the form $\ell_p L_{f_p} = \ell_p$. The commutativity of the rest follows from Fubini.

3.3. A perturbative expansion of the integral kernel

The integral kernel in Theorem 3.2 has the form of a product,

$$\prod_{p\in\Lambda} k_{w_p} \circ F_p(z_\Lambda, \xi_{\Lambda^c}), \tag{3.24}$$

in which $F_p = f_p + g_p$ with $g_p \in H_\theta$ and $|g_p|_\theta \leqslant \kappa$. For a given $p \in \Lambda$ and $w_p \in A[\rho]$ the factor, $k_{w_p} \circ F_p$, is singular at every point where $w_p = F_p(z_\Lambda)$. On the other hand, the integral in (3.16) is only carried out on the distinguished boundary Γ_Λ where these singularities are not present. It may be tempting to restrict the domain of definition for the z_Λ variable to Γ_Λ and as a consequence only retain information about continuity of the kernel. This is not good enough for the estimates below. One really needs analyticity in the variables z_q for q different from p (i.e. the image of the vertex operator, defined in (4.60), should be holomorphic in its variables for the subsequent bounds to be valid). The solution is then to 'mix' continuity in the variable z_p and analyticity in the z_q's. This leads us to consider function spaces of so-called weakly holomorphic functions. We will here deal with the full system and in Section 4.8 obtain the restriction to finite subsets (with boundary conditions as before) as a special case.

For $p \in \mathcal{F}$ and $S \subset \Omega$ (S need not be finite) the product domain,

$$D_{p,S} = A_p \times \Gamma_p \times \prod_{q\in S\setminus\{p\}} A_q, \tag{3.25}$$

is a compact subset of a complex manifold (a direct product of complex cylinders). We let $E_{p,S} = E(D_{p,S})$ be the corresponding Banach space of weakly holomorphic functions in the sup-norm (Definition B.1 in Appendix B). Thus, a function $\beta \in E_{p,S}$ is continuous in $(w_p, z_p, z_{S\setminus\{p\}}) \in D_{p,S}$ and holomorphic in each of $w_p \in \operatorname{Int} A_p$ and $z_q \in \operatorname{Int} A_q$, $q \in S \setminus \{p\}$ (the respective interior domains). Following the discussion above the variable z_p takes values only at the boundary Γ_p and no analytic behavior apart from continuity is associated with this variable. The spaces $(E_{p,\Lambda})_{\Lambda\in\mathcal{F}}$ are inductively ordered with natural inclusions (tacitly omitted in the notation) and their union is dense in $E_{p,\Omega}$.

LEMMA 3.4. – *Let* $0 \leqslant \kappa < (\lambda - 1)\rho$ *and define the constant*:

$$C_\beta = C_\beta(\rho, \lambda, \kappa) = \frac{e^{2\pi\kappa}}{e^{2\pi(\lambda-1)\rho} - e^{2\pi\kappa}} - \frac{1}{e^{2\pi(\lambda-1)\rho} - 1}. \tag{3.26}$$

We have the following $E_{p,\Omega}$ *convergent expansion,*

$$k_{w_p} \circ F_p = k_{w_p} \circ f_p + \sum_{V\in\mathcal{F}} \beta_{p,V}, \tag{3.27}$$

where the functions $\beta_{p,V} \in E_{p,V}$, $V \in \mathcal{F}$ $(V$ *need not contain* $p)$ *satisfy*:

$$\sum_{V\in\mathcal{F}} \theta^{-|V|} |\beta_{p,V}| \leqslant C_\beta \quad \textit{and} \tag{3.28}$$

$$\oint_{S^1} dw_p \beta_{p,V}(w_p, z_{V\cup\{p\}}) \equiv 0, \quad z_p \in \Gamma_p,\ z_{V\setminus\{p\}} \in A_{V\setminus\{p\}}. \tag{3.29}$$

Proof. – We need an analytic estimate for the kernel $k_w(z) = \frac{1}{2i} \cot \pi(z - w)$. Consider the case $r = \operatorname{Im}(w - z) > 0$ (the case $r < 0$ is treated similarly). One has the following expansion: $k_w(z) = \frac{1}{2} + \sum_{p\geqslant 1} e^{-2\pi i p(z-w)}$, hence, for $n \geqslant 1$ the bound

$$\left| \left(\frac{d}{dw}\right)^n k_w(z) \right| \leqslant (-1)^n \sum_{p\geqslant 1} (-2\pi p)^n e^{-2\pi p r} = (-1)^n \left(\frac{d}{dr}\right)^n \frac{1}{e^{2\pi r} - 1}. \tag{3.30}$$

Consider now the Taylor expansion:

$$k_w(z + u) = k_w(z) + \sum_{n\geqslant 1} a_n(w, z) u^n. \tag{3.31}$$

We may bound the sum provided $|u| \leqslant \kappa < r$. We have the following estimate:

$$\sum_{n\geqslant 1} |a_n| \kappa^n \leqslant \left(e^{2\pi(r-\kappa)} - 1\right)^{-1} - \left(e^{2\pi r} - 1\right)^{-1}. \tag{3.32}$$

Returning to the kernel composed with $f_p \in \mathcal{E}(\rho, \lambda)$, the properties of f_p imply that for $p \in \Omega$, $z_p \in \partial A_p$ and $w_p \in A_p$:

$$\left|\operatorname{Im}\left(w_p - f_p(z_p)\right)\right| \geqslant (\lambda - 1)\rho + \delta, \tag{3.33}$$

for some $\delta > 0$. Therefore, in the Taylor-expansion,

$$k_{w_p}\left(f_p(z_p) + u\right) = k_{w_p}\left(f_p(z_p)\right) + \sum_{n\geqslant 1} a_n\left(w_p, f_p(z_p)\right) u^n, \tag{3.34}$$

the $a_n = a_n(w_p, f_p(z_p))$'s verify (3.32) with $r = (\lambda - 1)\rho + \delta$. Inserting for u the interaction g_p we get the 'perturbed' kernel expansion:

$$k_{w_p} \circ F_p = k_{w_p} \circ f_p + \sum_{n \geqslant 1} a_n \cdot (g_p)^n \tag{3.35}$$

and since $|g_p| \leqslant |g_p|_\theta \leqslant \kappa < (\lambda - 1)\rho$ this sum converges uniformly in the Banach algebra $E_{p,\Omega}$. This is not yet in the form we want since the right hand side involves 'interactions' with, in general, infinitely many variables.

Using, however, that H_θ is a Banach algebra, we have $|(g_p)^n|_\theta \leqslant |g_p|_\theta^n \leqslant \kappa^n$. By Lemma C.3, for each $n \geqslant 1$ and any given $\varepsilon_n > 0$ we may find $\hat{g}_p^{(n)} = (\hat{g}_{p,V}^{(n)})_{V \in \mathcal{F}} \in [(g_p)^n] \subset \hat{H}_\theta$ such that:

$$\left|\hat{g}_p^{(n)}\right|_\theta = \sum_{V \in \mathcal{F}} \theta^{-|V|} \left|\hat{g}_{p,V}^{(n)}\right| \leqslant |g_p|_\theta^n + \varepsilon_n \leqslant \kappa^n + \varepsilon_n. \tag{3.36}$$

The second term in (3.34) may then be written as the sum over $V \in \mathcal{F}$ of

$$\beta_{p,V}(w_p, z_{V \cup \{p\}}) = \sum_{n \geqslant 1} a_n\big(w_p, f_p(z_p)\big) \hat{g}_{p,V}^{(n)}(z_V) \in E_{p,V}. \tag{3.37}$$

For $\varepsilon > 0$ (when $\kappa = 0$ we may take $\varepsilon = 0$) we may choose the ε_n's so that:

$$\sum_{V \in \mathcal{F}} \theta^{-|V|} |\beta_{p,V}| \leqslant \sum_{n \geqslant 1} |a_n| \sum_{V \in \mathcal{F}} \theta^{-|V|} |\hat{g}_{p,V}^{(n)}| \leqslant \sum_{n \geqslant 1} |a_n| \kappa^n + \varepsilon, \tag{3.38}$$

and the first term was bounded by (3.32) with $r = (\lambda - 1)\rho + \delta$. Finally, as $\delta > 0$, a suitable initial choice of ε yields the bound (3.28) with the desired constant C_β.

For $n \geqslant 1$ the function a_n is the nth order derivative of $k_w(z)$ with respect to z (times $1/n!$). The anti-symmetry of the kernel shows that $n! a_n = (\partial_z^n k_w) \circ f(z) = (-1)^n \partial_w^n (k_w \circ f(z))$. Thus, as a differential form (with respect to w) we have that $a_n \, dw = -da_{n-1}/n$. Now, S^1 is closed and therefore

$$\oint_{S^1} \mu(w_p) \, a_n(w_p, z_p) = -\frac{1}{n} \oint_{S^1} da_{n-1}(\cdot, z_p) = 0, \quad n \geqslant 1. \tag{3.39}$$

Uniform convergence in (3.37) implies

$$\oint_{S^1} dw_p \beta_{p,V}(w_p, z_{V \cup \{p\}}) \equiv 0, \tag{3.40}$$

valid for any $p \in \Omega$, $V \in \mathcal{F}$, $z_p \in \Gamma_p$ and $z_q \in A_q$, $q \in V - \{p\}$. $\square$

Remarks 3.5. – We note that for fixed $\lambda > 1$ and $\rho > 0$ the constant C_β tends to zero as $\kappa \to 0^+$.

4. The global Perron–Frobenius operator

Prior to the coming section on configurational operators we need to introduce the relevant family of Banach spaces for our operators to act upon.

4.1. Projective limits and modules

Given $\Lambda \in \mathcal{F}$ non-empty, let $z = (z_p)_{p\in\Lambda}$ be natural coordinates on the Λ-torus, S_Λ. For $K \subset \Lambda \in \mathcal{F}$ we define a linear operator, $\pi_{K,\Lambda} : E_\Lambda \to E_K$, by 'integrating away' the coordinates outside K:

$$\pi_{K,\Lambda}\phi_\Lambda(z_K) = \int_{S_{\Lambda-K}} dm_{\Lambda-K}(z_{\Lambda-K})\phi_\Lambda(z_\Lambda), \quad \phi_\Lambda \in E_\Lambda. \tag{4.41}$$

Such maps are norm-contracting and Fubini's Theorem shows that $\pi_{H,K}\pi_{K,\Lambda} = \pi_{H,\Lambda}$ whenever $H \subset K \subset \Lambda \in \mathcal{F}$. The family $(E_\Lambda, \pi_{K,\Lambda})$ is thus projective and we denote by $\mathcal{M}$ its projective limit. An element $\phi = (\phi_\Lambda)_{\Lambda\in\mathcal{F}} \in \mathcal{M}$ satisfies $\pi_{K,\Lambda}\phi_\Lambda = \phi_K$ whenever $K \subset \Lambda \in \mathcal{F}$. We write $\pi_\Lambda\phi$ for the natural projection $\phi \in \mathcal{M} \mapsto \phi_\Lambda \in E_\Lambda$. From the defining equation it is clear that if $K \subset \Lambda$, $a_K \in E_K$ and $\phi_\Lambda \in E_\Lambda$ then

$$\pi_{K,\Lambda}\big((j_{\Lambda,K}a_K)\phi_\Lambda\big) = a_K\pi_{K,\Lambda}\phi_\Lambda \tag{4.42}$$

simply because a_K does not depend on the variables which are 'integrated away'. (The natural inclusion $j_{\Lambda,K}$ was defined in Section 1.1.) We may therefore apply the procedure of Appendix C to conclude that the projective limit of the E_Λ's in a natural way becomes a module (the corresponding action is denoted by the symbol $\star$) over the inductive limit of the E_Λ's. In the same appendix, we introduce for $\vartheta \in (0,1]$ the 'norm':

$$\|\phi\|_\vartheta = \sup_{\Lambda\in\mathcal{F}} \vartheta^{|\Lambda|}|\phi_\Lambda| \in [0,+\infty], \quad \phi \in \mathcal{M}. \tag{4.43}$$

The subset, $\mathcal{M}_\vartheta \subset \mathcal{M}$, for which this norm is finite is then a Banach-module over the Banach algebra, H_ϑ, and the action satisfies the bound,

$$\|a \star \phi\|_\vartheta \leqslant |a|_\vartheta \|\phi\|_\vartheta, \tag{4.44}$$

whenever $a \in H_\vartheta$ and $\phi \in \mathcal{M}_\vartheta$. The norm of the natural projection $\pi_\Lambda : \mathcal{M}_\vartheta \to E_\Lambda$ is clearly given by

$$\|\pi_\Lambda\|_\vartheta = \vartheta^{-|\Lambda|}. \tag{4.45}$$

4.2. Configurations

Consider the local Perron Frobenius operator, L_Λ, relative to the set $\Lambda \in \mathcal{F}$. A perturbative expansion of this operator emerges if we insert the kernel expansion (3.27) into the product (3.24), and interchange products and sums. Uniform convergence allows us to do so. However, the boundary condition fixing the values of variables 'outside' Λ is rather artificial and, this is one of our main points, not necessary in a 'global' perturbative expansion of the operator. We pay a (small) price for dispensing from this Λ-confinement, namely that afterwards we have to show that our global operator really describes the dynamical system we started out with. Let us first introduce some terminology.

DEFINITION 4.1. – Let $S \in \mathcal{F}$ and let $V : S \to \mathcal{F}$ be a map. We call (S,V) a *branching pair* with S being the *singular set* and V the *vertex map*. We call $V[S] \equiv S \cup \bigcup_{p\in S} V(p)$ the *vertex set* for the branchings.

When $K \in \mathcal{F}$ and $S \subset K$ we say that (S,V) is a *branching* over K. We call $K - V[S]$ the *free set* and $K \cup V[S]$ the (S,V)*-expansion* of K.

Recall that in Lemma 3.4 above we wrote the 'coupled' kernel as a sum of an unperturbed kernel, $k_{w_p} \circ f_p$, and perturbative kernels, $\beta_{p,V} \in E_{p,V}$, with $p \in \Omega$ and $V \in \mathcal{F}$. Let $K \in \mathcal{F}$ and consider a branching (S, V) over K, expanding it into $H = K \cup V[S]$. We will introduce an integral kernel which will be a product of kernels-factors where the choice of each factor is determined by this branching.

For each point in the singular set, $p \in S$, we pick as factor the perturbation $\beta_{p,V}$ with $V = V(p)$ and for each point $q \in K - S$, the unperturbed kernel $k_{w_q} \circ f_q$. To the triple (K, S, V) we then associate the kernel:

$$G_{K,S,V}(w_K, z_H) = \prod_{p \in S} \beta_{p,V(p)}(w_p, z_{V(p)\cup\{p\}}) \prod_{q \in K-S} k_{w_q} \circ f_q(z_q), \tag{4.46}$$

and we define a bounded linear operator $L_{K,S,V} : \phi \in E_H \to E_K$ as follows:

$$L_{K,S,V}\phi(w) = \pm \int_{S_{H-K}} dm_{H-K}(z_{H-K}) \int_{\Gamma_K} \mu_K(z_K) G_{K,S,V}(w,z)\phi(z), \quad w \in A_K, \tag{4.47}$$

where $\pm$ is the sign of the orientation of F_K (cf. Section 3.2). In the case $K = \emptyset$ the branching pair is also empty. The associated operator acts on $E_\emptyset = \mathbb{C}$ as the identity map:

$$L_{\emptyset,\emptyset,\emptyset} c = c, \quad c \in \mathbb{C}. \tag{4.48}$$

LEMMA 4.2. – *Let (S, V) be a branching over $K \in \mathcal{F} - \{\emptyset\}$, expanding it into $H = K \cup V[S]$. If $p \in K$ then (cf. Fig. 1):*

$$\pi_{K-\{p\},K} L_{K,S,V} = \begin{cases} 0 & \textit{if } p \in S \quad (\textit{case } 1), \\ L_{K-\{p\},S,V} & \textit{if } p \in V[S] - S \quad (\textit{case } 2), \\ L_{K-\{p\},S,V}\pi_{H-\{p\},H} & \textit{if } p \in K - V[S] \quad (\textit{case } 3). \end{cases} \tag{4.49}$$

Proof. – The identity (3.29) shows that $\beta_{p,V(p)}$ is in the kernel of ℓ_p. The operator $\pi_{K-\{p\},K} : E_K \to E_{K-\{p\}}$ acts on (4.47) by integrating the variable w_p along the circle which

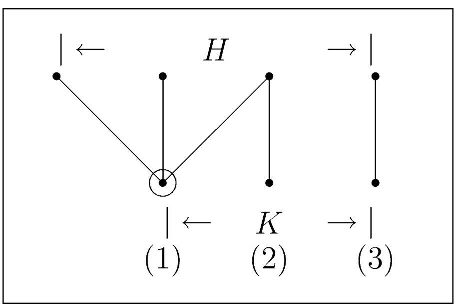

Fig. 1. A sample configuration which illustrates the three different cases (by removing the point (1), (2) or (3), respectively) in Lemma 4.2. Here, $|K| = 3$, $|H| = 4$ and there is one branching (the big circle) into a set of three points.

is precisely the action of ℓ_p. When $p \in S$, the result vanishes by Fubini, thereby proving the first case.

When $p \notin S$ the Perron–Frobenius identity, Lemma 3.3 last equation, shows that the action of ℓ_{w_p} is twofold: it removes the term $k_{w_p} \circ f_p(z_p)$ from the kernel (4.46) thus replacing it by $G_{K-\{p\},S,V}$ and it introduces the action of ℓ_{z_p} upon ϕ. When $p \in V[S] - S$ the latter action is already included in the toral integration (4.47), now over $S_{H-(K-\{p\})} = S_{H-K} \times S_{\{p\}}$ and the expansion of $K - \{p\}$ is still H. The second case follows.

Finally, if $p \in K - V[S]$ the (S,V)-expansion of $K - \{p\}$ is only $H - \{p\}$. We then have to insert the 'averaging' in the p-variable by hand, which amounts to pre-composing with $\pi_{H-\{p\},H}$ (the third case). □

[Note that if $K = \{p\}$ (i.e. contains only one point) there are only two possibilities: When $S = \{p\}$ we are in the first case while $S = \emptyset$ gives the third case and we have $\pi_{\emptyset,\{p\}} L_{\{p\},\emptyset,\emptyset} = \pi_{\emptyset,\{p\}}$].

