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MARK WILLIAMS

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# BOUNDARY LAYERS AND GLANCING BLOW-UP IN NONLINEAR GEOMETRIC OPTICS

BY MARK WILLIAMS [1]

ABSTRACT. – We construct rigorous geometric optics expansions of high order for semilinear hyperbolic boundary problems with oscillatory data. The errors approach zero in $L^\infty$ as the wavelength $\varepsilon \to 0$. To achieve such errors it is necessary to incorporate profiles of the *glancing, elliptic, and hyperbolic boundary layers* into the expansions. The analysis of the glancing boundary layer forces the introduction of a third scale $1/\sqrt{\varepsilon}$ in addition to the usual oscillatory $(1/\varepsilon)$ and spatial $(1)$ scales. The evolution of the leading part of the glancing profile is governed by a semilinear Schrödinger-type equation with nonhomogeneous boundary conditions. The description of the elliptic boundary layer involves complex phases and complex transport equations.

We also construct examples showing that when glancing modes of order at least 3 are present, the maximal time of existence $T_\varepsilon$ of the exact solution $u_\varepsilon$ can approach 0 as $\varepsilon \to 0$. The blow-up mechanism is different from the types of focusing known to occur in free space. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – On construit des développements de l'optique géométrique d'ordre supérieur pour des problèmes aux limites hyperboliques semi-linéaires avec données oscillantes. Les erreurs tendent vers zéro dans $L^\infty$ lorsque la longueur d'onde $\varepsilon$ tend vers 0. Pour atteindre de telles erreurs il est nécessaire d'incorporer les profils des couches limites *glancing*, *elliptiques* et *hyperboliques* dans les développements. L'analyse de la couche limite *glancing* nous amène à introduire une troisième échelle $1/\sqrt{\varepsilon}$ en plus des échelles oscillantes $(1/\varepsilon)$ et spatiales $(1)$ habituelles. L'évolution de la partie dominante du profil *glancing* est contrôlée par une équation semi-linéaire de type Schrödinger avec conditions non-homogènes au bord. La description de la couche limite elliptique fait intervenir des phases complexes et des équations de transport complexes.

Nous construisons aussi des exemples montrant qu'en présence de modes *glancing* d'ordre au moins 3, le temps maximal $T_\varepsilon$ d'existence de la solution exacte $u_\varepsilon$ tend vers 0 lorsque $\varepsilon \to 0$. Le mécanisme d'explosion est différent des types de focalisations qui se manifestent en l'absence de frontières. © 2000 Éditions scientifiques et médicales Elsevier SAS

# Part I
## Survey of the main results

### 1. Introduction

As in [17] we construct geometric optics expansions for a class of Kreiss well-posed semilinear boundary problems on $\overline{\mathbb{R}}_+^{N+1} = \{x = (x', x_N) = (x_0, \ldots, x_N): x_N \geqslant 0\}$

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$$
\begin{aligned}
&Lu_\varepsilon = f(u_\varepsilon, \bar{u}_\varepsilon),\\
&Bu_\varepsilon|_{x_N=0} = g_\varepsilon(x'),\\
&u_\varepsilon = 0 \quad \text{in } x_0 < 0,
\end{aligned}
\tag{1.1}
$$

and then show that the expansions are close to exact solutions for small wavelengths $\varepsilon$. Here $L$ is an $m \times m$ first-order operator, and $g_\varepsilon(x') = G(x', x' \cdot \xi'/\varepsilon)$, where $G(x', \theta_0)$ is smooth and periodic in $\theta_0$. The main difference is that in this paper, we produce expansions for which the errors approach zero in $L^\infty$, not merely in $L^2$ as in [17]. (By "error" we mean the difference between the exact solution and the approximate solution given by the expansion.) To achieve the smaller errors one must incorporate the profiles of the *glancing, elliptic, and hyperbolic boundary layers* into the expansions. The existence of the glancing and elliptic boundary layers, which are small in $L^2$ but not in $L^\infty$, was already evident in [17], but there they were absorbed into the error terms.

The analysis of the glancing boundary layer forces the introduction of a third scale $(1/\sqrt{\varepsilon})$, in addition to the usual oscillatory $(1/\varepsilon)$ and spatial $(1)$ scales. If only the usual scales are used, trouble arises since the equation for the glancing profile is then governed by a vector field everywhere tangent to the boundary. The profile is thus uniquely determined by the initial condition in $x_0 < 0$, so one can't impose a boundary condition. On the other hand, (generalized) eigenvectors associated to multiple real zeros of $\det L(\xi', \xi_N) = 0$ are needed to make the exact solution satisfy the boundary conditions (see Section 2). This inconsistency is one source of the large $L^\infty$ errors in [17].

As in [3,11,6,7] where different three-scale problems were considered (see Remark 8.9), the profile equations exhibit second-order terms in the intermediate scale when the original system (1.1) is first-order. The second-order derivatives are transverse to the boundary, so one can impose a boundary condition. Indeed, we obtain semilinear Schrödinger-type profile equations with *nonhomogeneous* boundary conditions (8.34). Schrödinger profile equations of a different sort were encountered in [3,11], while the second-order profile equations in [6,7] were of parabolic-hyperbolic type. In Section 2 a simple, explicit, linear example shows how the new scale and Schrödinger-type equations appear in connection with glancing modes.

The construction of the elliptic boundary layer also has some unusual features. Complex phases lead us to introduce periodic profiles that extend holomorphically into the upper complex half-plane. One must work with spaces of profiles that are invariant under complex conjugation and nonlinear functions (Remark 4.1). The transport equations associated to the complex phases involve complex vector fields, but these equations need merely be solved to high enough order at the boundary $x_N = 0$ (Proposition 9.4).

In contrast to [17], where a leading term expansion was constructed for solutions $u_\varepsilon$ to (1.1) with oscillatory data $g_\varepsilon$ defined by almost-periodic profiles, here we work with periodic profiles and construct expansions of arbitrarily high order under an appropriate generically valid small divisor hypothesis (Definition 4.1). It turns out that complete expansions can be constructed even without any hypotheses preventing rectification like the oddness hypotheses of [3].

The elliptic and glancing boundary layers are of width $\sim \varepsilon$ and $\sim \sqrt{\varepsilon}$, respectively, and appear in the leading term of the expansions. Nonlinear interactions cause a *hyperbolic boundary layer* of width $\sim \sqrt{\varepsilon}$ to appear as well in the higher order terms (Definition 8.1). One of the main tasks of this paper is to understand precisely how the three layers interact and evolve. This information is contained in the analysis of the profile equations. Much of it is summarized in Remark 8.4.

Once a sufficiently accurate approximate solution is constructed, the existence of a nearby exact solution follows from a Gues-type theorem for semilinear boundary problems (Theorem 6.2).

**Glancing blow-up**

Let $p(\xi) = \det L(\xi)$ be the scalar principal symbol of the $m \times m$ first-order operator $L$ in (1.1). To any $\xi' = (\xi_0, \ldots, \xi_{N-1}) \in \mathbb{R}^N \setminus 0$ there is associated a combination of glancing, hyperbolic, and elliptic phases corresponding to the zeros $\xi_N$ of $p(\xi', \xi_N) = 0$ which are respectively multiple real, simple real, and nonreal. The results of [17] were obtained under the assumption that there were no real zeros $\xi_N$ of multiplicity higher than 2. In Section 12 we show the necessity of this assumption by constructing examples where the presence of real zeros of multiplicity 3 results in a time of existence $T_\varepsilon \to 0$ for the exact solution $u_\varepsilon$ as $\varepsilon \to 0$. The examples involve constant coefficient operators and linear phases so they show that, in contrast to the situation for hyperbolic equations in free space, coherence (see [8]) does not suffice to prevent blow-up. The mechanism of blow-up is completely different from the types of focusing in free space identified in [8] (see Remark 12.3). Theorem 4.2 of [17] shows that high-order nonreal zeros do not cause blow-up as $\varepsilon \to 0$.

*Remark* 1.1. – In [20] we construct rigorous geometric optics expansions for perturbations of a stable planar shock produced by oscillations whose associated characteristic vector fields (3.6) all reflect strictly transversally off the shock (an oscillating free boundary). The results of this paper are intended partly as a step toward understanding the more general situation where elliptic and glancing boundary layers form near the shock.

## 2. Appearance of the new scale in Schrödinger-type profile equations

Consider the $2 \times 2$ example on $\overline{\mathbb{R}}^{2+1}_+ = \{(t, y, x)\colon\ x \geqslant 0\}$

$$\begin{aligned} &Lu_\varepsilon = 0, \\ &Bu_\varepsilon|_{x=0} = g_\varepsilon(t, y), \\ &u_\varepsilon = 0 \quad \text{in } t < 0, \end{aligned} \tag{2.1}$$

where

$$L = \partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_y,$$

and $B$ is a $1 \times 2$ constant, real matrix such that $(L, B)$ satisfies the uniform Kreiss condition (Definition 3.2). We take $g_\varepsilon(t, y) = b(t, y) e^{i\beta \cdot (t,y)/\varepsilon}$ where $g \equiv 0$ in $t < 0$ and $\beta = (\tau', \eta')$ with $|\tau'| = |\eta'|$.

The scalar symbol of $L$ is

$$p(\tau, \eta, \xi) = \tau^2 - (\xi^2 + \eta^2), \tag{2.2}$$

so $\xi = 0$ is a double real root of $p(\tau', \eta', \xi) = 0$. The associated glancing phase and glancing vector field are

$$\phi_g(t, y, x) = (\tau', \eta', 0) \cdot (t, y, x) \quad \text{and} \quad X_g = 2\tau' \partial_t - 2\eta' \partial_y \tag{2.3}$$

respectively.

Taking $\gamma > 0$ and ignoring powers of $2\pi$ we may write the exact solution $u_\varepsilon$ to (2.1) as

$$\begin{aligned} u_\varepsilon(t, y, x) = \int & e^{it(\tau - i\gamma) + iy\eta + ix\sqrt{(\tau - i\gamma)^2 - \eta^2}} \frac{r(\tau - i\gamma, \eta)}{Br(\tau - i\gamma, \eta)} \\ & \times \hat{b}\left(\tau - i\gamma - \frac{\tau'}{\varepsilon}, \eta - \frac{\eta'}{\varepsilon}\right) d\tau\, d\eta \end{aligned} \tag{2.4}$$

where "$\sqrt{\ }$" always denotes the square root with positive part, and $r$ is the eigenvector satisfying

$$L\big(\tau - i\gamma, \eta, \sqrt{(\tau - i\gamma)^2 - \eta^2}\big) r(\tau - i\gamma, \eta) = 0. \tag{2.5}$$

For $(\tau', \eta')$ fixed make the change of variable

$$\tilde{\tau} = \tau - \frac{\tau'}{\varepsilon}, \qquad \tilde{\eta} = \eta - \frac{\eta'}{\varepsilon} \tag{2.6}$$

in (2.4), set $X = (\tilde{\tau} - i\gamma + \tau'/\varepsilon, \tilde{\eta} + \eta'/\varepsilon)$, drop the tildes, and use $|\tau'| = |\eta'|$ to obtain

$$\begin{aligned} u_\varepsilon(t,x,y) = e^{i(t\tau' + y\eta')/\varepsilon} \int e^{i(t,y)(\tau - i\gamma, \eta) + i\frac{x}{\sqrt{\varepsilon}}\sqrt{2\tau'(\tau - i\gamma) - 2\eta'\eta + \varepsilon[(\tau - i\gamma)^2 - \eta^2]}} \frac{r(X)}{Br(X)} \\ \times \hat{b}(\tau - i\gamma, \eta)\, d\tau\, d\eta. \end{aligned} \tag{2.7}$$

(2.7) exhibits the new scale $x/\sqrt{\varepsilon}$. Observe that the glancing phase $\phi_g$ appears on the far left, and the first two terms under the square root give the symbol of $X_g$.

Now fix $(\tau', \eta') = (1, 1)$, and set

$$a_\varepsilon(\chi, t, y) e^{i\theta} = e^{i\theta} \int e^{i(t,y)(\tau - i\gamma, \eta) + i\chi\sqrt{2(\tau - i\gamma) - 2\eta}}\, \hat{b}(\tau - i\gamma, \eta) \frac{r(X)}{Br(X)}\, d\tau\, d\eta. \tag{2.8}$$

Consider the approximate solution

$$\tilde{u}_\varepsilon(t,y,x) = a_\varepsilon(\chi, t, y) e^{i\theta}\big|_{\chi = \frac{x}{\sqrt{\varepsilon}},\ \theta = \frac{\phi_g}{\varepsilon}}. \tag{2.9}$$

By Taylor expanding the square root in (2.7) and using $|e^{iw} - e^{iz}| \leqslant |w - z|$, it's easy to show

$$\big|e^{-i\frac{\phi_g}{\varepsilon}}(u_\varepsilon - \tilde{u}_\varepsilon)(t,y,x)\big|_{H^s_\gamma(t,y)} \leqslant Cx\sqrt{\varepsilon}|b|_{H^{s+(3/2)}_\gamma(t,y)}, \tag{2.10}$$

where $|v|_{H^s_\gamma(t,y)} \equiv |\langle \tau - i\gamma, \eta\rangle^s \hat{v}(\tau - i\gamma, \eta)|_{L^2(\tau,\eta)}$.

Finally, observe that $a_\varepsilon(\chi, t, y)$ satisfies a Schrödinger-type equation with nonhomogeneous boundary conditions:

$$\begin{aligned} &i\partial_\chi^2 a_\varepsilon + X_g a_\varepsilon = 0, \\ &B a_\varepsilon|_{\chi=0} = b(t,y), \\ &a_\varepsilon = 0 \quad \text{in } t < 0. \end{aligned} \tag{2.11}$$

*Remark* 2.1. – Later we'll encounter problems like (2.11) for the $n$th Fourier component $a_n(\chi, t, y, x)$ of a periodic profile where $i\partial_\chi^2$ is replaced by $i\partial_\chi^2/n$. The $n$-dependence is a source of difficulty in the analysis. To obtain linear estimates suitable for Picard iteration, we are led to introduce the cutoffs $\rho(\chi\langle D_\theta^{3/2}\rangle)$ (8.35) and to make use of Wiener algebras $\mathbb{A}_\theta(C(\chi, H^s))$ (7.7), (8.42).

## 3. Symbols, phases, the Kreiss condition, and regular boundary frequencies

We revert to the notation $\overline{\mathbb{R}}^{N+1}_+ = \{x = (x', x_N) = (x_0, x'') = (x_0, y, x_N)\colon\ x_N \geqslant 0\}$ of Section 1. Denote the dual variables by $\xi = (\xi', \xi_N) = (\xi_0, \eta, \xi_N)$.

**A.** In (1.1) $L = L(\partial) = \partial_{x_0} + \sum_{j=1}^{N} A_j \partial_{x_N}$ is an $m \times m$ first-order system, strictly hyperbolic with respect to $x_0$, and noncharacteristic with respect to the boundary $x_N = 0$. The corresponding symbols are

$$
\begin{aligned}
&L(\xi) = \xi_0 I + \sum_{j=1}^{N} A_j \xi_j = \xi_0 + A(\xi''),\\
&p(\xi) = \det L(\xi).
\end{aligned}
\tag{3.1}
$$

We assume the $A_j$ are real matrices. $B$ in (1.1) is a $\mu \times m$ real matrix where $\mu$ is determined from the Kreiss condition (Definition 3.2).

Define the matrix $\mathcal{A}(\xi')$ by the equation

$$
A_N^{-1} L(\xi) = \xi_N - \mathcal{A}(\xi'). \tag{3.2}
$$

**B.** If $\xi = (\xi', \xi_N) \in \mathbb{R}^N \times \mathbb{C}$, $x \in \overline{\mathbb{R}}_+^{N+1}$ we call $\xi$ (respectively $\xi \cdot x$) a *characteristic mode* (respectively *characteristic phase*) when $p(\xi) = 0$, and we set

$$
\operatorname{char} L = \{(\xi', \xi_N) \in \mathbb{R}^N \times \mathbb{C}\colon\ p(\xi) = 0\}. \tag{3.3}
$$

For $\xi \in$ char $L$ the mode (or associated phase) is called *glancing, hyperbolic*, or *elliptic* depending on whether the root $\xi_N$ of $p(\xi', \xi_N) = 0$ is multiple real, simple real, or nonreal.

**C.** Suppose $\xi = (\xi_0, \xi'') \in \operatorname{char} L \cap (\mathbb{R}^{N+1} \setminus 0)$. Strict hyperbolicity implies ker $L(\xi)$ is one dimensional. Indeed, for $\xi'' \in \mathbb{R}^N \setminus 0$, $A(\xi'')$ has $m$ distinct eigenvalues $-\xi_0^i(\xi'')$ satisfying

$$
\xi_0^1(\xi'') < \xi_0^2(\xi'') < \cdots < \xi_0^m(\xi'') \tag{3.4}
$$

with associated right eigenvectors $r_j(\xi'')$, $j = 1, \ldots, m$.

If $\underline{\xi} = (\underline{\xi}_0, \underline{\xi}'') \in \operatorname{char} L \cap (\mathbb{R}^{N+1} \setminus 0)$, $\underline{\xi}_0 = \xi_0^i(\underline{\xi}'')$ for some $i$ and we set $r(\underline{\xi}) = r_i(\underline{\xi}'')$. Let $\pi(\underline{\xi})$ denote projection of $\mathbb{C}^m$ onto ker $L(\underline{\xi})$ along range $L(\underline{\xi})$. This is the same as the projection onto $r_i(\underline{\xi}'')$ with respect to the decomposition

$$
\mathbb{C}^m = \operatorname{span} r_1(\underline{\xi}'') \oplus \cdots \oplus \operatorname{span} r_m(\underline{\xi}''). \tag{3.5}
$$

Set $p_i(\xi) = \xi_0 - \xi_0^i(\xi'')$. $\xi_0^i(\xi'')$ is real-analytic in $\xi''$ and the characteristic vector field associated to $\underline{\xi}$ is given by

$$
X_{\underline{\xi}} = \frac{\partial p_i}{\partial \xi}(\underline{\xi}) \cdot \partial_x \quad \bigl(\partial_x = (\partial_{x_0}, \ldots, \partial_{x_N})\bigr). \tag{3.6}
$$

$X_{\underline{\xi}}$ is called *glancing, outgoing*, or *incoming* when $\frac{\partial \xi_0^i}{\xi_N}(\underline{\xi}'')$ is respectively $= 0, < 0$, or $> 0$. $\underline{\xi}$ (respectively $\underline{\xi} \cdot x$) is then referred to as a glancing, outgoing, or incoming mode (respectively phase). The importance of $X_{\underline{\xi}}$ is connected to the well-known fact [12] that

$$
\begin{aligned}
&\pi(\underline{\xi}) A_k \pi(\underline{\xi}) = -\frac{\partial \xi_0^i}{\partial \xi_k}(\underline{\xi}'')\pi(\underline{\xi}) \quad \text{and thus}\\
&\pi(\underline{\xi}) L(\partial) \pi(\underline{\xi}) = X_{\underline{\xi}} \pi(\underline{\xi}).
\end{aligned}
\tag{3.7}
$$

**D.** When $\underline{\xi} = (\underline{\xi}_0, \underline{\xi}'') = (\underline{\xi}', \underline{\xi}_N) \in \operatorname{char} L$ and $\operatorname{Im} \underline{\xi}_N \neq 0$, we don't necessarily have a decomposition like (3.5) or a well-defined vector field $X_{\underline{\xi}}$ as in (3.6). This leads to

DEFINITION 3.1. – *The elliptic mode* $(\underline{\xi}', \underline{\xi}_N)$ *is* regular *if*

(a) $\underline{\xi}_N$ *is a simple root of* $p(\underline{\xi}', \underline{\xi}_N) = 0$.

(b) $A(\underline{\xi}'')$ *is diagonalizable.*

Condition (b) means there is a decomposition

$$\mathbb{C}^m = E_1(\underline{\xi}'') \oplus \cdots \oplus E_l(\underline{\xi}''), \quad l \leqslant m, \tag{3.8}$$

into eigenspaces of $A(\underline{\xi}'')$ associated to eigenvalues $-\xi_0^1(\underline{\xi}''), \ldots, -\xi_0^l(\underline{\xi}'') \in \mathbb{C}$. $\underline{\xi}_0 \in \mathbb{R}$ must equal one of these, say $\underline{\xi}_0 = \xi_0^1(\underline{\xi}'')$. We may now define $\pi(\underline{\xi})$ to be projection of $\mathbb{C}^m$ onto ker $L(\underline{\xi})$ along range $L(\underline{\xi})$. This is just projection onto $E_1$ with respect to the decomposition (3.8).

Since every eigenvector of $A(\underline{\xi}'')$ associated to $-\underline{\xi}_0$ is an eigenvector of $\mathcal{A}(\underline{\xi}')$ associated to $\underline{\xi}_N$ and vice versa (the same is not true for generalized eigenvectors!), condition (a) in Definition 3.1 and (3.8) imply $\underline{\xi}_0$ is simple as a root of $p(\underline{\xi}_0, \underline{\xi}'') = 0$. Thus, for $\xi'' = (\eta, \xi_N)$ near $(\underline{\eta}, \underline{\xi}_N)$ in $\mathbb{R}^{N-1} \times \mathbb{C}$, $\xi_0^1(\eta, \xi_N)$ is real-analytic in $\eta$ and analytic in $\xi_N$. Setting $p_1(\xi) = \xi_0 - \xi_0^1(\xi'')$ the (complex) characteristic vector field associated to the elliptic mode $\underline{\xi}$ may now be defined by

$$X_{\underline{\xi}} = \frac{\partial p_1}{\partial \xi}(\underline{\xi}) \cdot \partial_x. \tag{3.9}$$

*Remark* 3.1. –

(a) Since the matrices $A_j$ are real, if $(\xi', \xi_N)$ is a regular elliptic mode, so is $(\xi', \overline{\xi}_N)$. For a more general notion of regularity for elliptic modes see Definition 3.5.

(b) (3.7) holds also for regular elliptic modes $\underline{\xi}$.

**E. The uniform Kreiss condition and the spaces $\mathbf{E}^+(\xi')$.** For $\xi' \in \mathbb{R}^N$ and $\gamma > 0$ the strict hyperbolicity of $L$ implies that the eigenvalues $\xi_N^j(\xi_0 - i\gamma, \eta)$ of $\mathcal{A}(\xi_0 - i\gamma, \eta)$ (see (3.2)) have nonzero imaginary part. Denote by $E^+(\xi_0 - i\gamma, \eta)$ the direct sum of the generalized eigenspaces of $\mathcal{A}(\xi_0 - i\gamma, \eta)$ corresponding to the $\xi_N^j$ with $\operatorname{Im} \xi_N^j > 0$:

$$E^+(\xi_0 - i\gamma, \eta) = \bigoplus_{\operatorname{Im} \xi_N^j > 0} \ker\left[\left(\xi_N^j - \mathcal{A}(\xi_0 - i\gamma, \eta)\right)^{m_j}\right], \tag{3.10}$$

where $m_j$ is the multiplicity of $\xi_N^j$. The dimension $\mu = \sum_{\operatorname{Im} \xi_N^j > 0} m_j$ of $E^+(\xi_0 - i\gamma, \eta)$ is independent of $(\xi_0, \eta) \in \mathbb{R}^N$, $\gamma > 0$. Let

$$\mathcal{X} = \left\{(\xi_0, \eta, \gamma)\colon (\xi_0, \eta) \in \mathbb{R}^N,\ \gamma \geqslant 0,\ (\xi_0, \eta, \gamma) \neq 0\right\}.$$

The spaces $E^+(\xi_0 - i\gamma, \eta)$ form a $C^\infty$ subbundle of rank $\mu$ of the trivial $\mathbb{C}^m$ bundle over $\mathcal{X} \cap \{\gamma > 0\}$ which extends to a continuous subbundle $E^+$ of rank $\mu$ over $\mathcal{X}$ [2]. For $\xi' \in \mathbb{R}^N$ $E^+(\xi')$ denotes a fiber of this continuous extension.

DEFINITION 3.2. – *The pair* $(L, B)$ *satisfies the* uniform Kreiss condition *if the restriction of* $B$ *to* $E^+(\xi_0 - i\gamma, \eta)$ *is an isomorphism for all* $(\xi_0, \eta) \in \mathbb{R}^N$, $\gamma \geqslant 0$ *such that* $(\xi_0, \eta, \gamma) \neq 0$. *This forces* $B$ *to be a* $\mu \times m$ *matrix.*

**F. Decomposition of $\mathbf{E}^+(\xi')$.** For $\xi' \in \mathbb{R}^N \setminus 0$ let $\xi_N^i(\xi')$, $i = 1, \ldots, M(\xi') \leqslant m$ be the *distinct* roots of $\det(\xi_N - \mathcal{A}(\xi')) = 0$. Write the index set $\mathcal{M}(\xi') = \{1, \ldots, M(\xi')\}$ as a disjoint union of subsets $\mathcal{G}(\xi'), \mathcal{O}(\xi'), \mathcal{P}(\xi'), \mathcal{I}(\xi')$, and $\mathcal{N}(\xi')$ corresponding to the modes $\beta_i(\xi') \equiv (\xi', \xi_N^i(\xi'))$ that are respectively glancing, outgoing, such that $\operatorname{Im} \xi_N^i$ is positive, incoming, or such that $\operatorname{Im} \xi_N^i$ is negative.

