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ON THE KAUFFMAN BRACKET SKEIN ALGEBRA OF PARALLELIZED SURFACES

BY PIERRE SALLENAVE

ABSTRACT. – We establish a general link between the Kauffman bracket skein algebra of any parallelized, oriented compact surface F and two algebras built from the first homotopy group of the surface and its first homology group respectively. © 2000 Éditions scientifiques et médicales Elsevier SAS

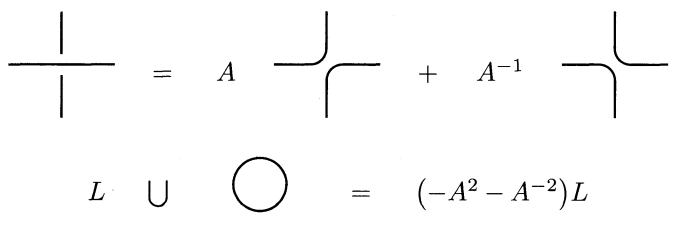
RÉSUMÉ. – Nous mettons en évidence une relation générale entre l'algèbre du crochet de Kauffman d'une surface compacte orientée munie d'une parallélisation et deux algèbres obtenues respectivement à l'aide du groupe fondamental de la surface et de son premier groupe d'homologie. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

1.1. Kauffman bracket skein algebras

Let R be a commutative ring with unit, A an invertible element of R and M an oriented three-manifold. We call "banded link" in $F \times I$ any unoriented link Γ equipped with a vector field nowhere tangent to Γ, called the "framing vector" of Γ; we denote by $\mathcal{L}$ the set of isotopy classes of banded links in M (including the empty link) and by $R[\mathcal{L}]$ the free R-module generated by the elements of $\mathcal{L}$. In the drawings, we take the following convention: the framing of a link shall be induced by the direction orthogonal to the sheet of paper so that the reader should think of a banded link L drawn on the paper as a band $L \times [0,1]$ in F, the core of which is the link L and the orientation of which is given by the vector normal to the sheet of paper and pointing upward; this particular choice of framing will be referred to as the "standard framing".

The Kauffman bracket skein module $\mathrm{K}(M)$ is now defined as the quotient of $R[\mathcal{L}]$ by the following skein relations:



Given any framed link L in M, its class in $\mathrm{K}(M)$ will be denoted by $\langle L \rangle$. The mapping class group of orientation preserving homeomorphisms of M acts in an obvious way on the skein module $\mathrm{K}(M)$.

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It is sometimes possible to put a structure of R-algebra on $\mathrm{K}(M)$. For example, when $A=-1$, the product of two links may be defined by the commutative relation $\langle L\rangle * \langle L'\rangle = \langle L \cup L'\rangle$ (where L and L' are disjoint). If M is homeomorphic to $F \times I$, with F a compact oriented surface, possibly with boundary components, one can proceed as follows: draw the banded links as projected on F and define the product $\langle L\rangle \cdot \langle L'\rangle$ of two links as their superposition (L over L').

One has the following result:

FACT 1 (Structure of algebra of $\mathrm{K}(F \times I)$). – *Given a compact oriented surface F and a multiplication $*$ in $\mathrm{K}(F \times I)$ defined as above, then $\mathrm{K}(F \times I)$ is an R-algebra. Moreover, the mapping class group of orientation preserving homeomorphisms of F acts on this algebra by automorphisms.*

Little is known about the structure of R-algebra of $\mathrm{K}(F \times I)$ in general. A full description is given in a recent paper [5] by J.H. Przytycki and A. Sikora when A has the special value -1, but only for a few examples do we know complete presentations by generators and relations if A has no specified value [2]. Moreover, these descriptions are in general difficult to handle.

To any simple closed unoriented curve drawn on F, one can associate a standard banded link, the framing of which is the standard framing. The structure of R-module of $\mathrm{K}(F \times I)$ is given by the following theorem [4]:

THEOREM 1. – *Let F be a compact oriented surface. Then $\mathrm{K}(F \times I)$ is a free R-module and the simple closed curves (not necessarily connected) with no trivial component in F, together with the empty link, induce a basis of this module.*

1.2. Relative skein algebras of surfaces with at least one boundary component

In [5], J.H. Przytycki and A. Sikora introduced the notion of relative skein algebras. We give here a slightly different definition, though equivalent.

Definition. – Let F be an oriented surface with boundary, p a point in ∂F and v a non-zero vector in $\mathrm{T}_P F \times [0,1]$. By a *special framed arc* in $F \times I$ we mean an embedded framed arc $\gamma : [0,1] \to F \times I$, such that $\gamma(0) = (p,0)$ and $\gamma(1) = (p,1)$ and the framing of which at the points $(p,0)$ and $(p,1)$ is given by v.

We can consider a special framed arc as a ribbon in $F \times I$, the ends of which lie exactly on $(p,0)$ and $(p,1)$, with v orthogonal to both ends and pointing from the same side of the ribbon. We now define a relative framed link in $F \times I$ to be the disjoint union of a special framed arc and a banded link in $F \times I$ and say that two relative framed links L and L' are ambient isotopic if there is an ambient isotopy of $F \times I$ which carries L to L' and is fixed on $F \times \{0\}$ and $F \times \{1\}$.

Let $\mathcal{L}^{\mathrm{rel}}$ denote the set of ambient isotopy classes of relative framed links in $F \times I$ and $R\mathcal{L}^{\mathrm{rel}}$ the R-module freely generated by the elements of $\mathcal{L}^{\mathrm{rel}}$. We define $\mathrm{K}^{\mathrm{rel}}(F \times I)$ to be the quotient of $R\mathcal{L}^{\mathrm{rel}}$ by the usual Kauffman bracket skein relations. This module is called the relative skein module of the surface F and it can be shown that this definition does not depend upon the choices of p and v we have done so far up to isomorphism (in particular, one should notice that for a fixed p the space of possible v is simply connected so that the relative skein algebra does not depend on the choice of v at all). We recall that $\mathrm{K}^{\mathrm{rel}}(F \times I)$ is a free R-module.

There is an obvious multiplication on $\mathrm{K}^{\mathrm{rel}}(F \times I)$ induced by the superposition in the same manner as for the Kauffman bracket skein module and this gives the relative skein module the structure of an R-algebra. This algebra has a unit which is represented by the "trivial special arc" K_0, i.e. the arc supported by $\{p\} \times I$ with constant framing vector, if v is not vertical and, otherwise, by $-A^{-3}\langle K_1\rangle$, where K_1 is the special arc $(\gamma(t), t)$, with constant vertical framing vector, for some trivial loop γ of degree $+1$.

Let G be the fundamental group of the surface F, i.e. $G = \pi_1(F)$. We shall use the group-ring $R(G)$.

J.H. Przytycki and A. Sikora proved the following two interesting facts:

THEOREM 2 (Relation with the Kauffman bracket). – *Let F be a surface with nonempty boundary and p a point in ∂F. There exists a natural algebra homomorphism*:

$$\eta : \mathrm{K}(F \times I) \to \mathrm{K}^{\mathrm{rel}}(F \times I)$$

such that for any link L lying in the interior of $F \times I$, $\eta(\langle L \rangle) = \langle L \cup K_0 \rangle$, if v is not vertical, or $\eta(\langle L \rangle) = -A^{-3}\langle L \cup K_1 \rangle$ if v is vertical, where K_0 and K_1 are as defined above. Moreover, if $A^2 + A^{-2}$ is invertible (actually it suffices that it be not a zero divisor) in R, then η is a monomorphism.