For an ordered couple $I \subset H \in \mathcal{F}$ we define an 'initial' projection, $Q_I^H : E_H \to E_H$,

(4.50)
$$Q_I^H \equiv \prod_{p \in I} (1 - \ell_p) \prod_{q \in H-I} \ell_p.$$

Using binomial expansion, we see that

(4.51)
$$\sum_{I \subset H} Q_I^H = \prod_{p \in H} \big((1 - \ell_p) + \ell_p\big) = \mathrm{id} : E_H \to E_H.$$

We observe that the ℓ_p operators correspond to the 'averaging' which is used when we defined the projective family of operators $\pi_{K,\Lambda}$. Therefore,

(4.52)
$$Q_I^H = j_{H,I} \circ \prod_{p \in I} (1 - \ell_p) \circ \pi_{I,H} = \sum_{J \subset I} (-1)^{|I-J|} j_{H,J} \circ \pi_{J,H},$$

and since $\pi_{I,H} \circ \pi_H = \pi_I$ we obtain the following identity:

(4.53)
$$Q_I^H \circ \pi_H = \sum_{J \subset I} (-1)^{|I-J|} j_{H,J} \circ \pi_J : \mathcal{M}_\vartheta \to E_H.$$

The important point in this sum-formula is that the initial projection in each term is to the in general smaller set $J (\subset I \subset H)$. This saves some crucial factors of ϑ in norm-estimates below.

DEFINITION 4.3. – Let $T \geqslant 1$ and let $(S_0, V_0), \ldots, (S_{T-1}, V_{T-1})$ be a T-tuple of branching-pairs. Also let $I \in \mathcal{F}$ be a finite subset (of Ω) which we denote an *initial* set. For $K \in \mathcal{F}$ the *expansion* of K along the T-tuple is a non-increaing sequence of finite sets,

$$K_T \subset K_{T-1} \subset \cdots \subset K_0,$$

where $K_T \equiv K$ and the remaining sets are defined recursively by:

(4.54)
$$K_{t-1} = K_t \cup V_{t-1}[S_{t-1}], \quad T \geqslant t \geqslant 1.$$

If for each $t = 0, \ldots, T-1$, we have $S_t \subset K_{t+1}$ (thus, (S_t, V_t) is a branching over K_{t+1}) and if $I \subset K_0$ then the T-tuple together with the initial set I is called a *time-T configuration over K*. The set of all time-T configurations over K is denoted $\mathcal{C}[K,T]$ (for an example, cf. Fig. 4).

Let $0 < \vartheta \leqslant 1$ be fixed for the moment and consider a time-T configuration $C \in \mathcal{C}[K,T]$ over the set $K \in \mathcal{F}$. We will construct a bounded linear operator from the space $\mathcal{M}_\vartheta$ of projective families in the ϑ-norm to the space E_K:

Let $K = K_T \subset K_{T-1} \subset \cdots \subset K_0$ be the expansion of K along C. First, the natural projection $\pi_{K_0} : \mathcal{M}_\vartheta \to E_{K_0}$ takes a projective family in $\mathcal{M}_\vartheta$ to the member defined over the set K_0. Second, we apply the initial projection $Q_I^{K_0} : E_{K_0} \to E_{K_0}$. This is well-defined since we assumed $I \subset K_0$. Third, for each $t = 0, \ldots, T-1$ the pair (S_t, V_t) is a branching over K_{t+1} and we may then let $L^{(t)} = L_{K_{t+1}, S_t, V_t} : E_{K_t} \to E_{K_{t+1}}$ denote the corresponding transfer operator defined by Eq. (4.47).

We combine all of the above and define the *configurational* operator,

$$L_K[C] \equiv L^{(T-1)} \circ \cdots \circ L^{(0)} \circ Q_I^{K_0} \circ \pi_{K_0} : \mathcal{M}_\vartheta \to E_{K_T} = E_K, \tag{4.55}$$

i.e. by the sequence of continuous operators:

$$\mathcal{M}_\vartheta \xrightarrow{\pi_{K_0}} E_{K_0} \xrightarrow{Q_I^{K_0}} E_{K_0} \xrightarrow{L^{(0)}} \cdots \xrightarrow{L^{(T-1)}} E_{K_T}. \tag{4.56}$$

Remarks 4.4. – In the case $K = \emptyset$ there exists but one configuration over K, namely the empty-configuration, $C = \emptyset$ which has no branchings and an empty initial set. The associated configurational operator is simply the projection of $\mathcal{M}_\vartheta$ to the empty set,

$$L_\emptyset[\emptyset] = \pi_\emptyset : \mathcal{M}_\emptyset \to E_\emptyset = \mathbb{C}. \tag{4.57}$$

The configurational operators satisfy the following:

PROPOSITION 4.5. – *Let $C \in \mathcal{C}[K,T]$ be a time-T configuration over $K \in \mathcal{F}$ and let $\alpha \subset K$. Then*

$$\pi_{\alpha,K} L_K[C] = \begin{cases} L_\alpha[C] & \text{if } C \in \mathcal{C}[\alpha,T], \\ 0 & \text{otherwise.} \end{cases} \tag{4.58}$$

Proof. – For $\alpha \subset \beta (\in \mathcal{F})$ we have $\mathcal{C}[\alpha,T] \subset \mathcal{C}[\beta,T]$ and also $\pi_{\alpha,\beta}\pi_{\beta,K} = \pi_{\alpha,K}$. It is therefore enough to show the above identity for a subset of K where only one point is omitted. The general case follows by induction. Fig. 2 illustrates the four possibilities described below.

Thus, let $p \in K$ and set $\alpha = K - \{p\}$. Let $K \equiv K_T \subset K_{T-1} \subset \cdots \subset K_0$ be the expansion of K along C and let $\alpha \equiv \alpha_T \subset \alpha_{T-1} \subset \cdots \subset \alpha_0$ be the expansion of α along C. When expanding the set α it may, or may not, happen that p is included at some stage. Let τ be the smallest number among $0, \ldots, T$ such that $p \notin \alpha_t$ whenever $t \geqslant \tau$. We will see what happens when carrying out the projection $\pi_{\alpha,K} L_K[C] = \pi_{\alpha_T,K_T} L_{K_T}[C]$.

By definition, p is not included in α_T. If $\tau < T$ then by Lemma 4.2 (case 3) and induction,

$$\pi_{\alpha_T,K_T} L_{K_T,S_{T-1},V_{T-1}} \cdots L_{K_{\tau+1},S_\tau,V_\tau} = L_{\alpha_T,S_{T-1},V_{T-1}} \cdots L_{\alpha_{\tau+1},S_\tau,V_\tau} \pi_{\alpha_\tau,K_\tau}. \tag{4.59}$$

When p is not included in the expansion, then $\tau = 0$ (i.e. $p \notin \alpha_0$) and it remains to evaluate $\pi_{\alpha_0,K_0} Q_I^{K_0} \pi_{K_0}$. We have the following dichotomy:

(1a) Either $I \subset \alpha_0$ (hence, $p \notin I$) and C is a configuration over α. Applying Fubini it follows that $\pi_{\alpha_0,K_0} Q_I^{K_0} \pi_{K_0} = Q_I^{\alpha_0} \pi_{\alpha_0} : \mathcal{M}_\vartheta \to E_{\alpha_0}$ and, indeed, we obtain $L_\alpha[C]$.

(1b) Or, $I \not\subset \alpha_0$ and C is not a configuration over α. In this case one must have $p \in I$. But $\ell_p(1-\ell_p) = 0$ implies $\ell_p Q_I^{K_0} = 0$, hence also $\pi_{\alpha_0,K_0} Q_I^{K_0} = 0$ and $\pi_{\alpha,K} L_K[C]$ vanishes as claimed.

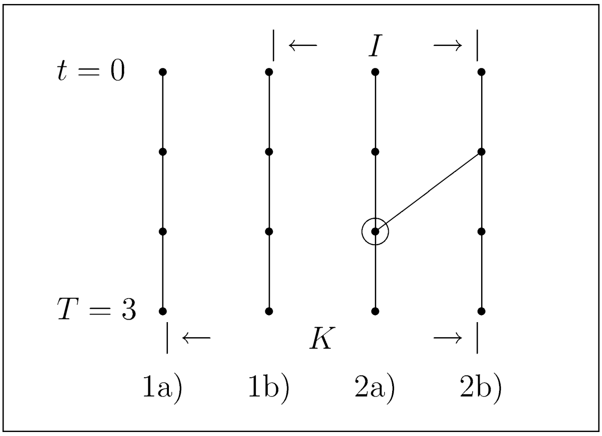

Fig. 2. A sample configuration which illustrates the four different cases (by removing the corresponding base point) in the proof of Proposition 4.5. Here, $|K| = 4$, $|I| = 3$ and there is one branching (the big circle) into a set of two points.

When p is eventually included in the expansion of α, then $\tau > 0$ and the dichotomy is as follows:

(2a) Either $p \in S_{\tau-1}$ and $(S_{\tau-1}, V_{\tau-1})$ can *not* be a branching over $\alpha_\tau = K_\tau - \{p\}$. Hence, C is *not* a configuration over $\alpha = K - \{p\}$. Indeed, Lemma 4.2 (case 1), shows that $\pi_{\alpha_\tau, K_\tau} L_{K_\tau, S_{\tau-1}, V_{\tau-1}}$ vanishes.

(2b) Or, $p \notin S_{\tau-1}$ and p must then belong to the set $V_{\tau-1}[S_{\tau-1}] - S_{\tau-1}$. Lemma 4.2 (case 2), shows that $\pi_{\alpha_\tau} L_{K_\tau, S_{\tau-1}, V_{\tau-1}} = L_{\alpha_\tau, S_{\tau-1}, V_{\tau-1}}$. When $t < \tau$ we have $\alpha_t = K_t$, C is a configuration over α and we end up with $L_\alpha[C]$ as claimed. □

[In terms of *chains* to be defined below, case (1a) and (2b) corresponds to the removal of end-chains. In case (1b) we are projecting away an initial-chain (yields zero) and in case (2a) the same is happening to an apex-chain.]

4.3. Chains and trees

Let $C \in \mathcal{C}[K, T]$ be a fixed time-T configuration over $K \in \mathcal{F}$ and let $K = K_T \subset \cdots \subset K_0$ be the expansion of K along C. By (S_t, V_t), $t = 0, \ldots, T-1$, we denote the branching pairs of the configuration.

DEFINITION 4.6. – The set of points $(q, t) \in \Omega \times \{0, \ldots, T\}$ for which $q \in K_t$ defines the *points of the configuration*. Such a point is called

- a *vertex* point, if $q \in V_t[S_t]$ (and $t < T$),
- an *apex* point, if $q \in S_{t-1} - V_t[S_t]$ (and $t > 0$),
- a *free* point, otherwise.

In Fig. 4 (left part) there are 3 apex points (there are 4 apexes but one apex belongs at the same time to a vertex set), 7 vertex points and 12 free points (22 points all together).

DEFINITION 4.7. – A *chain* in the configuration is a *maximal* sequence of points in the configuration, $\gamma = (q, t)_{t_1 \leqslant t \leqslant t_2}$, for which $0 \leqslant t_1 < t_2 \leqslant T$, $q \notin S_{t_2-1}$ and every intermediate point, (q, t), $t_1 < t < t_2$, is a free point. We say that the chain is positioned *over* q, *starts* at time t_1, *stops* at time t_2 and we define the *length* of the chain γ to be: $|\gamma| \equiv t_2 - t_1 \geqslant 1$.

DEFINITION 4.8. – The chain (of the previous definition) is called:

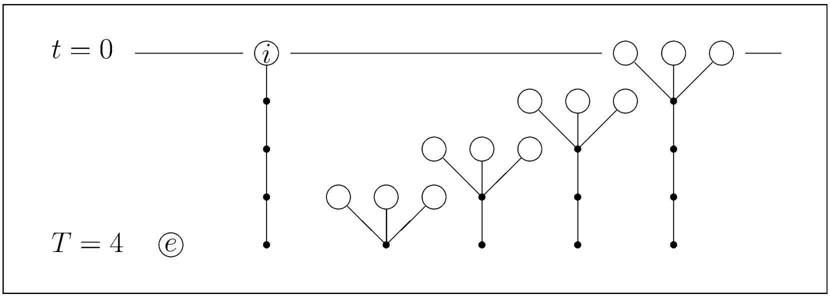

Fig. 3. Visualizing the six possible trees in $\mathcal{Y}[\cdot, 4]$ when $V \in \mathcal{F}$ is a fixed set (here containing 3 points). Encircled i and e indicates an initial-leaf and end-leaf, respectively. The empty circles symbolize (unspecified) trees at the given time-step. The last 3 trees contain apex-chains of length one, two and three, respectively.

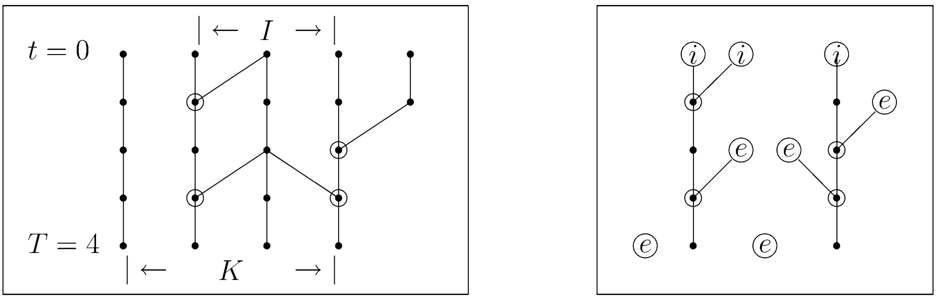

Fig. 4. (Left) A sample configuration over K (4 points) with initial set I (3 points). (Right) The corresponding tree structures. $i =$ initial leaf, $e =$ end leaf. (See also the text.)

(1) An *apex-chain*, if (q, t_1) is an apex point.
(2) An *initial-chain*, if $t_1 = 0$, $(q, 0)$ is a free point and q belongs to the initial set, I.
(3) An *end-chain*, otherwise (i.e. $q \in V_{t_1}[S_{t_1}]$ or $t_1 = 0$ and $q \in (K_0 - I) \cup V_0[S_0]$).

[Note that a chain can stop either because $t_2 = T$ and we say that the chain is rooted in K. Or, because $q \in V_{t_2}[S_{t_2}] - S_{t_2-1}$ meaning that the chain stops at the vertex of one or more branchings, but not at a point in a singular set.]

A 'typical' configuration looks pretty much like an intricate 'network'. We need to sum up operator bounds for these configurations (cf. below). In order to simplify the task we will map configurations into much simpler tree-structures. Let us first define what kind of trees to consider:

DEFINITION 4.9. – For each $T \geqslant 0$ and $p \in \Omega$ the collection of trees, $\mathcal{Y}[p, T]$, is defined as follows: Initially, $\mathcal{Y}[p, 0]$ has two elements, namely an *end-leaf* and an *initial-leaf*.

Recursively, we then define $\mathcal{Y}[p, t]$, $t > 0$, to be any of the trees created in the following way:
(1) an *end-leaf* (at (p, t)),
(2) an *initial-chain* of length t (from (p, t) to $(p, 0)$) followed by an *initial-leaf* (at $(p, 0)$) or
(3) an *apex-chain* of length $0 < k < t$ (from (p, t) to $(p, t - k)$) or nothing (in which case we set $k = 0$) followed by a branching over a set $V \in \mathcal{F}$ (at $(p, t - k - 1)$). To each point $q \in V \cup \{p\}$ we attach a tree $y_q^{t-k-1} \in \mathcal{Y}[q, t - k - 1]$.

The collection of leafs, chains and branchings is called the *components* of the tree. Every component of the tree is considered to be tagged by its position in time and in space as described in the parantheses above.

Let us proceed to describe how we will map a configuration $C \in \mathcal{C}[K,T]$ into tree-structures $(y_p^T)_{p\in K} \in \prod_{p\in K} \mathcal{Y}[p,T]$. Uniqueness in our construction below results after we fix arbitrarily a well-ordering of Ω (invoking the Axiom of Choice in the uncountable case).

During the construction at every time step $0 \leqslant t \leqslant T$ and to each point $q \in K_t$ (the time t expansion of K) we build a 'partial' tree $y_q^t \in \mathcal{Y}[q,t]$. These partial trees will be build recursively starting at $t = 0$ and ending at $t = T$. Initially, at time 0 we assign an *initial-leaf* to each point in the initial set I of the configuration and an *end-leaf* to each of the remaining points in $K_T - I$. Thus, each initial tree y_q^0, $q \in K_0$ is either an initial- or an end-leaf depending on whether $q \in I$ or not. Let $0 < t \leqslant T$. We will attach the partial trees (or leaves) in K_{t-1} to trees at time t and situated at the points of K_t. We do this recursively as follows (we take $V_T[S_T]$ to be the empty set):

(1) First, we go through the elements $q \in S_{t-1} \subset K_t$ (with respect to the given ordering of Ω). We let y_q^t be a branching into $|V_{t-1}(q)|$ trees. At each of the vertices $p \in V_{t-1}(q)$ we attach either the tree y_p^{t-1}, if it has not already been attached to another vertex, or, an *end-leaf*, in case it has.

(2) For each element $q \in K_t \cap V_t[S_t] - S_{t-1}$ we let y_q^t be an end-leaf.

(3) Finally for every $q \in K_t - V_t[S_t] - S_{t-1}$ we make the following distinction: If y_q^{t-1} is an end-leaf we let y_q^t be an end-leaf, too. If not, we let y_q^t be a link attached to the tree y_q^{t-1} (corresponding to an initial- or an apex-chain incrementing its size by one). In other words, we do not keep 'track' of the length of end-chains but we do for the other two types of chains.

[Note, that in (2) and (3) above, we are looking at vertices at time t but singular points at time $t-1$]. This completes the recursive description. It should be clear that each tree in K_{t-1} has been attached to precisely one tree in K_t.

The tree-mapping is designed so that each consecutive sequence of 'links' constitute a maximal chain, either apex or initial (end-chains have been replaced by end-leaves) of the configuration.