$E^+(\xi')$ may now be written as a direct sum

$$E^+(\xi') = \bigoplus_{i \in \mathcal{G}(\xi') \cup \mathcal{O}(\xi') \cup \mathcal{P}(\xi')} E^+\big(\beta_i(\xi')\big). \tag{3.11}$$

For $i \in \mathcal{P}(\xi')$ $E^+(\beta_i(\xi'))$ is the generalized eigenspace associated to the nonreal eigenvalue $\xi_N^i(\xi')$. For $i \in \mathcal{O}(\xi')$ $E^+(\beta_i(\xi')) = \ker L(\beta_i(\xi'))$, a one-dimensional space. Suppose finally that $i \in \mathcal{G}(\xi')$, $\xi' = (\xi_0, \eta)$, and that $m_i > 1$ is the multiplicity of the real root $\xi_N^i(\xi')$. For $\gamma > 0$ small $\xi_N^i(\xi_0, \eta)$ splits into $m_i$ roots $\xi_N^k(\xi_0 - i\gamma, \eta)$, $k = 1, \ldots, m_i$, with nonzero imaginary parts. Let $\mu_i(\xi')$ be the number of these with positive imaginary part. Then $\dim E^+(\beta_i(\xi')) = \mu_i$ and $E^+(\beta_i(\xi'))$ is spanned by generalized eigenvectors $w$ such that $[\xi_N^i(\xi') - \mathcal{A}(\xi')]^{\mu_i} w = 0$. The strict hyperbolicity and noncharacteristic boundary assumptions imply [2]

$$\begin{aligned} &\mu_i = \frac{m_i}{2} \quad \text{when } m_i \text{ is even.} \\ &\mu_i \text{ is either } \frac{m_i + 1}{2} \text{ or } \frac{m_i - 1}{2} \quad \text{when } m_i \text{ is odd.} \end{aligned} \tag{3.12}$$

DEFINITION 3.3. – *Let $P(\beta_i(\xi'))$ denote the projection of $E^+(\xi')$ onto $E^+(\beta_i(\xi'))$ with respect to the decomposition* (3.11).

*Remark* 3.2. – Suppose $\dim E^+(\beta_i(\underline{\xi}')) = 2$ for some $i \in \mathcal{G}(\underline{\xi}')$. For $\xi'$ near $\underline{\xi}'$, $E^+(\beta_i(\underline{\xi}'))$ can split into a direct sum of two nearly parallel eigenspaces. The projections onto those eigenspaces blow up as $\xi' \to \underline{\xi}'$, which raises the question of whether such unbounded projections can lead to a shrinking time of existence, $T_\varepsilon \to 0$, for the exact solution $u_\varepsilon$ to (1.1) as $\varepsilon \to 0$. Example 2 of Section 12 shows this to be the case. On the other hand Theorem 4.2 of [17] shows that elliptic modes of high multiplicity do not cause $T_\varepsilon \to 0$.

To avoid glancing blow-up we shall later fix $\xi' = (\xi_0, \eta) \in \mathbb{R}^N \setminus 0$ such that

$$\text{For all } i \in \mathcal{G}(\xi'), m_i = 2 \text{ and thus } \dim E^+\big(\beta_i(\xi')\big) = 1. \tag{3.13}$$

In this case $E^+(\beta_i(\xi'))$ is the eigenspace of $A(\eta, \xi_N^i(\xi'))$ associated to the eigenvalue $-\xi_0$.

DEFINITION 3.4. – *We call $\xi' = (\xi_0, \eta) \in \mathbb{R}^N \setminus 0$ a* regular boundary frequency *if $\xi'$ satisfies* (3.13) *and all the associated elliptic modes $\beta_i(\xi')$, $i \in \mathcal{P}(\xi') \cup \mathcal{N}(\xi')$, are regular* (*Definition* 3.1).

The results of this paper will assume that $g_\varepsilon(x') = G(x', x' \cdot \xi'/\varepsilon)$ in (1.1) oscillates with a fixed regular boundary frequency $\xi'$.

When $\xi'$ is regular, for each $i = 1, \ldots, M(\xi')$ the eigenspace of $A(\eta, \xi_N^i(\xi'))$ associated to $-\xi_0$ is one-dimensional. With $r(\xi', \xi_N^i(\xi'))$ denoting a corresponding basis vector, the decomposition (3.11) may now be written

$$E^+(\xi') = \bigoplus_{i \in \mathcal{G}(\xi') \cup \mathcal{O}(\xi') \cup \mathcal{P}(\xi')} \operatorname{span} r\big(\xi', \xi_N^i(\xi')\big). \tag{3.14}$$

*Example* 3.1. – Suppose

$$\det\bigl(\xi_0 + A(\xi'')\bigr) = \bigl(\xi_0^2 - c_1|\xi''|^2\bigr)\bigl(\xi_0^2 - c_2|\xi''|^2\bigr)\bigl(\xi_0^2 - c_3|\xi''|^2\bigr),$$

where $0 < c_1 < c_2 < c_3$. Then every $\xi' \in \mathbb{R}^N \setminus 0$ is a regular boundary frequency. In particular, if $\xi' = (\xi_0, \eta)$ satisfies $\xi_0^2 = c_2|\eta|^2$, then $\mathcal{G}(\xi'), \mathcal{O}(\xi'), \mathcal{P}(\xi')$ are all nonempty.

The following simple proposition will be helpful in the construction of the elliptic boundary layer.

PROPOSITION 3.1. – *Assume* $(L, B)$ *as in Part* **A** *satisfies the uniform Kreiss condition, and suppose* $\xi'$ *is a regular boundary frequency. Then* $\operatorname{card}\mathcal{P}(\xi') = \operatorname{card}\mathcal{N}(\xi')$ *and the restriction*

$$B: \bigoplus_{i \in \mathcal{G}(\xi') \cup \mathcal{O}(\xi') \cup \mathcal{N}(\xi')} \operatorname{span} r\bigl(\eta, \xi_N^i(\xi')\bigr) \to \mathbb{C}^\mu \tag{3.15}$$

*is an isomorphism. The* $r(\eta, \xi_N^i(\xi'))$ *can be chosen so that for each* $i \in \mathcal{P}(\xi')$ *there is a* $k \in \mathcal{N}(\xi')$ *such that* $\overline{\xi_N^i}(\xi') = \xi_N^k(\xi')$ *and* $\overline{r}(\eta, \xi_N^i(\xi')) = r(\eta, \xi_N^k(\xi'))$. *We denote by* $g: \mathcal{P}(\xi') \to \mathcal{N}(\xi')$ *the map that associates* $k \in \mathcal{N}(\xi')$ *to* $i \in \mathcal{P}(\xi')$.

*Proof.* – The statement follows immediately from the definitions, (3.14), and the fact that the matrices $A_j, B$ are real. □

*Remark* 3.3. – Elliptic modes $(\xi', \xi_N^i(\xi'))$ of multiplicity $> 1$ do not cause blow-up, so one can attempt to construct geometric optics expansions for $u_\varepsilon$ even when such modes are present. We shall not attempt to deal here with boundary frequencies $\xi'$ for which the geometric multiplicity of $\xi_N^i(\xi')$ as an eigenvalue of $\mathcal{A}(\xi')$ is less than the algebraic multiplicity (see [17], proof of Theorem 4.7c). But in Remark (9.3) we describe how the results extend to the more general case where elliptic modes of possibly high multiplicity are required to be regular in the weaker sense of the following definition.

DEFINITION 3.5. – *The elliptic mode* $(\underline{\xi}', \underline{\xi}_N) = (\underline{\xi}_0, \underline{\eta}, \underline{\xi}_N)$ *is* weakly regular *if*

(a) *there are neighborhoods* $\omega$ *of* $(\underline{\eta}, \underline{\xi}_N)$ *in* $\mathbb{R}^{N-1} \times \mathbb{C}$ *and* $\mathcal{O}$ *of* $(\underline{\xi}_0, \underline{\eta}, \underline{\xi}_N)$ *in* $\mathbb{R}^N \times \mathbb{C}$ *such that for each* $(\eta, \xi_N) \in \omega$ *there is exactly one point* $(\xi_0(\eta, \xi_N), \eta, \xi_N) \in \mathcal{O} \cap \operatorname{char} L$,
(b) *the algebraic and geometric multiplicities of* $\underline{\xi}_N$ *as an eigenvalue of* $\mathcal{A}(\underline{\xi}')$ *are equal,*
(c) $A(\underline{\eta}, \underline{\xi}_N)$ *is diagonalizable.*

**G.** In the analysis of the glancing boundary layer we shall need to work with Wiener algebras $\mathbb{A}_\theta(C(\chi, H^m(T)))$ (7.7) which are invariant under the function $f(u, \overline{u})$ in (1.1). This will be the case if we assume $f(u, v): \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^m$ is entire with $f(0,0) = 0$. One can easily consider more general entire functions $f(x, u, v) = \sum_{(\alpha,\beta)\neq 0} f_{\alpha,\beta}(x) u^\alpha v^\beta$ by imposing appropriate restrictions on the coefficients $f_{\alpha,\beta}(x)$ (see [17], (2.7)), but we'll refrain from doing so here.

## 4. Three-scale profile equations and small divisors

Using the notation and definitions of Section 3, Part **F** we fix a regular boundary frequency $\xi'$ and look for solutions $\tilde{u}_\varepsilon$ to (1.1) of the form

$$\tilde{u}_\varepsilon(x) = \bigl[a_0(\chi, x, \theta) + \sqrt{\varepsilon}\, a_1(\chi, x, \theta) + \cdots + (\sqrt{\varepsilon})^M a_M(\chi, x, \theta)\bigr]\big|_{\chi = \frac{x_N}{\sqrt{\varepsilon}},\ \theta = \frac{\phi(x)}{\varepsilon}}. \tag{4.1}$$

where the $M(\xi')$-tuple of phases $\phi(x)$ and placeholder $\theta$ are defined as follows.

*Notation* 4.1. – For a fixed regular boundary frequency $\xi'$ and $j\in\mathcal{G}(\xi')\cup\mathcal{O}(\xi')\cup\mathcal{I}(\xi')\cup\mathcal{P}(\xi')$

(a) $\phi_j(x)=(\xi',\xi^j_N(\xi'))\cdot x=\beta_j(\xi')\cdot x$, $\xi^j_N$ as in Section 3, Part **F**.
(b) $r_j=r(\beta_j(\xi'))$, a basis vector for ker $L(\beta_j(\xi'))$,
(c) $\pi_j=\pi(\beta_j(\xi'))$, projection onto ker $L(\beta_j(\xi'))$ along range $L(\beta_j(\xi'))$,
(d) $X_j=X_{\beta_j(\xi')}$, the characteristic vector field associated to $\beta_j(\xi')$,
(e) $\theta_j$ is the placeholder for $\phi_j/\varepsilon$.
(f) For $j\in\mathcal{N}(\xi')$, set $\phi_j(x)=(-\xi',-\overline{\xi^k_N}(\xi'))\cdot x$ for $k=g^{-1}(j)$ ($g$ as in Proposition 3.1), $\beta_j(\xi')=d\phi_j$, $r_j=\overline{r}_k$, $\pi_j=\pi(\beta_j(\xi'))$, and $X_j=X_{\beta_j(\xi')}$.
(g) For $j\in\mathcal{N}(\xi')$ $\theta_j=-\overline{\theta}_k$ for $k=g^{-1}(j)$.
(h) Set $\theta=(\theta_j)_{j=1,\dots,M(\xi')}$.

For $i\in\mathcal{G}(\xi')\cup\mathcal{O}(\xi')\cup\mathcal{I}(\xi')$ $\theta_i\in\mathbb{R}$, while for $i\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi')$ we have Im $\theta_i\geqslant 0$. Observe that for $i\in\mathcal{M}(\xi')\setminus\mathcal{N}(\xi')$, $\phi_i$ restricts when $x_N=0$ to the boundary phase $\phi_0(x)\equiv\xi'\cdot x$, while for $i\in\mathcal{N}(\xi')$, $\phi_i$ restricts to $-\phi_0(x)$. We denote by $\theta_0$ the placeholder for $\phi_0(x)/\varepsilon$.

Each profile $a_j(\chi,x,\theta)$ is associated to a smooth periodic function $\tilde a_j(\chi,x,\tilde\theta)$, $\tilde\theta\in\mathbb{R}^{M(\xi')}$, whose Fourier expansion has the special form (dropping the $j$)

$$\tilde a(\chi,x,\tilde\theta)=\sum_{\alpha\in Z(\xi')}a_\alpha(\chi,x)e^{i\alpha\tilde\theta},\tag{4.2}$$

where $Z(\xi')\subset\mathbb{Z}^{M(\xi')}$ is defined by

$$\begin{aligned}Z(\xi')=\big\{\alpha=(\alpha_i)_{i\in\mathcal{M}(\xi')}\colon\ &\alpha_i\in\mathbb{Z}\text{ if }i\in\mathcal{G}(\xi')\cup\mathcal{O}(\xi')\cup\mathcal{I}(\xi');\\ &\alpha_i\in\mathbb{Z}^+\text{ if }i\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi')\big\}.\end{aligned}\tag{4.3}$$

Here $\mathbb{Z}^+=\{0,1,2,\dots\}$. Note that since spec $\tilde a\subset Z(\xi')$, $\tilde a$ extends holomorphically in the variables $(\tilde\theta_i)_{i\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi')}$ to

$$\big\{(\tilde\theta_i)_{i\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi')}\colon\ \mathrm{Im}\,\tilde\theta_i\geqslant 0\big\}.$$

Denoting the holomorphic extension also by $\tilde a$ we define, for $\theta$ as in (4.1)(h),

$$a(\chi,x,\theta)=\tilde a(\chi,x,\tilde\theta)|_{\tilde\theta=\theta}.\tag{4.4}$$

PROPOSITION 4.1. – (a) *For $a(\chi,x,\theta)$ as in* (4.4), *there exists a function $\tilde c(\chi,x,\tilde\theta)$ of the form* (4.2) *such that*

$$\overline{a}(\chi,x,\theta)=\tilde c(\chi,x,\tilde\theta)|_{\tilde\theta=\theta}.$$

(b) *Let $f(u_1,\dots,u_n):(\mathbb{C}^m)^n\to\mathbb{C}^m$ be entire and let*

$$a_i(\chi,x,\theta)=\tilde a_i(\chi,x,\tilde\theta)|_{\tilde\theta=\theta},\quad i=1,\dots,n,$$

*for $\tilde a_i(\chi,x,\tilde\theta)$ as in* (4.2). *Then there exists a function $\tilde b(\chi,x,\tilde\theta)$ of the form* (4.2) *such that*

$$f(a_1,\dots,a_n)=\tilde b(\chi,x,\tilde\theta)|_{\tilde\theta=\theta}.$$

*Proof.* – (a) Write

$$a(\chi,x,\theta)=\sum_{\alpha\in Z(\xi')}a_\alpha(\chi,x)e^{i\alpha\theta}$$

and note that since $\theta_{g(i)} = -\overline{\theta}_i$ when $i \in \mathcal{P}(\xi')$ for $g$ as in Proposition 3.1, we have

$$\overline{i\alpha\theta} = i\beta\theta, \quad \text{for } \beta \in Z(\xi')$$

where

$$\beta_i = -\alpha_i \in \mathbb{Z}, \quad i \in \mathcal{G}(\xi') \cup \mathcal{O}(\xi') \cup \mathcal{I}(\xi'),$$

$$\beta_i = \alpha_{g(i)} \in \mathbb{Z}^+, \quad i \in \mathcal{P}(\xi'),$$

$$\beta_{g(i)} = \alpha_i \in \mathbb{Z}^+, \quad i \in \mathcal{P}(\xi').$$

(b) Just take $\tilde{b} = f(\tilde{a}_1, \ldots, \tilde{a}_n)$. $\square$

*Remark* 4.1. – (a) In particular for $a(\chi, x, \theta)$ as in (4.4) and $f(u, v): \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^m$ entire, we have

$$f\big(a(\chi, x, \theta), \overline{a}(\chi, x, \theta)\big) = \tilde{b}(\chi, x, \tilde{\theta})|_{\tilde{\theta}=\theta} \tag{4.5}$$

for a function $\tilde{b}$ of the form (4.2).

(b) In Section 9 we work with profiles $a(\chi, x, \theta)$ of the form

$$a(\chi, x, \theta) = M_\chi a(\chi, x, \theta) + M_\infty a(x, \theta), \tag{4.6}$$

where $M_\chi a$ is rapidly decreasing in $\chi$ and $M_\infty a$ is independent of $\chi$. For $f$ as in (4.5), $f(a(\chi, x, \theta), \overline{a}(\chi, x, \theta))$ also has the form (4.6) (Proposition 5.1).

*Notation* 4.2. – For functions $a(\chi, x, \theta) = \tilde{a}(\chi, x, \tilde{\theta})|_{\tilde{\theta}=\theta}$ as in (4.4) we'll often write $\partial_{\theta_i} a(\chi, x, \theta)$ instead of $[\partial_{\tilde{\theta}_i} \tilde{a}(\chi, x, \tilde{\theta})]|_{\tilde{\theta}=\theta}$.

Equations for the profiles $a(\chi, x, \theta)$ are obtained by plugging (4.1) into (1.1), Taylor expanding nonlinear functions of $\tilde{u}_\varepsilon$ about $a_0$, and setting coefficients of different powers of $\varepsilon$ equal to zero. With $L(\partial)$ and $L(\xi)$ as in Section 3, Part **A**, set

$$\mathcal{P}(\partial_\theta) = \sum_{i \in \mathcal{M}(\xi')} L\big(\beta_i(\xi')\big)\partial_{\theta_i}. \tag{4.7}$$

Let $\mathbb{I}_k$ (respectively $\mathbb{B}_k$) be the interior (respectively boundary) equations obtained by setting the coefficient of $\varepsilon^k$ equal to zero. The $\mathbb{I}_k$ equations on $x_N > 0$ are:

$$\begin{aligned}
&\mathbb{I}_{-1}\colon\ \mathcal{P}(\partial_\theta)a_0 = 0,\\
&\mathbb{I}_{-\frac{1}{2}}\colon\ \mathcal{P}(\partial_\theta)a_1 + A_N\partial_\chi a_0 = 0,
\end{aligned} \tag{4.8}$$

$$\mathbb{I}_0\colon\ \mathcal{P}(\partial_\theta)a_2 + A_N\partial_\chi a_1 + L(\partial)a_0 = f(a_0),$$

$$\mathbb{I}_{\frac{1}{2}}\colon\ \mathcal{P}(\partial_\theta)a_3 + A_N\partial_\chi a_2 + L(\partial)a_1 = f'(a_0)a_1,$$

$$\mathbb{I}_1\colon\ \mathcal{P}(\partial_\theta)a_4 + A_N\partial_\chi a_3 + L(\partial)a_2 = f'(a_0)a_2 + f''(a_0)(a_1, a_1),$$

$$\mathbb{I}_k,\ k > 1\colon\ \mathcal{P}(\partial_\theta)a_{2k+2} + A_N\partial_\chi a_{2k+1} + L(\partial)a_{2k} = f'(a_0)a_{2k} + \mathcal{F}(a_0, \ldots, a_{2k-1}).$$

Here we use the shorthand $f(a_0)$ for $f(a_0, \overline{a}_0)$ and do similarly for higher derivatives. Below we let $s(\theta_0)$ denote the $M(\xi')$-tuple such that

$$s(\theta_0)_i = \theta_0 \quad \text{if } i \in \mathcal{M}(\xi') \setminus \mathcal{N}(\xi'),$$

$$s(\theta_0)_i = -\theta_0 \quad \text{if } i \in \mathcal{N}(\xi').$$

The boundary equations on $x_N = 0$ are:

$$\begin{aligned} &\mathbb{B}_0\colon\ Ba_0(\chi, x, \theta)|_{\chi=0, x_N=0, \theta=s(\theta_0)} = G(x', \theta_0), \\ &\mathbb{B}_{\frac{k}{2}}, k > 0\colon\ Ba_k(\chi, x, \theta)|_{\chi=0, x_N=0, \theta=s(\theta_0)} = 0. \end{aligned} \tag{4.9}$$

*Remark* 4.2. – If $\mathcal{G}(\xi') = \emptyset$, then all odd profiles turn out to vanish, all even profiles are independent of $\chi$, and the equations $(\mathbb{I}_k, \mathbb{B}_k)$ reduce to the usual 2-scale equations (see Section 9).

An important step in solving the $(\mathbb{I}_k, \mathbb{B}_k)$ is solving equations like

$$\mathcal{P}(\partial_\theta) a(\chi, x, \theta) = F(\chi, x, \theta)$$

in algebras of functions of the form (4.4). This is generally impossible because of the occurrence of small divisors. As in [10] we'll impose a small divisor condition, but here the condition is slightly modified to take into account the elliptic boundary layer. Observe that for $a(\chi, x, \theta)$ as in (4.4),

$$\mathcal{P}(\partial_\theta) a(\chi, x, \theta) = i \sum_{\alpha \in Z(\xi')} \mathcal{P}(\alpha) a_\alpha(\chi, x) e^{i\alpha\theta}, \tag{4.10}$$

where

$$\mathcal{P}(\alpha) = L(\alpha \cdot d\phi),\ \phi(x) \quad \text{as in (4.1)}. \tag{4.11}$$

Denote the characteristic set by

$$\mathcal{C} = \left\{\alpha \in \mathbb{Z}^{M(\xi')}\colon \det P(\alpha) = 0\right\}.$$

DEFINITION 4.1. – *The boundary frequency $\xi'$ (or phase $\phi$) satisfies the* small divisor condition *if there exist $C > 0$ and $d \in \mathbb{R}$ such that*

$$|d_x(\alpha \cdot \phi)| \geqslant C|\alpha|^d \quad \textit{for all } \alpha \in \mathbb{Z}^{M(\xi')} \setminus 0, \tag{4.12}$$

$$|\det \mathcal{P}(\alpha)| \geqslant C|\alpha|^d \quad \textit{for all } \alpha \in Z(\xi') \setminus \mathcal{C}. \tag{4.13}$$

In what follows we shall always work with a fixed regular boundary frequency $\xi'$ chosen so that

$$\xi' \text{ satisfies the small divisor condition.} \tag{4.14}$$

*Remark* 4.3. – (a) Property (4.12) implies that the component phases of $\phi$ in (4.1) are $\mathbb{Q}$-independent, a point that will be important in the solution of the profile equations. The $\mathbb{Q}$-independence would not follow if $\mathbb{Z}^{M(\xi')} \setminus 0$ were replaced by $Z(\xi') \setminus 0$ in (4.12). Situations where rational relations hold among just the real phases can be handled by choosing an appropriate *adapted basis* (Definition 8.2) for the $\mathbb{Q}$-span of the real phases. This is done in [20] in the context of multidimensional shocks. It is possible to have relations involving a mixture of real and nonreal phases. For example, an integer combination of glancing and elliptic phases can equal a different elliptic phase. We shall not treat such situations here.

(b) Suppose card $(\mathcal{O}(\xi') \cup \mathcal{I}(\xi')) = m$ for all $\xi'$ in some open subset $\Omega \subset \mathbb{R}^N \setminus 0$, so the modes associated to each $\xi' \in \Omega$ are all hyperbolic. Propositions implying the validity of (4.12), (4.13) for almost every $\xi' \in \Omega$ are given in Section 4 of [20]. Similarly, one can consider lower dimensional submanifolds $\tilde{\Omega} \subset \mathbb{R}^N \setminus 0$ such that a fixed number of glancing modes, and possibly elliptic or hyperbolic modes as well, are associated to each $\xi' \in \tilde{\Omega}$. Examples where the small divisor properties hold for almost every $\xi' \in \tilde{\Omega}$ are given in Section 11.

## 5. Function spaces

Here we define the spaces needed to state the main results.

Fix a regular boundary frequency $\xi' \in \mathbb{R}^N \setminus 0$ and let $M(\xi')$ be as in Section 3, Part **F**. Let

$$\Omega_T = \{x \in \mathbb{R}^{N+1}; -\infty < x_0 < T,\ x_N \geqslant 0\}, \qquad \Omega_{T,\theta} = \Omega_T \times \mathbb{T}^{M(\xi')},$$

and set

$$b\Omega_T = \{x' \in \mathbb{R}^N\colon\ -\infty < x_0 < T\}, \qquad b\Omega_{T,\theta_0} = b\Omega_T \times \mathbb{T}^1.$$

We regard the functions $\tilde{a}(\chi, x, \tilde{\theta})$ in (4.2) as functions on $\overline{\mathbb{R}}_+ \times \Omega_T \times \mathbb{T}^{M(\xi')}$.

### 5.1. Spaces for profiles

(a) $\tilde{\mathbb{C}}^\infty_\chi(T,\xi') = \{\tilde{a}(\chi,x,\tilde{\theta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega_T \times \mathbb{T}^{M(\xi')})\colon \tilde{a}$ has the special form (4.2) and $\partial^\alpha_{(\chi,x,\tilde{\theta})}\tilde{a}$ is bounded for each $\alpha\}$.
(b) $\tilde{\mathbb{C}}^\infty(T,\xi') = \{\tilde{a}(x,\tilde{\theta}) \in C^\infty(\Omega_T \times \mathbb{T}^{M(\xi')})\colon \tilde{a}$ has the form (4.2) and $\partial^\alpha_{(x,\tilde{\theta})}\tilde{a}$ is bounded for each $\alpha$.$\}$.
(c) $\tilde{\Gamma}^\infty_\chi(T,\xi') = \{\tilde{a}(\chi,x,\tilde{\theta}) \in \tilde{\mathbb{C}}^\infty_\chi(T,\xi')\colon \chi^k\partial^\alpha_{(\chi,x,\tilde{\theta})}\tilde{a}$ is bounded for each pair $(\alpha,k)$, $k \geqslant 0\}$.
(d) $\Gamma^\infty_\chi(T,\xi') = \{a(\chi,x,\theta)\colon a = \tilde{a}|_{\tilde{\theta}=\theta}$, with $\tilde{a} \in \tilde{\Gamma}^\infty_\chi(T,\xi')$ and $\theta$ as in (4.1)(h)$\}$.
(e) $\mathbb{C}^\infty(T,\xi') = \{a(x,\theta)\colon a = \tilde{a}|_{\tilde{\theta}=\theta}$, with $\tilde{a} \in \tilde{\mathbb{C}}^\infty(T,\xi')$ and $\theta$ as in (4.1)(h)$\}$.
(f) $\mathbb{P}_\chi(T,\xi') = \{a(\chi,x,\theta)\colon a = b + c$, where $b(\chi,x,\theta) \in \Gamma^\infty_\chi(T,\xi')$ and $c \in \mathbb{C}^\infty(T,\xi')\}$.
(g) $\mathcal{C}^\infty(T) = \{a(x',\theta_0) \in C^\infty(b\Omega_{T,\theta_0})\colon \partial^\alpha_{(x',\theta_0)}a$ is bounded for each $\alpha\}$.
In (5.1)(d) $\tilde{a}$ has been holomorphically extended as in (4.4).