THEOREM 3 (Relative skein algebra for $A = -1$). – *If $A = -1$, the relative skein algebra $\mathrm{K}^{\mathrm{rel}}(F \times I)$ is isomorphic as an R-algebra to the algebra $H(G)$ defined as the quotient of $R(G)$ by the relation $h(g + g^{-1}) = (g + g^{-1})h$ for all g and h in G.*

In the first part of this article, we study how the choice of a parallelization for the surface F yields a natural homomorphism between $R(G)$ and the relative skein algebra for any value of A. In particular, we have the following:

THEOREM A. – *Let F be a compact oriented surface with boundary, p a point in ∂F and π a parallelization of F. There exists a natural surjective algebra homomorphism $\Psi : R(G) \to \mathrm{K}^{\mathrm{rel}}(F \times I)$ associated to π. Moreover, this homomorphism depends only on the isotopy class of π.*

1.3. A quantized version of the group ring of $\mathrm{H}_1(F)$

Let H be commutative group, the law of which we shall denote multiplicatively by $(\times)$. In practice, we shall take $H = \mathrm{H}_1(F, \mathbf{Z})$. We now consider the $\mathbf{Z}$-module RH freely generated by the elements of H. To avoid ambiguities, we write (h) any element h of H when considered as an element of RH. Let $\langle\, ,\, \rangle : H \otimes H \to \mathbf{Z}$ be the intersection form of F.

We put a multiplication $(*)$ on RH as follows:

$$\forall x, y \in H, \quad (x) * (y) = A^{\langle x, y \rangle}(x \times y),$$

and we extend this law to RH bilinearly. We observe that for all x in RH, $1_H * x = x * 1_H$, so that we may embed R in RH via the identification $1_H = 1_R = 1_{RH}$. This makes RH a R-algebra with unit which appears as an A-deformation of the group ring of H.

1.4. Weight of an oriented curve with regard to a specified parallelization of F

Recall that parallelizable compact surfaces are all surfaces with at least one boundary component plus the torus T^2.

Consider an oriented curve γ on the surface F. Let $\delta^\circ\gamma$ be the degree of γ with regard to the given parallelization $\pi : \mathrm{T}F \to F \times \mathbf{R}^2$. We recall that the degree of a connected curve may be obtained using the following procedure: let $\mathrm{d}\theta$ be the differential form on $\mathbf{R}^2 - \{0\}$ which detects variations of the argument in the complex plane $\mathbf{C} \simeq \mathbf{R}^2$. Now consider the sequence:

$$\theta : [0,1] \xrightarrow{\dot{\gamma}} \mathrm{T}F \xrightarrow{\pi} F \times (\mathbf{R}^2 - \{0\}).$$

One then has:

$$\delta^\circ\gamma = \frac{1}{2\pi}\oint_\gamma \mathrm{d}\theta(t) \in \mathbf{Z}.$$

If γ has several components γ_i, then $\delta\gamma = \sum_i \delta^\circ\gamma_i$.

Definition. – We define the weight $P(\gamma)$ of the curve γ with regard to the parallelization π as follows:

$$P(\gamma) = \left(-A^2\right)^{\delta^\circ\gamma}([\gamma]) \in RH,$$

where $[\gamma]$ stands for the homology class of γ.

Notice that this definition actually depends on the isotopy class of the parallelization π only. In the second part of this article we shall prove the following result relating the Kauffman bracket skein algebra $\mathrm{K}(F \times I)$ and the quantized group ring of $\mathrm{H}_1(F, \mathbf{Z})$.

THEOREM B. – *Let F be a parallelized oriented compact surface, possibly with boundary, given with a specified parallelization π, and let $H = \mathrm{H}_1(F, \mathbf{Z})$ be its first homology group. Let RH be the algebra built on H as above and $\mathrm{K}(F \times I)$ the Kauffman bracket skein algebra of the thickened surface.*

(1) *There exists a unique algebra homomorphism $\Phi : \mathrm{K}(F \times I) \to RH$ such that for any simple closed unoriented curve Γ on F, equipped with the standard framing, one has:*

$$\Phi\big(\langle\Gamma\rangle\big) = P(\gamma_1) + P(\gamma_2) = P(\gamma_1) + P(\gamma_1)^{-1},$$

where γ_1 and γ_2 are the two oriented curves supported by Γ.

(2) *The homomorphism Φ is injective if and only if F is a sphere with* 1, 2 *or* 3 *boundary components or a torus with no boundary.*

Eventually, we show that the homomorphism Φ may be naturally extended to a homomorphism from $\mathrm{K}^{\mathrm{rel}}(F \times I)$ to RH. Let $\hat\pi_1(F)$ be the subset of $\pi_1(F, p)$ of elements which can be represented by an embedded loop. For any smooth loop γ embedded in F, we denote by $\bar\gamma$ the special framed arc obtained as the lifting $\bar\gamma(t) = (\gamma(t), t)$ of γ, equipped with standard framing:

THEOREM C. – *Let F be a compact oriented surface with boundary component, p a point in ∂F and π a parallelization of F. There exist a unique map $\varepsilon : \hat\pi_1(F) \to \{-1, +1\}$ and a unique algebra homomorphism $\Theta : \mathrm{K}^{\mathrm{rel}}(F \times I) \to RH$ such that for any smooth loop γ embedded in* $\mathbf{F}$:

$$\Theta\big(\langle\bar\gamma\rangle\big) = A^{\varepsilon(\gamma)} P(\gamma).$$

Moreover, Θ is surjective, it depends only on the isotopy class of π, and $\Phi = \Theta \circ \eta$. The homomorphism Θ is bijective if and only if F is a sphere with one or two holes.

The link between Theorems A and C is enlightened by the following consequence:

THEOREM D. – *Let F be an oriented surface with boundary, p a point in ∂F and π a parallelization. Let $\Psi : R(G) \to \mathrm{K}^{\mathrm{rel}}(F \times I)$ and $\Theta : \mathrm{K}^{\mathrm{rel}}(F \times I) \to RH$ be algebra homomorphisms as in Theorems* A *and* C *above. Then there exists a unique map $\gamma : \pi_1(F, p) \to \mathbf{Z}$ such that for any g in $\pi_1(F, p)$ the following holds:*

$$\Theta \circ \Psi(g) = A^{\gamma(g)}[g],$$

where $[g]$ stands for the homology class corresponding to g.

We can summarize the results of Theorems A, B and C by the following diagram of algebra homomorphisms:

$$\begin{array}{ccccc} & & R(G) & & \\ & & \downarrow{\scriptstyle \Psi} & & \\ \mathrm{K}(F\times I) & \xrightarrow{\ \eta\ } & \mathrm{K}^{\mathrm{rel}}(F\times I) & \xrightarrow{\ \Theta\ } & RH \end{array}$$

2. Relative skein algebras of parallelized surfaces

2.1. Proof of Theorem A

Let F be a compact oriented surface with at least one boundary component and π a parallelization of F. Recall that we want to find a natural surjective algebra homomorphism $\Psi : R(G) \to \mathrm{K}^{\mathrm{rel}}(F \times I)$ associated to π. This will be performed in two steps.