A sample configuration and the associated tree structures are given in Fig. 4. Expanding the configuration over K, we obtain $K \equiv K_4 = K_3 = K_2 \subset K_1 = K_0$ where $|K_4| = 4$ and $|K_0| = 5$. The tree structures contain four trees, four branchings (the large circles), one initial-chain of length one (two initial-leaves are situated at vertices and have no chains associated) and three apex-chains, each of length one (one apex is situated at the vertex of another branching and does not give rise to a chain).

LEMMA 4.10. – *The above tree-mapping $C \in \mathcal{C}[K,T] \to (y_p^T)_{p\in K} \in \prod_{p\in K} \mathcal{Y}[p,T]$ is injective.*

Proof. – Each component of a tree-structure is labelled by its position (both in space and in time) and we may therefore map the tree-structure back into the configuration from which it originated. The collection of initial-leaves reconstitutes the corresponding initial set of the configuration. The only tricky point comes from the end-leaves (since the corresponding chains have been 'pruned' away). After having mapped all other components back into the configuration one considers each end-leaf. If its position in time is $< T$ and no chain is (yet) above this end-leaf then one inserts the longest possible end-chain which stops at this end-leaf. A moments reflection will show that we have reconstructed the configuration from which the tree-structure originated. □

DEFINITION 4.11. – We define the *size* of a tree, and we write $\mathrm{size}(y_q^T)$, to be the sum of sizes of each of its components: Each branching has size one, each initial- and end-leaf has size zero and finally each initial- or apex-chain has a size equaling its length.

When a metric on Ω is given we define the *interaction radius*, and we write $\mathrm{rad}(y_q^T)$, of a tree to be the sum of interaction radii, $\mathrm{rad}(p,V)$, of each of its branchings [cf. Section 1.4 and Definition 4.9(3)].

LEMMA 4.12. – *If a tree* $y_p^T \in \mathcal{Y}[p,T]$ *contains an initial-leaf then* $\mathrm{size}(y_p^T) \geqslant T$.

Proof. – A tree which contains an initial-leaf must necessarily contain a consequtive sequence of apex-chains and branchings (with zero or more repetitions), possibly followed by an initial-chain and finally an initial-leaf, connecting time T and time zero. Adding the sizes of the components involved we get at least T. □

4.4. Bounds for configurations

Given a configurational operator $L_K[C]$ (4.55) we shall use Fubini to re-shuffle the order of integration. For $\Lambda \in \mathcal{F}$ and $\beta_{p,V} \in E_{p,V}$ with $V \cup \{p\} \subset \Lambda$ (the set, V, need not contain p) we associate a vertex operator $M_{\Lambda,\beta_{p,V}} : E_\Lambda \to E_\Lambda$ given by

$$M_{\Lambda,\beta_{p,V}} \phi_\Lambda(w_p, z_{\Lambda-\{p\}}) = \pm \oint_\Gamma dz_p \beta_{p,V}(w_p, z_{V\cup\{p\}}) \phi_\Lambda(z_\Lambda), \tag{4.60}$$

with $\pm$ being the sign of orientation of f_p. This ‘interaction’ operator acts as a multiplication operator in the $V-\{p\}$-variables followed by an integration in the z_p variable along Γ.

Consider now $\Lambda \in \mathcal{F}$ and a point $p \in \Lambda$. The single-site map f_p gives rise to a single-site Perron–Frobenius operator $L_{f_p} : E(A[\rho]) \to E(A[\rho])$ which extends naturally to an operator (denoted by the same symbol) $L_{f_p} : E_\Lambda \to E_\Lambda$. Applying the identity (3.21) for the single site operator we may represent the iterated operator $L_{f_p}^n$ as an integral operator (the sign again representing the orientation of f_p):

$$\begin{aligned} &L_{f_p}^n \phi_\Lambda(w_p, z_{\Lambda-\{p\}}) \\ &= (\pm 1)^n \oint_\Gamma dz_p^{(n-1)} \ldots \oint_\Gamma dz_p^{(1)} \oint_\Gamma dz_p k_{w_p} \circ f_p\big(z_p^{(n-1)}\big) \ldots k_{z_p^{(1)}} \circ f_p(z_p)\phi_\Lambda(z_\Lambda). \end{aligned} \tag{4.61}$$

To each maximal chain $\gamma = (p,t)_{t_1 \leqslant t \leqslant t_2}$ over p and any set $\Lambda \in \mathcal{F}$ containing p we associate the chain-operator (omitting the explicit reference to Λ)

$$L_\gamma = (L_{f_p})^{|\gamma|} : E_\Lambda \to E_\Lambda. \tag{4.62}$$

Let C be a fixed time-T configuration over $K \in \mathcal{F}$ and let $ch(t)$ denote all the chains starting at time t with $0 \leqslant t < T$. We introduce the following auxiliary operators $\tilde{L}^{(t)} : E_{K_t} \to E_{K_{t+1}}$, $0 \leqslant t < T$:

$$\tilde{L}^{(t)} = \left(\prod_{\gamma \in \mathrm{ch(t)}} L_\gamma \right) \pi_{K_{t+1},K_t} \prod_{p \in S_t} M_{K_t,\beta_{p,V_t(p)}}. \tag{4.63}$$

We then have the following

LEMMA 4.13. – *The action of the configurational transfer operator* $L_K[C] : \mathcal{M}_\vartheta \to E_K$ *may be written as*:

$$L_K[C] = \tilde{L}^{(T-1)} \circ \cdots \circ \tilde{L}^{(0)} \circ j_{K_0,I} \circ \prod_{p \in I} (1-\ell_p) \circ \pi_I. \tag{4.64}$$

Proof. – A consequence of Fubinis theorem. More precisely, the same kernel factors and integrals are present in the two expressions for $L_K[C]$ so the only question is the validity of re-shuffling the 'time-ordering' when carrying out the integrals:

Due to the definition of the chains, each unperturbed kernel factor, say $k_{z_q^{t+1}} \circ f_q(z_q^t)$, occurs exactly once in some chain γ. We can thus identify this factor as a 'link' in the chain γ. If (q,t) is a free point of the configuration and $t>0$ then this link must necessarily be preceeded by another link in γ and since being a free point the only kernel factors which depend on the variable z_q^t are those associated to these two links. Collecting the terms along the chain yields a contribution which is precisely given by the integral expression (4.61) or in other words, the chain operator L_γ acting at the starting time of the chain. The vertex operator, $M_{K_t,\beta_{p,V_t(p)}}$, concatenates the multiplication with the interaction term and the corresponding integral in the singular variable. The π_{K_{t+1},K_t} operators provide the remaining toral integrals $\int_{S_{\tilde{K}-K}} dz_{\tilde{K}-K}$ in (4.46). Finally, for the initial projections we have used the identity (4.53). □

Each interaction operator satisfies the bound,

$$\|M_{\Lambda,\beta_{p,V}}\| \leqslant 2|\beta_{p,V}|, \tag{4.65}$$

where the factor 2 comes from $\oint_\Gamma |dz| = 2$. For the chain operators we may apply the results of Theorem A.1. Two possibilities arise. For an end-chain (of strictly positive, respectively, zero length) we have the *a priori* bounds,

$$\|L_\gamma\| \leqslant \left\|L_{f_p}^{|\gamma|}\right\| \leqslant c_h, \quad \|L_{|\gamma|=0}\| \leqslant 1, \tag{4.66}$$

but for an apex-chain or an initial-chain we are better off. In the case of an initial-chain its starting point is an element of the initial set I and at the same time a free point. Recalling the definition of the $Q_I^{K_T}$ operator (4.50) one realizes that the chain operator acts upon the image of $(1-\ell_p)$ which itself is in the kernel of ℓ_p (since this is a projection). In the case of an apex-chain the identity (3.29) of Lemma 3.4 shows that also an apex-chain operator acts upon an element in the kernel of ℓ_p. In both cases we therefore obtain the exponential bound,

$$\|L_\gamma\| \leqslant \left\|L^n_{f_p|\ker \ell_p}\right\| \leqslant c_r \eta^n, \quad n = |\gamma| \geqslant 1. \tag{4.67}$$

Each intermediate projection operator, π_{K_{t-1},K_t}, has norm one (and can thus safely be ignored). Finally, for the initial projection operator the bound and $\|\pi_J\| \leqslant \vartheta^{-|J|}$, Eq. (4.45), and the identity (4.53) implies:

$$\left\| j_{K_T,I} \circ \prod_{p\in I} (1-\ell_p) \circ \pi_I \right\| \leqslant \sum_{J\subset I} \|\pi_J\| \leqslant \left(1+\frac{1}{\vartheta}\right)^{|I|}. \tag{4.68}$$

This completes our description of how to bound the 'components' of the configurational operator $L_K[C]$. The operator itself is bounded by the product of the bounds over all the components of C.

4.5. Bounds for tree-structures

We will redistribute the bounds for the components of a configuration onto the components of the corresponding tree-structure. Thus, consider a configuration $C \in \mathcal{C}[K,T]$ which maps into the tree-structure $(y_p)_{p\in K} \in \prod_{p\in K} \mathcal{Y}[p,T]$. Each tree y_p is build of 'components' of the following types: end-leaves, initial-leaves, initial-chains, apex-chains and branchings.

To each chain and branching we will carry over the bounds from the configuration, i.e. (4.67) and (4.65). To an end-leaf we assign the bound (4.66) for a strictly positive, respectively, the zeroth time-step

$$\|\text{end-leaf}\|_{t>0} = c_h, \quad \|\text{end-leaf}\|_{t=0} = 1, \tag{4.69}$$

and finally to an initial-leaf we assign

$$\|\text{initial-leaf}\|_\vartheta = 1 + \frac{1}{\vartheta}. \tag{4.70}$$

We let $\|y_p\|_\vartheta$ be the product of bounds assigned to each of the components of the tree.

LEMMA 4.14. – *Let $(y_p)_{p\in K}$ be the tree-structure associated to the configuration $C \in \mathcal{C}[K,T]$. Then the bounded linear operator $L_K[C] : \mathcal{M}_\vartheta \to E_K$ satisfies the bound*:

$$\|L_K[C]\| \leqslant \prod_{p\in K} \|y_p\|_\vartheta. \tag{4.71}$$

Proof. – The number of initial-leaves in the tree-structure equals $|I|$. Hence, these contribute a factor $(2/\vartheta)^{|I|}$ as is needed in (4.68). Branchings, initial- and end-chains contribute in the same way for the configurations and for the tree-structures. Again the only difficulty arises from the end-leaves/end-chains. Each end-leaf is assigned the bound c_h (or 1) which bounds a possible corresponding end-chain. But this bound is also at least one in numerical value. Since the number of end-leaves in the tree-structure is either larger than or equals the number of end-chains (see also Fig. 4) the contributions from end-leaves will be an upper bound for the end-chains of the configuration. □

4.6. Time-renormalization of bounds

Let $\overline{\mathbb{R}}_+ = [0,+\infty]$ denote the non-negative reals together with plus infinity. We adopt the standard algebraic conventions for $\overline{\mathbb{R}}_+$. When μ is a positive measure on $\mathbb{R}$ a knowledge of finiteness of its Laplace transform may provide information about the 'tail'-distribution of that measure:

LEMMA 4.15 (Large deviations). – *Let μ_p, $p \in K$ (a finite index set) be a family of positive, not necessarily finite, Borel measures on $\mathbb{R}$ and let $\mu_K = \bigotimes_{p\in K} \mu_p$ denote their product measure. Suppose that there is $\gamma > 1$ and $M < \infty$ such that $\int d\mu_p(x_p)\gamma^{x_p} \leqslant M$ for all $p \in K$. Then for every $r \in \mathbb{R}$ we have*

$$\mu_K\left\{\sum_{p\in K} x_p \geqslant r\right\} \leqslant M^{|K|}\gamma^{-r}. \tag{4.72}$$

Proof. – Note that the left hand side does not exceed $\int d\mu_K \gamma^{\sum x_p - r}$. □

DEFINITION 4.16. – For $0 < \vartheta \leqslant 1$, $T \geqslant 0$ and $p \in \Omega$ we define a generating function (cf. Definition 4.11 and Section 4.5) for tree values through the following formal power series (which need not be convergent when $s \neq 0$):

$$u_p^T(s) \equiv \sum_{y\in\mathcal{Y}[p,T]} \|y\|_\vartheta s^{\text{size}(y)} \in \overline{\mathbb{R}}_+[s]. \tag{4.73}$$

PROPOSITION 4.17. – *Let again* $0 < \vartheta \leqslant 1$. *For* $p \in \Omega$ *one has* $u_p^0(s) = 2 + 1/\vartheta$ *and for any* $T \geqslant 1$ *and* $p \in \Omega$ *the generating functions verify the following recursive identities* (*the first defines auxiliary formal power series*):

$$b_p^k(s) \equiv s \sum_{V \in \mathcal{F}} 2|\beta_{p,V}| \prod_{q \in V \cup \{p\}} u_q^k(s), \quad k = 0, \ldots, T-1, \tag{4.74}$$

$$u_p^T(s) = c_h + \left(1 + \frac{1}{\vartheta}\right) c_r (\eta s)^T + b_p^{T-1}(s) + \sum_{k=1}^{T-1} c_r (\eta s)^k \, b_p^{T-k-1}(s). \tag{4.75}$$

Proof. – Recall Definition 4.9 of the tree-structure at time T and position $p \in \Omega$. For $T = 0$ a tree-structure is either an end- or an initial-leaf, hence by (4.69) and (4.70),

$$u_p^0(s) = 1 + (1 + 1/\vartheta).$$

For $T > 0$ the recursive definition yields the first and the second term in (4.75) for the end-leaf and the initial-chain (4.67) with its initial-leaf, respectively. For a branching at p into a set V we have two components: an apex-chain, bounded by (4.67), of either zero length (the third term) or greater than zero length (the fourth term) in both cases followed by the branching itself at time $T - k - 1$, $0 \leqslant k < T$. The vertex operator is bounded by $2|\beta_{p,V}|$, inequality (4.65), and since any possible tree may be attached to each point in $V \cup \{p\}$ we obtain precisely a product of generating functions at the given earlier time $T - k - 1$. The factors of s comes from the additivity in Definition 4.11 for the 'size' of a tree. □

Recall that the constant $C_\beta = C_\beta(\rho, \lambda, \kappa)$ was defined in (3.26) and that $c_h \geqslant 1$, c_r and $\eta < 1$ are the constants for the single site operator given in Theorem A.1. For $1 < \gamma < \eta^{-1}$ we set

$$\alpha_\gamma = 1 + \frac{c_r \eta \gamma}{1 - \eta\gamma}, \quad \alpha^1 = c_r \eta \quad \text{and} \quad \theta_c^{-1} = \max\{c_h + 2, c_h + (1 + c_h)\alpha^1\} \geqslant 3. \tag{4.76}$$

DEFINITION 4.18 (*Time-renormalizability of bounds*). – The parameters of the coupled map are said to satisfy condition TR (for time-renormalizability) with respect to the decay-constant $\gamma \in]1, \eta^{-1}[$ provided the following inequalities hold:

$$1 > \theta(c_h + 2) + 2\gamma C_\beta \alpha_\gamma \quad \text{and} \tag{4.77}$$

$$1 > \theta\big(c_h + \gamma\alpha^1(1 + c_h)\big) + 2\gamma C_\beta\big(1 + \gamma\alpha^1\alpha_\gamma\big). \tag{4.78}$$

Remarks 4.19. – When $\theta \geqslant \theta_c$ condition TR cannot hold for any values of $\gamma > 1$, $C_\beta \geqslant 0$ whereas for $\theta < \theta_c$ one can always find $\gamma > 1$ and $C_\beta > 0$ such that condition TR is fulfilled. In the latter case the corresponding value of the effective coupling constant, $\kappa = \kappa(\rho, \lambda, \theta) > 0$, is strictly larger than zero (cf. Remark 3.5). Thus, $\theta_c (\leqslant 1/3)$ plays the role of a critical parameter below which we may allow non-vanishing couplings (at least for our proofs to work).

We define the constants

$$\vartheta_c^{-1} = \frac{1}{\theta} - 2 \quad \text{and} \quad \vartheta_\infty^{-1} = c_h + \frac{2\gamma C_\beta}{\theta}\alpha_\gamma \tag{4.79}$$

and (4.77) then corresponds to the inequality:

$$0 < \vartheta_c < \vartheta_\infty \leqslant 1. \tag{4.80}$$

LEMMA 4.20 (Time-renormalization of bounds). – *Suppose that condition TR is satisfied for a decay-constant* $\gamma \in]1, \eta^{-1}[$. *Then there is* $\vartheta \in (\vartheta_c, \vartheta_\infty)$ *such that for all* $p \in \Omega$: $u_p^0(\gamma) \leqslant \theta^{-1}$ *and for* $T \geqslant 1$,

$$u_p^T(\gamma) \leqslant c_h + \left(1 + \frac{1}{\vartheta}\right) c_r(\eta\gamma)^T + \frac{2C_\beta \gamma}{\theta}\left(1 + \sum_{k=1}^{T-1} c_r(\eta\gamma)^k\right) \leqslant \theta^{-1}. \tag{4.81}$$

Proof. – For any $\vartheta \in (\vartheta_c, 1]$ the definition of ϑ_c implies $u_p^0(\gamma) < 2 + \vartheta_c^{-1} = \theta^{-1}$. We consider first the limiting value, $\vartheta = \vartheta_\infty$, for which we will prove a strict inequality by induction. Suppose that for $T > 0$, $u_p^k(\gamma) < \theta^{-1}$ for all $k = 0, \ldots, T-1$. For each of the latter k-values we then have the bound:

$$b_p^k(\gamma) \leqslant 2\gamma \sum_{V \in \mathcal{F}} |\beta_{p,V}| \theta^{-|V|-1} \leqslant 2\gamma C_\beta/\theta, \tag{4.82}$$

where we have used the inequality (3.28). Inserting this bound into the equation for u_p^T we find

$$u_p^T(\gamma) \leqslant c_h + \left(1 + \frac{1}{\vartheta}\right) c_r(\eta\gamma)^T + \frac{2\gamma C_\beta}{\theta}\left(1 + \sum_{k=1}^{T-1} c_r(\eta\gamma)^k\right). \tag{4.83}$$

Depending on parameters, this function is either increasing or decreasing with T. Hence we need only to consider the two limiting cases $T = 1$ and $T = +\infty$. Inserting $\vartheta = \vartheta_\infty$ and $T = 1$, the inequality above reads

$$u_p^1(\gamma) \leqslant c_h + \gamma\alpha^1(1 + c_h) + 2\gamma C_\beta\big(1 + \gamma\alpha^1\alpha_\gamma\big)/\theta$$

which by inequality (4.78) is strictly smaller than θ^{-1}. For the other limit we have $u_p^\infty(\gamma) = \vartheta_\infty^{-1} < \vartheta_c^{-1} < \theta^{-1}$. By continuity of these bounds we may find $\vartheta \in (\vartheta_c, \vartheta_\infty)$ such that $u_p^T(\gamma) \leqslant \theta^{-1}$ for all $T \geqslant 0$. The first part of the desired inequality is now given by (4.83). □

LEMMA 4.21. – *There is* $T_0 = T_0(\vartheta) < \infty$ *such that* $u_p^T(\gamma) \leqslant \vartheta^{-1}$ *for all* $T \geqslant T_0$ *and* $p \in \Omega$.