PROPOSITION 5.1. – *Suppose $f(u_1,\ldots,u_n)\colon (\mathbb{C}^m)^n \to \mathbb{C}^m$ is entire. Let $a_1,\ldots,a_n \in \mathbb{P}_\chi(T,\xi')$. Then $f(a_1,\ldots,a_n) \in \mathbb{P}_\chi(T,\xi')$. In particular, if $f(u,v)$ is entire and $a \in \mathbb{P}_\chi(T,\xi')$, then $f(a,\overline{a}) \in \mathbb{P}_\chi(T,\xi')$.*

*Proof.* – Let $a_i = b_i + c_i$, where $b_i \in \Gamma^\infty_\chi(T,\xi')$ and $c_i \in \mathbb{C}^\infty(T,\xi')$. Write

$$f(a_1,\ldots,a_n) = \big[f(b_1+c_1,\ldots,b_n+c_n) - f(c_1,\ldots,c_n)\big] + f(c_1,\ldots,c_n),$$

rewrite the first term on the right using Taylor's theorem, and apply Proposition 4.1. □

### 5.2. Spaces on $\Omega_T$

Let $H^k(\Omega_T)$, $W^{m,\infty}(\Omega_T)$ be the usual Sobolev spaces. For $m \in \{0,1,2,\ldots\}$, $\rho > 0$, $\varepsilon \in (0,1]$ let

$$\text{(a)}\quad \mathbb{B}^m_\rho(T) = \{u_\varepsilon(x) \in H^m(\Omega_T)\colon |\partial^\alpha u_\varepsilon|_{L^2(\Omega_T)} \leqslant \rho\varepsilon^{-|\alpha|} \text{ for } |\alpha| \leqslant m,\ \varepsilon \in (0,1]\}.$$

$\mathcal{B}^m_\rho(T)$ is the analogous space on $b\Omega_T$.

(b) $\mathbb{D}^m_\rho(T)=\{u_\varepsilon(x)\in W^{m,\infty}(\Omega_T)\colon |\partial^\alpha u_\varepsilon|_{L^\infty(\Omega_T)}\leqslant \rho\varepsilon^{-|\alpha|} \text{ for } |\alpha|\leqslant m,\ \varepsilon\in(0,1]\}.$

$\mathcal{D}^m_\rho(T)$ is the analogous space on $b\Omega_T$.

*Notation* 5.1. – The subscript "0" attached to any of the function spaces defined in this paper, e.g., $\mathbb{P}_{\chi,0}(T,\xi')$, will be used to denote the subspace of functions vanishing in $x_0<0$.

DEFINITION 5.1. – *For functions $a(\chi,x,\theta)\in\mathbb{P}_\chi(T,\xi')$ we define projections $M_\infty a=\lim_{\chi\to\infty} a\in\mathbb{C}^\infty(T,\xi')$ and $M_\chi a=(1-M_\infty)a\in\Gamma^\infty_\chi(T,\xi')$.*

*Remark* 5.1. – (a) $a(\chi,x,\theta)\in\mathbb{P}_\chi(T,\xi')$ implies $a(x/\sqrt{\varepsilon},x,\phi(x)/\varepsilon)\in\mathbb{D}^m_\rho(T)$ for all $m$, for some $\rho=\rho(m)>0$.

(b) Spaces like $\mathbb{B}^m_\rho(T)$ and $\mathbb{D}^m_\rho(T)$ were used in [4].

*Notation* 5.2. – (a) For functions $\tilde a(\chi,x,\tilde\theta)\in\tilde{\mathbb{C}}^\infty_\chi(T,\xi')$ we set $\underline{\tilde a}(\chi,x)$ equal to the mean of $\tilde a$, and $\tilde a^*=\tilde a-\underline{\tilde a}$. For $a(\chi,x,\theta)=\tilde a|_{\tilde\theta=\theta}$, we set $\underline{a}(\chi,x)=\underline{\tilde a}$ and $a^*=a-\underline{a}$.

(b) If $B$ is any space of periodic functions, $B^*$ will denote the subspace of functions with mean 0.

## 6. Main results

We shall construct exact solutions $u_\varepsilon$ of (1.1) in $H^{m_1}(\Omega_T)$ where $m_1>(N+1)/2$ by first constructing an approximate solution $\tilde u_\varepsilon$ of the form (4.1) with $M$ terms where $(M-1)/2\geqslant m_1$, and then proving a general Gues-type theorem for boundary problems to obtain a unique exact solution $u_\varepsilon$ nearby. The smooth profiles $a_j$ in (4.1) will be constructed to lie in $\mathbb{P}_{\chi,0}(T,\xi')$ (Definition 5.1(f)).

The analysis of the profile equations produces detailed information about the interaction and evolution of the boundary layers. Some of that information is summarized in Remark 8.4.

THEOREM 6.1. – *Fix a regular boundary frequency $\xi'\in\mathbb{R}^N\setminus 0$ (Definition 3.4) satisfying the small divisor condition (Definition 4.1). Consider the problem (1.1) where $(L,B)$ as in Section 3A satisfies the uniform Kreiss condition (Definition 3.2), $f:\mathbb{C}^m\times\mathbb{C}^m\to\mathbb{C}^m$ is entire with $f(0,0)=0$, and $g_\varepsilon(x')=G(x',x'\cdot\xi'/\varepsilon)$, where $G(x',\theta_0)\in\mathcal{C}^\infty(T)$ has compact support in $x'$ and $G\equiv 0$ in $x_0<0$ (so $G\in\mathcal{C}^\infty_0(T)$). Choose $(M,m_1)$ such that $(M-1)/2\geqslant m_1>(N+1)/2$. There exist $T_0$, $0<T_0\leqslant T$, and profiles $a_j(\chi,x,\theta)\in\mathbb{P}_{\chi,0}(T_0,\xi')$, for $j=1,\dots,M$ such that $\tilde u_\varepsilon$ as in (4.1) satisfies*

$$\begin{aligned}
&L\tilde u_\varepsilon=f(\tilde u_\varepsilon,\overline{\tilde u}_\varepsilon)+\varepsilon^{(M-1)/2}R_\varepsilon,\\
&B\tilde u_\varepsilon|_{x_N=0}=g_\varepsilon+\varepsilon^{(M-1)/2}r_\varepsilon,\\
&\tilde u_\varepsilon=0\quad \textit{in } x_0<0,
\end{aligned}\tag{6.1}$$

*where $R_\varepsilon, r_\varepsilon$ lie in $\mathbb{D}^{m_1}_{\rho,0}(T_0)\cap\mathbb{B}^{m_1}_{\rho,0}(T_0)$, $\mathcal{D}^{m_1}_{\rho,0}(T_0)\cap\mathcal{B}^{m_1}_{\rho,0}(T_0)$ respectively for some $\rho>0$.*

*Remark* 6.1. – (a) In fact for all $m\geqslant 0$, $R_\varepsilon, r_\varepsilon$ lie in $\mathbb{D}^m_{\rho(m),0}(T_0)\cap\mathbb{B}^m_{\rho(m),0}(T_0)$, $\mathcal{D}^m_{\rho(m),0}(T_0)\cap\mathcal{B}^m_{\rho(m),0}(T_0)$ respectively for some $\rho(m)>0$.

(b) Corner compatibility conditions in (6.1) hold to infinite order since $G\in\mathcal{C}^\infty_0(T)$ and $f(0,0)=0$.

**Exact solutions near approximate ones**

The following general Gues-type theorem also applies to boundary problems not involving oscillatory data.

THEOREM 6.2. – *Let* $(L,B)$ *and* $f$ *be as in Theorem* 6.1 *and choose* $M_1 \geqslant m_1 > (N+1)/2$. *Suppose* $\tilde{u}_\varepsilon \in \mathbb{D}^{m_1}_{\rho,0}(T)$ *satisfies*

$$\text{(a)}\quad \begin{aligned} &L\tilde{u}_\varepsilon = f(\tilde{u}_\varepsilon, \overline{\tilde{u}}_\varepsilon) + \varepsilon^{M_1} R_\varepsilon,\\ &B\tilde{u}_\varepsilon|_{x_N=0} = g_\varepsilon + \varepsilon^{M_1} r_\varepsilon,\\ &\tilde{u}_\varepsilon = 0 \quad \text{in } x_0 < 0, \end{aligned}$$

*where* $R_\varepsilon \in \mathbb{B}^{m_1}_{\rho,0}(T)$, $r_\varepsilon \in \mathcal{B}^{m_1}_{\rho,0}(T)$ *for some* $\rho > 0$.

*Then there exist* $\varepsilon_0 > 0$ *and* $\sigma > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *the problem*

$$\text{(b)}\quad \begin{aligned} &Lu_\varepsilon = f(u_\varepsilon, \overline{u}_\varepsilon),\\ &Bu_\varepsilon|_{x_N=0} = g_\varepsilon,\\ &u_\varepsilon = 0 \quad \text{in } x_0 < 0, \end{aligned}$$

*has a unique exact solution* $u_\varepsilon \in \tilde{u}_\varepsilon + \varepsilon^{M_1}\mathbb{B}^{m_1}_{\sigma,0}(T)$.

THEOREM 6.3 (Exact oscillatory solutions). – *Let* $(L,B)$, $f$, $g_\varepsilon$, $(M,m_1)$, $T_0$, *and the profiles* $a_j$ *be just as in Theorem* 6.1. *There exist* $\varepsilon_0 > 0$ *and* $\sigma > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$, *the problem* (1.1) *has a unique exact solution in* $H^{m_1}(\Omega_{T_0})$ *given by* $u_\varepsilon = \tilde{u}_\varepsilon + \varepsilon^{(M-1)/2}\mathbb{B}^{m_1}_{\sigma,0}(T)$ *where*

$$\tilde{u}_\varepsilon = a_0\left(\frac{x_N}{\sqrt{\varepsilon}}, x, \frac{\phi}{\varepsilon}\right) + \cdots + \left(\sqrt{\varepsilon}\right)^M a_M\left(\frac{x_N}{\sqrt{\varepsilon}}, x, \frac{\phi}{\varepsilon}\right). \tag{6.2}$$

*Proof.* – We have $\tilde{u}_\varepsilon \in \mathbb{D}^{m_1}_{\rho,0}(T_0)$, so (6.1) implies the hypotheses of Theorem 6.2 are satisfied with $M_1 = (M-1)/2$. An application of that theorem concludes the proof. □

*Remark* 6.2. – (a) A standard continuation principle and uniqueness for (1.1) imply that the solution $u_\varepsilon$ in Theorem 6.3 is in fact $C^\infty$.

(b) Theorems 6.1, 6.2, and 6.3 remain true exactly as stated when $f(u_\varepsilon, \overline{u}_\varepsilon)$ in (1.1) is replaced by certain more general forcing terms. See Remark 8.5.

(c) Cases where there are nontrivial rational relations (hence *resonances*) among the components of $\phi$, but which involve only the real phases, can be handled similarly using *adapted bases*. See Section 8.2.

(d) The treatment of boundary frequencies $\xi'$ for which the corresponding elliptic modes are of possibly high multiplicity, but still weakly regular in the sense of Definition 3.5, is described in Remark 9.3.

## 7. Spaces for constructing the glancing boundary layer

The leading term of the glancing boundary layer will be written as a sum of two pieces, $b_0 + c_0$ (8.35), (8.36). Different spaces are used in the Picard iterations for $b_0$ and $c_0$. Recall the notation $x = (x_0, x'') = (x_0, y, x_N) = (x', x_N)$ for $x \in \Omega_T$.

### 7.1. $b_0$-spaces

For $m \in \{0,1,2,\dots\}$, $T>0$, $\gamma>0$ let

(7.1)
$$H^m(T)=\Big\{u(x)\colon |u|_{m,T}=\sum_{|\alpha|\leqslant m}|\partial^\alpha u|_{L^2(\Omega_T)}<\infty\Big\}.$$

(7.2)
$$H^m_\gamma=\Big\{u(x)\colon |u|_{m,\gamma}=\sum_{|\alpha|\leqslant m}\gamma^{m-|\alpha|}|e^{-\gamma x_0}\partial^\alpha u|_{L^2(\overline{\mathbb{R}}^{N+1}_+)}<\infty\Big\}.$$

(7.3)
$$H^m_\gamma(T)=\Big\{u(x)\colon |u|_{m,\gamma,T}=\sum_{|\alpha|\leqslant m}\gamma^{m-|\alpha|}|e^{-\gamma x_0}\partial^\alpha u|_{L^2(\Omega_T)}<\infty\Big\}.$$

(7.4)
$$C\big(\chi,H^m(T)\big)=\big\{u(\chi,x)\colon u \text{ is a continuous, bounded function of } \chi\geqslant 0 \text{ with values in } H^m(T)\big\},$$
$$|u|_{\chi,m,T}=\sup_{\chi\geqslant 0}\big|u(\chi,\cdot)\big|_{m,T}.$$

(7.5)
$$C^k\big(\chi,H^m(T)\big)=\big\{u(\chi,x)\colon \partial^l_\chi u\in C(\chi,H^m(T)),\ l\leqslant k\big\}.$$

(7.6) The spaces and norms $C(\chi,H^m_\gamma)$, $|u|_{\chi,m,\gamma}$ and $C\big(\chi,H^m_\gamma(T)\big)$, $|u|_{\chi,m,\gamma,T}$ are defined similarly.

**Wiener algebras $\mathbb{A}_\theta(B)$.** It was observed in [9] that Wiener algebras are useful for proving estimates on profiles by mode-by-mode analysis. For a Banach space $B$ $\mathbb{A}_\theta(B)$ is the space of periodic $B$-valued functions of $\theta\in\mathbb{R}^p$ with absolutely summable Fourier coefficients. Here and elsewhere we suppress the "$p$" in the notation. Thus $V\in\mathbb{A}_\theta(B)$ if and only if

(7.7)
$$V=\sum_{n\in\mathbb{Z}^p}V_n e^{in\theta}\quad \text{with } |V|_{\mathbb{A}_\theta(B)}=|V|_{\theta;B}\equiv\sum_n|V|_B<\infty.$$

If $k\geqslant 1$ we define

$$\mathbb{A}^k_\theta(B)=\big\{V\colon \partial^\alpha_\theta V\in\mathbb{A}_\theta(B)\text{ for }|\alpha|\leqslant k\big\}$$

with

(7.8)
$$|V|_{\mathbb{A}^k_\theta(B)}=|V|_{k,\theta;B}\equiv\sum_n\langle n\rangle^k|V_n|_B.$$

A simple application of the triangle inequality shows that if $B$ is a Banach algebra satisfying $|uv|_B\leqslant C|u|_B|v|_B$ for some $C>0$, then for the same $C$

(7.9)
$$|UV|_{\theta;B}\leqslant C|U|_{\theta;B}|V|_{\theta;B}.$$

In particular (7.9) holds for $B=H^m(T)$ or $C(\chi,H^m(T))$ when $m>(N+1)/2$. The corresponding $|U|_{\theta;B}$ norms are written $|U|_{\theta;m,T}$, $|U|_{\theta;\chi,m,T}$ respectively.

Suppose $u,v\in C(\chi,H^m(T))\cap L^\infty(\overline{\mathbb{R}}_+\times\Omega_T)$ for some $m\geqslant 0$. We have the weighted Moser-type inequality [15]

(7.10)
$$|uv|_{\chi,m,\gamma,T}\leqslant C\big(|u|_{\chi,m,\gamma,T}|v|_{L^\infty}+|u|_{L^\infty}|v|_{\chi,m,\gamma,T}\big).$$

(7.10) implies, with the obvious notation,

$$|UV|_{\theta;\chi,m,\gamma,T} \leqslant C\big(|U|_{\theta;\chi,m,\gamma,T}|V|_{\theta;L^\infty} + |U|_{\theta;L^\infty}|V|_{\theta;\chi,m,\gamma,T}\big). \tag{7.11}$$

Suppose $f:\mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^m$ is entire with $f(0,0)=0$. If $B$ is a Banach algebra of $\mathbb{C}$-valued functions, straightforward use of the triangle inequality shows the map $u \to f(u,\overline{u})$ sends $B$ to itself and is bounded on bounded sets in $B$. Similarly, (7.9) implies the map

(7.12) $U \to f(U,\overline{U})$ sends $\mathbb{A}_\theta(B)$ to itself and is bounded on bounded sets in $\mathbb{A}_\theta(B)$.

For $U \in \mathbb{A}_\theta(C(\chi, H^m(T))) \cap L^\infty(\overline{\mathbb{R}}_+ \times \Omega_T))$, $m \geqslant 0$, (7.11) implies by the same argument

$$\big|f(U)\big|_{\theta;\chi,m,\gamma,T} \leqslant C\big(|U|_{\theta;L^\infty}\big)\big(|U|_{\theta;\chi,m,\gamma,T}\big). \tag{7.13}$$

### 7.2. $c_0$-spaces

With $a = \underline{a} + a^*$ as in Notation 5.2, set

$$L^{2*} = \big\{a(\chi,x'',\theta) \in L^2\big(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+^N \times \mathbb{T}^1\big)\colon \underline{a} = 0\big\}. \tag{7.14}$$

Let $\partial_\theta^{-1}$ be the map $L^{2*} \to L^{2*}$ which assigns to $a(\chi,x'',\theta)$ its periodic primitive. Then $D_\theta^{-1} = i\partial_\theta^{-1}$ is bounded and self-adjoint on $L^{2*}$. The operator $D_\chi^2 D_\theta^{-1}$, which will appear later in Schrödinger-type profile equations, is an unbounded self-adjoint operator on $L^{2*}$ with domain

$$\begin{aligned} D^* = \big\{a(\chi,x'',\theta) \in L^{2*}\colon\ & a \text{ is an } L^2 \text{ function of } (x'',\theta) \\ & \text{with values in } H^2(\overline{\mathbb{R}}_+) \cap H_0^1(\overline{\mathbb{R}}_+)\big\}. \end{aligned} \tag{7.15}$$

For $m \geqslant 0$ let

$$\Gamma^m = \left\{a(\chi,x'',\theta) \in L^2\big(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+^N \times \mathbb{T}^p\big)\colon |a|_{\Gamma^m} = \sum_{|\alpha,k| \leqslant m} \big|\chi^k \partial^\alpha_{(\chi,x'',\theta)} a\big|_{L^2} < \infty\right\}, \tag{7.16}$$

$$\Gamma^{m,*} = \big\{a(\chi,x'',\theta)\colon \chi^k \partial_\chi^l a \in D^*,\ |k,l| \leqslant m\big\}. \quad (\text{Here } \theta \in \mathbb{T}^1). \tag{7.17}$$

Spaces like $\Gamma^m$ were used in [3]. The following Proposition is an immediate consequence of Lemma 4.2 of [3].

PROPOSITION 7.1 (Moser inequality for $\Gamma^m$ spaces). – (a) *There exists $C$ such that for $a, b \in \Gamma^m \cap L^\infty$*

$$|ab|_{\Gamma^m} \leqslant C\big(|a|_{\Gamma^m}|b|_{L^\infty} + |a|_{L^\infty}|b|_{\Gamma^m}\big). \tag{7.18}$$

(b) *Suppose $f$ is $C^\infty$ and $a, b \in \Gamma^m$ satisfy for some $R > 0$*

$$|b|_{L^\infty} \leqslant R \quad \textit{and} \quad \big|\chi^k \partial^\alpha_{(\chi,x'',\theta)} a\big|_{L^\infty} \leqslant R \quad \textit{for all } |\alpha,k| \leqslant m. \tag{7.19}$$

*Then there is a constant $C(m,R,f)$ such that*

$$\big|f(a+b,\overline{a}+\overline{b}) - f(a,\overline{a})\big|_{\Gamma^m} \leqslant C|b|_{\Gamma^m}. \tag{7.20}$$

*In particular, if $f(0,0)=0$ we have*

$$\big|f(b,\overline{b})\big|_{\Gamma^m} \leqslant C\big(|b|_{L^\infty}\big)|b|_{\Gamma^m}. \tag{7.21}$$

For $m \geqslant 0$, $T > 0$ let

$$C(T,\Gamma^m) = \{a(\chi,x,\theta)\colon\ a \text{ is a bounded, continuous function of } x_0 \in (-\infty,T] \text{ with values in } \Gamma^m\},$$

$$|a|_{T,\Gamma^m} = \sup_{x_0\in(-\infty,T]} |a(\cdot,x_0,\cdot)|_{\Gamma^m}. \tag{7.22}$$

Spaces $C^k(T,\Gamma^m)$ are defined in the obvious way.

*Remark* 7.1. – Because of the initial condition in (1.1), functions with nonconstant dependence on $x_0$ in the following sections will always vanish identically in $x_0 < 0$.

## 8. Discussion of the proofs

### 8.1. Discussion of Theorem 6.1

We shall analyze the profile equations (4.8), (4.9) using the projections $M_\chi$, $M_\infty$ from Definition 5.1, as well as operators $\mathbb{E}$ which project onto ker $\mathcal{P}(\partial_\theta)$ and approximate inverses $\mathcal{Q}(\partial_\theta)$ such that $\mathcal{QP} = \mathcal{PQ} = I - \mathbb{E}$.

Recall the characteristic set

$$\mathcal{C} = \{\alpha \in \mathbb{Z}^{M(\xi')}\colon\ \det L(\alpha\cdot d\phi) = 0\}, \tag{8.1}$$

and for $j = 1,\dots,M(\xi')$ let

$$\mathcal{C}_j = \{\alpha \in \mathbb{Z}^{M(\xi')}\colon\ \alpha\cdot\phi \text{ is in the } \mathbb{R}\text{-span of } \phi_j\}. \tag{8.2}$$

We have $\mathcal{C} = \bigcup_{j=1}^{M(\xi')}\mathcal{C}_j$ and in view of the first small divisor condition (4.12),

$$\mathcal{C}_j = \{(0,\dots,0,\alpha_j,0,\dots,\dots,0) \in \mathbb{Z}^{M(\xi')}\colon\ \alpha_j \in \mathbb{Z}\}.$$

For $\alpha \in \mathbb{Z}^{M(\xi')}$ define $\pi_\alpha : \mathbb{C}^m \to \mathbb{C}^m$ by

$$\begin{aligned} &\text{(a)}\quad \pi_\alpha = 0 \quad \text{if } \alpha \notin \mathcal{C}, \qquad \pi_\alpha = \mathrm{Id} \quad \text{if } \alpha = 0.\\ &\text{(b)}\quad \text{If } \alpha \in \mathcal{C}_j\setminus 0, \quad \pi_\alpha = \pi_j. \end{aligned} \tag{8.3}$$

Next define $\mathbb{E} : \mathbb{P}_\chi(T,\xi') \to \mathbb{P}_\chi(T,\xi')$ (notation as in (f) of Section 5.1) by the following action on monomials.

$$\mathbb{E}\big(U_\alpha(\chi,x)e^{i\alpha\theta}\big) = (\pi_\alpha U_\alpha)e^{i\alpha\theta}, \quad \alpha \in Z(\xi'). \tag{8.4}$$

Note that $\mathbb{E} = \mathbb{E}_0 + \sum_{j=1}^{M(\xi')}\mathbb{E}_j$ where

$$\begin{aligned} &\mathbb{E}_j\big(U_\alpha e^{i\alpha\theta}\big) = (\pi_\alpha U_\alpha)e^{i\alpha\theta} \quad \text{if } \alpha \in \mathcal{C}_j\setminus 0, \qquad \mathbb{E}_j\big(U_\alpha e^{i\alpha\theta}\big) = 0 \quad \text{otherwise;}\\ &\mathbb{E}_0\big(U_\alpha e^{i\alpha\theta}\big) = U_\alpha \quad \text{if } \alpha = 0, \qquad \mathbb{E}_0\big(U_\alpha e^{i\alpha\theta}\big) = 0 \quad \text{otherwise.} \end{aligned} \tag{8.5}$$

By an argument of [10] the small divisor property (4.14) implies the existence of operators

$$\mathcal{Q}(\partial_\theta) : \mathbb{P}_\chi(T,\xi') \to \mathbb{P}_\chi(T,\xi')$$

satisfying

$$\text{(8.6)}\qquad \begin{aligned} &\text{(a)}\quad \mathcal{P}(\partial_\theta)\mathcal{Q}=\mathcal{Q}\mathcal{P}=I-\mathbb{E},\\ &\text{(b)}\quad \operatorname{range}\mathbb{E}=\ker\mathcal{P}=\ker\mathcal{Q},\\ &\text{(c)}\quad \ker\mathbb{E}=\operatorname{range}\mathcal{P}=\operatorname{range}\mathcal{Q}.\end{aligned}$$

Indeed, set $\mathcal{Q}(\partial_\theta)(U_\alpha e^{i\alpha\theta})=-i\mathcal{Q}(\alpha)U_\alpha e^{i\alpha\theta}$ where

$$\text{(8.7)}\qquad \begin{aligned} &\text{(a)}\quad \mathcal{Q}(\alpha)=L^{-1}(\alpha\cdot d\phi)\quad \text{if } \alpha\notin\mathcal{C},\\ &\text{(b)}\quad \mathcal{Q}(0)=0,\\ &\text{(c)}\quad \mathcal{Q}(\alpha)=\sum_{k\neq p}\frac{1}{\xi_0^p(\underline{\xi}'')-\xi_0^k(\underline{\xi}'')}\tilde{\pi}_k(\underline{\xi}'')\quad \text{if } \alpha\cdot d\phi=\big(\xi_0^p(\underline{\xi}''),\underline{\xi}''\big),\end{aligned}$$

where $-\xi_0^k(\underline{\xi}'')$, $k=1,\dots,l$, are the eigenvalues of $A(\underline{\xi}'')$, and $\tilde{\pi}_k$ the corresponding spectral projections. (8.6)(a) follows from

$$\text{(8.8)}\qquad \mathcal{P}(\alpha)\mathcal{Q}(\alpha)=\mathcal{Q}(\alpha)\mathcal{P}(\alpha)=1-\tilde{\pi}_p(\underline{\xi}'')=1-\pi_\alpha$$

with $\mathcal{P}(\alpha)$ as in (4.11).