(1) We first prove the existence of such a homomorphism Ψ.

We recall that a parallelization of F is equivalent to the data of a non-vanishing smooth vector field $V : F \to \mathrm{T}F$ on F. We can isotope π near p so that $V(p) = v$, where $p \in \partial F$ and $v \in (\mathrm{T}_p\partial F)^{\perp} \subset \mathrm{T}_pF$ are as specified in Part 1 of the present article (we also demand here that v be pointing outward of F). For any loop $\gamma : [0,1] \to F$ with $\gamma(0) = \gamma(1) = p$, we define a special framed arc $\Psi(\gamma)$ using the following procedure:

- put $\Psi(\gamma)(t) = (\gamma(t), t) \in F \times I$ for any t in $[0,1]$;
- the framing of $\Psi(\gamma)$ at the point $\Psi(\gamma)(t)$ is given by $w(t) = (V(\gamma(t)), 0) \in \mathrm{T}F \times I$. Notice that one can see this relative framed arc as an oriented band, the core of which is the arc $\Psi(\gamma)$, with the vector V pointing from the same side of the band for any value of t.

The proof that Ψ induces a homomorphism $\Psi : R(G) \to \mathrm{K}^{\mathrm{rel}}(F \times I)$ stands on the following lemma:

LEMMA 1. – *The ambient isotopy class of $\Psi(\gamma)$ depends on the class g of γ in $\pi_1(F,p)$ only.*

Proof. – Suppose given two loops γ_1 and γ_2 corresponding to the same element g of $\pi_1(F,p)$. There exists a homotopy

$$\Gamma : [0,1] \times [0,1] \to F,$$

leaving the ends fixed and such that $\Gamma(t,0) = \gamma_1(t)$ and $\Gamma(t,1) = \gamma_2(t)$ for all t in $[0,1]$. Then we obtain an ambient isotopy $\widetilde{\Gamma}$ from $\Psi(\gamma_1)$ to $\Psi(\gamma_2)$ by putting, for all (t,u) in $[0,1]^2$:

$$\widetilde{\Gamma}(t,u) = \big(\Gamma(t,u), t\big),$$
$$\widetilde{\Gamma}_u\big(w(t)\big) = \big(V\big(\Gamma(t,u)\big), 0\big).$$

The image via this map $\Psi : \pi_1(F) \to \mathrm{K}^{\mathrm{rel}}(F \times I)$ of the product of two elements g_1 and g_2 is clearly the superposition of their images. Hence Ψ is a group homomorphism which may be extended to $R(G)$ linearly so as to give rise to the desired algebra homomorphism. This ends the first part of the proof. □

(2) We now prove that the homomorphism Ψ is surjective.

We shall use the following result of Przytycki and Sikora [5]:

PROPOSITION 1. – *Let $\mathcal{B}$ be the set of special framed links (up to ambient isotopy) which are the disjoint union of some unoriented curve γ embedded in F with standard framing and of $\Psi(g)$ for some embedded loop g in F. Then $\mathcal{B}$ is a basis over R of the relative skein algebra.*

Hence, as an algebra, $\mathrm{K}^{\mathrm{rel}}(F \times I)$ is generated by the elements $\Psi(g)$ and the elements presented as the disjoint union of an embedded curve in F with standard framing and the trivial special framed arc. Hence we must prove that these elements also lie in the image of Ψ.

To begin with, let γ be a smooth embedded loop in F, with non-vanishing speed and its extremities at p. Let $\tilde{\gamma}$ be the oriented framed link given by γ with standard framing. Hence $\tilde{\gamma}$ is a ribbon embedded in F, $\tilde{\gamma}: \mathrm{S}^1 \times I \hookrightarrow F$. Let $U(t) = U = (0,1)$ be the constant vertical framing vector of $\tilde{\gamma}$ and let $\rho(t,\theta)$ be the rotation of angle $\frac{\varepsilon\theta\pi}{2}$ around $\dot{\gamma}(t)$ in $\mathbf{R}^3$, where ε is equal to ± 1 so that $\rho(0,1)(U) = (v,0)$. We define $\tilde{\gamma}'$ to be the banded link, with framing vector $U'(t) \in \mathrm{T}F$, obtained from $\tilde{\gamma}$ via the following isotopy Γ:

$$\Gamma_\theta(t) = \gamma(t),$$
$$\Gamma_\theta(U)(t) = \rho(t,\theta)\big(U(t)\big).$$

Now we define a special framed arc $\bar{\gamma}$, with framing vector $\overline{U}(t)$, as the "lifting" of $\tilde{\gamma}'$:

$$\bar{\gamma}(t) = \big(\gamma(t), t\big),$$
$$\overline{U}(t) = \big(U(t), 0\big).$$

The proof of Theorem A will be based on the following lemma:

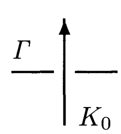
LEMMA 2. – *Let γ and $\bar{\gamma}$ be as above. Let g be the element of $\pi_1(F,p)$ represented by γ. Then, in $\mathrm{K}^{\mathrm{rel}}(F \times I)$, we have*:

$$\langle\bar{\gamma}\rangle = \big(-A^3\big)^{\delta^\circ\gamma}\,\Psi(g).$$

Proof. – It is clear that in the relative skein algebra $\bar{\gamma}$ and $\Psi(g)$ only differ by a certain number of twists. As t increases from 0 to 1, $U(t)$ winds in the "moving frame" given by V of an angle equal to 2π times the degree of γ. Therefore, we obtain $\bar{\gamma}$ by twisting $\Psi(g)$ of exactly $\delta^\circ\gamma$ positive twists. This is exactly the statement of the lemma. □

Apart from the use we make of this result right now, it has the technical advantage of allowing us to do computations about Ψ with planar drawings of diagrams with standard framing instead of three-dimensional ones taking the specific vector field V into account: one can represent $\Psi(g)$ by any smooth curve γ, the homotopy class of which is g, given with standard framing and with a corrective factor of $(-A^3)^{-\delta^\circ\gamma}$ corresponding to the specific choice of γ.

Now consider a special framed link L presented as the disjoint union of some unoriented embedded curve Γ in F with standard framing with the trivial framed arc K_0. Γ may be isotoped in $F \times I$ in such a way that one has a ball near $\big(p, \frac{1}{2}\big)$ where the following diagram appears (it is understood that both the vector v and the framing normal vector of Γ are pointing upward from the sheet of paper):



We apply the skein relation at this crossing and obtain that L satisfies the relation:

$$\langle L\rangle = A\langle\bar{\gamma}\rangle + A^{-1}\langle\overline{\gamma^{-1}}\rangle$$

for some embedded loop γ in F.

We see that, by Lemma 2, $\langle L\rangle$ lies in $\mathrm{Im}(\Psi)$ and, as Ψ is a homomorphism of algebras, it is surjective. Now it is clear from the construction that Ψ depends on the isotopy class of the parallelization π only. This completes the proof of Theorem A.

2.2. Influence of a change of parallelization

We can compare the values of Ψ for two non-isotopic parallelizations π and π' of F.