Proof. – Follows from $u_p^T(\gamma) \leqslant \vartheta_\infty^{-1} + \frac{2}{\vartheta} c_r(\eta\gamma)^T \to \vartheta_\infty^{-1}$, as $T \to \infty$ and the fact that $\vartheta_\infty^{-1} < \vartheta^{-1}$. □

4.7. The global operator

In the following, we will assume that condition TR is fulfilled and that a parameter ϑ and an initial time constant $T_0 = T_0(\vartheta)$ have been chosen according to Lemma 4.20 and Lemma 4.21.

LEMMA 4.22. – *For every* $K \in \mathcal{F}$ *and* $T \geqslant 1$, *one has*

$$\sum_{C \in \mathcal{C}[K,T]} \big\| L_K[C] \big\| \leqslant \theta^{-|K|}. \tag{4.84}$$

For $T \geqslant T_0$ *we have in addition*:

$$\sum_{C \in \mathcal{C}[K,T]} \big\| L_K[C] \big\| \leqslant \vartheta^{-|K|}. \tag{4.85}$$

Proof. – The tree-mapping $C \in \mathcal{C}[K,T] \to (y_p)_{p\in K} \in \prod_{p\in K} \mathcal{Y}[p,T]$ is injective, Lemma 4.10, and using the multiplicative bound $\|L_K[C]\| \leqslant \prod_{p\in K} \|y_p\|$ we obtain

$$\sum_{C\in\mathcal{C}[K,T]} \|L_K[C]\| \leqslant \prod_{p\in K} \sum_{y\in\mathcal{Y}[p,T]} \|y\| = \prod_{p\in K} u_p^T(s\equiv 1) \in \overline{\mathbb{R}}_+. \tag{4.86}$$

The two bounds follow from the previous two lemmas and monotonicity of $u_p^T(s)$ for $s \in [0,\gamma]$, $\gamma > 1$. □

Norm-convergence of the above sums permits us to define a family of bounded linear operators,

$$L_K^{(T)} = \sum_{C\in\mathcal{C}[K,T]} L_K[C] : \mathcal{M}_\vartheta \to E_K, \quad K \in \mathcal{F}. \tag{4.87}$$

Let $K \subset \Lambda \in \mathcal{F}$. By continuity of $\pi_{K,\Lambda} : E_\Lambda \to E_K$ and Proposition 4.5 on projectivity of the configurational operators:

$$\pi_{K,\Lambda} \sum_{C\in\mathcal{C}[\Lambda,T]} L_\Lambda[C] = \sum_{C\in\mathcal{C}[\Lambda,T]} \pi_{K,\Lambda} L_\Lambda[C] = \sum_{C\in\mathcal{C}[K,T]} L_K[C]. \tag{4.88}$$

When $\phi \in \mathcal{M}_\vartheta$ the family $(L_K^{(T)}\phi)_{K\in\mathcal{F}}$ is thus projective and verifies the bound, $\theta^{-|K|}|L_K^{(T)}\phi| \leqslant 1$, $K \in \mathcal{F}$, $T \geqslant 0$ (and with ϑ instead of θ when $T \geqslant T_0$). We have proved the following:

PROPOSITION 4.23. – *There are norm-contracting operators $L^{(T)} : \mathcal{M}_\vartheta \to \mathcal{M}_\theta$, $T \geqslant 0$ and $L^{(T)} : \mathcal{M}_\vartheta \to \mathcal{M}_\vartheta$, $T \geqslant T_0$ such that $\pi_K \circ L^{(T)} = L_K^{(T)}$ for $K \in \mathcal{F}$.*

These operators compose 'nicely' in the way that one should expect:

LEMMA 4.24. – *For $T \geqslant T_0$ and $t > 0$ the operator $L^{(t+T)} : \mathcal{M}_\vartheta \to \mathcal{M}_\vartheta \hookrightarrow \mathcal{M}_\theta$ (thus post-composed with the natural injection) equals $L^{(t)} \circ L^{(T)} : \mathcal{M}_\vartheta \to \mathcal{M}_\theta$. When in addition $t \geqslant T_0$, the equality $L^{(t+T)} = L^{(t)} \circ L^{(T)}$ holds as endomorphisms on $\mathcal{M}_\vartheta$.*

Proof. – Fix $K \in \mathcal{F}$. Take $C^t \in \mathcal{C}[K,t]$ and let $K = K_t \subset K_{t-1} \subset \cdots \subset K_0 \equiv \Lambda$ be the expansion of K along C^t. Now, let $C^T \in \mathcal{C}[\Lambda,T]$. There is a natural injective map taking a couple (C^t, C^T) into a configuration $C^{t+T} \in \mathcal{C}[K,t+T]$: We simply remove the initial set of C^t and concatenate configurations in the natural way. The resulting configuration is then the union of all branching pairs of the two configurations together with the initial set of C^T. Let us identify (by an equivalence relation) two configurations in $\mathcal{C}[K,t]$ if they differ only by their initial sets. Quotienting by this equivalence relation the above mapping becomes *bijective*. If C^t is in the equivalence class of $\tilde{C}^t$ and $I \subset K_t$ is its initial set we split the configurational operator (4.55) as follows: $L_K[C^t] = L_K[\tilde{C}^t] \circ Q_I^\Lambda \circ \pi_\Lambda$. One then has the identity

$$L_K\left[\tilde{C}^t\right] \circ L_{K_t}\left[C^T\right] = L_K\left[C^{t+T}\right]. \tag{4.89}$$

We wish to sum over all possible configurations. For the right hand side we obtain clearly $\pi_K \circ L^{(T+t)}$ with any of the two target spaces, $\mathcal{M}_\theta$ or $\mathcal{M}_\vartheta$. For the left hand side we note that when acting upon $\mathcal{M}_\vartheta$ we may sum over $C^T \in \mathcal{C}[\Lambda,T]$ (by Lemma 4.22 the sum is $\mathcal{M}_\vartheta$-convergent) to get $\sum_{C^T} L_\Lambda[C^T] = L_\Lambda^{(T)} = \pi_\Lambda \circ L^{(T)}$ (by Proposition 4.23) where

$L^{(T)}: \mathcal{M}_\vartheta \to \mathcal{M}_\vartheta$ is bounded. Using again Lemma 4.22 and the identity (4.51) we see that

$$\left(\sum_{\tilde{C}^t} L_K\left[\tilde{C}^t\right] \circ \pi_\Lambda\right) \circ L^{(T)} = \left(\sum_{C^t} L_K\left[C^t\right]\right) \circ L^{(T)} = \pi_K \circ L^{(t)} \circ L^{(T)} \tag{4.90}$$

is norm-convergent as well when we take $\mathcal{M}_\theta$ and $\mathcal{M}_\vartheta$ as target space for $t \geqslant 0$ and $t \geqslant T_0$, respectively. (We note that e.g. $L^{(1)} \circ L^{(1)}$ need not be defined.) □

A subspace of the projective ϑ-families of particular interest is the co-dimension one kernel space:

$$Z_\vartheta = \{\phi \in \mathcal{M}_\vartheta:\ \pi_\emptyset \phi = 0\}. \tag{4.91}$$

LEMMA 4.25. – *One has* $\pi_\emptyset L^{(T)} = \pi_\emptyset$ (*acting upon* $\mathcal{M}_\theta$ *and* $\mathcal{M}_\vartheta$ *for* $T \geqslant 0$ *and* $T \geqslant T_0$, *respectively*). *In particular,* $L^{(T)}$ *preserves the kernel of* $\pi_\emptyset$. *Both operators,* $L^{(T)}: Z_\vartheta \to Z_\theta$ $(T \geqslant 0)$ *and* $L^{(T)}: Z_\vartheta \to Z_\vartheta$ $(T \geqslant T_0)$, *are bounded in norm by* γ^{-T}.

Proof. – The first statement follows from Remark 4.4 by summing over configurations and using continuity to interchange the sum and the $\pi_\emptyset$ operator. If $C \in \mathcal{C}[K,T]$ is a configuration with an empty initial set then the corresponding operator $L_K[C]$ has $\phi \in Z$ in its kernel. In the corresponding tree-structure $(y_p^T)_{p\in K}$ at least one tree must have an initial leaf if one is to have a non-zero contribution. By Lemma 4.12 the size of such a tree is at least T. In the product of bounds $\prod_{p\in K} u_p^T(s=1)$ we should therefore only sum over coefficients to s^n for which n is at least T. For $T \geqslant 0$ (the case $T \geqslant T_0$ being similar) we write $u_p^T(s) = \int d\mu_p(x_p) s^{x_p}$ with μ_p being a positive measure with support on the non-negative integers. Using Lemma 4.20 we know that $u_p^T(s=\gamma) \leqslant \theta^{-1}$ and by Lemma 4.15 on large deviations we get for the product measure $\mu_K = \bigotimes_{p\in K} \mu_p$,

$$\mu_K\left(\sum_{p\in K} x_p \geqslant T\right) \leqslant \theta^{-|K|}\gamma^{-T}. \tag{4.92}$$

The above arguments now implies that

$$\left\|L_K^{(T)}\phi\right\|_\theta \leqslant \theta^{-|K|}\gamma^{-T}\|\phi\|_\vartheta, \quad \phi \in Z_\vartheta \tag{4.93}$$

and the statement follows. □

4.8. Finite box confinement

We can now make the necessary link between the global transfer operators defined above and the Λ-confined local Perron–Frobenius operator $L_\Lambda: E_\Lambda \to E_\Lambda$ considered in Section 3.2.

LEMMA 4.26. – *As linear operators from* $\mathcal{M}_\vartheta$ *to* E_K *we have the following norm-limit*:

$$\pi_K \circ L^{(T)} = \lim_{\Lambda\to\Omega} \pi_{K,\Lambda} \circ L_\Lambda^T \circ \pi_\Lambda. \tag{4.94}$$

Proof. – In order to prove this statement we need to go through the construction of $\pi_K \circ L^{(T)}$ as a convergent sum over configurations and see what happens when we consider the restriction to a finite subset $\Lambda \subset \mathcal{F}$.

Given finite subsets $K \subset \Lambda \in \mathcal{F}$ we consider the Tth iterate of the Perron–Frobenius operator, $\pi_{K,\Lambda} L_\Lambda^T$, where boundary conditions have been fixed outside Λ as in Section 3.2.

We define for $K \subset \Lambda$ and a branching pair (S,V) over K the Λ-confined (S,V)-expansion of K to be $H=(K\cup V[S])\cap\Lambda$. We then associate the following operator mapping $\phi\in E_H$ into E_K:

$$L^{(\Lambda)}_{K,S,V}\phi(w)=\pm\int_{S_{H-K}} dm_{H-K}(z_{H-K})$$

$$\times\int_{\Gamma_K}\mu_K(z_K)G_{K,S,V}(w,z_H,\xi_{(K\cup V[S])\cap\Lambda^c})\phi(z_H),\quad w\in A_K, \tag{4.95}$$

where as usual, $\pm$ denotes the sign of orientation of F_K (cf. Section 3.2). The kernel is the one defined in (4.46). The difference with the operator (4.47) is that we now fix coordinates outside Λ in the kernel. Note that when $K=\Lambda$ in (4.95) also $H=\Lambda$ and there is no initial toral integral. The operator $L^{(\Lambda)}_{\Lambda,S,V}$ then maps E_Λ into itself. Now, let us introduce the expansion (3.27) into the kernel (3.24) for the local operator L_Λ and interchange sums and products. We see that

$$L_\Lambda=\sum_{(S,V):\,S\subset\Lambda}L^{(\Lambda)}_{\Lambda,S,V}, \tag{4.96}$$

where the sum is over all possible branching pairs (S,V) over Λ.

Returning to the general situation, $K\subset\Lambda$, Lemma 4.2 holds also for the Λ-confined operator. The Λ-confined configurations, denoted $\mathcal{C}^{(\Lambda)}[K,T]$ are defined as in Definition 4.3, the only modification being that the sequence $K=K_T\subset\cdots\subset K_0$ is created through Λ-confined expansions. In particular, we have $K_0\subset\Lambda$. The Λ-confined configurational operator $L^{(\Lambda)}_K[C]:\mathcal{M}_\vartheta\to E_K$ is defined as in (4.55), but using the $L^{(\Lambda)}_{K_{t+1},S_t,V_t}$ operators in each time-step. Using (4.96) we obtain the following formula for the iterates of the Λ-confined local operator:

$$L^T_\Lambda\circ\pi_\Lambda=\sum_{C\in\mathcal{C}^{(\Lambda)}[\Lambda,T]}L^{(\Lambda)}_\Lambda[C]. \tag{4.97}$$

Proposition 4.5 also holds in the Λ-confined case (with the same proof) and

$$\pi_{K,\Lambda}\circ L^T_\Lambda\circ\pi_\Lambda=\sum_{C\in\mathcal{C}^{(K)}[K,T]}L^{(\Lambda)}_K[C] \tag{4.98}$$

is a consequence of that proposition and continuity of $\pi_{K,\Lambda}$.

[Note that $L^T_\Lambda=(L^1_\Lambda)^T$ holds for the local operators but, in general, not for the global operators.]

We now make the following observations:

First, there is a natural injective mapping from $\mathcal{C}^{(\Lambda)}[K,T]$ into $\mathcal{C}[K,T]$. One takes the same initial set (which is therefore a subset of Λ), the same branching pairs but simply creates the expansion of K without taking intersections with Λ.

Second, under this natural inclusion, our bound for $\|L_K[C]\|$ is also an upper bound for $\|L^{(\Lambda)}_K[C]\|$. To see this we observe that in our bound for the configurational operator $L_K[C]$ there might possibly be some additional elements, namely end-chains attached to vertices outside Λ. In our bound for $L_K[C]$ these end-chains enter as factors not smaller than one, which in the Λ-confined version are replaced by one.

Third, let $C\in\mathcal{C}[K,T]$ be any non-confined configuration and let K_0 denote the last expanded set without the Λ-confinement. If $K_0\subset\Lambda$ then C is also a Λ-confined configuration over K and moreover (!), $L_K^{(\Lambda)}[C] = L_K[C]$. Hence, in this case it makes no difference whether or not we apply the Λ-confinement.

Combining the above three observations with the known convergence of the full operator we may complete the proof. For a fixed configuration, $C \in \mathcal{C}^{(\Lambda)}[K,T]$, one has the 'point-wise' convergence:

$$L_K[C] = \lim_{\Lambda \to \Omega} L_K^{(\Lambda)}[C] \tag{4.99}$$

simply because the non-confined expansion of K will eventually be included in Λ. The injectivity of the inclusion of C into $\mathcal{C}[K,T]$ and the bounds for the corresponding operators also shows that the sum of $\|L_K^{(\Lambda)}[C]\|$ must be bounded by $\theta^{-|K|}$. Taking the limit $\Lambda \to \Omega$ every given configuration over K is eventually included as a Λ-confined configuration and therefore, by dominated convergence and (4.98)

$$\begin{aligned} \pi_K \circ L^{(T)} &= \sum_{C \in \mathcal{C}[K,T]} \lim_{\Lambda \to \Omega} L_K^{(\Lambda)}[C] \\ &= \lim_{\Lambda \to \Omega} \sum_{C \in \mathcal{C}^{(\Lambda)}[K,T]} L_K^{(\Lambda)}[C] = \lim_{\Lambda \to \Omega} \pi_{K,\Lambda} \circ L_\Lambda^T \circ \pi_\Lambda. \end{aligned} \tag{4.100}$$ □

4.9. Measures

For $\phi \in \mathcal{M}$ we introduce the variation-norm,

$$\|\phi\|_m = \sup_{\Lambda \in \mathcal{F}} m_\Lambda\big(|\phi_\Lambda(\cdot)|\big) \in [0, +\infty], \tag{4.101}$$

and we define the linear space

$$\mathcal{M}^m = \{\phi \in \mathcal{M}\colon \|\phi\|_m < \infty\}. \tag{4.102}$$

[Some care must be taken as this space is not complete.] We set $\mathcal{M}_\vartheta^m = \mathcal{M}^m \cap \mathcal{M}_\vartheta$.

Let $\phi \in \mathcal{M}^m$ and consider $b \in C(S_\Omega)$ and a net $b_K \in C(S_K)$, $K \in \mathcal{F}$ such that $j_K b_K \to b$, as $K \to \Omega$. Then (omitting here as in the rest of this section the natural inclusions):

$$\big|m_K(b_K\phi_K) - m_\Lambda(b_\Lambda\phi_\Lambda)\big| = \big|m_{K\cup\Lambda}\big((b_K - b_\Lambda)\phi_{K\cup\Lambda}\big)\big| \leqslant |b_K - b_\Lambda| \|\phi\|_m. \tag{4.103}$$

Therefore, $m_K(b_K\phi_K)$ converges in $\mathbb{C}$. The limit, $\nu_\phi(b) = \lim_{K\to\Omega} m_K(b_K\phi_K)$, does not depend on the choice of net and defines a bounded linear functional on $C(S_\Omega)$ whose norm is $\|\phi\|_m$. By Riesz, the functional $\nu_\phi \in C(S_\Omega)^*$ defines a complex measure, $d\nu_\phi$, on S_Ω whose total variation is precisely $\|\phi\|_m$.