**Decomposition of the profile equations**

In order to make use of the projections $M_\chi$, $M_\infty$ (Definition 5.1) to break up the profile equations into manageable pieces, we'll assume for the moment that solutions $a_k(\chi,x,\theta)\in\mathbb{P}_{\chi,0}(T_0,\xi')$ to (4.8), (4.9) do exist for some $T_0>0$. That assumption is verified to be true in Section 9.

Apply $\mathcal{Q}$ to the profile equation $\mathbb{I}_{-1}$ to obtain

$$\text{(8.9)}\qquad (I-\mathbb{E})a_0=0.$$

For all $k$ let us use

$$\mathbb{E}=\mathbb{E}_0+\sum_{j=1}^{M(\xi')}\mathbb{E}_j;\qquad I=M_\chi+M_\infty$$

to write

$$\text{(8.10)}\qquad \begin{aligned} &(\mathbb{E}a_k)(\chi,x,\theta)=\underline{a}_k(\chi,x)+\sum_{j=1}^{M(\xi')}\sigma_{k,j}(\chi,x,\theta_j)r_j,\\ &a_k=M_\infty a_k+M_\chi a_k.\end{aligned}$$

Applying $\mathbb{E}_0$, $\mathbb{E}_j$, and $\mathcal{Q}$ to $\mathbb{I}_{-\frac{1}{2}}$ we find, respectively,

$$\text{(8.11)}\qquad \begin{aligned} &\text{(a)}\quad A_N\partial_\chi\underline{a}_0=0\quad \text{so } \underline{a}_0=M_\infty\underline{a}_0,\\ &\text{(b)}\quad \partial_\chi\sigma_{0,j}=0\quad \text{for } j\in\mathcal{M}(\xi')\setminus\mathcal{G}(\xi'),\\ &\text{(c)}\quad (I-\mathbb{E})a_1=-\mathcal{Q}A_N\partial_\chi a_0.\end{aligned}$$

In (8.11)(b) we've used (3.7). Note that for $j\in\mathcal{G}(\xi')$, $\mathbb{E}_jA_N\mathbb{E}_j=0$.

Next apply $\mathbb{E}_0$, $M_\infty\mathbb{E}_0$, $\mathbb{E}_j$, and $M_\infty\mathbb{E}_j$ to $\mathbb{I}_0$ to obtain, respectively,

$$\text{(8.12)}\qquad \begin{aligned}&\text{(a)}\quad A_N\partial_\chi \underline{a}_1 + L(\partial)\underline{a}_0 = \underline{f(a_0)},\\ &\text{(b)}\quad L(\partial)\underline{a}_0 = M_\infty \underline{f(a_0)},\end{aligned}$$

$$\text{(c)}\quad \mathbb{E}_j A_N\partial_\chi a_1 + X_j\sigma_{0,j}r_j = \mathbb{E}_j f(a_0) \quad \text{(recall (3.7))}$$

$$\text{(d)}\quad X_j M_\infty \sigma_{0,j}r_j = M_\infty \mathbb{E}_j f(a_0).$$

The first observation is that (8.12)(b),(d) constitute a coupled semilinear system for $\underline{a}_0$, $M_\infty\sigma_{0,j}$, $j\in\mathcal{G}(\xi')\cup\mathcal{O}(\xi')\cup\mathcal{I}(\xi')$ since for any $k\in\mathcal{G}(\xi')\cup\mathcal{O}(\xi')\cup\mathcal{I}(\xi')$ the functions

$$\text{(8.13)}\qquad \begin{aligned}&M_\infty\underline{f(a_0)},\quad M_\infty\mathbb{E}_k f(a_0)\quad \text{depend only on } \underline{a}_0, M_\infty\sigma_{0,j},\\ &j\in\mathcal{G}(\xi')\cup\mathcal{O}(\xi')\cup\mathcal{I}(\xi')\end{aligned}$$

(see Proposition 9.1). They are independent of

$$\sigma_{0,j},\ j\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi') \quad\text{and}\quad M_\chi\sigma_{0,j},\ j\in\mathcal{G}(\xi').$$

To find the corresponding boundary conditions use the boundary equation $\mathbb{B}_0$ to write (with $s(\theta_0)$ as in (4.9))

$$\text{(8.14)}\qquad \begin{aligned}&\text{(a)}\quad B\underline{a}_0|_{x_N=0} = \underline{G}(x'),\\ &\text{(b)}\quad B\left(\sum_{j=1}^{M(\xi')}\sigma_{0,j}r_j\right)\Big|_{\chi=0,\ x_N=0,\ \theta=s(\theta_0)} = G^*(x',\theta_0).\end{aligned}$$

**Determination of boundary data for $a_0^*$**

We now show how the Kreiss condition and Proposition 3.1 are used to obtain the boundary values of

$$a_0^* = \sum_{j\in\mathcal{M}(\xi')\setminus\mathcal{G}(\xi')}\sigma_{0,j}(x,\theta_j)r_j + \sum_{j\in\mathcal{G}(\xi')}\sigma_{0,j}(\chi,x,\theta_j)r_j$$

from (8.14)(b). Letting $\sigma_{0,j,n}$, $G_n^*$ denote the $n$th Fourier coefficients of $\sigma_{0,j}$ (respectively $G^*$), we obtain the following equations from (8.14)(b). For $n>0$, $\chi=0$, $x_N=0$,

$$\text{(8.15)}\qquad B\Bigg(\sum_{j\in\mathcal{G}\cup\mathcal{O}\cup\mathcal{P}}\sigma_{0,j,n}r_j\Bigg) = G_n^* - B\sum_{j\in\mathcal{I}}\sigma_{0,j,n}r_j,$$

while for $n<0$, $\chi=0$, $x_N=0$

$$\text{(8.16)}\qquad B\Bigg(\sum_{j\in\mathcal{G}\cup\mathcal{O}}\sigma_{0,j,n}r_j + \sum_{j\in\mathcal{N}}\sigma_{0,j,-n}r_j\Bigg) = G_n^* - B\Bigg(\sum_{j\in\mathcal{I}}\sigma_{0,j,n}r_j\Bigg)$$

(recall $s(\theta_0)_j = -\theta_0$ for $j\in\mathcal{N}$ since $\phi_j|_{x_N=0} = -\phi_0$ for $j\in\mathcal{N}$).

The Kreiss condition implies $\{Br_j\}_{j\in\mathcal{G}\cup\mathcal{O}\cup\mathcal{P}}$ is a basis for $\mathbb{C}^\mu$, so the Fourier coefficients on the left in (8.15) can be expressed as a linear function of the coefficients on the right. Similarly, Proposition 3.1 implies $\{Br_j\}_{j\in\mathcal{G}\cup\mathcal{O}\cup\mathcal{N}}$ is a basis of $\mathbb{C}^\mu$, so the Fourier coefficients on the left in (8.16) can be expressed as a linear function of the coefficients on the right.

If $b$ is a periodic function of $\theta_0$ with $\underline{b}=0$, let $b^\pm$ denote the pieces with positive (respectively negative) spectra. The above discussion implies in particular the existence of constant matrices

$\mathbb{M}^{\pm}$ such that

$$(8.17)\qquad \big(M_\infty\sigma^{\pm}_{0,j}|_{x_N=0,\ \theta_j=\theta_0},\ j\in\mathcal{O}(\xi')\big)=\mathbb{M}^{\pm}\big(G^{*,\pm},\sigma^{\pm}_{0,j}|_{x_N=0,\ \theta_j=\theta_0},\ j\in\mathcal{I}(\xi')\big).$$

In the solution of the semilinear system (8.12)(b),(d) the boundary values of $M_\infty\sigma_{0,j}$, $j\in\mathcal{G}(\xi')\cup\mathcal{I}(\xi')$ are determined by transport. Kreiss-type estimates (Lemma 9.2) for the linearized system are readily obtained since $(L,B)$ satisfies the Kreiss condition and the $X_j$ are just vector fields.

After this semilinear system is solved by iteration, the values of

$$\sigma_{0,j}|_{\chi=0,\ x_N=0},\ j\in\mathcal{G}(\xi')\quad\text{and}\quad M_\infty\sigma_{0,j}|_{x_N=0},\ j\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi')$$

are then read off from (8.15) and (8.16) as explained in the above discussion. For $j\in\mathcal{G}(\xi')$ set

$$M_\chi\sigma_{0,j}|_{\chi=0,\ x_N=0}=(\sigma_{0,j}-M_\infty\sigma_{0,j})|_{\chi=0,\ x_N=0}.$$

To complete the determination of $a_0$ it remains to find

$$\sigma_{0,j}=M_\infty\sigma_{0,j},\ j\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi')\quad\text{and}\quad M_\chi\sigma_{0,j},\ j\in\mathcal{G}(\xi').$$

The elliptic modes are determined by solving the coupled system (8.12)(d) to sufficiently high order at $x_N=0$ (see Proposition 9.4) with the boundary data already determined.

*Remark* 8.1. – For $k\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi')$ $M_\infty\mathbb{E}_k f(a_0)$ depends on $\underline{a}_0$, $M_\infty\sigma_{0,j}$, $j\in\mathcal{G}(\xi')\cup\mathcal{O}(\xi')\cup\mathcal{I}(\xi')$, which have already been determined, as well as $M_\infty\sigma_{0,j}$, $j\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi')$. $M_\infty\mathbb{E}_k f(a_0)$ is independent of $M_\chi\sigma_{0,j}$, $j\in\mathcal{G}(\xi')$ (Proposition 9.3).

The leading part of the elliptic boundary layer is given by

$$(8.18)\qquad \sum_{j\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi')}\sigma_{0,j}(x,\theta_j)r_j|_{\theta_j=\frac{\phi_j}{\varepsilon}}.$$

**Leading part of the glancing boundary layer**

To determine $M_\chi\sigma_{0,j}$, $j\in\mathcal{G}(\xi')$ first apply $M_\chi$ to (8.12)(c) and then rewrite it using (8.11) and $\mathbb{E}_jA_N\mathbb{E}_j=0$ (for $j\in\mathcal{G}(\xi')$) as

$$(8.19)\qquad \mathbb{E}_jA_N\mathcal{Q}A_N\partial_\chi^2(M_\chi\sigma_{0,j})r_j+X_jM_\chi\sigma_{0,j}r_j=M_\chi\mathbb{E}_jf(a_0).$$

Write $M_\chi\sigma_{0,j}(\chi,x,\theta_j)=\sum_n U_n(\chi,x)e^{in\theta_j}$ and observe that

$$(8.20)\qquad \mathcal{Q}(\partial_\theta)U_ne^{in\theta_j}=-i\mathcal{Q}(n\underline{\alpha}_j)U_ne^{in\theta_j}=-i\mathcal{Q}(\underline{\alpha}_j)D^{-1}_{\theta_j}\big(U_ne^{in\theta_j}\big)$$

where $\underline{\alpha}_j=(0,\dots,0,1,0,\dots,0)$ with 1 in the $j$th slot. The following Lemma proved in Section 9 implies that the first term in (8.19) is also given by a *scalar* operator.

LEMMA 8.1. – *For $j\in\mathcal{G}(\xi')$ let $(\xi',\xi_N^j(\xi'))=(\xi_0^k(\xi''),\xi'')$, with $\xi_0^k$ as in* (3.4). *We have*

$$(8.21)\qquad \pi_jA_N\mathcal{Q}(\underline{\alpha}_j)A_N\pi_j=\frac{1}{2}\pi_j\frac{\partial^2\xi_0^k}{\partial\xi_N^2}(\xi'')\pi_j\equiv\frac{1}{c_j}\pi_j.$$

$c_j\in\mathbb{R}\setminus 0$ *since $(\xi',\xi_N^j(\xi'))$ is a glancing mode of order* 2.

For $j \in \mathcal{G}(\xi')$, $c_j$ as in (8.21) define the scalar operator

$$\mathcal{L}_j = -i\frac{D_\chi^2 D_{\theta_j}^{-1}}{c_j} + X_j. \tag{8.22}$$

(8.19) becomes

$$\mathcal{L}_j(M_\chi \sigma_{0,j})r_j = M_\chi \mathbb{E}_j f(a_0), \quad j \in \mathcal{G}(\xi'). \tag{8.23}$$

(8.23) is a coupled system for the $M_\chi \sigma_{0,j}$, $j \in \mathcal{G}(\xi')$ with boundary data determined by (8.14)(b) as described above, whose solution we'll discuss shortly.

*Remark* 8.2. – (a) The nonlinear term $M_\chi \mathbb{E}_j f(a_0)$ in (8.23) depends only on the known functions

$$\underline{a}_0, \quad \sigma_{0,k} = M_\infty \sigma_{0,k}, \quad k \in \mathcal{O}(\xi') \cup \mathcal{I}(\xi'), \qquad M_\infty \sigma_{0,k}, \quad k \in \mathcal{G}(\xi'),$$

and the unknowns $M_\chi \sigma_{0,k}$, $k \in \mathcal{G}(\xi')$ (Proposition 9.5).

(b) The value of $M_\chi \sigma_{0,j}$, $j \in \mathcal{G}$ at $\chi = 0$ is independent of $x_N$. The $M_\chi \sigma_{0,j}$ inherit nontrivial $x_N$ dependence in $\chi > 0$ from the forcing term.

The leading part of the glancing boundary layer is given by

$$\sum_{j \in \mathcal{G}(\xi')} M_\chi \sigma_{0,j}(\chi, x, \theta_j) r_j \big|_{\chi = \frac{x_N}{\sqrt{\varepsilon}},\ \theta_j = \frac{\phi_j}{\varepsilon}}. \tag{8.24}$$

**Determination of** $a_1, a_2, \ldots, a_M$

With $a_0 \in \mathbb{P}_\chi(T_0, \xi')$ thereby constructed, we read off $(I - \mathbb{E})a_1$ from (8.11)(c).

$\partial_\chi \underline{a}_1 \in \Gamma_\chi^\infty(T_0, \xi')$ is determined by (8.12)(a) and we recover $M_\chi \underline{a}_1 \in \Gamma_\chi^\infty(T_0, \xi')$ using the following obvious Lemma.

LEMMA 8.2. – *Suppose $b(\chi, x, \theta) \in \Gamma_\chi^\infty(T_0, \xi')$ and set*

$$B(\chi, x, \theta) = -\int_\chi^\infty b(\chi', x, \theta)\, d\chi'.$$

*Then $B \in \Gamma_\chi^\infty(T_0, \xi')$ and $\partial_\chi B = b$.*

To determine $M_\chi \sigma_{1,j}$, $j \in \mathcal{O}(\xi') \cup \mathcal{I}(\xi') \cup \mathcal{P}(\xi') \cup \mathcal{N}(\xi')$ apply $M_\chi$ to (8.12)(c) and use (8.11)(c) to get

$$\mathbb{E}_j A_N \partial_\chi \mathbb{E} a_1 + \mathbb{E}_j A_N \partial_\chi (I - \mathbb{E}) a_1 + X_j M_\chi \sigma_{0,j} r_j = M_\chi \mathbb{E}_j f(a_0). \tag{8.25}$$

For these $j$, $\mathbb{E}_j A_N \mathbb{E}_j = d_j \mathbb{E}_j$ for some $d_j \in \mathbb{C} \setminus 0$ by (3.7), so $\partial_\chi M_\chi \sigma_{1,j} \in \Gamma_\chi^\infty(T_0, \xi')$ can be read off from (8.25). Application of Lemma (8.2) yields the functions $M_\chi \sigma_{1,j} \in \Gamma_\chi^\infty(T_0, \xi')$.

Next apply $\mathbb{E}_0$, $M_\infty \mathbb{E}_0$, $\mathbb{E}_j$, and $M_\infty \mathbb{E}_j$ to $\mathbb{I}_{\frac{1}{2}}$ to obtain respectively:

$$\begin{aligned} &\text{(a)} \quad A_N \partial_\chi \underline{a}_2 + L(\partial) \underline{a}_1 = \underline{f'(a_0) a_1}, \\ &\text{(b)} \quad L(\partial) M_\infty \underline{a}_1 = M_\infty \underline{f'(a_0) a_1}, \end{aligned} \tag{8.26}$$

$$\text{(c)}\quad \mathbb{E}_j A_N \partial_\chi a_2 + X_j \sigma_{1,j} r_j = \mathbb{E}_j f'(a_0) a_1 + \mathbb{E}_j \mathcal{F},$$

$$\text{(d)}\quad X_j M_\infty \sigma_{1,j} r_j = M_\infty \mathbb{E}_j f'(a_0) a_1 + M_\infty \mathbb{E}_j \mathcal{F},$$

where $\mathcal{F}$ here and henceforth represents an already determined element of $\mathbb{P}_\chi(T_0, \xi')$.

Parallel to an earlier argument we observe that (8.26)(b),(d) constitute a linear system for $M_\infty \underline{a_1}$, $M_\infty \sigma_{1,j}$, $j \in \mathcal{G}(\xi') \cup \mathcal{O}(\xi') \cup \mathcal{I}(\xi')$ since $M_\infty \underline{f'(a_0) a_1}$ and $M_\infty \mathbb{E}_j f'(a_0) a_1$ depend only on $M_\infty \underline{a_1}$, $M_\infty \sigma_{1,k}$, $k \in \mathcal{G}(\xi') \cup \mathcal{O}(\xi') \cup \mathcal{I}(\xi')$, and the known function $a_0$.

To find the boundary data for this system, rewrite $\mathbb{B}_{\frac{1}{2}}$ in (4.9) as

$$\text{(8.27)}\qquad \begin{aligned} &\text{(a)}\quad B M_\infty \underline{a_1}|_{x_N=0} = -B M_\chi \underline{a_1}|_{\chi=0,\ x_N=0}, \\ &\text{(b)}\quad B\mathbb{E} a_1 = B \sum_{j=1}^{M(\xi')} (M_\infty \sigma_{1,j} + M_\chi \sigma_{1,j}) r_j = -B(I-\mathbb{E}) a_1, \\ &\qquad \text{on } x_N = 0,\ \chi = 0,\ \theta = s(\theta_0). \end{aligned}$$

The functions

$$M_\chi \underline{a_1}, \qquad M_\chi \sigma_{1,j}, \quad j \in \mathcal{O}(\xi') \cup \mathcal{I}(\xi') \cup \mathcal{P}(\xi') \cup \mathcal{N}(\xi')$$

and $(I-\mathbb{E}) a_1$ are already known. The procedure followed earlier yields boundary values first for

$$M_\infty \sigma_{1,j}, \quad j \in \mathcal{G}(\xi') \cup \mathcal{O}(\xi') \cup \mathcal{I}(\xi'),$$

and then for the remaining pieces of $a_1$, namely

$$M_\chi \sigma_{1,j}, \quad j \in \mathcal{G}(\xi') \quad \text{and} \quad M_\infty \sigma_{1,j}, \quad j \in \mathcal{P}(\xi') \cup \mathcal{N}(\xi').$$

(8.26)(d) gives a linear system for the $M_\infty \sigma_{1,j}$, $j \in \mathcal{P}(\xi') \cup \mathcal{N}(\xi')$ which is solved to sufficiently high order at $x_N = 0$ with the boundary data just determined. For these $j$, $M_\infty \mathbb{E}_j f'(a_0) a_1$ depends on the known functions

$$a_0, \qquad M_\infty \underline{a_1}, \qquad M_\infty \sigma_{1,k}, \quad k \in \mathcal{G}(\xi') \cup \mathcal{O}(\xi') \cup \mathcal{I}(\xi')$$

and the unknowns $M_\infty \sigma_{1,k}$, $k \in \mathcal{P}(\xi') \cup \mathcal{N}(\xi')$.

It remains to find the $M_\chi \sigma_{1,j}$, $j \in \mathcal{G}(\xi')$. First apply $\mathcal{Q}$ to $\mathbb{I}_0$ to get

$$\text{(8.28)}\qquad (I-\mathbb{E}) a_2 = -\mathcal{Q} A_N \partial_\chi a_1 + \mathcal{F}.$$

Applying $M_\chi$ to (8.26)(c) and using (8.28) yields

$$\text{(8.29)}\quad \mathbb{E}_j A_N \partial_\chi \mathbb{E} a_2 + \mathbb{E}_j A_N \partial_\chi (I-\mathbb{E}) a_2 + X_j M_\chi \sigma_{1,j} r_j = M_\chi \mathbb{E} f'(a_0) a_1 + M_\chi \mathbb{E}_j \mathcal{F}.$$

For $j \in \mathcal{G}(\xi')$, (8.29) may be rewritten using (8.28), $\mathbb{E}_j A_N \mathbb{E}_j = 0$, and Lemma 8.1 as

$$\text{(8.30)}\qquad \mathcal{L}_j (M_\chi \sigma_{1,j}) r_j = M_\chi \mathbb{E}_j f'(a_0) a_1 + M_\chi \mathbb{E}_j \mathcal{F}.$$

The linear system (8.30) is solved in Section 9 with the boundary data determined above.

*Remark* 8.3. – The construction of $a_2, a_3, \ldots, a_{M-2}$ is exactly parallel to the construction of $a_1$. Note that for any $k$, making $\mathbb{I}_k$ hold (to sufficiently high order at $x_N = 0$) requires complete determination of $a_0, \ldots, a_{2k}$, but only determination of

$$(I-\mathbb{E}) a_{2k+1}, \qquad M_\chi \underline{a_{2k+1}}, \qquad M_\chi \sigma_{2k+1,j}, \quad j \in \mathcal{O}(\xi') \cup \mathcal{I}(\xi') \cup \mathcal{P}(\xi') \cup \mathcal{N}(\xi'),$$

and $(I-\mathbb{E})a_{2k+2}$. To achieve (6.1) we need to solve $(\mathbb{I}_0,\mathbb{B}_0),\dots,(\mathbb{I}_{\frac{M-2}{2}},\mathbb{B}_{\frac{M-2}{2}})$ so $a_{M-1},a_M$ need only be partially determined.

DEFINITION 8.1. – *For any $K$ the* glancing, elliptic, and hyperbolic boundary layers of order *$K$ are, respectively,*

$$\sum_{i=0}^{K}\left(\sqrt{\varepsilon}\right)^i\sum_{j\in\mathcal{G}(\xi')}M_\chi\sigma_{i,j}(\chi,x,\theta_j)r_j\big|_{\chi=\frac{x_N}{\sqrt{\varepsilon}},\ \theta_j=\frac{\phi_j}{\varepsilon}},\tag{8.31}$$

$$\sum_{j\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi')}\sigma_{0,j}(x,\theta_j)r_j\big|_{\theta_j=\frac{\phi_j}{\varepsilon}}+\sum_{i=1}^{K}\left(\sqrt{\varepsilon}\right)^i\sum_{j\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi')}\sigma_{i,j}(\chi,x,\theta_j)r_j\big|_{\chi=\frac{x_N}{\sqrt{\varepsilon}},\ \theta_j=\frac{\phi_j}{\varepsilon}},\tag{8.32}$$

*and*

$$\sum_{i=1}^{K}(\sqrt{\varepsilon})^i\sum_{j\in\mathcal{O}(\xi')\cup\mathcal{I}(\xi')}M_\chi\sigma_{i,j}(\chi,x,\theta_j)r_j\big|_{\chi=\frac{x_N}{\sqrt{\varepsilon}},\ \theta_j=\frac{\phi_j}{\varepsilon}}\tag{8.33}$$

*with $\sigma_{i,j}$ as in* (8.10).

*Remark* 8.4. – (a)

$$a_0(\chi,x,\theta)=\underline{a_0}(x)+\sum_{j\in\mathcal{G}(\xi')}\sigma_{0,j}(\chi,x,\theta_j)r_j+\sum_{j\in\mathcal{M}(\xi')\setminus\mathcal{G}(\xi')}\sigma_{0,j}(x,\theta_j)r_j.$$

For $k\geqslant 1$, $\mathbb{E}a_k(\chi,x,\theta)=\underline{a_k}(\chi,x)+\sum_{j\in\mathcal{M}(\xi')}\sigma_{k,j}(\chi,x,\theta_j)r_j$.