Let π be a parallelization of F and Ψ_π the corresponding homomorphism $R(G) \to \mathrm{K}^{\mathrm{rel}}(F \times I)$. We recall that to any parallelization π' of F corresponds a cohomology class σ in $\mathrm{H}^1(F, \mathbf{Z})$, depending on the isotopy class of π' only, in such a way that the set of isotopy classes of parallelizations of F is an affine space over $\mathrm{H}^1(F, \mathbf{Z})$; hence for all smooth curve γ, one has

$$\delta^\circ_{\pi'}\gamma = \delta^\circ_{\pi+\sigma}\gamma = \delta^\circ_\pi\gamma + \langle \sigma, [\gamma] \rangle,$$

where $[\gamma]$ is the homology class of γ and $\langle\, ,\, \rangle$ is the evaluation map. This leads to the following result in a straightforward manner.

PROPOSITION 2. – *If Ψ_π denotes the homomorphism associated to a parallelization π of the surface F and $\sigma \in \mathrm{H}^1(F, \mathbf{Z})$, the following relation holds*:

$$\forall g \in G = \pi_1(F, p), \quad \Psi_{\pi+\sigma}(g) = \left(-A^3\right)^{\langle \sigma, [g] \rangle} \Psi_\pi(g).$$

In the case $A = -1$, Theorem 4 gives a description of the kernel of the homomorphism Ψ as generated by the elements of the form

$$(*) \qquad h\left(g + g^{-1}\right) - \left(g + g^{-1}\right)h.$$

One can easily show that for an unspecified value of A the center of the relative skein algebra is very different. Therefore, we cannot expect some equivalent result and it has so far remained unclear whether there exists a generalization.

2.3. A natural orientation of embedded loops

We saw in Subsection 2.1 that for a special framed link L presented as the disjoint union of some embedded curve Γ with standard framing with the trivial framed arc K_0, there exist oriented loops γ and γ^{-1} such that $\langle L \rangle = A\langle \bar{\gamma} \rangle + A^{-1}\langle \bar{\gamma}^{-1} \rangle$. We can give a more accurate result:

PROPOSITION 3. – *There exists a unique map $\varepsilon : \hat{\pi}_1(F) \to \{-1, +1\}$ such that for any L, γ and γ^{-1} as above*:

$$\langle L \rangle = A^{-\varepsilon(\gamma)}\langle \bar{\gamma} \rangle + A^{-\varepsilon(\gamma^{-1})}\langle \bar{\gamma}^{-1} \rangle.$$

Proof. – Let $S \subset \partial F$ be the component of ∂F which contains p. The orientation of F yields a compatible orientation of S. Let g be the generator of $\pi_1(S, p)$ corresponding to this orientation.

Let $\widetilde{F} \xrightarrow{\sigma} F$ be a covering space of F such that σ_* is an isomorphism of $\pi_1(\widetilde{F}, \tilde{p})$ to $\pi_1(S, p)$. By lifting of paths, to any x in $\pi_1(F, p)$ there is associated a unique element $\tilde{x}$ of $\pi_1(S, p)$. Write $\tilde{x} = \tilde{\varepsilon}(x)g$. The map defined by $\varepsilon(x) = \mathrm{sgn}(\tilde{\varepsilon}(x))$ is the one we desire. □

This result will be of some interest for the proof of Theorem C where the map ε will be the one we have just built.

We now study how the data of a parallelization π of a compact oriented surface F yields algebra homomorphisms between $\mathrm{K}(F \times I)$ or $\mathrm{K}^{\mathrm{rel}}(F \times I)$ (if the surface has non empty boundary in this case) and the A-deformation RH of the group ring $R(\mathrm{H}_1(F, \mathbf{Z}))$.

3. Relation between RH and $\mathrm{K}(F\times I)$ and $\mathrm{K}^{\mathrm{rel}}(F\times I)$: proof of Theorems B and C

In this part, we build two maps $\Phi:\mathrm{K}(F\times I)\to RH$ and $\Theta:\mathrm{K}^{\mathrm{rel}}(F\times I)\to RH$ (when F has non empty boundary in this case) meeting the requirements of Theorems B and C. We first define two new skein modules $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ based on oriented diagrams drawn on F together with R-module homomorphisms ϕ, ψ, τ, $\tilde{\eta}$ and $\widetilde{\psi}$ such that the following diagram commutes:

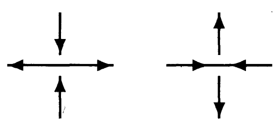
$$\begin{array}{ccccc}
\mathrm{K}(F\times I) & \xrightarrow{\phi} & \mathcal{M} & \xrightarrow{\psi} & RH \\
\downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\tilde{\eta}} & & \downarrow{\scriptstyle\mathrm{Id}} \\
\mathrm{K}^{\mathrm{rel}}(F\times I) & \xrightarrow{\tau} & \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{\psi}} & RH
\end{array}$$

Afterwards, we prove that the R-module homomorphisms $\Phi=\psi\circ\phi$ and $\Theta=\psi\circ\tau$ are actually R-algebra homomorphisms.

3.1. Skein modules based on oriented diagrams drawn on F

We recall that F is a compact oriented surface equipped with a parallelization π. Let D be a link diagram drawn on F in general position, hence D is a diagram whose only singular points correspond to crossings of two branches and such that for every crossing the branch passing over the other is specified. We say that the arcs between the crossings of D are the arcs of D. For each arc α of D we denote by ε_α and $\bar{\varepsilon}_\alpha$ the two possible orientations of α. We allow some of the arcs, say $\alpha_1,\ldots,\alpha_k$, of D to have a pre-specified orientation $\nu_{\alpha_1},\ldots,\nu_{\alpha_k}$ and call again D the data of D together with the pre-specified orientations.

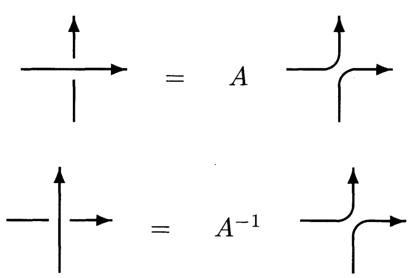
Definition (*Admissible orientations*). – Let $D=(D,(\alpha_1,\nu_{\alpha_1}),\ldots,(\alpha_k,\nu_{\alpha_k}))$ be a diagram. Let $\nu:\{\text{arcs}\}\to\bigcup_\alpha\{\varepsilon_\alpha,\bar{\varepsilon}_\alpha\}$ be the data of an orientation for each arc of a diagram D. We say that ν is *admissible for* D if $\nu(\alpha_i)=\nu_{\alpha_i}$ for all pre-oriented arc α_i and if for every crossing one has exactly two arcs oriented inward and two arcs oriented outward. A crossing where the orientations of arcs belonging to a same branch are not the same is called *exotic* (*see* figure below).

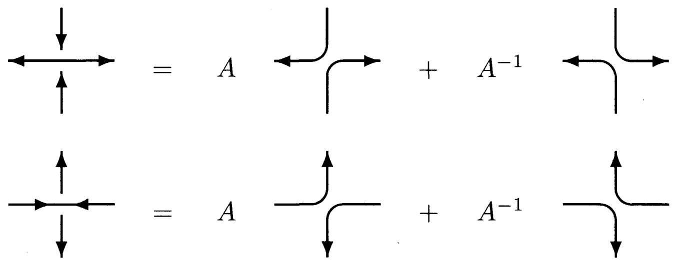


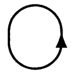
Exotic crossings

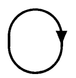
A diagram D, together with an admissible orientation ν is said to be *oriented* and denoted by (D,ν). If there is no exotic crossing in ν, then D is *totally oriented*.