One important class of examples comes from the set of 'positive' densities:

$$\mathcal{M}^+ = \{\phi \in \mathcal{M}\colon \phi_{\Lambda|S_\Lambda} \geqslant 0,\ \Lambda \in \mathcal{F}\}. \tag{4.104}$$

Projectivity implies $m_\Lambda(|\phi_\Lambda(\cdot)|) = m_\Lambda(\phi_\Lambda(\cdot)) = \pi_\emptyset\phi$, hence $\|\phi\|_m \leqslant \pi_\emptyset\phi < \infty$ and $\mathcal{M}^+ \subset \mathcal{M}^m$. In particular, when $\pi_\emptyset\phi = 1$, $\phi \in \mathcal{M}^+$ gives rise to a probability measure ν_ϕ on S_Ω.

LEMMA 4.27. – *Let $a \in H_\vartheta$ and $\phi \in \mathcal{M}_\vartheta^m$.*

(1) *Then $a * \phi \in \mathcal{M}_\vartheta^m$, $\|a * \phi\|_m \leqslant |a|\ \|\phi\|_m$, $\|a \star \phi\|_\vartheta \leqslant |a|_\vartheta\ \|\phi\|_\vartheta$ and $\nu_\phi(a) = \pi_\emptyset(a \star \phi)$. When $b \in C(S_\Omega)$ then $\nu_{a*\phi}(b) = \nu_\phi(ab)$.*

(2) *When $b \in C_\vartheta$ then $|\nu_\phi(b)| \leqslant |b|_{C_\vartheta} \|\phi\|_\vartheta$.*

Proof. – For $a_\Lambda \in E_\Lambda$ one has $\pi_K(a_\Lambda \star \phi) = \pi_{K,\Lambda\cup K}(a_\Lambda \phi_{\Lambda\cup K})$. When $b_K \in C(S_K)$ then

$$m_K(b_K(a_\Lambda \star \phi)_K) = m_{K\cup\Lambda}(b_K a_K \phi_{K\cup\Lambda}) = \nu_\phi(b_K a_\Lambda). \tag{4.105}$$

This is bounded by $|b_K||a_\Lambda|\|\phi\|_m$. Repeating the previous arguments equality (4.105) extends by continuity to $b \in C(S_\Omega)$. When $a = \sum_\Lambda a_\Lambda$ has finite ϑ-norm then so does $a \star \phi$. As the sum is uniformly convergent the equality (4.105) also extends to $b \in C(S_\Omega)$ and $a \in H_\vartheta$. Setting $b \equiv 1$ yields $\nu_\phi(a) = \pi_\emptyset(a \star \phi)$. For the second bound we simply note that when $b_K \in C(S_K)$ we have $\nu_\phi(b_K) \leqslant |b_K||\phi_K| \leqslant \vartheta^{-|K|}|b_K|\|\phi\|_\vartheta$ and we may then sum over sets $K \in \mathcal{F}$. □

For $K \subset \Lambda$ and $b \in C(S_K)$, the defining relation (3.10) for the local Perron–Frobenius operator and projectivity imply

$$m_\Lambda\big((j_{\Lambda,K} b_K) \circ F_\Lambda^T \cdot \phi_\Lambda\big) = m_\Lambda\big((j_{\Lambda,K} b_K) L_\Lambda^T \phi_\Lambda\big) = m_K\big(b_K \pi_{K,\Lambda}\big(L_\Lambda^T \phi_\Lambda\big)\big). \tag{4.106}$$

We wish to take the limit of this equality as $\Lambda \to \Omega$. On the left hand side we may do so as long as $\phi \in \mathcal{M}^m$ while on the right hand side we need $\phi \in \mathcal{M}_\vartheta$ (in order to apply Lemma 4.26) to get something sensible in the limit. Combining the two we get:

LEMMA 4.28. – *For $T \geqslant 0$ the mapping $L^{(T)} : \mathcal{M}_\vartheta^m \to \mathcal{M}_\theta^m$ (replacing θ by ϑ when $T \geqslant T_0$) is contracting both in the projective $(\vartheta$ to $\theta)$ norm and the m-norm. When $\phi \in \mathcal{M}_\vartheta^m$ and $b \in C(S_\Omega)$ then*

$$\nu_\phi\big(b \circ F^T\big) = \nu_{L^{(T)}\phi}(b), \quad T \geqslant 0. \tag{4.107}$$

Proof. – Let $b_K \in C(S_K)$, $K \in \mathcal{F}$ be a net such that $j_K b_K$ converges to $b \in C(S_\Omega)$. First, let $K \in \mathcal{F}$ be fixed. The net $j_\Lambda \circ ((j_{\Lambda,K} b_K) \circ F_\Lambda^T)$ converges to $(j_K b_K) \circ F^T$ as $\Lambda \to \Omega$. Since $\phi \in \mathcal{M}^m$, we may take the limit on the left hand side (LHS) in (4.106) to obtain $\nu_\phi((j_K b_K) \circ F^T)$ which tends to $\nu_\phi(b \circ F^T)$ as $K \to \Omega$. Again for fixed $K \in \mathcal{F}$ we may by Lemma 4.26 and the assumption $\phi \in \mathcal{M}_\vartheta$ take the limit $\Lambda \to \Omega$ on the right hand side (RHS) of the same equation to get

$$\nu_\phi\big((j_K b_K) \circ F^T\big) = m_K\big(b_K \pi_K\big(L^{(T)}\phi\big)\big). \tag{4.108}$$

From this last identity we may draw the wanted conclusions. First, the LHS is bounded by $|b_K|\|\phi\|_m$, whence so is the RHS. We must therefore have $L^{(T)}\phi \in \mathcal{M}^m$ with

$$\big\|L^{(T)}\phi\big\|_m \leqslant \|\phi\|_m.$$

Hence, we may write the RHS as $\nu_{L^{(T)}\phi}(j_K b_K)$ and take the limit as $K \to \Omega$ to obtain (4.107). □

Remarks 4.29. – The lemma also shows that the operators $L^{(T)}$ are positive and since $\pi_\emptyset L^{(T)} = \pi_\emptyset$ they even preserve the affine subspace of probability measures in $\mathcal{M}_\vartheta^+$ (when $T \geqslant T_0$).

5. Proof of Theorem 2.1

Recall that we only consider the time-independent case here (leaving the 'easy' extension to the time-dependent case to the reader). Let $h^0 \equiv (1_\Lambda)_{\Lambda\in\mathcal{F}} \in M_\vartheta^+$ denote the projective family in which each Λ-component is identically 1.

For $T \geqslant T_0$ the sequence, $h^T \equiv L^{(T)} h^0$, is in M_ϑ^+ and by Lemma 4.25 we have $\pi_\emptyset h^T = 1$ for all T. The same lemma combined with Lemma 4.24 now implies:

$$\left\| h^{t+T} - h^{\tau+T} \right\|_\vartheta = \left\| L^{(T)}(h^t - h^\tau) \right\|_\vartheta \leqslant 2\gamma^{-T}, \quad T, t, \tau \geqslant T_0, \tag{5.109}$$

which shows that the sequence is Cauchy in M_ϑ^+ (hence also in M_θ^+). The limit, $h \in \mathcal{M}_\vartheta$, is clearly positive and $\pi_\emptyset h = 1$. Hence, the associated measure ν_h is a probability measure on S_Ω. By Lemma 4.28 and Lemma 4.24, $\nu_h(b \circ F^T) = \nu_{L^{(T_0)}h}(b \circ F^T) = \nu_{L^{(T+T_0)}h}(b) = \nu_h(b)$ for all $T \geqslant 0$, $b \in C(S_\Omega)$. Now, let $a \in H_\vartheta$, $b \in C_\theta$ and set $c = \nu_h(a) = \pi_\emptyset(a \star h)$. We note that $\nu_h(b \circ F^T \cdot a) = \nu_{a \star h}(b \circ F^T) = \nu_{L^{(T)}(a \star h)}(b)$. Therefore, $\nu_h(b \circ F^T \cdot a) - \nu_h(b)\nu_h(a) = \nu_\psi(b)$ where $\psi = L^{(T)}(a \star h) - ch = L^{(T)}((a - c1) \star h)$. Since $\pi_\emptyset((a - c1) \star h) = \pi_\emptyset(a \star h) - c = 0$ our kernel lemma implies that $\|\psi\|_\theta \leqslant 2\gamma^{-T}|a|_\vartheta$. Finally, by Lemma 4.27 we obtain

$$\left|\nu_\psi(b)\right| \leqslant 2\gamma^{-T}|b|_{C_\theta}|a|_\vartheta$$

which gives Eq. (2.5) of the theorem.

In order to see that ν_h is a natural measure it suffices to show that for a continuous function $b \in C(S_\Omega)$ the 'time-shifted' Birkhoff average $1/n \sum_{k=1}^n b \circ F^{T_0+k}(x)$ converges Lebesgue almost surely to $\nu_h(b)$ as $n \to \infty$. On the probability space (S_Ω, m_Ω) (with its standard Borel σ-algebra) we denote by $E(X) = \int_{S_\Omega} X \, dm_\Omega$ the expectation of an L^1 random variable X and by $\operatorname{Cov}(X, Y) = E((X - E(X))(Y - E(Y)))$ the covariance of square integrable random variables X and Y. Let us recall

THEOREM 5.1 (The strong law of large numbers). – *Let X_n, $n = 1, 2, \ldots,$ be a sequence of random variables on the probability space (S_Ω, m_Ω). We denote by $S_n = X_1 + \cdots + X_n$ the partial sums and we assume the following two properties*:

(1) $\lim_{n\to\infty} \frac{1}{n} E(S_n) = c$.

(2) $\operatorname{Cov}(X_k, S_n) \leqslant C$ (*uniformly in k and n*).

Then S_n/n converges m_Ω-a.s. to c.

Proof. – Subtracting $E(X_n)$ from each X_n we may assume that each expectation $E(X_n)$ vanishes and that $c = 0$. The estimate

$$\int \left(\frac{1}{n} S_n\right)^2 dm = \frac{1}{n^2} \sum_{k=1}^n \operatorname{Cov}(X_k, S_n) \leqslant C/n$$

implies that the subsequence $(\frac{1}{k^2} S_{k^2})_{k \geqslant 1}$ is $L^2(m_\Omega)$-summable, hence tends to zero m_Ω-a.s. (but at the moment not the a.s. convergence of S_n/n since $\sum 1/n = \infty$). A standard trick is now to introduce $\phi_n = \min\{k^2 \colon n \leqslant k^2\}$ and write:

$$\frac{1}{n} S_n = \frac{1}{\phi_n} S_{\phi_n} \cdot \frac{\phi_n}{n} + \frac{S_n - S_{\phi_n}}{n}.$$

The bound $n \leqslant \phi_n \leqslant n + 2\sqrt{n}$ ensures the m_Ω a.s.-convergence of the first term and for the second term we note that

$$\int \left(\frac{S_n - S_{\phi_n}}{n}\right)^2 dm \leqslant \frac{\phi_n - n}{n^2} 2C \leqslant 4C/n^{3/2}.$$

This is now summable, whence implies the m_Ω-a.s. convergence to zero of the second term as well. □

Returning to our dynamical system consider a smooth observable $a \in H_\vartheta \subset C(S_\theta)$ and the associated sequence of random variables $X_m = a \circ F^{m+T_0}$, $m = 1, 2, \ldots$. We set $c_m = E(X_m) = \nu_{h^0}(a \circ F^{m+T_0}) = \nu_{h^{m+T_0}}(a)$ and note that $c_m = \pi_\emptyset(a \star h^{m+T_0}) \to \nu_h(a)$. This implies the first property in the above Theorem. For the second we may use Lemma 4.28 to rewrite the covariance as follows (for $k \geqslant 0$ and $m \geqslant 1$):

$$\operatorname{Cov}(X_{m+k}, X_m) = \nu_{L^{(k)}((a-c_m)\star h^{m+T_0})}(a - c_{m+k}). \tag{5.110}$$

Here $(a - c_m) \star h^{m+T_0} \in Z_\vartheta$ (this is why we shifted the time-averaging by T_0) and using Lemmas 4.27 and 4.25 we see that the right hand side is bounded by $4\gamma^k |a|_\theta\, |a|_\vartheta$. Therefore,

$$\bigl|\operatorname{Cov}(X_k, S_n)\bigr| \leqslant 4|a|_\theta |a|_\vartheta \bigl(1 + 2/(1-\gamma)\bigr) \tag{5.111}$$

independently of k and n. By the strong law of large numbers the time-shifted Birkhoff average of a converges Lebesgue almost surely to $\nu_h(a)$ for a smooth observable $a \in H_\vartheta$. Since H_ϑ is dense in $C(S_\Omega)$ and the operator is positive a standard 'squeezing' argument shows that the same is true for $a \in C(S_\Omega)$ and we conclude that the measure ν_h is indeed natural. □

Remarks 5.2. – In the time-dependent case one introduces an initial time τ_i and final time τ_f with $T = \tau_f - \tau_i > 0$. The set of configurations $\mathcal{C}[K, \tau_i, \tau_f]$ over a set K now depends on both initial and final time. (the configurations doesn't really but the associated configurational operators do). The tree-mappings, bounds and renormalization maps are as before (no time-dependence). The bounds for the global operators, which we may denote $L^{(\tau_f, \tau_i)}$, are also as above. The projective family for the (time-dependent) natural measure is now obtained by letting the initial time tend to minus infinity: $h(\tau_f) = \lim_{\tau_i \to -\infty} L^{(\tau_f,\tau_i)} h^0$. The remaining changes are straight-forward and left to the reader.

5.1. Spatial bounds. Proof of Theorem 2.5

We suppose now that (Ω, d) is a metric space and that there is $0 < \xi < 1$ so that the ξ-spatial effective coupling strength is bounded by κ (for the definition cf. Section 1.4) We may then choose the β's (Lemma 3.4, repeating the proof using the $|\cdot|_{\theta,p,\xi}$-norm) so that

$$\sum_{V \in \mathcal{F}} |\beta_{p,V}| \theta^{-|V|} \xi^{-\operatorname{rad}(p,V)} \leqslant C_\beta. \tag{5.112}$$

We define $h^0 \equiv 1$, $h^T = L^{(T)} h^0$, $T \geqslant T_0$ as above and we have the expansion

$$h_K^T = \sum_{C \in \mathcal{C}[K,T]} L_K[C] h^0.$$

For fixed $K, J \in \mathcal{F} \setminus \{\emptyset\}$ and $T > 0$ we will first study the spatial correlation function,

$$h_{K\cup J}^T - h_K^T h_J^T = \sum_{C \in \mathcal{C}[K\cup J,T]} L_{K\cup J}[C] h^0 - \sum_{(C',C'')} L_K[C'] h^0 \cdot L_J[C''] h^0, \tag{5.113}$$

where the last sum extends over $(C', C'') \in \mathcal{C}[K,T] \times \mathcal{C}[J,T]$. On the right hand side a number of terms are going to cancel, namely those where configurations do not 'connect' the sets K and J. More precisely, we make the following

DEFINITION 5.3. – Let $K \in \mathcal{F}$ and $J \in \mathcal{F}$ be disjoint and consider time-T configurations $C' \in \mathcal{C}[K,T]$ and $C'' \in \mathcal{C}[J,T]$. Let

$$K = K_T \subset K_{T-1} \subset \cdots \subset K_0 \quad \text{and} \quad J = J_T \subset J_{T-1} \subset \cdots \subset J_0$$

be the respective expansions. We say that C' and C'' are (K,J)-disconnected when K_0 and J_0 are disjoint (and otherwise connected).

Let C' and C'' be (K,J)-disconnected. We may then in the natural way concatenate them into a configuration $C = C' \cup C'' \in \mathcal{C}[K \cup J, T]$. Conversely, we may say that $C \in \mathcal{C}[K \cup J, T]$ is (K,J)-disconnected iff it is the result of a concatenation of (K,J)-disconnected configurations over K and J, respectively. Since $h^0 \in \mathcal{M}_\vartheta$ 'splits' according to $h^0_{K_T \cup J_T} = h^0_{K_T} h^0_{J_T}$ (everything equals one!) we have (using Fubini) for the disconnected case the identity

$$L_{K \cup J}[C]h^0 = L_K[C']h^0 \cdot L_J[C'']h^0. \tag{5.114}$$

Returning to the sum (5.113) we see that all disconnected contributions cancel and we are left with the task of estimating the remaining contributions from the (K,J)-connected configurations. Again we may do so using a simple large deviation estimate.

DEFINITION 5.4. – For $0 < \vartheta \leqslant 1$, $T \geqslant 0$ and $p \in \Omega$ we define a generating function (cf. Definition 4.11 and Section 4.5) for tree values through the following formal power series:

$$u_p^T(s,x) \equiv \sum_{y \in \mathcal{Y}[p,T]} \|y\|_\vartheta s^{\mathrm{size}(y)}\, x^{\mathrm{rad}(y)}. \tag{5.115}$$

We replace $u_p^T(s)$ by $u_p^T(s,x)$ and note that Proposition 4.17 holds for these time-space generating functions when we replace (4.74) by

$$b_p^k(s,x) = s \sum_{V \in \mathcal{F}} 2|\beta_{p,V}| x^{\mathrm{rad}(p,V)} \prod_{q \in V \cup \{p\}} u_p^k(s,x). \tag{5.116}$$

Also the remaining lemmas hold with $u_p^T(\gamma, \xi^{-1})$ instead of $u_p^T(\gamma)$. The proof of these generalizations (left for the reader) only needs that $\mathrm{rad}(p,\cdot)$ is positive and sub-additive, i.e. that $0 \leqslant \mathrm{rad}(p, K \cup V) \leqslant \mathrm{rad}(p,K) + \mathrm{rad}(p,V)$ when $K, V \in \mathcal{F}$. Suppose now that the configuration $C \in \mathcal{C}[K \cup J, T]$ (similarly for the pair $(C', C'') \in \mathcal{C}[K,T] \times \mathcal{C}[J,T]$) is (K,J)-connected. Then there must be trees in the corresponding tree-structure for which the sum of interaction radii is not smaller than $d(K,J) = \min\{d(p,q)\colon\ p \in K, q \in J\}$. Summing over connected trees only (indicated in the sum by a prime) we get by the Large deviation Lemma 4.15 (cf. also the proof of Lemma 4.25):

$$\sum_C{}' \left|L_{K \cup J}[C]h^0\right| \leqslant \xi^{d(K,J)} \prod_{p \in K \cup J} u_p^T(\gamma, 1) \leqslant \vartheta^{-|K|-|J|} \xi^{d(K,J)} \tag{5.117}$$

(with precisely the same bound for the double sum). Therefore,

$$\left|h^T_{K \cup J} - h^T_K \cdot h^T_J\right| \leqslant 2\vartheta^{-|K|-|J|} \xi^{d(K,J)}. \tag{5.118}$$

We may now take the (uniform) limit $T \to \infty$, and deduce the claim in Theorem 2.5. □

Remarks 5.5. – Through a more careful study of the generating function, $u_p^T(s,x)$, one may infer simultaneous decay of space and time correlations. We omit the details.