(b) For any $k\geqslant 0$ the pieces of $a_k$ are determined in the following order:

$$\begin{array}{lll}
(1) & (I-\mathbb{E})a_k, & \\
(2) & M_\chi\underline{a_k}, & M_\chi\sigma_{k,j},\quad j\in\mathcal{M}(\xi')\setminus\mathcal{G}(\xi'),\\
(3) & M_\infty\underline{a_k}, & M_\infty\sigma_{k,j},\quad j\in\mathcal{G}(\xi')\cup\mathcal{O}(\xi')\cup\mathcal{I}(\xi'),\\
(4) & M_\infty\sigma_{k,j}, & j\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi'),\\
(5) & M_\chi\sigma_{k,j}, & j\in\mathcal{G}(\xi').
\end{array}$$

(c) The ordering in (b) reflects the boundary and interior coupling (via the forcing term) of the different pieces of $a_k$.

The term in (1) is determined directly from knowledge of the previous profiles. The terms in (2) can be read off from knowledge of (1) and the previous profiles. The terms in (3) are determined by a system whose forcing term depends only on the terms in (3), (1) and previous profiles, while the forcing term for (4) depends on the terms in (4), (3), (1) and previous profiles. Finally, the forcing term for (5) depends on the terms in (5), (3), $M_\chi\underline{a_k}$ and $M_\chi\sigma_{k,j}$, $j\in\mathcal{O}(\xi')\cup\mathcal{I}(\xi')$ from (2), (1), and previous profiles.

These systems are semilinear when $k=0$ and linear for $k\geqslant 1$. The time of existence $T_0$ is determined by the systems for (3) and (5) when $k=0$.

The coupling on the boundary is somewhat different. For example, the boundary data for the terms in (1), (2) is needed to determine that for the terms in (3), (4). On the other hand while there is interior coupling among the terms in (4), there is no boundary coupling in the sense that for a given $j\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi')$, $M_\infty\sigma_{k,j}|_{x_N=0}$ can be read off from an equation like

(8.27)(b) independently of $M_\infty\sigma_{k,i}|_{x_N=0}$, $i\neq j$. The same applies to the terms of (5). However, $M_\infty\sigma_{k,j}|_{x_N=0}$, $j\in\mathcal{G}(\xi')$ is needed to determine $M_\chi\sigma_{k,j}|_{\chi=0}$, $j\in\mathcal{G}(\xi')$.

(d) For any fixed $k$ there is no interior coupling or boundary coupling (in the sense used in (c)) between the elliptic and glancing boundary layer terms given by (4) and (5) respectively. The evolution of these terms is influenced by the glancing and hyperbolic pieces in (3), but as noted in (c) the influence is not mutual.

(e) In particular ($k=0$) *there is no interior or boundary coupling between the leading parts of the elliptic* (8.18) *and glancing* (8.24) *boundary layers.*

*Remark* 8.5. – The same analysis handles the case when $f(u_\varepsilon,\overline{u_\varepsilon})$ in (1.1) is replaced, for example, by $f(x,u_\varepsilon,\overline{u_\varepsilon})+F(x_N/\sqrt{\varepsilon},x,\phi/\varepsilon)$, provided $F(\chi,x,\theta)\in\mathbb{P}_{\chi,0}(T,\xi')$ and $f(x,u,v)$ is smooth in $x$, entire in $(u,v)$, vanishes in $x_0<0$, and satisfies $f(x,0,0)=0$. Remark 8.4 continues to apply. Note that when $F(\chi,x,\theta)\equiv 0$, we have $M_\infty\sigma_{k,j}\equiv 0$, $j\in\mathcal{G}(\xi')$, for all $k$.

**Construction of the glancing boundary layer**

We must solve the system (8.23) with specified data at $\chi=0$. Set $s_0(\chi,x,\theta)=(s_{0,j}(\chi,x,\theta_j))=(M_\chi\sigma_{0,j})$, $j\in\mathcal{G}(\xi')$. Remark 8.2(a) implies this system has the form, with $\mathcal{L}_j$ as in (8.22),

$$\begin{aligned}&\mathcal{L}_j(s_{0,j})r_j=\mathbb{E}_j h(x,\omega,s_0),\quad j\in\mathcal{G}(\xi'),\ \omega=(\theta_k),\ k\in\mathcal{G}\cup\mathcal{O}\cup\mathcal{I},\\&s_{0,j}|_{\chi=0}=g_j(x',\theta_j),\\&s_{0,j}=0\quad\text{in } x_0<0,\end{aligned}\tag{8.34}$$

where the $h,g_j$ are smooth in $x,\omega,\theta_j$, vanish in $x_0<0$, and by finite propagation speed have compact support in $x'$ on $[0,T]$. $h$ is entire in $(s_0,\overline{s_0})$, $h(x,\omega,0)=0$, and $h$ has compact support in $x_N$ on $[0,T]$ if it is not independent of $x_N$ (see Proposition 9.5).

Choose a cutoff function $\rho(r)\in C_0^\infty(\overline{\mathbb{R}}_+)$, supp $\rho\subset[0,1]$, $\rho\equiv 1$ near $r=0$. We will construct $s_0$ in the form $s_0=b_0+c_0$ (see Remark 8.7) where $b_0,c_0$ satisfy respectively

$$\begin{aligned}&\mathcal{L}_j(b_{0,j})r_j=\rho\big(\chi\langle D_{\theta_j}^{3/2}\rangle\big)\mathbb{E}_j h(x,\omega,b_0),\quad j\in\mathcal{G}(\xi'),\\&b_{0,j}|_{\chi=0}=g_j,\\&b_{0,j}=0\quad\text{in } x_0<0,\end{aligned}\tag{8.35}$$

$$\begin{aligned}&\mathcal{L}_j(c_{0,j})r_j=\mathbb{E}_j h(x,\omega,b_0+c_0)-\rho\big(\chi\langle D_{\theta_j}\rangle^{3/2}\big)\mathbb{E}_j h(x,\omega,b_0),\\&c_{0,j}|_{\chi=0}=0,\\&c_{0,j}=0\quad\text{in } x_0<0.\end{aligned}\tag{8.36}$$

Here $\rho(\chi\langle D_{\theta_j}^{3/2}\rangle)$ acts on periodic functions of $\theta_j$ as the multiplier $\rho(\chi\langle n\rangle^{3/2})$.

These two systems are solved by separate Picard iterations corresponding in the obvious way to the following scalar (hence no $j$) linear problems:

$$\begin{aligned}&\mathcal{L}b_0=\rho\big(\chi\langle D_\theta\rangle^{3/2}\big)h(\chi,x,\theta),\quad h\in\Gamma^\infty_{\chi,0}(T,\xi'),\ \theta\in\mathbb{T}^1,\\&b_0|_{\chi=0}=g(x',\theta),\quad g\in C_0^\infty(T),\\&b_0=0\quad\text{in } x_0<0,\end{aligned}\tag{8.37}$$

$$\begin{aligned}&\mathcal{L}c_0=k(\chi,x,\theta),\quad k\in\Gamma^\infty_{\chi,0}(T,\xi'),\\&c_0|_{\chi=0}=0,\\&c_0=0\quad\text{in } x_0<0.\end{aligned}\tag{8.38}$$

*Remark* 8.6. – $k$ vanishes to infinite order at $\chi=0$, and $h,k,g$ have compact support in $x'$ on $[0,T]$, with $h,k$ having compact support in $x_N$ if not independent of $x_N$. We also let $h\in\Gamma^\infty_{\chi,0}(T+1,\xi')$, $g\in\mathcal{C}^\infty_0(T+1)$ denote extensions of $h,g$ vanishing in $x_0>T+1/2$ with $\underline{h}=\underline{k}=\underline{g}=0$.

(8.37) is solved for each Fourier coefficient. The problem for the $n$th coefficient $b_{0,n}(\chi,x)$ is, with $c,X$ corresponding to $c_j,X_j$ in (8.22),

$$\begin{aligned}&\left(\frac{i\partial_\chi^2}{cn}+X\right)b_{0,n}=\rho\big(\chi\langle n\rangle^{3/2}\big)h_n,\\ &b_{0,n}|_{\chi=0}=g_n(x'),\\ &b_{0,n}=0\quad\text{in } x_0<0.\end{aligned}\tag{8.39}$$

Let $\mathbb{X}(\xi_0-i\gamma,\xi'')$ denote the symbol of $\frac{1}{i}X$. For $\gamma>0$ the partial Fourier transform of the solution is

$$\begin{aligned}b^\wedge_{0,n}(\chi,\xi_0-i\gamma,\xi'',x_N)=&\int_0^\chi e^{i(\chi-\chi')\sqrt{cn\mathbb{X}(\xi_0-i\gamma,\xi'')}}\\ &\left(\int_{\chi'}^\infty e^{-i(\chi'-\chi'')\sqrt{cn\mathbb{X}(\xi_0-i\gamma,\xi'')}}icn\rho\big(\chi''\langle n\rangle^{3/2}\big)h^\wedge_n(\chi'',\xi_0-i\gamma,\xi'',x_N)\,d\chi''\right)d\chi'\\ &+e^{i\chi\sqrt{cn\mathbb{X}(\xi_0-i\gamma,\xi'')}}\hat g_n(\xi_0-i\gamma,\xi''),\end{aligned}\tag{8.40}$$

where $\sqrt{\ }$ denotes the square root with positive imaginary part. (8.40) leads to the estimates (notation as in Section 7)

$$|b_{0,n}|_{C(\chi,H^m_\gamma(T))}\leqslant\frac{C}{\sqrt{\gamma}\langle n\rangle}|h_n|_{C(\chi,H^m_\gamma(T))}+\langle g_n\rangle_{H^m_\gamma(T)},\tag{8.41}$$

$$|b|_{\mathbb{A}^1_\theta(C(\chi,H^m_\gamma(T)))}\leqslant\frac{C}{\sqrt{\gamma}}|h|_{\mathbb{A}_\theta(C(\chi,H^m_\gamma(T)))}+\langle g\rangle_{\mathbb{A}^1_\theta(H^m_\gamma(T))}.\tag{8.42}$$

Iteration together with a continuation principle proved using the Moser-type inequality for Wiener algebras (7.13) yields a solution $b_0\in\Gamma^\infty_{\chi,0}(T_0,\xi')$ to (8.35) for some $T_0>0$ (Proposition 9.6).

*Remark* 8.7. – (a) (8.42) is immediate from (8.41).

(b) The factor $1/\langle n\rangle$ in (8.41) is there because supp $\rho(\chi\langle n\rangle^{3/2})\subset[0,1/\langle n\rangle^{3/2}]$. Without the cutoff $1/\langle n\rangle$ would be replaced by $\langle n\rangle$, and the resulting estimate for $b_0$ would not be suitable for Picard iteration. Having 1 in place of $1/\langle n\rangle$ would make iteration possible, but we need the gain of one $\theta$ derivative in (8.42) to deduce infinite regularity in $\theta$ of the solution to (8.35).

We can write down the solution to (8.38) using the unitary group corresponding to the self-adjoint operator $D^2_\chi D^{-1}_\theta/c_j$ with domain $D^*\subset L^{2*}$ (7.15). Integrating along characteristics of the glancing vector field $X=\partial_{x_0}+v\cdot\partial_{x''}$, we obtain

$$c_0(\chi,x,\theta)=\int_0^{x_0}e^{i(x_0-s)\frac{D^2_\chi D^{-1}_\theta}{c_j}}k\big(\chi,x+(s-x_0)(1,v,0),\theta\big)\,ds\tag{8.43}$$

which yields the estimate

$$|c_0(\cdot,x_0,\cdot)|_{L^2} \leqslant \int_0^{x_0} |k(\cdot,s,\cdot)|_{L^2}\,ds. \tag{8.44}$$

To get $\Gamma^m$ estimates, $m > N/2$ we use the following commutator result, proved in Section 9.

PROPOSITION 8.1. – *Suppose $k(\chi,x,\theta) \in L^{2*}$ is such that $\chi^k \partial_\chi^l k \in D^*$, $|k,l| \leqslant 1$. Then*

$$\left[\chi, e^{it\frac{D_\chi^2 D_\theta^{-1}}{c_j}}\right]k = 2te^{it\frac{D_\chi^2 D_\theta^{-1}}{c_j}}\frac{D_\chi D_\theta^{-1}}{c_j}k. \tag{8.45}$$

This readily gives the key estimate

$$\left|c_0(\cdot,x_0,\cdot)\right|_{\Gamma^m} \leqslant \int_0^{x_0} |k(\cdot,s,\cdot)|_{\Gamma^m}\,ds \quad \text{for } k \in \Gamma^{m,*} \text{ (7.17).} \tag{8.46}$$

Iteration together with a continuation principle obtained from the Moser-type inequality for $\Gamma^m$ spaces (7.21) produces a solution $c_0 \in \Gamma^\infty_{\chi,0}(T_0,\xi')$ to (8.36) for a possibly smaller $T_0 > 0$.

The equations for higher order glancing profiles are similar but linear (recall (8.30)). They are solved in Section 9.4.

### 8.2. Real resonances and adapted bases

A *resonance* occurs when there is a relation

$$\sum_{i\in\mathcal{M}(\xi')} \alpha_i\phi_i = 0,$$

where at least 3 of the $\alpha_i \in \mathbb{Z}$ are nonzero. Suppose $\xi'$ is such that all resonances involve only the real phases. That is,

$$\sum_{i\in\mathcal{M}(\xi')} \alpha_i\phi_i = 0 \Rightarrow \alpha_i = 0, \quad i \in \mathcal{P}(\xi') \cup \mathcal{N}(\xi'). \tag{8.47}$$

This situation can be handled by constructing an *adapted basis* $B(\xi')$ for the $\mathbb{Q}$-span of $\{\phi_j\}_{j\in\mathcal{G}(\xi')\cup\mathcal{O}(\xi')\cup\mathcal{I}(\xi')}$.

Suppose $B(\xi') = \{\psi_k(x)\}_{k=1,\dots,b}$ is a basis for the $\mathbb{Q}$-span of $\{\phi_j\}_{j\in\mathcal{G}\cup\mathcal{O}\cup\mathcal{I}}$. With $\psi = (\psi_1,\dots,\psi_b)$ and $j \in \mathcal{G}\cup\mathcal{O}\cup\mathcal{I}$ let

$$\mathcal{D}_j = \left\{\alpha \in \mathbb{Z}^b\colon\ \alpha\cdot\psi \text{ is in the } \mathbb{R}\text{-span of } \phi_j\right\}, \tag{8.48}$$

and let $\alpha_j^*$ be the element of minimal length in $\mathcal{D}_j$ such that $\alpha_j^*\cdot\psi$ is a positive multiple of $\phi_j$. Observe that every element of $\mathcal{D}_j$ is some integer multiple of $\alpha_j^*$.

DEFINITION 8.2. – *A basis $B(\xi') = \{\psi_k\}_{k=1,\dots,b}$ for the $\mathbb{Q}$-span of $\{\phi_j\}_{j\in\mathcal{G}\cup\mathcal{O}\cup\mathcal{I}}$ in $C^\infty(\mathbb{R}_x^{N+1})$ is adapted if:*

(a) *for each $k$, $\psi_k|_{x_N=0} = m_k\phi_0$, $m_k \in \mathbb{Z}$;*

(b) *for each $j \in \mathcal{G}(\xi')\cup\mathcal{O}(\xi')\cup\mathcal{I}(\xi')$, $\alpha_j^*\cdot\psi = \phi_j$.*

*Remark* 8.8. – (a) If $f(\theta)$ is a periodic function of $\theta \in \mathbb{R}^b$ with spectrum contained in $\mathcal{D}_j$, there is a uniquely determined periodic function $f^*(\omega_j)$ of $\omega_j \in \mathbb{R}$ such that

$$f(\theta) = f^*|_{\omega_j = \alpha_j^* \cdot \theta}.$$

To see this write

$$f(\theta) = \sum_{\alpha \in \mathcal{D}_j} f_\alpha e^{i\alpha\theta} = \sum_{l \in \mathbb{Z}} f_{l\alpha_j^*} e^{il\alpha_j^* \theta}.$$

(b) No matter what relations exist among the $\{\phi_j\}_{j \in \mathcal{G} \cup \mathcal{O} \cup \mathcal{I}}$, an adapted basis can always be chosen. The construction and use of adapted bases is explained in detail in [20]. The definition there is slightly more complicated since the shock is a free oscillating boundary.

If we now take

$$\phi(x) = \big((\psi_k)_{k=1,\dots,b}, (\phi_j)_{j \in \mathcal{P}(\xi') \cup \mathcal{N}(\xi')}\big)$$

and assume $\phi(x)$ satisfies a small divisor property similar to Definition 4.1, a construction just like that outlined above produces approximate solutions $\tilde{u}_\varepsilon$ of the form (6.2).

### 8.3. Discussion of Theorem 6.2

Consider the linearized problem corresponding to (1.1):

$$\begin{aligned} &Lu = f, \\ &Bu|_{x_N=0} = g, \\ &u = 0 \quad \text{in } x_0 < 0, \end{aligned} \tag{8.49}$$

where $f \in L^2(\Omega_T)$, $g \in L^2(b\Omega_T)$ and both vanish in $x_0 < 0$. The starting point is the classic inequality for systems $(L, B)$ satisfying the uniform Kreiss condition:

There exist $C > 0$, $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$,

$$|u|_{0,\mu,\lambda} + \frac{1}{\sqrt{\lambda}} \langle u \rangle_{0,\mu,\lambda} \leqslant C\left(\frac{|f|_{0,\mu,\lambda}}{\lambda} + \frac{\langle g \rangle_{0,\mu,\lambda}}{\sqrt{\lambda}}\right). \tag{8.50}$$

Here we use Gues's weighted norms

$$|w|_{m,\mu,\lambda} = \sum_{0 \leqslant |\alpha| \leqslant m} \mu^{m-|\alpha|} |e^{-\lambda x_0} \partial^\alpha w|_{L^2(\Omega_T)}, \tag{8.51}$$

and denote boundary norms by $\langle \, \rangle_{m,\mu,\lambda}$.

As in [4] the idea is to take advantage of the factor $\varepsilon^{M_1}$ in hypothesis (a) of Theorem 6.2 to find an exact solution

$$u_\varepsilon = \tilde{u}_\varepsilon + w_\varepsilon$$

as in (b) of the theorem. $w_\varepsilon$ is constructed by the iteration scheme (some epsilons are suppressed):

$$\begin{aligned} &Lw_{n+1} = f(\tilde{u} + w_n) - f(\tilde{u}) - \varepsilon^{M_1} R_\varepsilon, \\ &Bw_{n+1}|_{x_N=0} = -\varepsilon^{M_1} r_\varepsilon, \\ &w_{n+1} = 0 \quad \text{in } x_0 < 0. \end{aligned} \tag{8.52}$$

The key estimate is proved in Section 10 by differentiating (8.52) and applying (8.50). Here we set

$$\|u\|_{m,\mu,\lambda} = |u|_{m,\mu,\lambda} + \frac{1}{\sqrt{\lambda}}\langle u\rangle_{m,\mu,\lambda}. \tag{8.53}$$

PROPOSITION 8.2. – *Let $m_1 > (N+1)/2$ and $\rho$ be as in Theorem 6.2. Suppose $w_n \in H_0^{m_1}(\Omega_T)$ satisfies*

$$|w_n|_{L^\infty(\Omega_T)} \leqslant K. \tag{8.54}$$

*There exist $C(K,\rho) > 0$, $\lambda_0(K,\rho) > 0$ and a positive function $\phi(\lambda)$ such that for $\lambda \geqslant \lambda_0(K,\rho)$*

$$\|w_{n+1}\|_{m_1,\,\lambda/\varepsilon,\,\lambda} \leqslant \frac{C(K,\rho)}{\lambda}\|w_n\|_{m_1,\,\lambda/\varepsilon,\,\lambda} + \varepsilon^{M_1-m_1}\phi(\lambda). \tag{8.55}$$

(8.55) easily implies the existence of $\lambda_1(K,\rho) \geqslant \lambda_0(K,\rho)$ and $\varepsilon_1(\lambda)$ such that for $\lambda \geqslant \lambda_1$, $\varepsilon \in (0, \varepsilon_1]$ we have

$$\begin{aligned} &\text{(a)} \quad \|w_n\|_{m_1,\,\lambda/\varepsilon,\,\lambda} \leqslant 2\varepsilon^{M_1-m_1}\phi(\lambda) \quad \text{for all } n,\\ &\text{(b)} \quad |w_n|_{L^\infty(\Omega_T)} \leqslant K \quad \text{for all } n. \end{aligned} \tag{8.56}$$

Convergence of the iterates is proved by a similar argument.

*Remark* 8.9. – (a) In [6,7] Gues considers semilinear Dirichlet problems for viscous perturbations $\mathcal{H} + \varepsilon\mathcal{E}$ of a first-order symmetric hyperbolic operator $\mathcal{H}$ ($\mathcal{E}$ is second-order, elliptic). He shows that the solution $u_\varepsilon$ converges in $H^s$, $s < 1/2$, to the solution $u_0$ of a maximal dissipative boundary problem for $\mathcal{H}$. The obstruction to convergence for $s > 1/2$ is a boundary layer of size $\varepsilon$. When the boundary is charac teristic, the layer has another piece of size $\sqrt{\varepsilon}$. The profile equations in [6,7] are of parabolic-hyperbolic type.

(b) [3] and [11] construct rigorous geometric optics expansions valid for times of order $1/\varepsilon$ for nonlinear hyperbolic initial value problems in free space. The profiles $a(\varepsilon x, x, \beta \cdot x/\varepsilon)$ involve a third (slow) scale ($\varepsilon$). Some of the profile equations are of Schrödinger type and have the form

$$\frac{\partial a}{\partial T} - P(\partial_Y, \partial_y, \partial_\theta)a = f(a), \tag{8.57}$$

where $a = a(T, Y, y, \theta)$, $x = (t, y)$, $X = (T, Y) = (\varepsilon t, \varepsilon y)$. Here $P$ is anti-self-adjoint and second-order only in the $\partial_y$ derivatives.

These equations differ from our profile equations in 2 respects. The second-order derivatives do not occur in variables $X = \varepsilon x$ corresponding to the new scale. More significantly, those derivatives do not involve the time variable, so the solution to the linearized initial value problem for (8.57) is given by the operator $e^{TP}$. Because our profile equations involve second-order derivatives in $\chi = x_N/\sqrt{\varepsilon}$ and nonhomogeneous boundary conditions on $x_N = 0$, we were forced to do mode-by-mode analysis involving the cutoffs $\rho(\chi\langle n\rangle^{3/2})$ and the use of Wiener algebras.

# Part II
## The proofs

### 9. Theorem 6.1: Construction of the approximate solution

#### 9.1. Determination of $\underline{a_0}$, $M_\infty\sigma_{0,j}$, $j \in \mathcal{G}(\xi') \cup \mathcal{O}(\xi') \cup \mathcal{I}(\xi')$

To solve the system (8.12)(b,d) with boundary conditions (8.14)(a) and (8.17), we'll need the following spaces of profiles.

DEFINITION 9.1. – *Let $\Omega_{T,\theta} = \Omega_T \times \mathbb{T}^p$, $b\Omega_{T,\theta_0} = b\Omega_T \times \mathbb{T}^1$. For $m \in \{0,1,2,\ldots\}$, $T > 0$, $\sigma > 0$ let*

(a) $\mathbb{H}^m(T) = \{U(x,\theta)\colon |U|_{m,T} = \sum_{|\alpha| \leqslant m} |\partial^\alpha_{(x,\theta)} U|_{L^2(\Omega_{T,\theta})} < \infty\}$;

(b) $\mathcal{H}^m(T) = \{u(x',\theta_0)\colon \langle u\rangle_{m,T} = \sum_{|\alpha| \leqslant m} \langle \partial^\alpha_{(x',\theta_0)} u\rangle_{L^2(b\Omega_{T,\theta_0})} < \infty\}$;

(c) $\mathbb{H}^m_\sigma(T) = \{U(x,\theta)\colon |U|_{\sigma,m,T} = \sum_{|\alpha| \leqslant m} \sigma^{m-|\alpha|} |e^{-\sigma x_0} \partial^\alpha_{(x,\theta)} U|_{L^2(\Omega_{T,\theta})} < \infty\}$. *$\mathcal{H}^m_\sigma(T)$ is defined similarly on $b\Omega_{T,\theta_0}$.*

The following proposition clarifies an earlier assertion about the nonlinear terms in (8.12)(b,d).

*Notation* 9.1. – Suppose $u(\chi, x, \theta) \in \mathbb{P}_\chi(T, \xi')$ satisfies $\mathbb{E}u = u$, so

$$u = \underline{u}(\chi, x) + \sum_{j=1}^{M(\xi')} \sigma_j(\chi, x, \theta_j) r_j$$

for some $\underline{u}$, $\sigma_j$. For $f$ as in Theorem 6.1 we sometimes write

$$\begin{aligned} f(u,\overline{u}) &= f\big(M_\chi \underline{u}; M_\infty \underline{u}; (M_\chi \sigma_j),\ j \in \mathcal{M}(\xi'); (M_\infty \sigma_j),\ j \in \mathcal{M}(\xi')\big) \\ &= f\big(M_\chi \underline{u}, M_\infty \underline{u}, M_\chi u_{\mathcal{G}(\xi')}, M_\infty u_{\mathcal{G}(\xi')}, \ldots, M_\chi u_{\mathcal{P}\cup\mathcal{N}}, M_\infty u_{\mathcal{P}\cup\mathcal{N}}\big), \end{aligned} \tag{9.1}$$

where $\mathcal{M}(\xi') = \mathcal{G}(\xi') \cup \mathcal{O}(\xi') \cup \mathcal{I}(\xi') \cup \mathcal{P}(\xi') \cup \mathcal{N}(\xi')$.