Let $\mathcal{M}'$ be the R-module freely generated by the set of oriented diagrams in F modulo isotopy of diagrams. We consider the following skein relations for elements of $\mathcal{M}'$:





 $= -A^2$

 $= -A^{-2}$

It should be emphasized that the distinction between two orientations of a circle embedded in F makes sense precisely because we have excluded the case of a sphere with no hole by requiring the surface to be parallelizable. Noticeably, these relations make sense for all oriented compact surfaces except the sphere with no hole.

The class of an element D of $\mathcal{M}'$ modulo these relations is denoted by $[D]$. Let $\mathcal{M}$ be the quotient of $\mathcal{M}'$ by these skein relations. One can easily see that $\mathcal{M}$ is a free R-module, a basis of which is given by the embedded oriented curves with no trivial component in F.

Similarly, if F has a boundary and a base point $p \in \partial F$, we define a "pointed" version $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ to be the quotient by the same skein relations of the free module generated by oriented diagrams in the interior of F, except at most one component which then contains p, up to isotopy leaving p fixed. Obviously, there is a natural injective homomorphism $\tilde{\eta}: \mathcal{M} \hookrightarrow \widetilde{\mathcal{M}}$ given by $\tilde{\eta}(D) = D$.

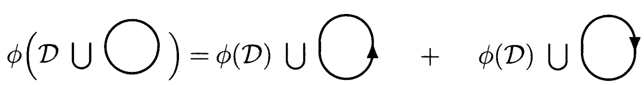
We now construct a natural homomorphism between $\mathrm{K}(F \times I)$ and $\mathcal{M}$.

PROPOSITION 4. – *Let D be an unoriented link diagram on F corresponding to some banded link L in $F \times I$. Let $E(D)$ be the set of all admissible orientations for D. Then there is a natural homomorphism of R-modules $\phi: \mathrm{K}(F \times I) \to \mathcal{M}$ such that for any banded link L:*

$$\phi(\langle L \rangle) = \sum_{\nu \in E(D)} [(D, \nu)].$$

Proof. – Given an unoriented link diagram $\mathcal{D}$ on F, one can obviously define $\phi(\mathcal{D})$ as in the proposition without ambiguity. This definition is clearly invariant under isotopy of diagrams. To show that it actually leads to a homomorphism from $\mathrm{K}(F \times I)$ to $\mathcal{M}$, we now prove that it is also compatible with the Kauffman bracket skein relations, and therefore invariant under type 2 and type 3 Reidemeister moves.

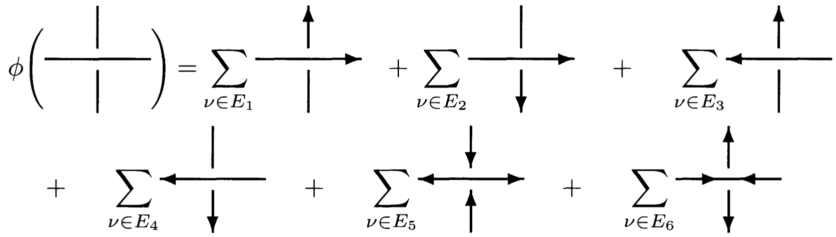
– First consider the "trivial link" skein relation: any embedded circle in F can be equipped with two different orientations, corresponding to values of $+1$ and -1 of its degree, so:



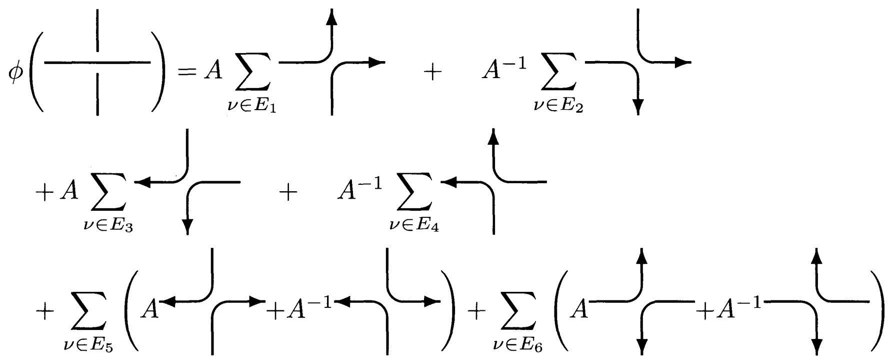
Hence, applying the skein relations in $\mathcal{M}$:

$$\phi\big(\mathcal{D} \cup \bigcirc\big) = (-A^2 - A^{-2})\phi(\mathcal{D}) = \phi\big((-A^2 - A^{-2})\mathcal{D}\big)$$

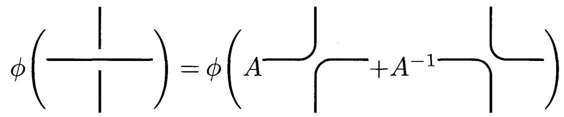
– Furthermore if we look at the "crossing" skein relation near one specified crossing p, we obtain the following identity:



where each sum in the above equation is extended to the set E_i of admissible orientations of the link diagram which have the desired value for the crossing p we are watching.
Now apply the skein relations to the oriented diagrams in each of these sums:



and rearrange the summation according to the unoriented support of the oriented branches having appeared. It turns out that each term corresponds to an admissible orientation for its support, giving the following:



which means exactly that ϕ is compatible with both Kauffman bracket skein relations. □

Equivalently, we can define a homomorphism $\tau : \mathrm{K}^{\mathrm{rel}}(F \times I) \to \widetilde{\mathcal{M}}$.

Given any relative framed link L in $F \times I$, it can be ambient isotoped to a smooth relative framed link L', the framing of which is the standard framing everywhere except in small neighborhoods of $(p,0)$ and $(p,1)$ where it has an arbitrary small twist to meet the requirement that the framing at these points be given by the horizontal vector v. Let $D(L')$ be the link diagram on F obtained as the projection of L' (the relative framed arc in L' projects as component of the link diagram). We give $D(L')$ the pre-specified orientation of the unique arc containing p obtained as the projection of the "upward" orientation of the corresponding relative framed arc.

PROPOSITION 5. – *There exists a natural R-module homomorphism*:

$$\tau : \mathrm{K}^{\mathrm{rel}}(F \times I) \to \widetilde{\mathcal{M}},$$

such that for all relative framed link L and diagram D as above, the following holds:

$$\tau(\langle L \rangle) = \sum_{\nu} \big[(D, \nu)\big],$$

where the sum runs for all admissible orientations ν of D.