6. Proof of the examples

It is clear that our first example verifies the conditions of the Theorem for ε small enough. The problem is to give sensible bounds on ε. The situation is simplified by our choice of uncoupled expanding map. For $f(z) = 2z \bmod \mathbb{Z}$ the uncoupled operator takes the form $L\phi(z) = (\phi(z/2) + \phi((1+z)/2))/2$ (where, strictly speaking, only the sum of the two terms is well-defined for $z \in \mathbb{C}/\mathbb{Z}$). We have $c_h = 1$ and for the following calculations we may also assume that $\gamma = 1$ and $\lambda = 2$.

Let $Z = \ker \ell$. Using e.g. Fourier expansion of functions in $E(A[\rho])$ we find for $T \geqslant 1$:

$$\|L^T_{|Z}\| \leqslant \frac{1}{\exp(2\pi(2^T - 1)\rho) - 1} + \frac{1}{\exp(2\pi(2^T + 1)\rho) - 1}. \tag{6.119}$$

To check condition TR we may instead of (sums of) $c_r \eta^n$ use the more precise values, $\alpha^1 = \|L_{|Z}\|$ and $\alpha_{\gamma=1} = 1 + \sum_{k \geqslant 1} \|L^k_{|Z}\|$. The relation between ε and the coupling strength is: $\kappa = \kappa(\rho, \theta, \varepsilon) = \varepsilon \cosh(2\pi\rho)/(2\pi\theta)$ and finally, $C_\beta = C_\beta(\rho, \kappa)$ was given in Eq. (3.26). All that remains is to find θ, ρ and ε so that

$$1 > 3\theta + 2C_\beta \alpha_{\gamma=1} \quad \text{and} \quad 1 > \theta\big(1 + 2\alpha^1\big) + 2C_\beta\big(1 + \alpha^1 \alpha_{\gamma=1}\big). \tag{6.120}$$

(Strictly speaking, here we also need to verify that all the values $\theta \cdot u^T_p(\gamma)$ remain smaller than one.) The first inequality turns out to be the harder to satisfy. By trial and error I found that choosing $\rho = 0.34622$, $\theta = 0.20525$ and $\varepsilon_0 = 0.03239374$ (which is larger than $1/31$) ensured the Condition TR. The above value of ε_0 is probably optimal to within a few per cent for the proofs to work.

For the second example we shall employ a little trick, renormalizing the Euclidean norm, and define a (logarithmic) distance, $d(p,q) = \log(1 + |p - q|)$, $p, q \in \mathbb{Z}^d$. Recall that

$$N_\varrho = \left(\sum_{q \neq 0} \frac{1}{|p - q|}\right)^{-1}.$$

Setting $0 < \delta = \log(1/\xi) < \varrho$ we have: $\xi^{d(p,q)} = (1 + |p - q|)^{-\delta} \leqslant |p - q|^{-\delta}$ and therefore,

$$|g_p|_{\theta,p,\xi} \leqslant \frac{\varepsilon N_\varrho}{N_{\varrho-\delta}} \frac{\cosh(2\pi\rho)}{2\pi\theta}. \tag{6.121}$$

All we need for a spatial decay is that this ξ-spatial coupling strength is bounded by the κ value from before and this is the case provided $\varepsilon N_\varrho / N_{\varrho-\delta} < \varepsilon_0$ (with ε_0 as above). Theorem 2.5 now shows that indeed the densities of the natural measure exhibit a polynomial spatial decay,

$$\vartheta^{|K|+|\Lambda|} |h_{K \cup \Lambda} - h_K h_\Lambda| \leqslant 2\xi^{-d(K,\Lambda)} = 2\big(1 + \operatorname{dist}(K, \Lambda)\big)^{-\delta}. \tag{6.122}$$

Appendix A. Expanding analytic circle maps

In the following let $\rho > 0$ and $\lambda > 1$ be fixed. We will establish uniform bounds for invariant natural measures and the exponential decay of correlations for a sequence of single-site maps in the class $\mathcal{E}(\rho, \lambda)$. For a fixed map the existence of constants as in the theorem below is standard, the uniformity over the class, however, is not. Our proof is based on the study of a suitable cone in the function space $E(A[\rho])$. It is interesting that all quantities below may be given explicit bounds in terms of the parameters, ρ and λ. For the map, $f(z) = 2z \bmod \mathbb{Z}$, used in our examples better bounds may be obtained using other techniques.

Recall that for $\phi \in E(A[\rho])$ the holomorphic Perron–Frobenius operator associated with f is given by the expression

$$L_f\phi_y = \pm \sum_{x \in f^{-1}y} \frac{1}{f'_x}\phi_x, \quad y \in A[\rho], \tag{A.123}$$

where the sign is the sign of the orientation of f restricted to S^1 (we will establish below that this sign is well-defined and that f does not have any critical values in the domain considered). The Lebesgue integral along S^1 is denoted $\ell(\phi) = \oint_{S^1} dz\phi(z)$. By definition of the Perron–Frobenius operator we have $\ell(L_f\phi) = \ell(\phi)$.

THEOREM A.1. – *Let $f_1, f_2, \ldots \in \mathcal{E}(\rho,\lambda)$ be a sequence of (ρ,λ)-expanding maps and let $L^{(n)} = L_{f_1}\cdots L_{f_n}$ denote the product of the corresponding Perron–Frobenius operators acting upon $E = E(A[\rho])$.*

There are constants, $\eta < 1$, c_h and c_r depending on ρ and λ only such that:

$$\|L^{(n)}\| \leqslant c_h, \tag{A.124}$$

$$\|L^{(n)}_{|\ker \ell}\| \leqslant c_r\eta^n, \quad n \geqslant 0. \tag{A.125}$$

Remarks A.2. – In fact, we have $L^{(n)}\phi \to h \cdot \ell(\phi)$ where the convergence is exponentially fast and $h > 0$ is in the cone C_σ defined below. The exponential convergence follows from the observation that $L\phi - \phi \in \ker \ell$ for any $\phi \in E$. Hence

$$\|L^{(n+1)} - L^{(n)}\| \leqslant \|L^{(n)}_{|\ker \ell}\| \cdot \|L_{f_{n+1}} - 1\| \leqslant c_r\eta^n(c_h + c_r\eta + 1).$$

The ‘strip’ $\widehat{A}[\rho] = \{z \in \mathbb{C}\colon\ |\operatorname{Im} z| \leqslant \rho\}$ is a universal cover of $A[\rho]$. Now, $z \mapsto \tanh \pi z/4\rho$ maps conformally and bijectively the interior of the strip onto the unit disc $\mathbb{D}$. Pulling back the Poincaré metric $2|dz|/(1-|z|^2)$ we obtain the hyperbolic metric for (the interior of) the annulus and the strip:

$$ds_\rho(z;dz) = \frac{\pi}{2\rho}\frac{|dz|}{\cos(\frac{\pi}{2\rho}\operatorname{Im} z)}. \tag{A.126}$$

By direct computation we see that $ds_\rho(z;dz) \geqslant \lambda ds_{\lambda\rho}(z;dz)$ which means that with respect to the hyperbolic metrics the injection $A[\rho] \hookrightarrow A[\lambda\rho]$ is Lipschitz contracting with the constant $1/\lambda < 1$. Since $1 - \frac{2}{\pi}x \leqslant \cos(x) \leqslant \frac{\pi}{2} - x$ for $0 \leqslant x \leqslant \pi/2$ we have also the inequalities

$$\frac{|dz|}{r} \leqslant ds_{\lambda\rho} \leqslant \frac{\pi}{2}\frac{|dz|}{r}, \quad r = \lambda\rho - |\operatorname{Im} z|. \tag{A.127}$$

Consider the map $f \in \mathcal{E}(\rho,\lambda)$. We will first show that f has no critical points and is expanding. Let $\widehat{f}\colon \widehat{A}[\rho] \to \mathbb{C}$ denote the lift of f for which $\operatorname{Re}\widehat{f}(0) \in [0,1)$. The lift satisfies:

$$\widehat{f}(z+1) = \widehat{f}(z) + p, \quad z \in \widehat{A}[\rho], \tag{A.128}$$

for some $p \in \mathbb{Z}$. For $k \in \mathbb{N}$ let

$$U_k = \{z \in \mathbb{C}\colon\ |\operatorname{Im} z| \leqslant \rho,\ -k+1 \leqslant \operatorname{Re} z \leqslant k\} \tag{A.129}$$

and let H_k and V_k denote the horizontal (parallel to the real axis) and vertical boundaries, respectively, of U_k. If p were zero then the assumptions on f implies that the boundary of fU_1 can not be contained in the image under f of the boundary of U_1. This is impossible as f is holomorphic and hence, p is non-zero. By the assumption on f, hence its lift $\widehat{f}$, we know that $\widehat{f}H_k$ does not intersect $\widehat{A[\rho]}$ for any $k \in \mathbb{N}$. Looking at the two components of $U_{k+1} - U_k$ we see that the images under $\widehat{f}$ are simply $\pm pk$ translations of U_1 (periodicity) which implies that both images have finite (Euclidean) diameters independent of k (pre-compactness and periodicity). For $w \in \widehat{A}[\lambda\rho]$ we may then choose k_0 large enough to ensure that the two components of V_{k_0} are mapped to opposite sides of $\operatorname{Re} z = w$. It then follows that for any $k \geqslant k_0$ the winding number of $\widehat{f}\partial U_k$ around w equals 1. Therefore $\widehat{f}$ has a unique (necessarily real-analytic) inverse, $\widehat{\psi} : \widehat{A}[\lambda\rho] \to \widehat{A}[\rho]$ which (at least) does not expand distances with respect to the hyperbolic metrics. By composing with the injection we see that the map $\widehat{\psi} : \widehat{A}[\lambda\rho] \to \widehat{A}[\lambda\rho]$ in fact contracts the hyperbolic metric with a factor $\lambda^{-1} < 1$.

Returning to f we conclude that it has no critical values in $A[\rho]$. In particular, when restricted to the circle, we see that f_{S^1} has degree $\deg(f) = |p|$. Apart from a constant, the Euclidean metric is the same as the hyperbolic metric on the circle. Hence, f_{S^1} expands also the Euclidean metric by a factor λ which therefore has to satisfy $|p| \geqslant \lambda > 1$. The local inverses of f on $A[\lambda\rho]$ are precisely given by the (only locally defined) projections of the maps $z \mapsto \widehat{\psi}(z+k)$ on $\widehat{A}[\lambda\rho]$, $k = 0, 1, \ldots, |p|$.

In the rest of this section we let $d = d_{\lambda\rho}$ denote the hyperbolic metric on the annulus/strip of half-width $\lambda\rho$. We shall consider the (smaller) annulus/strip $A[\rho]$ not with its own hyperbolic metric but with the metric, $d_{\lambda\rho}$, inherited from the larger annulus. By

$$D = D(\rho, \lambda) = \max\bigl\{d(x,y)\colon\ x \in S^1, y \in A[\rho]\bigr\}$$

we denote the maximal distance between points on the circle and points in the closed annulus. One may even calculate this quantity explicitly through a mapping to the unit disc. With $\alpha = |\tanh \frac{\pi}{4\lambda\rho}(\frac{1}{2} + \mathrm{i}\rho)| < 1$ one has (we omit the details), $D = \log|\frac{1+\alpha}{1-\alpha}|$.

Let $E^{\mathbb{R}} \subset E$ denote the subset of real-analytic functions of $E = E(A[\rho])$. For $\sigma > 0$ we introduce the convex cone

$$C_\sigma = \bigl\{\phi \in E^{\mathbb{R}} \setminus \{0\}\colon\ |\phi_y - \phi_x| \leqslant \phi_x\bigl(\mathrm{e}^{\sigma d(x,y)} - 1\bigr),\ x \in S^1,\ y \in A[\rho]\bigr\} \tag{A.130}$$

and we define for $\phi_1, \phi_2 \in C_\sigma$ a functional β_σ and then Hilbert's projective metric θ_σ by

$$\beta_\sigma(\phi_1, \phi_2) = \inf\bigl\{\mu \in \mathbb{R}_+ \cup \{\infty\}\colon\ \mu\phi_1 - \phi_2 \in C_\sigma\bigr\}, \tag{A.131}$$

$$\theta_\sigma(\phi_1, \phi_2) = \log\bigl(\beta_\sigma(\phi_1, \phi_2)\beta_\sigma(\phi_2, \phi_1)\bigr). \tag{A.132}$$

We may relate the projective metric and the standard Banach norm on $E^{\mathbb{R}}$ through the following

LEMMA A.3. –
(1) *For* $\phi_1, \phi_2 \in C_\sigma \cap \{\ell = 1\}$ *we have* $|\phi_1 - \phi_2| \leqslant 2\mathrm{e}^{\sigma D}(\mathrm{e}^{\theta_\sigma(\phi_1,\phi_2)} - 1)$.
(2) *For* $u \in E^{\mathbb{R}}$ *one has*: $\theta_\sigma(1+u, 1) \leqslant (2 + \frac{8\lambda}{\pi\sigma})|u| + o(|u|)$.

Proof. – For $\phi \in C_\sigma$ we have the bounds

$$|\phi| \geqslant \ell(\phi) = \oint_{S^1} \phi_x\, dx \geqslant \mathrm{e}^{-\sigma D}|\phi|. \tag{A.133}$$

Let $\phi_1, \phi_2 \in C_\sigma \cap \{\ell = 1\}$ and set $\mu = \beta(\phi_1, \phi_2)$. Then $0 \leqslant \ell(\mu\phi_1 - \phi_2) = \mu - 1$ shows that $\mu \geqslant 1$. We then obtain

$$|\phi_1 - \phi_2| \leqslant |\mu\phi_1 - \phi_2| + (\mu - 1)|\phi_1| \leqslant 2\mathrm{e}^{\sigma D}(\mu - 1) \leqslant 2\mathrm{e}^{\sigma D}\big(\mathrm{e}^{\theta_\sigma(\phi_1,\phi_2)} - 1\big)$$

showing the first claim.

First, let $M < 1$. Given $u \in E^{\mathbb{R}}$, $|u| \leqslant M$, we lift it to a function $\hat{u}$ on the cover $\widehat{A}[\rho]$. Then for fixed $x \in \mathbb{R}$,

$$\frac{|\hat{u}_y - \hat{u}_x|}{|1 + \hat{u}_x|} \cdot \frac{1}{y - x} \tag{A.134}$$

is holomorphic in $y \in \widehat{A}[\rho]$ and bounded by $\mu = 2M/(\rho(1 - M))$. Hence,

$$|\hat{u}_y - \hat{u}_x| \leqslant (1 + \hat{u}_x)\mu|y - x| \leqslant (1 + \hat{u}_x)\mu\frac{2\lambda\rho}{\pi}\hat{d}(x, y), \tag{A.135}$$

which on the annulus implies:

$$|u_y - u_x| \leqslant (1 + u_x)\big(\mathrm{e}^{\frac{2\lambda\rho\mu}{\pi}d(x,y)} - 1\big). \tag{A.136}$$

In order for $1 + u$ to be in the cone we need that $2\lambda\rho\mu/\pi \leqslant \sigma$ and this is assured when we set $M = \pi\sigma/(4\lambda + \pi\sigma)$. Thus, the ball centered at 1 and having radius M belongs to the cone. For $|u| < M$ this implies:

$$\Theta_\sigma(1 + u, 1) \leqslant \frac{M + |u|}{M - |u|} \leqslant \frac{2}{M}|u| + \mathrm{o}(|u|), \tag{A.137}$$

which yields the desired bound. □

For $z \in \operatorname{Int} \widehat{A}[\lambda\rho]$ the function $\widehat{\psi}$ is univalent in a disk of radius (at least) $r = \lambda\rho - |\operatorname{Im} z|$ centered at z. Koebe distortion [7, Theorem 1.5] then implies $|\widehat{\psi}''_z/\widehat{\psi}'_z| \leqslant 4/r$ and a calculation now shows that

$$\left|\frac{d}{ds}\log\widehat{\psi}'_z\right| = \left|\frac{\widehat{\psi}''_z}{\widehat{\psi}'_z}\right|\frac{|dz|}{ds} \leqslant 4. \tag{A.138}$$

Since $|\mathrm{e}^z - 1| \leqslant \mathrm{e}^{|z|} - 1$ for $z \in \mathbb{C}$ we may integrate the above inequality and obtain for $u, f(u) \in A[\rho]$ and $v \in S^1$:

$$\left|\frac{f'(v)}{f'(u)} - 1\right| \leqslant \mathrm{e}^{4d(f(u),f(v))} - 1. \tag{A.139}$$

LEMMA A.4. – *Let* $\tilde{\sigma} = 4 + \lambda^{-1}\sigma$ *and suppose that* $\tilde{\sigma} < \sigma$. *Then* $L = L_f : C_\sigma \to C_{\tilde{\sigma}}$ *and there is a constant* $\eta = \eta(\rho, \lambda, \sigma) < 1$ *such that*

$$\theta_\sigma(L\phi_1, L\phi_2) \leqslant \eta\theta_\sigma(\phi_1, \phi_2), \quad \textit{for } \phi_1, \phi_2 \in C_\sigma. \tag{A.140}$$

Proof. – Let $x \in S^1$ and $y \in A[\rho]$. We want to compare $L\phi_x$ and $L\phi_y$. In the following we will sum over the preimages u and v of y and x, respectively. We may pair these preimages so that every pair satisfy $d(u, v) \leqslant \lambda^{-1}d(x, y)$. Let $d = d_{\lambda\rho}(x, y)$. Then using (A.139)

$$|L\phi_y - L\phi_x| \leqslant \sum \left| \frac{1}{f'_u}\phi_u - \frac{1}{f'_v}\phi_v \right| \tag{A.141}$$

$$\leqslant \sum \frac{1}{|f'_u|}|\phi_u - \phi_v| + \left|\frac{1}{f'_u} - \frac{1}{f'_v}\right|\phi_v \tag{A.142}$$

$$\leqslant \left(\mathrm{e}^{4d}\left(\mathrm{e}^{\lambda^{-1}\sigma d} - 1\right) + \left(\mathrm{e}^{4d} - 1\right)\right) L\phi_x \tag{A.143}$$

$$= \left(\mathrm{e}^{\tilde{\sigma} d} - 1\right) L\phi_x \tag{A.144}$$

shows that $L\phi \in C_{\tilde{\sigma}}$.