PROPOSITION 9.1. – *Suppose $a \in \mathbb{P}_\chi(T, \xi')$ satisfies (8.9) and (8.11)(b). Then for $j \in \mathcal{G}(\xi') \cup \mathcal{O}(\xi') \cup \mathcal{I}(\xi')$*

$$\begin{aligned} &\text{(a)} \quad M_\infty \underline{f(a)} = \mathbb{E}_0 f(0, \underline{a}, 0, M_\infty a_{\mathcal{G}}, 0, a_{\mathcal{O}\cup\mathcal{I}}, 0), \\ &\text{(b)} \quad M_\infty \mathbb{E}_j f(a) = \mathbb{E}_j f(0, \underline{a}, 0, M_\infty a_{\mathcal{G}}, 0, a_{\mathcal{O}\cup\mathcal{I}}, 0). \end{aligned}$$

The proof makes use of the following Lemma, which shows that elliptic modes are never cancelled by multiplication with a nonzero element of $\mathbb{P}_\chi(T, \xi')$.

LEMMA 9.1. – *Let $a, b \in \mathbb{P}_\chi(T, \xi')$ and suppose $\mathbb{E}_j a = a$ for some $j \in \mathcal{P}(\xi') \cup \mathcal{N}(\xi')$. Then for $k = 0$, and $k \in \mathcal{G}(\xi') \cup \mathcal{O}(\xi') \cup \mathcal{I}(\xi')$ we have $\mathbb{E}_k(a \cdot b) = 0$.*

*Proof.* – Write out the Fourier series of $a$ and $b$, observe that if $a \neq 0$, the spectrum of $a$ contains only positive integers while spec $b \subset Z(\xi')$, and use the definition of $\mathbb{E}_k$. □

*Proof of Proposition 9.1.* – Note that

$$\underline{a} = M_\infty \underline{a}, \quad a_{\mathcal{O}\cup\mathcal{I}\cup\mathcal{P}\cup\mathcal{N}} = M_\infty a_{\mathcal{O}\cup\mathcal{I}\cup\mathcal{P}\cup\mathcal{N}}.$$

Now

$$f(a)-f(0,\underline{a},0,M_\infty a_{\mathcal{G}},0,a_{\mathcal{O}\cup\mathcal{I}},0)=f(0,\underline{a},M_\chi a_{\mathcal{G}},M_\infty a_{\mathcal{G}},0,a_{\mathcal{O}\cup\mathcal{I}},0,a_{\mathcal{P}\cup\mathcal{N}})$$
$$-f(0,\underline{a},0,M_\infty a_{\mathcal{G}},0,a_{\mathcal{O}\cup\mathcal{I}},0)$$
$$=(M_\chi a_{\mathcal{G}})g_1+(a_{\mathcal{P}\cup\mathcal{N}})g_2$$
$$=\mathcal{F}_1+\mathcal{F}_2 \tag{9.2}$$

for $g_1, g_2$ coming from Taylor's formula. We have $M_\infty \mathcal{F}_1=0$, since $M_\chi a_{\mathcal{G}}$ is rapidly decreasing in $\chi$ and $g_1\in\mathbb{P}_\chi(T,\xi')$ by Proposition 5.1. Also, $g_2\in\mathbb{P}_\chi(T,\xi')$ so $\mathbb{E}_j\mathcal{F}_2=0$ for $j=0$ or $j\in\mathcal{G}(\xi')\cup\mathcal{O}(\xi')\cup\mathcal{I}(\xi')$ by Lemma 9.1. (a) and (b) now follow since the $\mathbb{E}_j$ commute with $M_\infty$. □

Proposition 9.1 leads us to consider the system

$$\begin{aligned}
&L(\partial)\underline{a_0}=\mathbb{E}_0 f(0,\underline{a_0},0,M_\infty a_{0,\mathcal{G}},0,M_\infty a_{0,\mathcal{O}\cup\mathcal{I}},0),\\
&X_j M_\infty \sigma_{0,j} r_j=\mathbb{E}_j f(0,\underline{a_0},0,M_\infty a_{0,\mathcal{G}},0,M_\infty a_{0,\mathcal{O}\cup\mathcal{I}},0),\quad j\in\mathcal{G}\cup\mathcal{O}\cup\mathcal{I},\\
&B\underline{a_0}|_{x_N=0}=\underline{G}(x'),\\
&\big(M_\infty\sigma^{\pm}_{0,j}|_{x_N=0,\ \theta_j=\theta_0},\ j\in\mathcal{O}\big)=\mathbb{M}^{\pm}\big(G^{*,\pm},M_\infty\sigma^{\pm}_{0,j}|_{x_N=0,\ \theta_j=\theta_0},\ j\in\mathcal{I}\big).
\end{aligned}\tag{9.3}$$

PROPOSITION 9.2. – *The semilinear system* (9.3) *has solutions* $\underline{a_0}$, $M_\infty\sigma_{0,j}$, $j\in\mathcal{G}\cup\mathcal{O}\cup\mathcal{I}$ *belonging to* $\mathbb{C}^\infty_0(T_0,\xi')$ *for some* $T_0>0$ *and with compact support in* $x$.

*Proof.* – Convergence of the usual iteration scheme will follow from an a priori estimate for the following linear system for the unknowns $\underline{u}$, $\sigma_j(x,\theta_j)$, $j\in\mathcal{G}\cup\mathcal{O}\cup\mathcal{I}$:

$$\begin{aligned}
&\text{(a)}\quad L(\partial)\underline{u}=\underline{f}(x),\\
&\text{(b)}\quad X_j\sigma_j(x,\theta_j)=f_j(x,\theta_j),\quad j\in\mathcal{G}\cup\mathcal{O}\cup\mathcal{I},
\end{aligned}\tag{9.4}$$

$$\begin{aligned}
&\text{(c)}\quad B\underline{u}|_{x_N=0}=\underline{G}(x'),\\
&\text{(d)}\quad \big(\sigma_j^{\pm}|_{x_N=0,\ \theta_j=\theta_0},\ j\in\mathcal{O}\big)=\mathbb{M}^{\pm}\big(G^{*,\pm},\sigma_j^{\pm}|_{x_N=0,\ \theta_j=\theta_0},\ j\in\mathcal{I}\big),
\end{aligned}$$

where $\mathbb{M}^\pm$ is defined as in (8.17), all functions vanish in $x_0<0$, and the data $(\underline{f},f_j)$, $G$ is smooth with compact support in $\Omega_T$, $b\Omega_T$ respectively.

Set

$$W(x,\theta)=\underline{u}+\sum_{j\in\mathcal{G}\cup\mathcal{O}\cup\mathcal{I}}\sigma_j(x,\theta_j)r_j$$

and

$$F(x,\theta)=\underline{f}+\sum_{j\in\mathcal{G}\cup\mathcal{O}\cup\mathcal{I}}f_j(x,\theta_j)r_j.$$

LEMMA 9.2. – *Let* $m\in\{0,1,2,\dots\}$. *There exist* $\sigma_0>0$, $C>0$ *such that for* $\sigma>\sigma_0$,

$$|W|_{\sigma,m,T}+\frac{1}{\sqrt{\sigma}}\langle W\rangle_{\sigma,m,T}\leqslant C\left(\frac{1}{\sigma}|F|_{\sigma,m,T}+\frac{1}{\sqrt{\sigma}}\langle G\rangle_{\sigma,m,T}\right).\tag{9.5}$$

*Proof.* – Write $X_p=\partial_{x_0}+\sum_{j=1}^{N-1}a_{p,j}\partial_{x_j}+a_{p,N}\partial_{x_N}$, where $a_{p,N}>0$ (respectively $<0$, $=0$) if $X_p$ is outgoing (respectively incoming, glancing). Assume now that $X_p$ is outgoing, $X_q$ incoming, and $X_r$ glancing. $e^{-\sigma x_0}W$ satisfies a system just like (9.4) except $\partial_{x_0}$ in $L(\partial)$ and $X_j$ is replaced by $(\partial_{x_0}+\sigma)$, and $F,G$ are multiplied by $e^{-\sigma x_0}$.

Multiply the $\sigma_p$ equation in this new system by $e^{-\sigma x_0}\sigma_p(x,\theta_p)$, integrate over $\Omega_T \times \mathbb{T}^1$, and do similarly for the $\sigma_q$ and $\sigma_r$ equations to obtain:

$$
(9.6)\qquad
\begin{aligned}
&\text{(a)}\quad \sigma|\sigma_p|^2_{\sigma,0,T} \leqslant a_{p,N}\langle\sigma_p\rangle^2_{\sigma,0,T} + |f_p|_{\sigma,0,T}|\sigma_p|_{\sigma,0,T},\\
&\text{(b)}\quad \sigma|\sigma_q|^2_{\sigma,0,T} + |a_{q,N}|\langle\sigma_q\rangle^2_{\sigma,0,T} \leqslant |f_q|_{\sigma,0,T}|\sigma_q|_{\sigma,0,T},\\
&\text{(c)}\quad \sigma|\sigma_r|^2_{\sigma,0,T} \leqslant |f_r|_{\sigma,0,T}|\sigma_r|_{\sigma,0,T}.
\end{aligned}
$$

Use (9.4)(d) in (9.6)(a) to write $\sigma_p|_{x_N=0}$ in terms of $(G^*, \sigma_j, j \in \mathcal{I})$. For $\underline{u}$ we have the Kreiss estimate

$$
(9.7)\qquad |\underline{u}|_{\sigma,0,T} + \frac{1}{\sqrt{\sigma}}\langle\underline{u}\rangle_{\sigma,0,T} \leqslant C\left(\frac{1}{\sigma}|\underline{f}|_{\sigma,0,T} + \frac{1}{\sqrt{\sigma}}\langle\underline{G}\rangle_{\sigma,0,T}\right).
$$

Summing over all modes and absorbing terms in the usual way by taking $\sigma$ large, we obtain the estimate (9.5) for $|\ |_{\sigma,0,T}$ norms. Differentiating (9.4) and applying the $L^2$ estimate yields (9.5). □

For $M > (N+2)/2$, $f(a_0)$ as in (8.12), and $W$ as above, since

$$
(9.8)\qquad |UV|_{m,T} \leqslant C\big(|U|_{L^\infty}|V|_{m,T} + |U|_{m,T}|V|_{L^\infty}\big),
$$

$$
(9.9)\qquad \big|f(W)\big|_{m,T} \leqslant C\big(|W|_{L^\infty}\big)|W|_{m,T}, \quad \text{and}
$$

$$
(9.10)\qquad \big|\mathbb{E}_j f(W)\big|_{m,T} \leqslant C\big|f(W)\big|_{m,T}, \quad j=0 \text{ or } j \in \mathcal{G}\cup\mathcal{O}\cup\mathcal{I},
$$

a standard iteration argument and continuation principle based on (9.5) yield solutions $\underline{a_0}$, $M_\infty\sigma_{0,j}$, $j \in \mathcal{G}\cup\mathcal{O}\cup\mathcal{I}$ to the semilinear system belonging to $\mathbb{H}^\infty_0$ for some $T_0 > 0$. Their support in $x$ is compact by finite propagation speed. This concludes the proof of Proposition 9.2. □

### 9.2. Determination of $\sigma_{0,j}$, $j \in \mathcal{P}(\xi')\cup\mathcal{N}(\xi')$

The next Proposition is proved just like Proposition 9.1.

PROPOSITION 9.3. – *Suppose $a \in \mathbb{P}_\chi(T,\xi')$ satisfies* (8.9) *and* (8.11)(b). *Then for $j \in \mathcal{P}(\xi')\cup \mathcal{N}(\xi')$,*

$$
M_\infty\mathbb{E}_j f(a) = \mathbb{E}_j f(0,\underline{a},0,M_\infty a_{\mathcal{G}},0,a_{\mathcal{O}\cup\mathcal{I}},0,a_{\mathcal{P}\cup\mathcal{N}}).
$$

This proposition and (8.12)(d) lead us to consider the system

$$
(9.11)\qquad
\begin{aligned}
&\text{(a)}\quad X_j\sigma_{0,j}r_j = \mathbb{E}_j f(0,\underline{a_0},M_\infty a_{0,\mathcal{G}},0,a_{0,\mathcal{O}\cup\mathcal{I}},0,a_{0,\mathcal{P}\cup\mathcal{N}}) \equiv \mathcal{F}_j,\ j \in \mathcal{P}\cup\mathcal{N},\\
&\text{(b)}\quad \sigma_{0,j}|_{x_N=0,\ \theta_j=s(\theta_0)_j} = g_j(x',\theta_0),
\end{aligned}
$$

where the $g_j \in \mathcal{C}^\infty_0(T_0)$ are obtained from (8.14)(b) using the Kreiss condition, and $s(\theta_0)$ is as in (4.9).

PROPOSITION 9.4. – *Let $T_0$ be as in Proposition* 9.2. *There exist functions $\sigma_{0,j}$, $\mathcal{R}_{0,j}(x,\theta_j)$, $j \in \mathcal{P}(\xi')\cup\mathcal{N}(\xi')$ all belonging to $\mathbb{C}^\infty_0(T_0,\xi')$ with compact support in $x$, such that*

$$
(9.12)\qquad
\begin{aligned}
&X_j\sigma_{0,j}r_j = \mathcal{F}_j + x_N^{[(M-1)/2]}\mathcal{R}_{0,j}(x,\theta_j),\\
&\sigma_{0,j}|_{x_N=0,\ \theta_j=s(\theta_0)_j} = g_j(x',\theta_0).
\end{aligned}
$$

*Here $[(M-1)/2]$ is the smallest integer $\geqslant (M-1)/2$.*

*Proof.* – The coefficient of $\partial_{x_N}$ in the complex vector field $X_j$ is nonzero. Requiring (9.11)(a) to hold just at $x_N = 0$ determines $\partial_{x_N}\sigma_{0,j}|_{x_N=0}$, $j \in \mathcal{P} \cup \mathcal{N}$. Similarly, requiring (9.11)(a) to hold to order $[(M-1)/2]-1$ at $x_N = 0$ determines $\partial^l_{x_N}\sigma_{0,j}|_{x_N=0}$, $l \leqslant [(M-1)/2]$. Choosing elements of $\mathbb{C}_0^\infty(T_0,\xi')$ with compact support in $x$ with these traces at $x_N = 0$ determines the $\mathcal{R}_{0,j}$. □

### 9.3. Determination of $M_\chi\sigma_{0,j}$, $j \in \mathcal{G}(\xi')$

Although the next proof uses an argument from [3], we include it here to clarify how $\mathcal{L}_j$ as in (8.22) arises from a glancing mode $(\xi', \xi_N^j(\xi'))$.

**Proof of Lemma 8.1**

*Proof.* – For $\underline{\alpha}_j$ as in (8.21)

$$\underline{\alpha}_j \cdot d\phi = \left(\xi', \xi_N^j(\xi')\right) = \left(\xi_0^k(\xi''), \xi''\right)$$

for some $k$. We'll write $\mathcal{Q}(\underline{\alpha}_j) = \mathcal{Q}(\xi_0^k, \xi'')$. As in (3.7)

$$L(\xi_0^k, \xi'')\pi\left(\xi_0^k, \xi''\right) = 0$$

implies

$$\pi\left(\xi_0^k(\xi''), \xi''\right)\left(\frac{\partial \xi_0^k}{\partial \xi_N} + A_N\right)\pi = 0. \tag{9.13}$$

Differentiate (9.13) to get

$$\partial_{\xi_N}\pi\left(\frac{\partial \xi_0^k}{\partial \xi_N} + A_N\right)\pi + \pi\frac{\partial^2 \xi_0^k}{\partial \xi_N^2}\pi + \pi\left(\frac{\partial \xi_0^k}{\partial \xi_N} + A_N\right)\partial_{\xi_N}\pi = 0. \tag{9.14}$$

Differentiate $(I - \pi) = L(\xi_0^k, \xi'')\mathcal{Q}(\xi_0^k, \xi'')$ (respectively $I - \pi = \mathcal{Q}L$) and multiply on the left (respectively right) by $\pi$ to get

$$-\pi \cdot \partial_{\xi_N}\pi = \pi\left(\frac{\partial \xi_0^k}{\partial \xi_N} + A_N\right)\mathcal{Q}, \tag{9.15}$$

respectively

$$-\partial_{\xi_N}\pi \cdot \pi = \mathcal{Q}\left(\frac{\partial \xi_0^k}{\partial \xi_N} + A_N\right)\pi. \tag{9.16}$$

Multiply (9.14) on the left and right by $\pi$ and use (9.15), (9.16) to find

$$\begin{aligned} &-\pi\left(\frac{\partial \xi_0^k}{\partial \xi_N} + A_N\right)\mathcal{Q}\left(\frac{\partial \xi_0^k}{\partial \xi_N} + A_N\right)\pi + \pi\frac{\partial^2 \xi_0^k}{\partial \xi_N^2}\pi \\ &\quad - \pi\left(\frac{\partial \xi_0^k}{\partial \xi_N} + A_N\right)\mathcal{Q}\left(\frac{\partial \xi_0^k}{\partial \xi_N} + A_N\right)\pi = 0. \end{aligned} \tag{9.17}$$

$(\xi_0^k, \xi'')$ is glancing so $\partial\xi_0^k/\partial\xi_N = 0$ and (9.17) implies (8.21). □

The same argument as for Proposition 9.1 gives

PROPOSITION 9.5. – *Suppose* $a \in \mathbb{P}_\chi(T,\xi')$ *satisfies* (8.9) *and* (8.11)(b). *Then for* $j \in \mathcal{G}(\xi')$

$$M_\chi \mathbb{E}_j f(a) = \mathbb{E}_j\big[f(0,\underline{a},M_\chi a_{\mathcal{G}},M_\infty a_{\mathcal{G}},0,a_{\mathcal{O}\cup\mathcal{I}},0) - f(0,\underline{a},0,M_\infty a_{\mathcal{G}},0,a_{\mathcal{O}\cup\mathcal{I}},0)\big].$$

This proposition and (8.23) lead us to consider

$$\begin{aligned}
&\mathcal{L}_j(M_\chi \sigma_{0,j} r_j) = \mathbb{E}_j\big[f(0,\underline{a_0},M_\chi a_{0,\mathcal{G}},M_\infty a_{0,\mathcal{G}},0,a_{0,\mathcal{O}\cup\mathcal{I}},0)\\
&\qquad\qquad - f(0,\underline{a_0},0,M_\infty a_{0,\mathcal{G}},0,a_{0,\mathcal{O}\cup\mathcal{I}},0)\big], \quad j\in\mathcal{G},\\
&M_\chi \sigma_{0,j}|_{\chi=0,\ \theta_j=\theta_0} = g_j(x',\theta_0),
\end{aligned}\tag{9.18}$$

where the $g_j \in C_0^\infty(T)$ are obtained from (8.14)(b) as before.

The system (9.18) is of the form (8.26) and is solved by constructing $b_0$ as in (8.35) and then $c_0$ as in (8.36).

PROPOSITION 9.6. – *There exist* $T_0 > 0$ *and a solution* $b_0 \in \Gamma^\infty_{\chi,0}(T_0,\xi')$ *to the semilinear system* (8.35).

*Proof.* – **1.** The first step is to establish the estimate (8.41) on the $n$th Fourier coefficient $b_{0,n}(\chi,x)$ and (8.42). Define $\zeta_n^+,\beta_n$ by rewriting (8.40) as

$$b^\wedge_{0,n}(\chi,\xi_0-i\gamma,\xi'',x_N) = \int_0^\chi e^{i(\chi-\chi')\zeta_n^+}\beta_n(\chi',\cdot)\,d\chi' + e^{i\chi\zeta^+}g_n^\wedge = B^\wedge_{n,1}+B^\wedge_{n,2}.\tag{9.19}$$

Integrate by parts twice in

$$B^\wedge_{n,1} = \int_0^\chi \frac{\partial_{\chi'}(e^{i(\chi-\chi')\zeta_n^+})}{-i\zeta_n^+}\beta_n(\chi',\cdot)\,d\chi'$$

to get

$$\begin{aligned}
2B^\wedge_{n,1}(\chi,\cdot) &= \frac{e^{i(\chi-\chi')\zeta_n^+}}{-i\zeta_n^+}\beta_n(\chi',\cdot)|_0^\chi + \int_0^\chi \frac{e^{i(\chi-\chi')\zeta_n^+}}{i\zeta_n^+}\big(-icn\rho(\chi'\langle n\rangle^{3/2})\big)h_n^\wedge(\chi',\cdot)\,d\chi'\\
&= \frac{i}{\zeta_n^+}\Bigg[\int_\chi^\infty e^{-i(\chi-\chi')\zeta_n^+}icn\rho(\chi'\langle n\rangle^{3/2})h_n^\wedge(\chi',\cdot)\,d\chi'\\
&\quad - e^{i\chi\zeta_n^+}\int_0^\infty e^{i\chi'\zeta_n^+}icn\rho(\chi'\langle n\rangle^{3/2})h_n^\wedge(\chi',\cdot)\,d\chi'\\
&\quad + \int_0^\chi e^{i(\chi-\chi')\zeta_n^+}icn\rho(\chi'\langle n\rangle^{3/2})h_n^\wedge(\chi',\cdot)\,d\chi'\Bigg].
\end{aligned}\tag{9.20}$$

Now $\zeta^+ = \sqrt{cn\mathbb{X}(\xi_0-i\gamma,\xi'')}$, $|\mathrm{Im}\,\mathbb{X}|=\gamma$, and

$$\operatorname*{supp}_{\chi'}\rho(\chi'\langle n\rangle^{3/2}) \subset [0,\langle n\rangle^{-3/2}],$$

so we read off from (9.19), (9.20) the estimate

$$|b_{0,n}|_{C(\chi,\,H^m_\gamma)} \leqslant \frac{C}{\sqrt{\gamma}\langle n\rangle}|h_n|_{C(\chi,\,H^m_\gamma)} + \langle g_n\rangle_{H^m_\gamma}, \tag{9.21}$$

for $C$ independent of $n$, $H^m_\gamma$ as in (7.2). (9.21) implies that $b_{0,n}$ must vanish in $x_0 < T$ if $h_n$ and $g_n$ do, so we may replace the $H^m_\gamma$ norms in (9.21) by $H^m_\gamma(T)$ norms (7.3). This gives (8.41) and summing over $n$ we obtain

$$|b_0(\chi,x,\theta)|_{\mathbb{A}^1_\theta(C(\chi,\,H^m_\gamma(T)))} \leqslant \frac{C}{\sqrt{\gamma}}|h|_{\mathbb{A}_\theta(C(\chi,\,H^m_\gamma(T)))} + \langle g\rangle_{\mathbb{A}^1_\theta(H^m_\gamma(T))}. \tag{9.22}$$

**2.** There exist constants $C_1, C_2$ independent of $\gamma > 0$, $T > 0$ such that

$$C_1|u(x)|_{H^m_\gamma(T)} \leqslant \sum_{|\alpha|\leqslant m} |e^{-\gamma x_0}\partial^\alpha u|_{L^2(\Omega_T)} \leqslant C_2|u|_{H^m_\gamma(T)}. \tag{9.23}$$

Indeed, the second inequality is obvious. When $\Omega_T = \overline{\mathbb{R}}^{N+1}_+$ the first inequality follows from the Plancherel theorem and the fact that

$$(D_t - i\gamma)^k\big(e^{-\gamma t}u\big) = e^{-\gamma t}D^k_t u.$$

Reduce to this case using the fact that for $u \in H^m_\gamma(T)$ there exist Seeley extensions $\tilde{u} \in H^m_\gamma$ such that

$$|\tilde{u}|_{m,\gamma} \leqslant C|u|_{m,\gamma,T}. \tag{9.24}$$

**3.** Fix $m > (N+1)/2$. Using (9.23) we may set $\gamma = 1/T$ in (9.22) to obtain

$$|b_0|_{\mathbb{A}^1_\theta(C(\chi,\,H^m(T)))} \leqslant C\sqrt{T}|h|_{\mathbb{A}_\theta(C(\chi,\,H^m(T)))} + \langle g\rangle_{\mathbb{A}^1_\theta(H^m(T))}. \tag{9.25}$$

Since

$$|\mathbb{E}_j h(x,\omega,b_0)|_{\mathbb{A}_\theta(C(\chi,\,H^m(T)))} \leqslant C|h|_{\mathbb{A}_\theta(C(\chi,\,H^m(T)))},$$

we may use (7.9), (7.12) and Picard iteration to find a solution $b_0(\chi,x,\theta) \in \mathbb{A}^1_\theta(C(\chi,H^m(T_0)))$, for some $T_0 > 0$. A continuation principle based on (7.13) yields $b_0 \in \mathbb{A}^1_\theta(C(\chi,H^\infty(T_0)))$ in the usual way.