Moreover, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{K}(F \times I) & \xrightarrow{\ \phi\ } & \mathcal{M} \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \tilde{\eta}} \\ \mathrm{K}^{\mathrm{rel}}(F \times I) & \xrightarrow{\ \tau\ } & \widetilde{\mathcal{M}} \end{array}$$

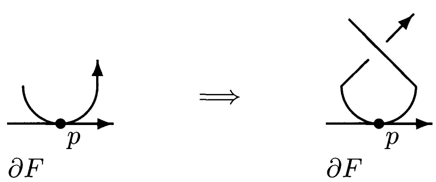
Proof. – The proof that τ is a well defined homomorphism follows exactly the same steps as for Proposition 4. The fact that $\tilde{\eta} \circ \phi = \tau \circ \eta$ is a straightforward consequence of the definition of η and $\tilde{\eta}$. □

It is so far unclear whether there exists or not a multiplication on $\mathcal{M}$ such that the map ϕ should be a homomorphism of algebras ... However, we shall use ϕ to prove Theorems B and C. To begin with, we define a basis of $\mathrm{K}^{\mathrm{rel}}(F \times I)$ with which the computations will prove easier.

3.2. An appropriate basis of $\mathrm{K}^{\mathrm{rel}}(F \times I)$

We wish to define a suitable basis of $\mathrm{K}^{\mathrm{rel}}(F \times I)$ for computations with link diagrams drawn on F with standard framing. In particular, as we want to look at products, we want the elements of this new basis to glue together properly at p when they are superposed. By Proposition 1, a basis of $\mathrm{K}^{\mathrm{rel}}(F \times I)$ is given by $\mathcal{B}$.

Let x be an element $\Gamma \cup \Psi(\gamma)$ of $\mathcal{B}$ for some embedded smooth loop γ in F. If $\varepsilon(\gamma) = 1$ we can modify γ near p so that $\dot{\gamma}(0) = \dot{\gamma}(1)$ be in the opposite sense to that of the oriented boundary component S as defined in Subsection 2.3 (*see* figure below)



We call γ' the special framed arc obtained by doing the equivalent local modification on $\bar{\gamma}$ if $\varepsilon(\gamma) = 1$ and equal to $\bar{\gamma}$ otherwise. Clearly, x and $\Gamma \cup \gamma'$ only differ by a certain number of twists. Hence the set $\mathcal{B}'$ of elements $\Gamma \cup \gamma'$ is also a basis of $\mathrm{K}^{\mathrm{rel}}(F \times I)$.

We recall that we want to obtain algebra homomorphisms $\Phi : \mathrm{K}(F \times I) \to RH = RH_1(F)$ and $\Theta : \mathrm{K}^{\mathrm{rel}}(F \times I) \to RH$ such that:

– for any embedded simple closed curve Γ in F, with possible oriented versions γ_1 and γ_2,

$$\Phi(\langle\Gamma\rangle) = P(\gamma_1) + P(\gamma_2);$$

– for any embedded loop γ,

$$\Theta(\langle\bar{\gamma}\rangle) = A^{\varepsilon(\gamma)} P(\gamma),$$

where $P(\gamma)$ is the "weight" of the oriented curve γ with regard to the parallelization π of the surface F.

For convenience of notation we define $Q(\gamma)$ to be equal to $P(\gamma)$ if γ does not contain p and to $A^{\varepsilon(\gamma_1)}P(\gamma)$ if γ has a component γ_1 which contains p.

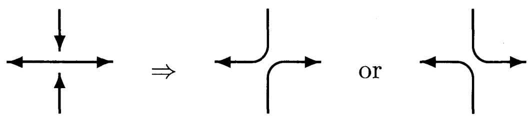
The proof of Theorems B and C will follow two steps: first we construct an R-module homomorphism $\widetilde{\psi}:\widetilde{\mathcal{M}} \to RH$ and define Θ to be $\widetilde{\psi} \circ \tau$ and Φ to be $\psi \circ \phi = \widetilde{\psi} \circ \tilde{\eta} \circ \phi$; then we show that these maps are actually algebra homomorphisms.

3.3. A homomorphism $\widetilde{\psi}:\widetilde{\mathcal{M}} \to RH$

Definition (*Path choices on oriented diagram*). – Let D be an oriented diagram drawn on F. We call "*path choice*" for D the specification of a generically embedded path c supported by D and compatible with its orientation.

In other words, a path choice for a diagram D is the choice of a smoothing compatible with the orientation for each crossing.

In particular, only exotic crossings actually provide a choice between two possible paths (*see* figure below) so that there exists only one possible path choice for any totally oriented diagram.



path choices near an exotic crossing

Given a path choice c for an oriented diagram D and a crossing p of D, we put

$$m(c,p) \stackrel{\text{def}}{=} A$$

if the choice c at p is to turn left when one follows the upper branch of the local diagram towards the crossing, and

$$m(c,p) \stackrel{\text{def}}{=} A^{-1}$$

if the choice is to turn right.

To any oriented diagram D and compatible path choice c, one clearly has an associated weight $P(D,c) \in RH$.

PROPOSITION 6. – *There exists a unique R-module homomorphism $\widetilde{\psi}:\widetilde{\mathcal{M}} \to RH$ such that for any embedded curve γ in F, $\widetilde{\psi}(\gamma) = Q(\gamma)$. For any oriented diagram D, $\widetilde{\psi}([D])$ is given by the following formula*:

$$\widetilde{\psi}([D]) = \sum_c \left[\prod_p m(c,p)\right] Q(D,c),$$

where the sum runs over all possible path choices c for the oriented diagram D and the product over all crossings of D. Furthermore, the compositions:

$$\Phi = \widetilde{\psi} \circ \tilde{\eta} \circ \phi = \psi \circ \phi : \mathrm{K}(F \times I) \to RH,$$
$$\Theta = \widetilde{\psi} \circ \tau : \mathrm{K}^{\mathrm{rel}}(F \times I) \to RH$$

are the unique R-module homomorphisms between $\mathrm{K}(F \times I)$ (respectively $\mathrm{K}^{\mathrm{rel}}(F \times I)$) and RH such that for any embedded unoriented curve $\Gamma = \Gamma^1 \cup \cdots \cup \Gamma^n$ in F, each Γ^i with oriented versions γ_1^i and γ_2^i (respectively any embedded curve $\gamma = \gamma^1 \cup \Gamma^2 \cup \cdots \cup \Gamma^n$, with γ^1 the component containing p):

$$\Phi(\langle \Gamma \rangle) = \prod_{i=1}^{n} \left[P(\gamma_1^i) + P(\gamma_2^i) \right],$$
$$\Theta(\langle \bar{\gamma} \rangle) = Q(\gamma^1) \prod_{i=2}^{n} \left[P(\gamma_1^i) + P(\gamma_2^i) \right].$$

Proof. –
- *Unicity*: the requirements of the proposition specify the values of $\widetilde{\psi}$, Φ, and Θ over bases of $\widetilde{\mathcal{M}}$, $\mathrm{K}(F \times I)$ and $\mathrm{K}^{\mathrm{rel}}(F \times I)$ respectively. It is also clear that Θ is surjective.
- *Verification of the formula*: it is clearly true for embedded curves, and, for general oriented diagrams, the reader may easily check that it is compatible with the skein relations in $\widetilde{\mathcal{M}}$. □

Eventually the homomorphisms Φ and Θ appear as state summations: let D_1 be a diagram for some banded link Γ_1 and D_2 a diagram for some special framed link Γ_2, then

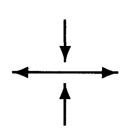
$$\Phi(\langle \Gamma_1 \rangle) = \sum_{(\nu,c)} \left[\prod_{p} m(c,p) \right] P(D_1, \nu, c) = \sum_{(\nu,c)} \widetilde{\phi}(D_1, \nu, c),$$
$$\Theta(\langle \Gamma_2 \rangle) = \sum_{(\nu,c)} \left[\prod_{p} m(c,p) \right] Q(D_2, \nu, c) = \sum_{(\nu,c)} \widetilde{\phi}(D_2, \nu, c),$$

where the sums run over all pairs of admissible orientation and path choice.