When $\tilde{\sigma} < \sigma$ it is clear that $C_{\tilde{\sigma}} \subset C_\sigma$. In order to bound the projective diameter of $C_{\tilde{\sigma}}$ in C_σ consider $\phi^1, \phi^2 \in C_{\tilde{\sigma}}$. We wish to find $\mu > 0$ such that $\phi = \mu\phi^1 - \phi^2 \in C_\sigma$. Let $x \in S^1$, $y \in A[\rho]$. The properties of $C_{\tilde{\sigma}}$ imply that

$$|\phi_x - \phi_y| \leqslant \mu\left|\phi^1_x - \phi^1_y\right| + \left|\phi^2_x - \phi^2_y\right| \leqslant \left(\mu\phi^1_x + \phi^2_x\right)\left(\mathrm{e}^{\tilde{\sigma} d} - 1\right) \tag{A.145}$$

and this does not exceed $\phi_x(\mathrm{e}^{\sigma d} - 1)$ provided

$$\frac{\mu\phi^1_x + \phi^2_x}{\mu\phi^1_x - \phi^2_x} \leqslant \frac{\mathrm{e}^{\sigma d} - 1}{\mathrm{e}^{\tilde{\sigma} d} - 1}. \tag{A.146}$$

Now, the right hand side is bounded from below by $\sigma/\tilde{\sigma}$ and it therefore suffices to take

$$\mu = \frac{\sigma + \tilde{\sigma}}{\sigma - \tilde{\sigma}} \sup_{x \in S^1} \frac{\phi^1_x}{\phi^2_x}. \tag{A.147}$$

Then

$$\theta_{\tilde{\sigma}}(\phi_1, \phi_2) \leqslant 2\log\frac{\sigma + \tilde{\sigma}}{\sigma - \tilde{\sigma}} + \sup_{x,y \in S^1} \log\frac{\phi^1_x \phi^2_y}{\phi^1_y \phi^2_x}. \tag{A.148}$$

Since $\phi^1_y \leqslant \mathrm{e}^{\tilde{\sigma} d(x,y)}\phi^1_x \leqslant \mathrm{e}^{\tilde{\sigma}\pi/2\lambda\rho}\phi^1_x$ for $x, y \in S^1$ (and the same for ϕ^2) we obtain the bound

$$\mathrm{diam}_{C_\sigma}(C_{\tilde{\sigma}}) \leqslant \Delta = 2\log\frac{\sigma + \tilde{\sigma}}{\sigma - \tilde{\sigma}} + \frac{\pi\tilde{\sigma}}{\lambda\rho} < \infty. \tag{A.149}$$

We may now let $\eta = \tanh\frac{\Delta}{4} = \tilde{\sigma}/\sigma$ and apply Birkhoffs Theorem [3] (we refer also to Liverani [17] and [11] for similar applications to dynamical systems), to obtain (A.140).

Proof of the theorem. – There is a canonical decomposition $E = E^{\mathbb{R}} \oplus \mathrm{i}E^{\mathbb{R}}$ given by

$$\phi(z) = \frac{1}{2}\left(\phi(z) + \overline{\phi(\bar{z})}\right) + \frac{1}{2}\left(\phi(z) - \overline{\phi(\bar{z})}\right). \tag{A.150}$$

The two projections have both norm one and it is therefore enough to prove the theorem for the subspace of real-analytic functions $E^{\mathbb{R}}$.

Let $L^{(n)}$ be as in the theorem, choose $\sigma > \tilde{\sigma} = 4 + \lambda^{-1}\sigma$ and let $\eta < 1$ be as in Lemma A.4. For $u \in E^{\mathbb{R}} \cap \ker \ell$ with $|u| = 1$ we have $\theta_\sigma(1, 1+u) \leqslant (2 + 8\lambda/\pi\sigma)|u| + \mathrm{o}(|u|) \equiv c|u| + \mathrm{o}(|u|)$. By contraction we have $\theta_\sigma(L^{(n)}1, L^{(n)}(1+u)) \leqslant \eta^n c|u| + \mathrm{o}(|u|)$. By Lemma A.3,

$$\left|L^{(n)}u\right| = \left|L^{(n)}(1+u) - L^{(n)}1\right| \leqslant 2\mathrm{e}^{\sigma D}\eta^n c|u| + \mathrm{o}(|u|) \equiv c_r\eta^n|u| + \mathrm{o}(|u|), \tag{A.151}$$

and by linearity, the term $o(|u|)$, disappears in the desired bound.

Since $L^{(n)}1 \in C^\sigma$ and $\ell(L^{(n)}1) = \ell(1) = 1$ we obtain by Lemma A.3, $|L^{(n)}1| \leqslant \mathrm{e}^{\sigma D}$. Setting $c_h = \mathrm{e}^{\sigma D} + c_r$ we obtain a uniform bound for $\|L^{(n)}\|$. All constants may be chosen to depend on λ and ρ only, and the conclusion of the Theorem follows. □

Appendix B. Analytic function spaces

When we compute bounds for individual configurations one essential ingredient is to 'integrate away' some factors while retaining analyticity/continuity in the remaining variables. In order to formalize this procedure we will define appropriate function spaces. A 'standard' approach is to take an open and non-empty subset D of a complex manifold and consider the space of functions,

$$C^\omega(D) \cap C^0(\mathrm{Cl}\, D), \tag{B.152}$$

holomorphic in D and having continuous extension to the boundary. This is not adequate for our purposes as some of our domains involves products with closed sets without interior. Instead we shall consider what may be called *weakly holomorphic functions*.

DEFINITION B.1. – Let M be a complex manifold, $S \subset M$ a subset thereof and let Y be a Banach space or a complex manifold. We write $C^\omega(\mathbb{D}; S)$ for the set of holomorphic maps from the unit disc to M such that the image is a subset of S.

A map $\phi: S \to Y$ is said to be *weakly holomorphic* if it is continuous and if $\phi \circ \psi : \mathbb{D} \to Y$ is holomorphic whenever $\psi \in C^\omega(\mathbb{D}; S)$.

When Y is a Banach space we let $E(S;Y)$ denote the Banach space (in the sup-norm) of Y-valued weakly holomorphic functions over S. We write $E(S)$ instead of $E(S;\mathbb{C})$.

In the case of $E_\Lambda = E(A_\Lambda)$ the above definition reduces to the one given in (B.152) because A_Λ is the closure of its own interior. More generally, one has an obvious inclusion,

$$E(F;Y) \subset C^0(F;Y) \cap C^\omega(\mathrm{Int}\, F;Y). \tag{B.153}$$

(The reader may try to find general conditions under which equality holds.)

Example B.2. – For $\rho > 0$ let $F_1 = A[\rho] \subset \mathbb{C}C$ be a closed annulus and $F_2 = \partial A[\rho]$ its boundary. Then $E(F_1 \times F_2)$ are the $\mathbb{C}$-valued continuous functions $\phi(z_1, z_2)$ in $(z_1, z_2) \in F_1 \times F_2$ for which the map

$$z_1 \in \mathrm{Int}\, A[\rho] \mapsto \phi(z_1, z_2) \in \mathbb{C} \tag{B.154}$$

is holomorphic for any fixed $z_2 \in \partial A[\rho]$. (This is the prototype of example which is used in Section 3.3.)

THEOREM B.3. – *Let F and K be closed, respectively compact, subsets of complex manifolds and let Y be a Banach space. The Banach spaces $E(F \times K;Y)$ and $E(F;E(K;Y))$ are then isomorphic.*

Proof. – Let $\phi \in E(F \times K;Y)$. For each $z \in F$,

$$r_z\phi(w) \equiv \phi(z,w), \quad w \in K, \tag{B.155}$$

defines a map $r_z : E(F \times K;Y) \to E(K;Y)$. As K is compact (needed!) and ϕ is continuous, $r_z\phi$ is uniformly continuous on compact subsets and it follows that $z \in F \to r_z\phi \in E(K;Y)$ is a continuous map. If $\psi^K \in C^\omega(\mathbb{D};K)$ and $\psi^F \in C^\omega(\mathbb{D};F)$ then our assumption implies that the map

$$u \in \mathbb{D},\ v \in \mathbb{D} \mapsto \phi\big(\psi^F(u), \psi^K(v)\big) \in Y \tag{B.156}$$

is separately analytic in u and v, hence jointly analytic as ϕ is continuous. We therefore have a Y-convergent expansion

$$r_{\psi^F(u)}\phi(w) = \sum_{k\geqslant 0} c_k(w)u^k, \tag{B.157}$$

where each coefficient $c_k \in E(K;Y)$. Thus, $u \in \mathbb{D} \mapsto r_{\psi^F(u)}\phi \in E(K;Y)$ is holomorphic and we conclude that $r\phi \in E(F;E(K;Y))$.

Conversely, given an element $r\phi \in E(F;E(K;Y))$ the same identification (B.155) defines a map $\phi: F \times K \to Y$ which is clearly continuous. Let $\psi = (\psi_F, \psi_K) \in C^\omega(\mathbb{D}; F \times K)$. Then

$$u \in \mathbb{D} \mapsto r_{\psi_F(u)}\phi \in E(K;Y) \tag{B.158}$$

is holomorphic, whence so is

$$u \in \mathbb{D} \mapsto r_{\psi_F(u)}\phi\big(\psi_K(u)\big) = \phi \circ \psi(u) \in Y. \qquad \square \tag{B.159}$$

Let M be a bounded linear operator on $E(K;Y)$. It extends in a natural way to an operator $\hat{M}$ acting upon $\phi \in E(F \times K;Y)$. First note that since the map $z \in F \mapsto r_z\phi \in E(K;Y)$ is continuous, so is $z \in F \mapsto Mr_z\phi \in E(K;Y)$. Similarly if $\psi^F \in C^\omega(\mathbb{D};F)$ then $r_{\psi^F(u)}\phi$ has a convergent series expansion in $E(K;Y)$, hence so has $Mr_{\psi^F(u)}\phi$. By a slight abuse of notation we write $Mr\phi$ for the resulting $E(K;Y)$ valued function of F and we conclude that $Mr\phi \in E(F;E(K;Y))$. Composing with the inverse shows that:

$$\hat{M} = r^{-1}Mr \in L\big(E(F \times K;Y)\big). \tag{B.160}$$

It is also clear that the norms of the two operators are the same. In particular, we have shown the following

THEOREM B.4. – *Every* $M \in L(E(K;Y))$ *extends isometrically to an operator* $\hat{M} \in L(E(F \times K;Y))$ *for which*

$$r\hat{M} = Mr. \tag{B.161}$$

THEOREM B.5. – *Let* Y *be a Banach space and* $\rho > 0$*. Then* $E(A[\rho]) \otimes Y$ *is dense in* $E(A[\rho];Y)$.

Proof. – Let $\Phi \in E(A[\rho];Y)$. We wish to approximate Φ by a finite linear combination of elements of Y times functions in $E(A[\rho])$. With $k_w(z) = 1/2\mathrm{i}\cot(\pi(z-w))$ we have for $|\operatorname{Im} w| < \rho$:

$$\Phi(w) = \oint_{\Gamma_+\cup\Gamma_-} k_w(z)\Phi(z)\,dz = \Phi_+(w) + \Phi_-(w), \tag{B.162}$$

where Γ_+ and Γ_- are the two connected parts of $\partial A[\rho]$. Clearly, Φ_+ extends holomorphically for $\operatorname{Im} w < \rho$ and Φ_- similarly for $\operatorname{Im} w > -\rho$. Then $\Phi_+(w) = \Phi(w) - \Phi_-(w)$ is the difference of two functions which extends continuously to $\operatorname{Im} w = \rho$, whence $\Phi_+(w)$ itself extends continuously to $\operatorname{Im} w \leqslant \rho$ (similarly for Φ_-). By uniform continuity it follows that

$$\Phi_\varepsilon(w) = \Phi_+(w - \mathrm{i}\varepsilon) + \Phi_-(w + \mathrm{i}\varepsilon), \quad \varepsilon > 0, \tag{B.163}$$

converges uniformly to Φ in the $E(A[\rho];Y)$-norm as $\varepsilon \to 0^+$. By construction

$$\Phi_\varepsilon \in E\big(A[\rho+\varepsilon];Y\big)$$

and its Fourier series converges uniformly (in fact exponentially fast) in the $E(A[\rho];Y)$-norm. Taking finite truncations of these Fourier series we obtain an element of $E(A[\rho]) \otimes Y$ which is arbitrarily close to $\Phi \in E(A[\rho];Y)$. □

COROLLARY B.6. – *For Λ an index set, $\bigotimes_{p\in\Lambda} E(A[\rho])$ is dense in $E(\prod_{p\in\Lambda} A[\rho])$.*

Proof. – When Λ is finite this follows from induction. Denoting $A_\Lambda = \prod_{p\in\Lambda} A[\rho]$, Theorem B.3 and Theorem B.5 implies that

$$E(A_p) \otimes E(A_{\Lambda-p}) \sim E\big(A_p; E(A_{\Lambda-p})\big)$$

is dense in $E(A_\Lambda)$. Also $E(A_q) \otimes E(A_{\Lambda-p-q})$ is dense in $E(A_{\Lambda-p})$ and inductively we see that $E(A_p) \otimes E(A_q) \otimes E(A_{\Lambda-p-q})$ is dense in $E(A_\Lambda)$, etc. When Λ is infinite, an element in the direct tensor product has only finitely many components different from one. Since A_Λ is compact, a function in $E(A_\Lambda)$ being continuous may be approximated arbitrarily well by functions depending on finitely many variables, hence by elements in the tensor product. □

COROLLARY B.7. – *Let $p,q \in \Lambda$ (an index set) and let Q_p and Q_q be bounded linear operators on $E(A[\rho])$. Then Q_p and Q_q extends isometrically to operators $\hat{Q}_p$ and $\hat{Q}_q$ on $E(A_\Lambda)$. When $p \neq q$ the extended operators commute while if $p = q$ the extension of the product, $\widehat{Q_pQ_q}$, equals the product of the extensions, $\hat{Q}_p\hat{Q}_q$.*

Proof. – Let $\hat{Q}_p$ be the (isometric) extension of Q_p as in Theorem B.4 (and similarly for Q_q). The action of $\hat{Q}_p$ on a direct tensor product, $\phi_\Lambda = \phi_p \otimes \phi_{\Lambda-p} \in E(A_p) \otimes E(A_{\Lambda-p})$ amounts to setting

$$\hat{Q}_p\phi_\Lambda \equiv (Q_p\phi_p) \otimes \phi_{\Lambda-p}. \tag{B.164}$$

When $p = q$ the action of $\hat{Q}_p\hat{Q}_q$ on such a direct product clearly equals the action of the extension of Q_pQ_q. This extends (isometrically) by continuity and linearity to the whole of $E(A_\Lambda)$ since, by the previous corollary, the tensor product $E(A_p) \otimes E(A_{\Lambda-p})$ is dense in that space. When $p \neq q$, and $\phi = \phi_p \otimes \phi_q \otimes \phi_{\Lambda-p-q}$ is a direct tensor product then

$$\hat{Q}_p\hat{Q}_q\phi = (Q_p\phi_p) \otimes (Q_q\phi_q) \otimes \phi_{\Lambda-p-q} \tag{B.165}$$

shows that the extended operators commute when acting on direct tensor products and again, this property extends by continuity, linearity and density. The last statement is clear. □

Appendix C. Banach algebras and modules

The function spaces (of infinitely many variables) we need are obtained by taking certain limits, an inductive and a projective, which are then equipped with suitable norms. Out of the inductive and the projective limits emerge then families of Banach algebras and families of Banach modules (over the Banach algebras), respectively. The construction is perhaps best understood without explicit reference to variables so let us proceed with a slightly more abstract version, emphazising the simple algebraic properties rather than the particular realization of those properties in the context of coupled maps.

C.1. Inductive limits, projective limits and modules

As usual we denote by $\mathcal{F}$ the family of finite subsets of Ω. It is a directed set under inverse inclusions. Let $(E_\Lambda)_{\Lambda\in\mathcal{F}}$ be a family of Banach algebras each equipped with a unity element and such that $E_\emptyset = \mathbb{C}$.

For each pair $K \subset \Lambda \in \mathcal{F}$ we assume that there are mappings

$$\begin{aligned} &j_{\Lambda,K} : E_K \to E_\Lambda \quad \text{(inclusion)} \\ &\pi_{K,\Lambda} : E_\Lambda \to E_K \quad \text{(projection)} \end{aligned} \tag{C.166}$$

such that $j_{\Lambda,\Lambda}$ and $\pi_{\Lambda,\Lambda}$ are identity mappings on E_Λ and such that $j_{\Lambda,K} \circ j_{K,J} = j_{\Lambda,J}$ and $\pi_{J,K} \circ \pi_{K,\Lambda} = \pi_{J,\Lambda}$ whenever $J \subset K \subset \Lambda \in \mathcal{F}$. We will also assume that $j_{\Lambda,K}$ is an isometry (onto its image) preserving the algebraic structure and that $\pi_{K,\Lambda}$ is norm-contracting and a left inverse of $j_{\Lambda,K}$.