**4.** Differentiate (8.35) to obtain (with obvious notation)

$$\begin{aligned} &\mathcal{L}\partial_\theta b_0 = \rho\mathbb{E}_j\left[\frac{\partial h}{\partial\omega}(x,\omega,b_0) + \frac{\partial h}{\partial b_0}\frac{\partial b_0}{\partial\theta}\right] \quad \big(\omega = (\theta_k), k \in \mathcal{G}\cup\mathcal{O}\cup\mathcal{I}\big),\\ &\partial_\theta b_0|_{\chi=0} = \partial_\theta g,\\ &\partial_\theta b_0 = 0 \quad \text{in } x_0 < 0. \end{aligned} \tag{9.26}$$

Apply (9.25) and use (7.9), (7.12) again to deduce

$$\partial_\theta b_0 \in \mathbb{A}^1_\theta\big(C(\chi,H^\infty(T_0))\big).$$

Repeating the argument we obtain

$$b_0 \in \mathbb{A}^\infty_\theta\big(C(\chi,H^\infty(T_0))\big).$$

Eq. (8.35) then gives

$$b_0 \in \mathbb{A}^\infty_\theta\big(C^\infty(\chi,H^\infty(T_0))\big). \tag{9.27}$$

**5.** To see $b_0 \in \Gamma^\infty_{\chi,0}(T_0,\xi')$ let $b_0^{(k)}$ denote the $k$th iterate in the Picard scheme and write

$$b_0^{(k)}(\chi,x,\theta) = B_1^{(k)} + B_2^{(k)}, \quad b_0 = B_1 + B_2 \tag{9.28}$$

where the $B_i^{(k)}$ correspond in the obvious way to the terms $B_{n,i}$ of (9.19). Observe that $B_2^{(k)}$ is the same for all $k$. Since $G \in \mathcal{C}_0^\infty(T)$ and

$$\operatorname{Im} \zeta_n^+ = \operatorname{Im} \sqrt{cn\mathbb{X}(\xi_0 - i\gamma, \xi'')} \geqslant C\sqrt{|n|}\sqrt{\gamma} \quad \text{for } \gamma \text{ large}, \tag{9.29}$$

it follows easily that $B_2 = B_2^{(k)} \in \Gamma^\infty_{\chi,0}(T_0,\xi')$.

For each $n$, $B_{n,1}$ in (9.19) is constant in $\chi$ for $\chi \geqslant 1$ since $\operatorname{supp}_\chi \rho(\chi\langle n\rangle^{3/2}) \subset [0, \langle n\rangle^{-3/2}]$. We have

$$\mathcal{L}B_{n,1} = \left(\frac{i\partial_\chi^2}{cn} + X\right)B_{n,1} = \rho\big(\chi\langle n\rangle^{3/2}\big)h_n = 0 \quad \text{in } \chi \geqslant 1. \tag{9.30}$$

Thus, $XB_{n,1} = 0$ in $\chi \geqslant 1$. Since $X$ is glancing and $B_{n,1} = 0$ in $x_0 < 0$ we deduce $B_{n,1} \equiv 0$ in $\chi \geqslant 1$. Thus $B_1 \in \Gamma^\infty_{\chi,0}(T_0,\xi')$ and we conclude $b_0 \in \Gamma^\infty_{\chi,0}(T_0,\xi')$. □

The following $L^2$ estimate for linear Schrödinger-type equations with *homogeneous* boundary conditions will be used in the construction of $c_0$ (8.36) and also in the solution of the higher profile equations. $H^1(x', L^{2*}(\chi, x_N, \theta))$ denotes the space of $H^1$ functions of $x'$ with values in the space of $L^2$ functions of $(\chi, x_N, \theta)$ with mean 0.

PROPOSITION 9.7. – *Let $a(\chi,x,\theta) = (a_j(\chi,x,\theta_j), j \in \mathcal{G}(\xi'))$ belong to $L^2_0((-\infty,T]: D^*) \cap H^1(x', L^{2*}(\chi, x_N, \theta))$ and satisfy*

$$\mathcal{L}_j a_j = \mathbb{E}_j(ba) + d_j, \quad j \in \mathcal{G}(\xi'), \tag{9.31}$$

*where $b(\chi,x,\theta) \in \mathbb{P}_{\chi,0}(T,\xi')$, $d_j(\chi,x,\theta) \in \Gamma^\infty_{\chi,0}(T,\xi')$. Then for $0 \leqslant x_0 \leqslant T$*

$$|a(\cdot,x_0,\cdot)|^2_{L^2(\chi,x'',\theta)} \leqslant \int_0^{x_0} e^{M(x_0-s)} |d(\cdot,s,\cdot)|^2_{L^2}\, ds, \tag{9.32}$$

*where $M$ depends on $|b|_{L^\infty}$.*

*Proof.* – Note that $a_j|_{\chi=0} = 0$ and $a_j|_{x_0<0} = 0$. In the usual way take the $L^2(\chi, x'', \theta_j)$ inner product of both sides of (9.31) with $a_j$ and integrate by parts, using the anti-self-adjointness of $iD_\chi^2 D_\theta^{-1}$. To conclude use $|\mathbb{E}_j(ba)|_{L^2} \leqslant C|b|_{L^\infty}|a|_{L^2}$ and Gronwall's inequality. □

*Remark* 9.1. – Similar estimates for $|\partial^k_{x_0} a(\cdot,x_0,\cdot)|^2_{L^2(\chi,\, H^m(x'',\theta))}$ follow for sufficiently regular $a$ by differentiation of (9.31) with respect to $(x_0, x'', \theta)$ (not $\chi$). These estimates, while useful for the linear higher order profile equations, don't help with the semilinear problem for $a_0$ because they give no $L^\infty$ control in $\chi$.

PROPOSITION 9.8. – *With $T_0$ as in Proposition 9.6 there exist $T_0'$, $0 < T_0' \leqslant T_0$ and a solution $c_0 \in \Gamma^\infty_{\chi,0}(T_0',\xi')$ to the semilinear system* (8.36). *$c_0$ vanishes to infinite order at $\chi = 0$.*

*Proof.* – **1.** In this step we prove Proposition 8.1, which implies the key estimate (8.46).

Let $W_1$ (respectively $W_2$) equal the left (respectively right) side of (8.45). Straightforward computation shows that both $W_i$ satisfy

$$
\begin{aligned}
&\left(\partial_t - i\frac{D_\chi^2 D_\theta^{-1}}{c_j}\right) W_i = 2e^{it\frac{D_\chi^2 D_\theta^{-1}}{c_j}} \frac{D_\chi D_\theta^{-1}}{c_j} k,\\
&W_i|_{\chi=0} = 0, \qquad (9.33)\\
&W_i = 0 \quad \text{in } x_0 < 0.
\end{aligned}
$$

$W_1 - W_2$ satisfies the hypotheses of Proposition 9.7, and since (9.32) continues to hold if $X_j$ is replaced by $\partial_t$ in the definition of $\mathcal{L}_j$, we have $W_1 = W_2$.

**2.** The function $k(\chi, x, \theta)$ in (8.38) lies in $\Gamma^\infty_{\chi,0}(T, \xi')$ and vanishes to infinite order at $\chi = 0$. From (8.46) and the fact that

$$
e^{ix_0\frac{D_\chi^2 D_\theta^{-1}}{c_j}} : D^* \to D^*
$$

we deduce the same properties for $c_0$.

**3.** With $c_0 = (c_{0,j}(\chi, x, \theta_j), j \in \mathcal{G}(\xi'))$ in (8.36) and $m > (N+2)/2$, use the estimate (8.46), the Moser-type estimates (7.18), (7.20), and iteration to find a solution $c_0 \in C_0(T_0', \Gamma^m)$ (7.22) for some $0 < T_0' < T_0$. A continuation principle based on (7.21) yields $c_0 \in C_0(T_0', \Gamma^\infty)$, and then the equation gives $c_0 \in C_0^\infty(T_0', \Gamma^\infty) \subset \Gamma^\infty_{\chi,0}(T_0', \xi')$. $c_0$ vanishes to infinite order at $\chi = 0$ because all the iterates do. □

This completes the determination of the $M_\chi \sigma_{0,j} \in \Gamma^\infty_{\chi,0}(T_0', \xi')$ and hence the construction of the leading profile $a_0(\chi, x, \theta) \in \mathbb{P}_\chi(T_0', \xi')$.

### 9.4. Construction of the higher order profiles

The construction of the higher profiles parallels the construction of $a_0$, but is much simpler since the equations are all linear. For example, there is no need for the cutoff $\rho(\chi\langle D_\theta\rangle^{3/2})$ or for the use of Wiener algebras $\mathbb{A}_\theta$. Thus, we'll focus here mainly on the final step in the construction of $a_1$, the solution of (8.30) for $M_\chi \sigma_{1,j}$, $j \in \mathcal{G}(\xi')$ with boundary data at $\chi = 0$ determined from (8.27). At this stage the other pieces of $a_1$ have already been determined and lie in $\mathbb{P}_{\chi,0}(T_0, \xi')$, where $T_0$ is the time of existence of $a_0$.

PROPOSITION 9.9. – *The system* (8.30) *for* $M_\chi \sigma_{1,j}$, $j \in \mathcal{G}(\xi')$, *with* $M_\chi \sigma_{1,j}|_{\chi=0} = g_j(x', \theta_j) \in C_0^\infty(T_0)$ *given has a unique solution in* $\Gamma^\infty_{\chi,0}(T_0, \xi')$, *where* $T_0$ *is the time of existence of* $a_0$.

*Proof.* – The first term on the right in (8.30) may be written

$$
M_\chi \mathbb{E}_j f'(a_0) a_1 = \mathbb{E}_j f'(a_0)\left(\sum_{j \in \mathcal{G}(\xi')} M_\chi \sigma_{1,j} r_j\right) + \mathcal{F}_j(\chi, x, \theta_j), \tag{9.34}
$$

where $\mathcal{F}_j \in \Gamma^\infty_{\chi,0}(T_0, \xi')$ is known. Thus, the problem has the same form as (9.31), but with boundary conditions $a_j|_{\chi=0} = g_j$. Write $a_j = t_j + u_j$, where

$$
\begin{aligned}
&\mathcal{L}_j t_j = 0,\\
&t_j|_{\chi=0} = g_j,
\end{aligned} \tag{9.35}
$$

$$
\begin{aligned}
&\mathcal{L}_j u_j = \mathbb{E}_j[b(t+u)] + d_j,\\
&u_j|_{\chi=0} = 0.
\end{aligned} \tag{9.36}
$$

The solution $t$ to (9.35) can be written down using the Fourier transform (see e.g. (8.40)) and lies in $\Gamma^{\infty}_{\chi,0}(T_0,\xi')$. Standard linear arguments using (9.32) and the higher derivative estimates described in Remark 9.1 yield a solution $u$ to (9.36) in

$$C_0^{\infty}\big((-\infty,T_0]:L^2\big(\chi,H^{\infty}(x'',\theta)\big)\big) \quad \text{(new but obvious notation).}$$

Eq. (9.36) then gives

$$u\in C_0^{\infty}\big((-\infty,T_0]:H^{\infty}(\chi,x'',\theta)\big).$$

Finally, to see $u\in C_0^{\infty}((-\infty,T_0]:\Gamma^{\infty})\subset \Gamma^{\infty}_{\chi,0}(T_0,\xi')$ we commute powers of $\chi$ through (9.36) using

$$[\chi, D_\chi^2 D_\theta^{-1}]u = 2\partial_\chi D_\theta^{-1}u, \tag{9.37}$$

and apply the above argument to deduce successively $\chi u\in C_0^{\infty}((-\infty,T_0:H^{\infty}(\chi,x'',\theta))$, $\chi^2 u\in C_0^{\infty}((-\infty,T_0):H^{\infty})$, etc. □

*Remark* 9.2. – (a) The profiles $a_2,\dots,a_M$ are constructed by the same analysis as for $a_1$. Note that in the interior equations for $M_\infty\sigma_{k,j}$, $j\in\mathcal{P}(\xi')\cup\mathcal{N}(\xi')$ the term $x_N^{[(M-1)/2]}\mathcal{R}_{0,j}(x,\theta_j)$ in (9.12) may be replaced by $x_N^{[(M-1)/2]-(k/2)}\mathcal{R}_{k,j}(x,\theta_j)$. As observed in Remark 8.30(a) we need only determine

$$(I-\mathbb{E})a_{M-1},\qquad M_\chi \underline{a_{M-1}},\qquad M_\chi\sigma_{M-1,j},\quad j\in\mathcal{O}(\xi')\cup\mathcal{I}(\xi')\cup\mathcal{P}(\xi')\cup\mathcal{N}(\xi'),$$

and $(I-\mathbb{E})a_M$ in order to solve $\mathbb{I}_0,\dots,\mathbb{I}_{\frac{M-2}{2}}$.

(b) By finite propagation speed every profile has compact support in $x'$ and is compactly supported in $x_N$ if not independent of $x_N$.

### 9.5. Analysis of the remainders

The proof of Theorem 6.1 is completed in the next Proposition.

PROPOSITION 9.10. – *The remainders $R_\varepsilon(x)$, $r_\varepsilon(x')$ in (6.1) lie in $\mathbb{D}^{m_1}_{\rho,0}(T_0)\cap\mathbb{B}^{m_1}_{\rho,0}(T_0)$, $\mathcal{D}^{m_1}_{\rho,0}(T_0)\cap\mathcal{B}^{m_1}_{\rho,0}(T_0)$ respectively for some $\rho>0$.*

*Proof.* – The error $L\tilde{u}_\varepsilon - f(\tilde{u}_\varepsilon,\overline{\tilde{u}})$ may be written as a sum of two pieces

$$x_N^{(M-1)/2}\mathcal{R}\big(x,(\theta_j)_{j\in\mathcal{P}\cup\mathcal{N}}\big)\big|_{\theta_j=\frac{\phi_j}{\varepsilon}} + \varepsilon^{(M-1)/2}\mathcal{S}_\varepsilon(\chi,x,\theta). \tag{9.38}$$

The first piece is the error in solving the complex transport equations for elliptic modes (e.g. (9.12)), and the second is the usual error coming from Taylor remainders and reflecting the failure of $\mathbb{I}_{\frac{M-1}{2}}$, $\mathbb{I}_{\frac{M}{2}}$ to hold. Note that $\mathcal{R}\in\mathbb{C}_0^{\infty}(T,\xi')$, $\mathcal{S}_\varepsilon\in\mathbb{P}_\chi(T,\xi')$ with smooth, bounded dependence on $\sqrt{\varepsilon}$, and both have the support property described in Remark 9.2(b). Thus,

$$\mathcal{S}_\varepsilon|_{\chi=\frac{x_N}{\sqrt{\varepsilon}},\,\theta=\frac{\phi}{\varepsilon}}\in\mathbb{D}^{m_1}_{\rho,0}(T_0)\cap\mathbb{B}^{m_1}_{\rho,0}(T_0)$$

for some $\rho$.

Since

$$x_N^{(M-1)/2}\mathcal{R}=\varepsilon^{(M-1)/2}\left(\frac{x_N}{\varepsilon}\right)^{(M-1)/2}\mathcal{R}\quad\text{and}\quad \operatorname{Im}\frac{\phi_j}{\varepsilon}=b_j\left(\frac{x_N}{\varepsilon}\right)$$

with $b_j > 0$ for $j \in \mathcal{P} \cup \mathcal{N}$, we have as well

$$\left(\frac{x_N}{\varepsilon}\right)^{(M-1)/2} \mathcal{R}\left(x, \left(\frac{\phi_j}{\varepsilon}\right)_{j \in \mathcal{P} \cup \mathcal{N}}\right) \in \mathbb{D}^{m_1}_{\rho,0}(T_0) \cap \mathbb{B}^{m_1}_{\rho,0}(T_0) \tag{9.39}$$

for some $\rho$.

The boundary remainder $r_\varepsilon$ is handled similarly. □

*Remark* 9.3 (*Weakly regular boundary frequencies*). – Suppose $\xi' = (\underline{\xi_0}, \underline{\eta}) \in \mathbb{R}^N \setminus 0$ satisfies (3.13), but all the associated elliptic modes $\beta_j(\xi') = (\xi', \xi^j_N(\xi'))$, $j \in \mathcal{P}(\xi') \cup \mathcal{N}(\xi')$ are only required to be *weakly regular* (Definition 3.5). Let $\underline{\xi} = (\underline{\xi_0}, \underline{\eta}, \underline{\xi_N}) = (\xi', \xi^j_N(\xi'))$ for some $j \in \mathcal{P}(\xi') \cup \mathcal{N}(\xi')$. Part (c) of Definition 3.5 allows us to define $\pi_j$ as before (Notation 4.1). The function $\xi_0(\eta, \xi_N)$ in part (a) of Definition 3.5 is real analytic in $\eta$ and analytic in $\xi_N$ near $(\underline{\eta}, \underline{\xi_N})$, so the complex vector field $X_{\underline{\xi}}$ may be defined as in (3.9) and (3.7) is still valid. Part (b) of Definition 3.5 implies that $E^+(\xi')$ is spanned by eigenvectors (not just generalized eigenvectors) of $\mathcal{A}(\xi')$, a point that is important in our solution of the profile equations.

The operators $\mathbb{E}$, $\mathbb{E}_j$, and $\mathcal{Q}$ are defined just as before. Instead of (8.10) we now have

$$\mathbb{E}a_k(\chi, x, \theta) = \underline{a_k}(\chi, x) + \sum_{j \in \mathcal{G} \cup \mathcal{O} \cup \mathcal{I}} \sigma_{k,j}(\chi, x, \theta_j) r_j + \sum_{j \in \mathcal{P} \cup \mathcal{N}} A_{k,j}(\chi, x, \theta_j) \tag{9.40}$$

where $A_{k,j}$ takes values in the eigenspace of $\mathcal{A}(\xi')$ associated to $\underline{\xi_N}$ (which equals the eigenspace of $A(\underline{\eta}, \underline{\xi_N})$ associated to $-\underline{\xi_0}$). In (9.11)(a), for example, $\sigma_{0,j} r_j$ should be replaced by $A_{0,j}$. Since the elliptic transport equations are again governed by complex vector fields like $X_{\underline{\xi}}$, their analysis goes through with no substantial changes.

## 10. Theorem 6.2: Exact solutions near approximate ones

The proof is similar to an argument in [4] but simpler. We give a brief version here for the sake of completeness.

*Notation* 10.1. – (a) For $\alpha \in \{0, 1, 2, \ldots\}$ $\partial^\alpha_x$ is any operator of the form $\partial^{\alpha_0}_{x_0} \cdots \partial^{\alpha_N}_{x_N}$ such that $\alpha_0 + \cdots + \alpha_N = \alpha$. $\partial^\alpha_{x'}$ is an analogous tangential operator.

(b) For $\alpha \in \{1, 2, 3, \ldots\}$ denote by $\partial^{\langle\alpha\rangle}\phi$ any product of the form $(\partial^{\alpha_1}\phi_{i_1}) \cdots (\partial^{\alpha_r}\phi_{i_r})$ where $1 \leqslant r \leqslant \alpha$, $\alpha_1 + \cdots + \alpha_r = \alpha$, $\alpha_i \geqslant 1$. If $\alpha = 0$ set $\partial^{\langle 0\rangle}\phi = 1$.

(c) $\|w\|'_{m,\mu,\lambda}$ is the analogue of (8.54) defined by replacing $\partial^\alpha_x$ with $\partial^\alpha_{x'}$.

(d) Let $|u|_* = |u|_{L^\infty}$.

LEMMA 10.1 ([4]). – *Let $\alpha_1 + \cdots + \alpha_r \leqslant k \leqslant m$, $\alpha_i \in \{0, 1, 2, \ldots\}$, and $u_i \in H^m_0(\Omega_T) \cap L^\infty(\Omega_T)$. For $1 \leqslant \lambda \leqslant \mu$*

$$\mu^{m-1} \left|(\partial^{\alpha_1} u_1) \cdots (\partial^{\alpha_r} u_r)\right|_{0,\mu,\lambda} \leqslant C \sum_{i=1}^r |u_i|_{m,\mu,\lambda} \left(\prod_{j \neq 1} |u_i|_*\right).$$

*Proof of Proposition* (8.52)(a). – **1.** For $0 \leqslant k \leqslant m_1$ we will apply the $L^2$ estimate (8.50) to the problem satisfied by $\mu^{m-k}\partial^k_{x'} w_{n+1}$, and then use the equation to estimate $\partial_{x_N}$ derivatives.

Below we'll write $m = m_2$, $w_{n+1} = w$, $w_n = b$, $C$ will always be a constant depending on $(K, \rho)$, $\phi$ is always some $C^\infty$ function of its arguments, and $0 < \varepsilon \leqslant 1 \leqslant \varepsilon\mu$.

We will show

$$\|w\|_{m,\mu,\lambda} \leqslant \frac{C}{\lambda}\|b\|_{m,\mu,\lambda} + \sum_{k=0}^{m} \varepsilon^{M_1-m}(\varepsilon\mu)^{m-k}\rho \tag{10.1}$$

for $\lambda \geqslant \lambda_0(K,\rho)$, and then set $\mu = \lambda/\varepsilon$ to get (8.55). The first step is to show

$$\|w\|'_{m,\mu,\lambda} \leqslant \frac{C}{\lambda}\|b\|_{m\mu,\lambda} + \sum_{k=0}^{m} \varepsilon^{M_1-m}(\varepsilon\mu)^{m-k}\rho \tag{10.2}$$

for $\lambda \geqslant \lambda'_0(K,\rho)$.

**2.** Set $\partial^k_{x'} = \partial^k$. For $0 \leqslant k \leqslant m$ the problem satisfied by $\partial^k w$ is

$$\begin{aligned} &L\partial^k w = \partial^k\{[f(\tilde{u}+b) - f(\tilde{u})] - \varepsilon^{M_1}R_\varepsilon\},\\ &Bw|_{x_N=0} = -\varepsilon^{M_1}\partial^k r_\varepsilon,\\ &w = 0 \quad \text{in } x_0 < 0. \end{aligned} \tag{10.3}$$

Write

$$f(\tilde{u}+b) - f(\tilde{u}) = g(\tilde{u},b)b.$$

Then $|\mu^{m-k}\partial^k[f(\tilde{u}+b) - f(\tilde{u})]|_{0,\mu,\lambda}$ yields terms of the form

$$\mu^{m-k}\big|\phi(\tilde{u},b)\partial^{\langle\alpha\rangle}\tilde{u}\partial^{\langle\beta\rangle}b\partial^\gamma b|_{0,\mu,\lambda}, \tag{10.4}$$

where $\alpha+\beta+\gamma = k$. $\tilde{u} \in \mathbb{D}^m_{\rho,0}(T)$, so

$$(10.4) \leqslant \mu^{m-k+\alpha}(\varepsilon\mu)^{-\alpha}C|\partial^{\langle\beta\rangle}b\partial^\gamma b|_{0,\mu,\lambda}, \tag{10.5}$$

with $\beta+\gamma = k-\alpha$. Lemma 10.2 implies

$$(10.5) \leqslant C|b|_{m,\mu,\lambda}. \tag{10.6}$$

We have

$$\mu^{m-k}\big|\partial^k(\varepsilon^{M_1}R_\varepsilon)\big|_{0,\mu,\lambda} \leqslant \varepsilon^{M_1-m}(\varepsilon\mu)^{m-k}\rho$$

since $R_\varepsilon \in \mathbb{B}^m_{\rho,0}(T)$, and a similar estimate holds for $\varepsilon^{M_1}\partial^k r_\varepsilon$. This gives (10.2). Using the equation to estimate normal derivatives gives (10.1). □

PROPOSITION 10.1 (Boundedness of the iterates $\|w_n\|_{m_1,\,\lambda/\varepsilon,\,\lambda}$). – *There exist* $\lambda_1(K,\rho) \geqslant \lambda_0(K,\rho)$ *and* $\varepsilon_1(\lambda)$ *such that for* $\lambda \geqslant \lambda_1$, $\varepsilon \in (0,\varepsilon_1]$ *the* $w_n$ *defined by* (8.52) *satisfy*

$$\begin{aligned} &\text{(a)} \quad \|w\|_{m_1,\,\lambda/\varepsilon,\,\lambda} \leqslant 2\varepsilon^{M_1-m_1}\phi(\lambda) \quad \textit{for all } n,\\ &\text{(b)} \quad |w_n|_* \leqslant K \quad \textit{for all } n. \end{aligned} \tag{10.7}$$

*Proof.* – (a) follows immediately from (8.55) for $\lambda \geqslant \lambda_1$ large enough. (b) follows from (a) for $\varepsilon_1$ small enough, since for $u \in H^{m_1}_0(\Omega_T)$, $0 < \delta < m_1 - ((N+1)/2)$, we have the Sobolev inequality [4]

$$|u|_* \leqslant \mu^{-\delta}C(\lambda)|u|_{m_1,\mu,\lambda}. \tag{10.8}$$

□

PROPOSITION 10.2 (Convergence of $(w_n)$). – *There exist $\lambda_2(K,\rho) \geqslant \lambda_1(K,\rho)$ and $\varepsilon_2(\lambda) \leqslant \varepsilon_1(\lambda)$ such that for $\lambda \geqslant \lambda_2$ the sequence $(w_n)$ converges in the $\| \|_{m_1,\ \lambda/\varepsilon,\ \lambda}$ norm, uniformly with respect to $\varepsilon \in (0,\varepsilon_2]$. The limit $w$ satisfies*

$$\|w\|_{m_1,\ \lambda/\varepsilon,\ \lambda} \leqslant 2\varepsilon^{M_1-m_2}\phi(\lambda). \tag{10.9}$$

*Hence $(w, w|_{x_N=0}) \in \varepsilon^{M_1}\mathbb{B}^{m_1}_{\sigma,0}(T) \times \varepsilon^{M_1}\mathcal{B}^{m_1}_{\sigma,0}(T)$ for some $\sigma > 0$.*

*Proof.* – $\Delta_{n+1} = w_{n+1} - w_n$ satisfies

$$\begin{aligned} &L\Delta_{n+1} = f(\tilde{u} + w_n) - f(\tilde{u} + w_{n+1}) = \Delta_n g(\tilde{u}, w_n, w_{n-1}),\\ &B\Delta_{n+1}|_{x_N=0} = 0,\\ &\Delta_{n+1} = 0 \quad \text{in } x_0 < 0. \end{aligned} \tag{10.10}$$

By the argument that produced (10.1) we use (10.7)(a) to show

$$\|\Delta_{n+1}\|_{m_1,\ \lambda/\varepsilon,\ \lambda} \leqslant \frac{C}{\lambda}|\Delta_n|_{m_1,\ \lambda/\varepsilon,\ \lambda} + C\phi_1(\lambda)|\Delta_n|_* \tag{10.11}$$

for $\lambda$ large and $\varepsilon \in (0, \varepsilon_1(\lambda)]$. (10.11) and (10.8) allow us to choose $\lambda_2(K,\rho)$, $\varepsilon_2(\lambda)$ so that

$$\|\Delta_{n+1}\|_{m_1,\ \lambda/\varepsilon,\ \lambda} \leqslant \frac{1}{2}\|\Delta_n\|_{m_1,\ \lambda/\varepsilon,\ \lambda}$$

for $\lambda \geqslant \lambda_2$, $\varepsilon \in (0,\varepsilon_2]$. □

## 11. Generic validity of the small divisor properties

We first recall a result from [20] which gives generic validity of the small divisor properties in the hyperbolic region

$$\mathcal{H} = \{\xi' \in \mathbb{R}^N \setminus 0\colon\ \mathcal{M}(\xi') = \mathcal{O}(\xi') \cup \mathcal{I}(\xi')\}.$$

For $\xi' \in \mathcal{H}$ we have

$$\phi(x) = \big(x'\xi' + x_N \xi_N^i(\xi')\big)_{i \in \mathcal{O}(\xi') \cup \mathcal{I}(\xi')}.$$

Set $k_i(\xi') = \xi_N^i(\xi')$.