3.4. Proof of the first part of Theorem B

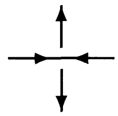
The end of the proof of Theorem B now stands on three technical lemmas regarding the image via ψ of a superposition of two embedded curves in F: we will eventually prove that for such an element $\langle \Gamma \rangle = \langle \Gamma_1 \rangle \langle \Gamma_2 \rangle$ in $\mathrm{K}(F \times I)$ the value of $\Phi(\langle \Gamma \rangle)$ may be worked out by considering only total orientations of the associated banded link diagram.

LEMMA 3. – *Let D be a diagram given as a superposition of two embedded curve diagrams and let ν be an admissible orientation for D. Suppose that there exists at least one exotic crossing p on a given branch B of D. Then the branch B meets at least one exotic crossing of the following type*:



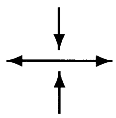
In fact the number of exotic crossings of each type met by any branch of D is exactly the same.

Proof. – Suppose the given exotic crossing is not of that type, i.e.

$p \;=$ 

Then follow one of the involved branches in any direction. As Γ_1 and Γ_2 are closed curves this branch must be closed so that you come back to p along the same branch from the direction opposite to that chosen. The orientation ν of this branch differs from one side of p to the other; hence there must be at least another exotic crossing on the branch. Let p' be the first exotic crossing encountered when following the branch starting from p, then p' is necessarily of the desired type and exotic crossings of either type actually appear alternatively. □

LEMMA 4. – *Let ν be an admissible orientation for D with an exotic crossing p of the following type*:

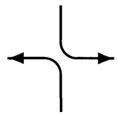


Let c be a compatible path choice and c' the path choice everywhere equal to c except at the crossing p. Then the following identity holds:

$$\widetilde{\phi}(D,\nu,c') = -\widetilde{\phi}(D,\nu,c).$$

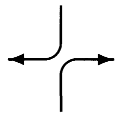
Consequently, if a banded link diagram D is the superposition of two simple closed curves in F, then one can compute $\Phi(\langle D\rangle)$ by considering only total orientations of D.

Proof. – For convenience of notations, suppose that the path choice c near p is the following:



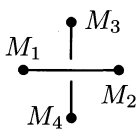
$$m(c,p) = A^{-1}$$

so that c' is characterized near p by:



$$m(c,p) = A$$

- We first notice that the homology class $[(D,\nu,c)]$ associated to the oriented curve (D,ν,c) does not depend on c. Hence (D,ν,c) and (D,ν,c') induce the same element in $\mathrm{H}_1(F)$.
- Furthermore, we can compare the respective degrees of these two embedded curves. We put a natural Riemannian metric on F and we start by isotoping D_1 and D_2 near p so that they exactly coincide with the right angle drawing:



We recall the expression of the degree of an oriented curve γ as an integrated function along the curve given in the introduction of the present article:

$$\delta^\circ\gamma = \frac{1}{2\pi}\oint_\gamma \mathrm{d}\theta(t) \in \mathbf{Z}.$$

In a contractible neighborhood of p, the parallelization π is isotopically standard. Therefore, it can be modified inside this neighborhood in such a way that θ be constant along each branch of the crossing, equal to 0, $\frac{\pm\pi}{2}$ or π modulo 2π according to the direction followed, and as (D,ν,c) and (D,ν,c') only differ near p, we have:

$$\delta^\circ(D,\nu,c) - \delta^\circ(D,\nu,c') = \frac{1}{2\pi}\left[\left(\int_{M_4}^{M_1}\mathrm{d}\theta + \int_{M_3}^{M_2}\mathrm{d}\theta\right) - \left(\int_{M_4}^{M_2}\mathrm{d}\theta + \int_{M_3}^{M_1}\mathrm{d}\theta\right)\right].$$

Hence

$$\delta^\circ(D,\nu,c) - \delta^\circ(D,\nu,c') = \frac{1}{2\pi}\left[\left(\frac{\pi}{2}+\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}-\frac{\pi}{2}\right)\right] = 1.$$

– We finally obtain the following identity in RH:

$$\begin{aligned}\widetilde{\phi}(D,\nu,c) &= \left(\prod_{p'\neq p} m(c,p')\right)\cdot m(c,p)\cdot\left(-A^2\right)^{\delta^\circ(D,\nu,c)}[(D,\nu,c)]\\ &= \left(\prod_{p'\neq p} m(c',p')\right)\cdot\left(A^{-2}m(c',p)\right)\cdot\left(-A^2\right)^{\delta^\circ(D,\nu,c')+1}[(D,\nu,c')]\\ &= -\widetilde{\phi}(D,\nu,c').\end{aligned}$$

□

We now focus our attention on the total orientations of D. We have the following property:

LEMMA 5. – *Let (D,ν) be a totally oriented diagram given as the superposition of two oriented closed curves embedded in F, (D_1,ν_1,c_1) and (D_2,ν_2,c_2). Let c be the only compatible path choice for D, then*:
– $(\prod_p m(c,p)) = A^{\langle[(D_1,\nu_1,c_1)],[(D_2,\nu_2,c_2)]\rangle}$, *where the product runs for all crossings p of D*;
– $[(D,\nu,c)] = [(D_1,\nu_1,c_1)]\cdot[(D_2,\nu_2,c_2)]$ *in* $\mathrm{H}_1(F)$;
– $\delta^\circ(D,\nu,c) = \delta^\circ(D_1,\nu_1,c_1) + \delta^\circ(D_2,\nu_2,c_2)$.

COROLLARY. – *The following holds in RH*: $\widetilde{\phi}(D,\nu,c) = \widetilde{\phi}(D_1,\nu_1,c_1) * \widetilde{\phi}(D_2,\nu_2,c_2)$.

Proof. –
– We begin by proving the last assertion: it follows immediately from the additiveness property of the integration in the formula we use to compute degrees.
– The second assertion is a basic result of homology theory.
– Eventually, the first assertion stands on the fact that the intersection form $\langle[x],[y]\rangle$ counts the algebraic number of crossings of two oriented curves x and y, by counting a crossing where the pair (x,y) is direct as $+1$ and a crossing where it is indirect as -1. □

End of the proof of the first part of Theorem B

By Proposition 5, any algebra homomorphism satisfying the requirements of Theorem B must be equal to Φ. We now prove that Φ is in fact compatible with the multiplicative structures. For this purpose, it is obviously sufficient to study the image via Φ of the product of two curves embedded in F.