Finally, we shall assume that if $a_K \in E_K$ and $\phi_\Lambda \in E_\Lambda$ then:

$$\pi_{K,\Lambda}\big((j_{\Lambda,K} a_K)\phi_\Lambda\big) = a_K \pi_{K,\Lambda}\phi_\Lambda. \tag{C.167}$$

We denote by $\mathcal{M} = \operatorname{proj}\lim E_\Lambda$ the projective limit of the projective system $(E_\Lambda, \pi_{K,\Lambda})$. An element $\phi \in \mathcal{M}$ is a family $\phi_\Lambda \in E_\Lambda$, $\Lambda \in E_\Lambda$ for which $\phi_K = \pi_{K,\Lambda}\phi_\Lambda$ whenever $K \subset \Lambda \in \mathcal{F}$. We write $\pi_K : \phi \mapsto \phi_K$ for the canonical mapping $\mathcal{M} \to E_K$.

Likewise, let $\mathcal{H} = \operatorname{ind}\lim E_\Lambda$ denote the inductive limit of the inductive system $(E_\Lambda, j_{\Lambda,K})$, i.e. their disjoint union quotiented by the obvious equivalence relation: If $J, K \in \mathcal{F}$ and $J \cup K \subset L \in \mathcal{F}$ then

$$\phi_K \sim \phi_J \Leftrightarrow j_{L,K}\phi_K = j_{L,J}\phi_J. \tag{C.168}$$

We write $j_K : E_K \to \mathcal{H}$ for the canonical mapping.

LEMMA C.1. – *$\mathcal{M}$ is an $\mathcal{H}$-module. Writing $\star$ for the corresponding bi-linear action, the mapping $(a,\phi) \in \mathcal{H} \times \mathcal{M} \mapsto a \star \phi \in \mathcal{M}$ is defined as follows*: *Choose $a_\Lambda \in E_\Lambda$ for which $a = j_\Lambda a_\Lambda$. For $K \in \mathcal{F}$ let $L \in \mathcal{F}$ be any subset containing $K \cup \Lambda$. Then*:

$$(a \star \phi)_K = \pi_{K,L}\big((j_{L,\Lambda} a_\Lambda)(\pi_L \phi)\big). \tag{C.169}$$

Proof. – We need to verify three things:

First, the definition does not depend on the choice of $L \supset K \cup \Lambda$. This follows by noting that the right hand side equals

$$\pi_{K,K\cup\Lambda}\pi_{K\cup\Lambda,L}\big((j_{L,K\cup\Lambda}(j_{K\cup\Lambda,\Lambda} a_\Lambda))(\pi_L\phi)\big)$$

which reduces to $\pi_{K,K\cup\Lambda}((j_{K\cup\Lambda,\Lambda} a_\Lambda)(\pi_{K\cup\Lambda}\phi))$. Second, the family $(a \star \phi)_K$, $K \in \mathcal{F}$ is really projective: If $J \subset K$ and $L \supset K \cup \Lambda$ then also $L \supset J \cup \Lambda$ and

$$\pi_{J,K}(a \star \phi)_K \pi_{J,K}\pi_{K,L}\big((j_{L,\Lambda} a_\Lambda)(\pi_L\phi)\big) = (a \star \phi)_J.$$

Third, if $a_\Lambda \sim a_N$ are two representatives for $a \in \mathcal{H}$ then they give rise to the same $(a \star \phi)_K$. This follows by taking $L = \Lambda \cup N \cup K$ in (C.169) and note that $j_{L,\Lambda} a_\Lambda = j_{L,N} a_N$. $\square$

C.2 The ϑ spaces

Remarks C.2 (*Generalization*). – For the construction of ϑ spaces, the ‘weight’ $\vartheta^{-|\Lambda|}$ which enters in (C.171) and (its reciprocal) in (C.180) may be replaced by any sub-multiplicative function $\omega(\Lambda)$, i.e. for which $1 \leqslant \omega(K \cup \Lambda) \leqslant \omega(K)\omega(\Lambda)$ whenever $K, \Lambda \in \mathcal{F}$.

As the j’s are isometries the E_Λ-norms pass down to the quotient and we may therefore consider the completion H of $\mathcal{H}$ with respect to this norm. H is then a Banach algebra.

Let $0 < \vartheta \leqslant 1$. For a given value of this parameter we introduce norms on the inductive and the projective limits, respectively. In the following we omit for simplicity the natural inclusions, $E_K \hookrightarrow \mathcal{H}$, from the notation.

First, let $\hat{\mathcal{H}} = \sum E_\Lambda$ denote the algebraic direct sum of the spaces E_Λ, $\Lambda \in \mathcal{F}$. Thus, an element $\hat{a}$ is a family of a_Λ with only finitely many components being non-zero. The linear mapping $\hat{\mathcal{H}} \to \mathcal{H}$,

$$\hat{a} \mapsto a = \sum_{\Lambda \in \mathcal{F}} \hat{a}_\Lambda \tag{C.170}$$

defines an equivalence relation on $\hat{\mathcal{H}}$, an equivalence class being the preimage of $a \in \mathcal{H}$ and denoted $[a]$. To $\hat{a} \in \hat{\mathcal{H}}$ we assign the ‘weighted’ norm

$$|\hat{a}|_\vartheta = \sum_{\Lambda \in \mathcal{F}} \vartheta^{-|\Lambda|} |\hat{a}_\Lambda| \tag{C.171}$$

and then for $a \in \mathcal{H}$:

$$|a|_\vartheta = \inf\big\{ |\hat{a}|_\vartheta \colon \hat{a} \in [a] \big\}. \tag{C.172}$$

Then $|\cdot|_\vartheta$ defines norms on both $\mathcal{H}$ and $\hat{\mathcal{H}}$. By taking completions we obtain Banach spaces H_ϑ and $\hat{H}_\vartheta$, respectively.

As $\vartheta \leqslant 1$ we have for $a \in \mathcal{H}$ and $\hat{a} \in [a]$:

$$|a| \leqslant \sum_{\Lambda \in \mathcal{F}} |\hat{a}_\Lambda| \leqslant |\hat{a}|_\vartheta, \tag{C.173}$$

an inequality which is preserved when we take infimum over representations and also after completions.

$$|a| \leqslant |a|_\vartheta, \quad a \in H_\vartheta. \tag{C.174}$$

For $\vartheta = 1$ the two norms are in fact the same. The family of Banach spaces H_ϑ, $0 < \vartheta \leqslant 1$, is ordered increasingly with $H_1 = H$.

A multiplication on $\hat{\mathcal{H}}$ is defined if for $\hat{a} \in [a]$ and $\hat{b} \in [b]$ we set:

$$(\hat{a} \cdot \hat{b})_\Lambda = \sum_{\Lambda_1 \cup \Lambda_2 = \Lambda} \hat{a}_{\Lambda_1} \hat{b}_{\Lambda_2}. \tag{C.175}$$

The mapping $\hat{\mathcal{H}} \to \mathcal{H}$ is then a (ring) homomorphism, i.e. the calculation:

$$\sum_\Lambda (\hat{a} \cdot \hat{b})_\Lambda = \sum_{\Lambda_1} \sum_{\Lambda_2} \hat{a}_{\Lambda_1} \hat{b}_{\Lambda_2} = ab \tag{C.176}$$

shows that $\hat{a}\hat{b} \in [ab]$. Since $\Lambda \subset \Lambda_1 \cup \Lambda_2$ and the weight is sub-multiplicative we get

$$|\hat{a} \cdot \hat{b}|_\vartheta = \sum_\Lambda \vartheta^{-|\Lambda|} |(\hat{a} \cdot \hat{b})_\Lambda| \leqslant \sum_{\Lambda_1} \sum_{\Lambda_2} \vartheta^{-|\Lambda_1|-|\Lambda_2|} |\hat{a}_{\Lambda_1}||\hat{b}_{\Lambda_2}| = |\hat{a}|_\vartheta |\hat{b}|_\vartheta. \tag{C.177}$$

Taking infimum over representations, we obtain:

$$|ab|_\vartheta \leqslant |a|_\vartheta |b|_\vartheta, \quad a, b \in \mathcal{H}. \tag{C.178}$$

This inequality also holds under the completion and shows that H_ϑ is a Banach algebra.

We will need the following

LEMMA C.3 (Approximation). – *Given $a \in H_\vartheta$ and $\varepsilon > 0$ we may find $\hat{a} \in \hat{H}_\vartheta$ with $\hat{a} \in [a]$ such that*

$$|\hat{a}|_\vartheta \leqslant |a|_\vartheta + \varepsilon. \tag{C.179}$$

In other words, we may find a representation for a as a countable sum of elements in $(E_\Lambda)_{\Lambda \in \mathcal{F}}$ which approximates a arbitrarily well in the $|\cdot|_\vartheta$ norm.

Proof. – Let $(a^{(k)})_{k \geqslant 1}$ be a Cauchy-sequence in $\mathcal{H}$ which converges in H_ϑ to $a \in H_\vartheta$. By extracting a sub-sequence we may assume that $|a^{(k+1)} - a^{(k)}|_\vartheta < \varepsilon/2^{k+1}$. For each k we may find $\hat{b}^{(k+1)} \in [a^{(k)} - a^{(k+1)}] \subset \hat{H}_\vartheta$ with $|\hat{b}^{k+1}|_\vartheta < \varepsilon/2^{k+1}$. Finally let $\hat{b}^{(1)} \in [a^{(1)}] \subset \hat{H}_\vartheta$ be such that $|\hat{b}^{(1)}|_\vartheta < |a^{(1)}|_\vartheta + \varepsilon/2$. Then $\hat{a} = \sum_k \hat{b}^{(k)} \in \hat{H}_\vartheta$ is well-defined and satisfies the requirements. □

We will now proceed to the projective limit. For $\phi \in \mathcal{M}$ we simply pose:

$$\|\phi\|_\vartheta = \sup_{\Lambda \in \mathcal{F}} \vartheta^{|\Lambda|} |\phi_\Lambda| \in [0, \infty] \tag{C.180}$$

and we define $\mathcal{M}_\vartheta \subset \mathcal{M}$ to be the Banach space of projective elements in this norm for which the norm is finite. We note that when $\phi \in \mathcal{M}_\vartheta$ then $|\pi_\Lambda \phi| \leqslant \vartheta^{-|\Lambda|} \|\phi\|_\vartheta$. The $\star$-action defined earlier satisfies for $a_\Lambda \in E_\Lambda$:

$$\left|(a_\Lambda \star \phi)_K\right| \leqslant |a_\Lambda| \vartheta^{-|K \cup \Lambda|} \|\phi\|_\vartheta \leqslant |a_\Lambda| \vartheta^{-|K|-|\Lambda|} \|\phi\|_\vartheta. \tag{C.181}$$

Hence,

$$\|a_\Lambda \star \phi\|_\vartheta \leqslant \vartheta^{-|\Lambda|} |a_\Lambda| \|\phi\|_\vartheta. \tag{C.182}$$

If $\hat{a} \in [a]$, $a \in \mathcal{H}$, then using linearity we see that

$$\|a \star \phi\|_\vartheta \leqslant |\hat{a}|_\vartheta \|\phi\|_\vartheta. \tag{C.183}$$

We may then take infimum over representations and limits to see that the $\star$-action extends to a continuous bi-linear action of H_ϑ upon $\mathcal{M}_\vartheta$. We have proved:

PROPOSITION C.4. – *For each $0 < \vartheta \leqslant 1$, $\mathcal{M}_\vartheta$ is a Banach module over the Banach algebra H_ϑ. For $a \in H_\vartheta$ and $\phi \in \mathcal{M}_\vartheta$ the action verifies*:

$$\|a \star \phi\|_\vartheta \leqslant |a|_\vartheta \|\phi\|_\vartheta. \tag{C.184}$$

Remarks C.5. – (1) A projective family $\phi = (\phi_\Lambda)_{\Lambda \in \mathcal{F}} \in \mathcal{M}_\vartheta$ is said to be of ϑ-bounded density. The E_Λ norm of the element ϕ_Λ may grow with increasing index set $\Lambda \in \mathcal{F}$ but not faster than (a constant times) the exponential of the set-weight function, i.e.:

$$|\phi_\Lambda| \leqslant \|\phi\|_\vartheta \vartheta^{-|\Lambda|}. \tag{C.185}$$

Appendix D. A fixed point theorem

Let S be an index set and let $\mathcal{U} = \prod_{p \in S} \mathbb{D}$ be a product of unit disks, each equipped with the Poincaré metric d_D. The so-called Kobayashi metric for the product domain may be defined as:

$$d_\mathcal{U}(x, y) = \sup_{p \in S} d_D(x_p, y_p). \tag{D.186}$$

For each $p \in S$ we assume that $\psi_p \in E(\mathcal{U})$ is a weakly holomorphic function whose image is confined within a subset $U_p \subset \mathbb{D}$. The proof of the following contracting mapping theorem is essentially due to Douady [8].

THEOREM D.1. – *Suppose that the inclusion map $U_p \hookrightarrow \mathbb{D}$ is L-Lipschitz with $L < 1$, uniformly in p. Then, the mapping $\psi : \mathcal{U} \to \mathcal{U}$ is L-Lipschitz for the Kobayashi metric.*

Proof. – Given $x, y \in \mathcal{U}$, set $\alpha = d_\mathcal{U}(x, y)$ and let $r \in [0, 1)$ be the unique value for which $d_D(0, r) = \alpha$. For each $p \in S$ (Axiom of Choice in the uncountable case) we may find a holomorphic map $j_p : \mathbb{D} \to \mathbb{D}$ for which $j_p(0) = x_p$ and $j_p(r) = y_p$ (since $d_D(x_p, y_p) \leqslant \alpha$). The family $j = (j_p)_{p \in S}$ maps $\mathbb{D}$ holomorphically into $\mathcal{U}$ and since each ψ_p is a uniform limit of holomorphic functions in finitely many variables it follows that the composed map

$$\psi_p \circ j : \mathbb{D} \to U_p \tag{D.187}$$

is holomorphic, hence it can not be expanding in the hyperbolic metrics. Composing with the contracting inclusion, $\mathcal{U}_p \hookrightarrow D$, we see that $\psi_p \circ j : \mathbb{D} \to \mathbb{D}$ is L-Lipschitz. But then:

$$d_\mathcal{U}(\psi x, \psi y) = \sup_{p \in S} d_D(\psi_p x, \psi_p y) \leqslant L\alpha = L d_\mathcal{U}(x, y). \qquad \square \tag{D.188}$$

COROLLARY D.2. – *Under the conditions above, if S is finite* [3] *then ψ has a unique fixed point in $\mathcal{U}$. If ψ depends analytically on a parameter in a complex manifold, then so does the fixed point.*

Proof. – Take $x, \psi(x) \in \mathcal{U}$. As S is finite the Kobayashi distance between x and $\psi(x)$ is automatically finite. As $(\mathcal{U}, d_\mathcal{U})$ is a complete metric space, $x^* = \lim_{n \to \infty} \psi^n(x)$ yields the unique fixed point. When ψ depends analytically on parameters, so does $\psi^n(x)$. Uniform convergence now implies analyticity of x^* in the parameters. $\square$

Appendix E. Expanding properties of the coupled maps

Let F be a coupled map with parameters $(\rho, \lambda, \theta, \kappa)$ and fix a non-empty subset $\Lambda \in \mathcal{F}$. As in Section 3.2 we write $F_\Lambda = q_\Lambda \circ F \circ i_\Lambda$ for the Λ-confined dynamical system. In each

[3] Under suitable extra conditions one may generalize to $|S| = \infty$.

coordinate we lift the map, F_p, $p \in \Lambda$, to a map, $\tilde{F}_p = \tilde{f}_p + \tilde{g}_p : A[\rho] \to \mathbb{C}$, on the universal cover. By $\hat{\psi}_p : \widehat{A}[\lambda\rho] \to \widehat{A}[\rho]$ we denote the inverse of each of the uncoupled maps, $\hat{f}_p$, on the larger annulus, cf. Appendix A.

LEMMA E.1. – *For each $p \in \Lambda$ and $0 < r < \lambda\rho$, we have that $\hat{\psi}_p \widehat{A}[r] \subset \widehat{A}[r/\lambda]$.*

Proof. – The imaginary part $v(x,y) = \operatorname{Im} \hat{\psi}_p(x + \mathrm{i}y)$ is a bounded harmonic function of (x,y) in the annulus. As $\hat{\psi}$ is real-analytic, $v(x,0) \equiv 0$ and the contracting properties of $\hat{\psi}_p$ assures that $|v(x, \pm\lambda\rho)| \leqslant \rho$. The lemma follows by the maximum principle applied to $v(x,y) \pm \rho y$. □

Given $w = w_\Lambda \in \widehat{A}_\Lambda$ we want to find the set of preimages $\tilde{F}_\Lambda^{-1}(y)$. Let $n = n_\Lambda \in \mathbb{Z}^\Lambda$ be an integer vector. By hypothesis $\kappa = \sup_{p \in \Omega} |g_p| < (\lambda - 1)\rho$ and the above lemma shows that for each $p \in \Lambda$ $\phi_p^{n,w}(x) := \hat{\psi}_p(w_p + n_p - g_p(x))$ is real-analytic (when w_Λ is real) and maps $\widehat{A}[\rho]_\Lambda$ into $\widehat{A}[(\rho + \kappa)/\lambda]$. The natural inclusion of $\widehat{A}[(\rho + \kappa)/\lambda]$ into $\widehat{A}[\rho]_\Lambda$ is Lipschitz contracting with constant $(\rho + \kappa)/(\lambda\rho) < 1$. The contracting mapping theorem above applies also to mappings of annuli when we replace the Poincaré metric by the hyperbolic metric. By the previous section the map, $\phi_\Lambda^{n,y} = (\phi_p^{n,y})_{p \in \Lambda} : A_\Lambda \to A_\Lambda$, is Lipschitz contracting in the Kobayashi metric. The unique fixed point is clearly a preimage of F_Λ and depends real-analytically on w_Λ. A homotopy argument (letting the coupling go to zero) shows that by varying n_Λ we obtain exactly $\prod_{p \in \Lambda} d^o(f_p)$ preimages.

Acknowledgements

I am grateful to Jean-Baptiste Bardet and Viviane Baladi for pointing out errors and omissions in earlier versions of the paper.

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(Manuscript received March 13, 2001;
accepted, after revision, October 12, 2001.)

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