Take a skein element $\langle \Gamma \rangle$ in $\mathrm{K}(F \times I)$ which is the product of $\langle \Gamma_1 \rangle$ and $\langle \Gamma_2 \rangle$ for two curves embedded in F and given with standard framing. Represent Γ by the superposition D of the associated diagrams D_1 and D_2. By Lemma 4, one can compute $\Phi(\langle \Gamma \rangle)$ by taking into account total orientations only:

$$\Phi\bigl(\langle \Gamma \rangle\bigr) = \sum_{\nu} \widetilde{\phi}(D, \nu, c),$$

where the sum runs over all total orientations ν and for each ν, c is the only possible path choice.

The set of total orientations ν of D corresponds exactly to the set of pairs (ν_1, ν_2) of total orientations of D_1 and D_2 respectively by splitting of ν in two parts, and, by Lemma 3, $\widetilde{\phi}(D, \nu, c) = \widetilde{\phi}(D_1, \nu_1, c_1) * \widetilde{\phi}(D_2, \nu_2, c_2)$ so that:

$$\Phi\bigl(\langle \Gamma \rangle\bigr) = \sum_{(\nu_1, \nu_2)} \widetilde{\phi}(D_1, \nu_1, c_1) * \widetilde{\phi}(D_2, \nu_2, c_2).$$

In other words:

$$\Phi\bigl(\langle \Gamma \rangle\bigr) = \Biggl(\sum_{\nu_1 \in \mathrm{TO}(D_1)} \widetilde{\phi}(D_1, \nu_1, c_1) \Biggr) * \Biggl(\sum_{\nu_2 \in \mathrm{TO}(D_2)} \widetilde{\phi}(D_2, \nu_2, c_2) \Biggr),$$

where $\mathrm{TO}(D_i)$ is the set of total orientations of D_i.

And, as all orientations of an embedded curve are total orientations, we finally get the desired result:

$$\Phi\bigl(\langle \Gamma_1 \rangle \langle \Gamma_2 \rangle\bigr) = \Phi\bigl(\langle \Gamma_1 \rangle\bigr) * \Phi\bigl(\langle \Gamma_2 \rangle\bigr).$$

This completes the proof that Φ is a homomorphism of R-algebras.

3.5. Proof of the first part of Theorem C

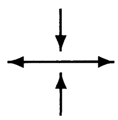
The proof will be very similar to that given for Theorem B. We shall prove that for a diagram D corresponding to a superposition $x = x_1 x_2$ of elements of $\mathcal{B}'$, one can compute $\Theta(x)$ by considering only total orientations of D.

Let D be such a diagram and ν an admissible orientation of D. Call D' the component corresponding to the special framed arc in x. The crossings of D correspond to crossings between different components of D or self-crossings of D'. By Lemmas 3 and 4, we know that if ν has exotic crossings on crossings of the first type (between two components) then $\widetilde{\psi}([(D, \nu)]) = 0$. We therefore can concentrate on orientations having no exotic crossings on crossings of two components. We now look at such orientations and focus on self-crossings of D'.

LEMMA 6. – *Let D' be as above. If ν has an exotic crossing on a self-crossing of D', then* $\widetilde{\psi}([(D, \nu)]) = 0$.

Therefore, one can compute $\Theta(x_1 x_2)$ by considering only total orientations of the associated diagram.

Proof. – Follow D' starting from p in the "upward" sense. Then the first exotic crossing is necessarily of the following type:



Therefore $\widetilde{\psi}([(D,\nu)])$ is zero by Lemma 4. □

As in the case of Lemma 5, we now focus our attention on total orientations of a diagram D which is associated to the product in $\mathrm{K}^{\mathrm{rel}}(F \times I)$ of two elements of $\mathcal{B}'$.

LEMMA 7. – *Let (D,ν) be a totally oriented diagram as above. Let (D_1,ν_1) and (D_2,ν_2) be the totally oriented diagrams corresponding to the elements x_1 and x_2 of $\mathcal{B}'$. Let c be the only possible path choice for D. Then*:
- $(\prod_q m(c,q)) = A^{\langle[(D_1,\nu_1,c_1)],[(D_2,\nu_2,c_2)]\rangle-1}$ *where the product runs for all crossings q of D*;
- $[(D,\nu,c)] = [(D_1,\nu_1,c_1)].[(D_2,\nu_2,c_2)]$ *in* $\mathrm{H}_1(F)$;
- $\delta^\circ(D,\nu,c) = \delta^\circ(D_1,\nu_1,c_1) + \delta^\circ(D_2,\nu_2,c_2)$.

COROLLARY. – *The following holds in RH*:

$$\widetilde{\phi}(D,\nu,c) = \widetilde{\phi}(D_1,\nu_1,c_1) * \widetilde{\phi}(D_2,\nu_2,c_2).$$

Proof. – The two last assertions are exactly equivalent to those of Lemma 5. On the other hand, we know again that $\langle[(D_1,\nu_1,c_1)],[(D_2,\nu_2,c_2)]\rangle$ counts the algebraic number of crossings of the oriented curves D_1 and D_2. Now, the special framed arc x' in x is built as the superposition of the special framed arc x_1' and x_2', where x_1' and x_2' correspond to smooth embedded loops with equal speed at their extremities on p. To obtain the associated component D' of D, one must isotope x' in a neighborhood of $(p,\frac{1}{2})$, where x_1' and x_2' are glued so that D' meets p only at its extremities. By considering the possible relative positions of D_1' and D_2' near p, one can see that this operation always leads to a change of -1 of the algebraic number of crossings. This is exactly the result stated in the lemma.

The corollary is now a consequence of the fact that after smoothing all the crossings, the components of D, D_1 and D_2 which contain p all give the value $+1$ to the map ε. □

We now proceed to the second part of Theorems B and C. We already know that Θ is surjective.
- The structure of the Kauffman bracket skein algebra of the sphere with one, two or three holes is already well known (*see* [1] for example) and is:
  (1) isomorphic to R if there is one hole,
  (2) isomorphic to the polynomial algebra $R[C]$ if there are two boundary components C and C',
  (3) isomorphic to the polynomial algebra $R[C,C',C'']$ if there are three boundary components C, C' and C''.

  The reader will easily check that the homomorphism Φ is actually injective in these cases and so is Θ except for a sphere with three holes. On the other hand, if a sphere F is given with more than three holes, then one can find two non-isotopic embedded curves in F having the same homology class. Hence Φ and Θ cannot be injective.
- Suppose that F is a surface of genus g at least equal to two or a torus with at least one hole, then one can find a non trivial embedded curve in F the homology class of which is null. Hence Φ and Θ cannot be injective in this case either.
- The last possible parallelized surface is the torus with no boundary component. An equivalent result was already established in [3] and [6], using presentation by generators and

relations of the skein algebra. However, for the sake of consistency of this article, we give here another proof of this fact: recall that the first homology group of the torus is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ and that to any element h of $\mathbf{Z} \oplus \mathbf{Z}$ there is associated an oriented embedded curve (up to isotopy), the homology class of which is h. Consequently, one can see the image of Φ as the submodule of RH generated by the elements $(-A^2)^{\delta^\circ h}(h) + (-A^2)^{-\delta^\circ h}(-h)$ in $\mathbf{Z} \oplus \mathbf{Z} \simeq RH$ and an easy induction shows that these elements are free in RH. Hence Φ maps a basis of $\mathrm{K}(\mathrm{T}^2 \times I)$ onto a free family of elements of RH.

This completes the proof of Theorems B and C.

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