PROPOSITION 11.1. – *Let $\mathcal{O} \subset \mathcal{H}$ be open. Suppose for each $j \in \{1,\dots,m\}$ there exists $l \neq j$ such that the $m-2$ vectors*

$$d_{\xi'}\left(\frac{k_i - k_j}{k_l - k_j}\right), \quad i \in \{1,\dots,m\} \setminus \{l,j\},$$

*are linearly independent in $\mathbb{R}^N$ for each $\xi' \in \mathcal{O}$. Then for almost every $\xi' \in \mathcal{O}$, $\xi'$ (or $\phi$) satisfies the small divisor condition* (4.12), (4.13).

In this section, as in this whole paper, we are mainly interested in $\xi'$ that correspond to glancing and elliptic modes as well as hyperbolic modes. To simplify the exposition we'll focus here on a $6 \times 6$ example where all three types of modes are present simultaneously. It will be clear that similar propositions can be formulated and proved in the same way for systems of any dimension. Proposition 11.1 represents the simplest case in this collection of results.

The proof depends on the following classical result in simultaneous diophantine approximation.

THEOREM 11.1 ([1]). – (a) *Fix* $\varepsilon > 0$. *For almost all* $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ *there exists a constant* $C_\theta > 0$ *such that*

$$|\alpha_1\theta_1 + \cdots + \alpha_n\theta_n - \alpha_{n+1}| \geqslant C_\theta |\alpha|^{-n-\varepsilon} \tag{11.1}$$

*for all* $\alpha \in \mathbb{Z}^{n+1} \setminus 0$.

(b) *In particular* (11.1) *holds when* $1, \theta_1, \dots, \theta_n$ *are algebraic numbers linearly independent over the rationals.*

*Notation* 11.1. – (a) Let $L$ as in (1.1) be a $6 \times 6$ system, $p(\xi) = p(\xi_0, \eta, \xi_N) = \det L(\xi)$, and $\mathcal{B} \subset \mathbb{R}^{N-1} \setminus 0$ an open set satisfying $\mathcal{B} \cap -\mathcal{B} = \emptyset$. Suppose $\xi_0 : \mathcal{B} \to \mathbb{R}$ is a $C^\infty$ function homogeneous of degree 1 such that for $\xi'(\eta) = (\xi_0(\eta), \eta)$, $\eta \in \mathcal{B}$,

$$\begin{aligned} p(\xi'(\eta), \xi_N) &= \big(\xi_N - k_1(\xi')\big)^2 \big(\xi_N - k_2(\xi')\big)\big(\xi_N - k_3(\xi')\big) \\ &\quad \times \big(\xi_N - k_4^+(\xi')\big)\big(\xi_N - \overline{k}_4^+(\xi')\big). \end{aligned} \tag{11.2}$$

Here the $k_i$ are $C^\infty$ functions on the submanifold of $\mathbb{R}^N$ parametrized by $\xi'(\eta)$, $k_1, k_2, k_3$ are real and distinct for each $\eta \in \mathcal{B}$, and $\operatorname{Im} k_4 > 0$ on $\mathcal{B}$. For $\eta \in \mathcal{B}$ define $k_i(-\xi'(\eta)) = -k_i(\xi'(\eta))$, $i = 1, 2, 3$, and $k_4(-\xi'(\eta)) = -\overline{k}_4(\xi'(\eta))$. Note that (11.2) holds now with $\xi'(\eta)$ replaced by $-\xi'(\eta)$.

(b) For $i = 1, \dots, 4$ and $\xi'(\eta) = (\xi_0(\eta), \eta)$,

$$\phi_i(x) = x'\xi' + x_N k_i\big(\xi'(\eta)\big); \qquad \phi_5(x) = -x'\xi' - x_N \overline{k}_4\big(\xi'(\eta)\big).$$

Write $k_4 = \sigma_4 + i\tau_4$.

PROPOSITION 11.2. – (a) *In the notation above suppose there exists* $l \in \{1, 2, 3\}$ *such that the* 2 *vectors*

$$d_\eta\left(\frac{k_i - \sigma_4}{k_l - \sigma_4}\right), \quad i \in \{1, 2, 3, 4\} \setminus \{l, 4\}, \tag{11.3}$$

*are linearly independent in* $\mathbb{R}^{N-1}$ *for each* $\eta \in \mathcal{B}$. *Then for almost every* $\eta \in \mathcal{B}$, $\xi' = (\xi_0(\eta), \eta)$ *satisfies the first small divisor condition* (4.12).

(b) *Assume* (11.3). *Suppose also that for each* $j \in \{1, 2, 3\}$ *there exists* $l \in \{1, 2, 3\} \setminus \{j\}$ *such that the vector*

$$d_\eta\left(\frac{k_i - k_j}{k_l - k_j}\right), \quad i \in \{1, 2, 3\} \setminus \{j, l\}, \tag{11.4}$$

*is nonzero for each* $\eta \in \mathcal{B}$. *Then for almost every* $\eta \in \mathcal{B}$, $\xi' = (\xi_0(\eta), \eta)$ *satisfies the second small divisor condition* (4.13).

*Proof.* – (a) Fix $\varepsilon > 0$. For $\alpha = (\alpha_1, \dots, \alpha_5) \in \mathbb{Z}^5 \setminus 0$

$$\begin{aligned} d_x(\alpha \cdot \phi) &= \big((\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \alpha_5)\xi', \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 \\ &\quad + (\alpha_4 - \alpha_5)\sigma_4 + i(\alpha_4 + \alpha_5)\tau_4\big) \\ &\equiv \beta(\alpha, \eta). \end{aligned} \tag{11.5}$$

If $\alpha_1+\alpha_2+\alpha_3+\alpha_4-\alpha_5\neq 0$ or $\alpha_4+\alpha_5\neq 0$ then $|\beta(\alpha,\eta)|\geqslant C|\eta|$. If both equal 0, then $(\alpha_1,\alpha_2,\alpha_3)\neq 0$ and with $l=1$ say in (11.3)

$$\begin{aligned}|\beta(\alpha,\eta)|&=\left|\alpha_1(k_1-\sigma_4)+\alpha_2(k_2-\sigma_4)+\alpha_3(k_3-\sigma_4)\right|\\&=|k_1-\sigma_4|\left|\alpha_1+\alpha_2\frac{k_2-\sigma_4}{k_1-\sigma_4}+\alpha_3\frac{k_3-\sigma_4}{k_1-\sigma_4}\right|.\end{aligned}\tag{11.6}$$

Define the $C^\infty$ map $f:\mathcal{B}\to\mathbb{R}^2$ by

$$f(\eta)=\left(\frac{k_2-\sigma_4}{k_1-\sigma_4},\frac{k_3-\sigma_4}{k_1-\sigma_4}\right).$$

Let $E$ be the set of measure zero in $\mathbb{R}^2$ where (11.1) fails to hold. By hypothesis $f$ is a submersion at every $\eta\in\mathcal{B}$, so the preimage $f^{-1}(E)$ has measure zero. Thus, for almost every $\eta\in\mathcal{B}$ there exists $C_\eta>0$ such that the right side of (11.6) $\geqslant C_\eta|\alpha_1,\alpha_2,\alpha_3|^{-2-\varepsilon}$.

(b) Fix $\varepsilon>0$. We now consider $\alpha\in Z(\xi')\setminus 0$, so $\alpha_4\geqslant 0$, $\alpha_5\geqslant 0$. It suffices to prove the result with $\mathcal{B}$ replaced by an arbitrary open set $\mathcal{O}\subset\subset\mathcal{B}$.

Define $S=\{\xi=(\xi',\xi_N)\in\mathbb{R}^N\times\mathbb{C}\colon \operatorname{Im}\xi_N\geqslant 0, |\xi|=1\}$ and let $P:(\mathbb{R}^N\times\mathbb{C})\setminus 0\to S$ be the map $P(\xi)=\xi/|\xi|$. For $\delta>0$ define

$$\mathcal{E}_\delta=\left\{\xi\in S\colon d(\xi,p^{-1}(0)\cap S)\geqslant\delta\right\},\tag{11.7}$$

where $d$ indicates distance measured on $S$.

There exists $C_\delta>0$ such that

$$|p(\xi)|\geqslant C_\delta\quad\text{for }\xi\in\mathcal{E}_\delta.\tag{11.8}$$

For $\eta\in\mathcal{O}$ let $\xi'=(\xi_0(\eta),\eta)$ or $(-\xi_0(\eta),-\eta)$. Provided $\delta_1>\delta_2>0$ are chosen small enough, if $P(\xi',\xi_N)\in(\mathcal{E}_{\delta_2})^c$ there exists an unique $l\in\{1,2,3,4\}$ and $C>0$ such that

$$\begin{aligned}&\text{(a)}\quad P\big(\xi',\xi_N+s(k_l(\xi')-\xi_N)\big)\in(\mathcal{E}_{\delta_1})^c\quad\text{for }0\leqslant s\leqslant 1,\\&\text{(b)}\quad\left|\frac{\partial p}{\partial\xi_N}\right|>C\text{ at }P(\xi',\xi_N)\quad\text{if }l\neq 1,\\&\text{(c)}\quad\left|\frac{\partial^2 p}{\partial\xi_N^2}\right|>C\text{ at }P(\xi',\xi_N)\quad\text{if }l=1,\\&\text{(d)}\quad\operatorname{Im}\xi_N=0\quad\text{if }l\in\{1,2,3\}\quad\text{and}\quad(\xi',\xi_N)=\beta(\alpha,\eta).\end{aligned}\tag{11.9}$$

Suppose first that $P(\beta(\alpha,\eta))\in\mathcal{E}_{\delta_2}$. Then by (11.8)

$$\left|p\big(\beta(\alpha,\eta)\big)\right|\geqslant C_{\delta_2}|\beta|^6.\tag{11.10}$$

Note that if both $\alpha_1+\alpha_2+\alpha_3+\alpha_4-\alpha_5=0$ and $\alpha_4+\alpha_5=0$, then $\alpha_1+\alpha_2+\alpha_3=0$ now. Thus, the argument of part (a) shows that except for $\eta$ in a set of measure zero

$$|\beta|^6\geqslant C_{\delta_2}\big(C_\eta|\alpha|^{-1-\varepsilon}\big)^6\tag{11.11}$$

in this case.

Suppose next that $P(\beta(\alpha,\eta)) \in (\mathcal{E}_{\delta_2})^c$, but $\beta \notin p^{-1}(0)$, and apply (11.9)(a) with $\beta(\alpha,\eta)$ in place of $(\xi',\xi_N)$. If $l=1$ in (11.9)(a) Taylor's formula gives, with $\beta(\alpha,\eta)=(\beta_1,\beta_2)$ and $M=\alpha_1+\alpha_2+\alpha_3+\alpha_4-\alpha_5$,

$$\begin{aligned} p\big(\beta(\alpha,\eta)\big) &= p\big(\beta(\alpha,\eta)\big) - p\big(M\xi', k_1(M\xi')\big) \\ &= \frac{1}{2}\frac{\partial^2 p}{\partial \xi_N^2}\big(\beta_1, k_1(\beta_1)+s\big[\beta_2-k_1(\beta_1)\big]\big)\cdot\big(\beta_2-k_1(\beta_1)\big)^2 \end{aligned} \tag{11.12}$$

for some $0<s<1$. By (11.9)(c) the first factor on the right in (11.12) had modulus $\geqslant C|\eta|^{m-2}$. The second factor, in the case where $M>0$ equals

$$\big[\alpha_2(k_2-k_1)+\alpha_3(k_3-k_1)\big]^2 \tag{11.13}$$

since (11.9)(d) implies $\alpha_4=\alpha_5=0$. Arguing just as in (a) we see that except for $\eta$ in a set of measure zero, this factor has modulus $\geqslant C_\eta|\alpha|^{-2-\varepsilon}$. The case $M<0$ is treated similarly (see below).

When $l=2$ or 3 in (11.9)(a), argue as above using (11.9)(b) instead of (11.9)(c).

Finally, if $l=4$ in (11.9)(a), one obtains a product similar to (11.12) in which the first factor has modulus $\geqslant C|\eta|^{m-1}$ by (11.9)(b). The second factor $(\beta_2-k_4(\beta_1))$ in the case $M>0$ equals

$$\big[\alpha_1(k_1-\sigma_4)+\alpha_2(k_2-\sigma_4)+\alpha_3(k_3-\sigma_4)+i\tau_4(2\alpha_5-\alpha_1-\alpha_2-\alpha_3)\big]. \tag{11.14}$$

If $2\alpha_5-\alpha_1-\alpha_2-\alpha_3\neq 0$, the modulus of (11.14) is $\geqslant C|\eta|$. If $2\alpha_5-\alpha_1-\alpha_2-\alpha_3=0$, then $(\alpha_1,\alpha_2,\alpha_3)\neq 0$ and $|\beta_2-k_4(\beta_1)|$ reduces again to (11.6).

When $M<0$ write

$$k_4(\beta_1)=k_4(M\xi')=k_4\big(-|M|\xi'\big)=-|M|\overline{k}_4(\xi').$$

The second factor now equals

$$\big[\alpha_1(k_1-\sigma_4)+\alpha_2(k_2-\sigma_4)+\alpha_3(k_3-\sigma_4)+i\tau_4(2\alpha_4+\alpha_1+\alpha_2+\alpha_3)\big] \tag{11.15}$$

and is handled like (11.14). □

*Remark* 11.1. – (a) In the analogue of Proposition 11.2 for the case when precisely $k$ glancing modes are present instead of just one, $\mathcal{B}$ should be replaced by an open subset of $\mathbb{R}^{N-k}\setminus 0$.

(b) Part (b) of Theorem 11.1 may be used to identify explicit choices of $\xi'$ for which the small divisor condition holds.

# Part III
## Examples of blow-up

### 12. Examples of blow-up caused by high-order glancing modes

In Theorems 6.1 and 6.3 we assumed that $p(\xi',\xi_N)=0$ had real roots $\xi_N$ of multiplicity at most two. Here we present examples showing that when real roots of order $\geqslant 3$ are present, the time of existence $T_\varepsilon$ can shrink to 0 as $\varepsilon\to 0$.

*Example* 1. – We work on $\overline{\mathbb{R}}^{2+1}_{+}=\{(x,t,y)\colon x\geqslant 0\}$ and denote dual variables by $(\xi,\tau,\eta)$. In this example part of the boundary data oscillates at a boundary frequency $(\tau_0,\eta_0)$ such that $p(\xi,\tau_0,\eta_0)=0$ has a triple real root $\xi_0$.

Let $P(D)$ be a third-order scalar operator with total symbol

$$p(\xi,\tau,\eta)=(\xi-\eta)^3+3\tau(\xi-\eta)^2+\left(3\tau\eta^2-\tau^3\right) \tag{12.1}$$

and consider the semilinear system for $u_\varepsilon, v_\varepsilon$

$$\begin{aligned}
&\text{(a)}\quad P(D)u_\varepsilon=0,\\
&\qquad u_\varepsilon|_{x=0}=0,\\
&\qquad D_x u_\varepsilon|_{x=0}=\varepsilon G(t,y)e^{i(\tau_0,\eta_0)\cdot(t,y)/\varepsilon},\\
&\qquad u_\varepsilon=0\quad \text{in } t<-T\quad (T>0),\\
&\text{(b)}\quad (\partial_t+\partial_x)v_\varepsilon=|\partial_x^2 u_\varepsilon|^\beta v_\varepsilon^2,\\
&\qquad v_\varepsilon|_{x=0}=H(t,y)e^{i(\tau_1,\eta_1)\cdot(t,y)/\varepsilon},\quad H(0)>0,\\
&\qquad v_\varepsilon=0\quad \text{in } t<-T.
\end{aligned} \tag{12.2}$$

Here $(\tau_0,\eta_0)=(0,1)$, $(\tau_1,\eta_1)\neq 0$ is arbitrary, $H$ and $G$ are supported in $-T\leqslant t\leqslant T$, and for $\gamma<1$ to be chosen we take

$$G(t,y)=e^{\gamma t}g_1(t)g_2(y), \tag{12.3}$$

where $g_1(t)$, $g_2(y)$ are smooth functions, and

$$\begin{aligned}
&\text{(a)}\quad \hat g_1(\tau)\geqslant 0,\quad \hat g_1>1\quad \text{on } [-b,b]\\
&\text{(b)}\quad \hat g_2(\eta)\geqslant 0,\quad \operatorname{supp}\hat g_2(\eta)\subset\left\{|\eta|\leqslant\gamma^{1+\delta}\right\}\quad \text{for some}\\
&\qquad 0<\delta<1/3,\quad \hat g_2(\eta)>1\quad \text{on } |\eta|\leqslant\gamma^{1+\delta}/2.
\end{aligned} \tag{12.4}$$

LEMMA 12.1 (Cardano). – *Consider the cubic $p(\zeta)=\zeta^3+3p\zeta+q$ where $p,q\in\mathbb{R}$. If $q^2+4p^3<0$ (respectively $>0$, $=0$), then $p(\zeta)$ has 3 distinct real (respectively, 1 real and 2 complex conjugate, repeated real) roots.*

The lemma applied to (12.1) implies that $P(D)$ is strictly hyperbolic with respect to $t$. Thus, (12.2) is a system with 2 blocks, with each block satisfying the Kreiss condition and noncharacteristic with respect to $x=0$.

*Remark* 12.1. – To see that two is the correct number of boundary conditions for $u_\varepsilon$ in (12.2), observe that for $\gamma>0$ $p(\xi,-i\gamma,0)=0$ has 2 roots with Im $\xi>0$, one with Im $\xi<0$. The same therefore holds for the roots of $p(\xi,\tau-i\gamma,\eta)=0$ for any $(\tau,\eta)\neq 0$.

LEMMA 12.2. – *Regarded as a cubic in $\xi$, $p(\xi,\tau,\eta)$ as in (12.1) has 3 distinct real (respectively, 1 real and 2 complex conjugate, repeated real) roots, when*

$$-\tau^2\left(\tau^2-3\eta^2\right)\left(\tau^2+\eta^2\right)<0\quad (\textit{respectively}>0,=0).$$

*Indeed, if $\tau=0$, $\xi=\eta$ is a triple root, while if $\tau\neq 0$, $\tau^2-3\eta^2=0$, then $\xi=\eta$ is a double root and $\xi=\eta-3\tau$ a simple root.*

*Proof.* – Set $\beta=\xi-\eta$ in (12.1) and note that $\beta^3+3\tau\beta^2+(3\tau\eta^2-\eta^3)$ takes the form $\zeta^3+3p\zeta+q$ as in Lemma 12.1 if $\zeta=\beta+\tau$ and $p=-\tau^2$, $q=\tau^3+3\tau\eta^2$. Thus,

$$q^2+4p^3=-3\tau^2\left(\tau^2-3\eta^2\right)\left(\tau^2+\eta^2\right).$$

The last sentence of the Lemma is obvious. □

*Remark* 12.2. – $\beta_0 = (\xi_0, \tau_0, \eta_0) = (1, 0, 1)$ is a glancing mode of order 3 for $P(D)$. That is, $\xi_0 = 1$ is a triple root of $p(\xi, \tau_0, \eta_0) = 0$.

The solution to (12.2)(a) is given by

$$u_\varepsilon(x,t,y) = \int e^{it(\tau - i\gamma + \frac{\tau_0}{\varepsilon}) + iy(\eta + \frac{\eta_0}{\varepsilon})} \varepsilon \hat{G}(\tau - i\gamma, \eta) \frac{e^{ix\xi_1(X)} - e^{ix\xi_2(X)}}{\xi_1(X) - \xi_2(X)} \, d\tau \, d\eta. \tag{12.5}$$

Here $X = ((\tau_0, \eta_0)/\varepsilon) + (\tau - i\gamma, \eta)$, $\xi_1(X)$, $\xi_2(X)$ are the roots of $p(\xi, X) = 0$ with positive imaginary parts, and we ignore factors of $1/2\pi$. For $a > 0$, $b_1 > 0$, $b_2 > 0$ small and $|t| \leqslant b_1 \varepsilon^a$, $x \sim \varepsilon^{(2+a)/3}$, $|y| \leqslant b_2$, we estimate

$$|\partial_x^2 u_\varepsilon| = e^{t\gamma} \left| \int e^{it\tau + iy\eta} \varepsilon \hat{g}_1(\tau) \hat{g}_2(\eta) Q(X) \, d\tau \, d\eta \right| \tag{12.6}$$

from below. In (12.6)

$$Q(X) = \frac{\xi_1^2(X) e^{ix\xi_1(X)} - \xi_2^2(X) e^{ix\xi_2(X)}}{\xi_1(X) - \xi_2(X)}. \tag{12.7}$$

*Remark* 12.3. – Blow-up will be caused by the fact that for $\tau, \gamma, \varepsilon$ small enough, $|\xi_1(X) - \xi_2(X)| \sim \varepsilon^{-2/3}$ since $\beta_0$ is glancing of order 3. If $\beta_0$ were hyperbolic or glancing of order 2, we would have $\varepsilon^{-1}$ instead. Along with the factor of $\varepsilon$ in (12.6), this would exactly cancel the $1/\varepsilon^2$ from $\xi_1^2(X)$ and $\xi_2^2(X)$, and blow-up would not occur. Roughly speaking, the boundary data in (12.2)(a) is oscillating with a single linear phase that "resonates" with a structural defect in the solution operator defined by (12.5). This mechanism is completely different from the focusing mechanisms in free space identified in [8]. The latter mostly involve phases with critical points.

**1.** With $Y_0 = (\tau_0, \eta_0)$, $Y = (\tau_0, \eta_0) + \varepsilon(\tau - i\gamma, \eta)$ we use a classical argument (e.g., Sakamoto [16]) to compare the zeros of $p(\xi, Y)$ to those of $p(\xi, Y_0)$. We have

$$p(\xi, Y) = (\xi - \xi_0)^3 + h_2(Y)(\xi - \xi_0)^2 + h_1(Y)(\xi - \xi_0) + h_0(Y), \tag{12.8}$$

where $h_j(Y_0) = 0$, $j = 0, 1, 2$. Here $h_0(Y) = p(\xi_0, Y)$ satisfies

$$\frac{\partial h_0}{\partial \tau}(Y_0) = 3\eta_0^2 \equiv C_0 > 0. \tag{12.9}$$

Write

$$h_0(Y) = C_0 \varepsilon(\tau - i\gamma) + \frac{\partial h_0}{\partial \eta}(Y_0)\varepsilon\eta + \mathrm{O}\big(|\varepsilon(\tau - i\gamma, \eta)|^2\big)$$

and set $q(\xi, Y) = (\xi - \xi_0)^3 + C_0 \varepsilon(\tau - i\gamma)$. For $\gamma < 1$, $|\eta| \leqslant \gamma^{1+\delta}$ (as is the case on supp $\hat{g}_2(\eta)$), $C_1 > C_0$, and $|\xi - \xi_0| \leqslant (C_1 \varepsilon |\tau, \gamma|)^{1/3}$ we obtain

$$\big|p(\xi, Y) - q(\xi, Y)\big| \leqslant C\big(\varepsilon^{4/3}|\tau, \gamma|^{4/3} + \varepsilon\gamma^{1+\delta} + \varepsilon^2|\tau, \gamma|^2\big), \tag{12.10}$$

where $C$ will always denote a constant independent of $(\varepsilon, \tau, \gamma, \eta)$.

Suppose now

$$|\tau| < \varepsilon^{-a} \quad \text{for some } 0 < a < \frac{1}{3}. \tag{12.11}$$

(12.10) implies there exists an $\varepsilon_0(\gamma)$ such that for $\varepsilon < \varepsilon_0(\gamma)$

$$|p(\xi,Y) - q(\xi,Y)| \leqslant 3C\varepsilon\gamma^{1+\delta}. \tag{12.12}$$

Write the roots of $q(\xi,Y)$ as

$$\tilde{\xi}_j = \xi_0 + \left[C_0\varepsilon(i\gamma-\tau)\right]_j^{1/3}, \quad j=1,2,3,$$

where $\operatorname{Im}\tilde{\xi}_j > 0$ for $j=1,2$. Let $k \in \{1,2,3\}$, fix $M>0$, and note that for $\gamma(M) < 1$ small enough, if $|\xi - \tilde{\xi}_k| = M\varepsilon^{1/3}\gamma^{(1/3)+\delta}$, then $|\xi - \xi_0| \leqslant (C_1\varepsilon|\tau,\gamma|)^{1/3}$ and $|\xi - \tilde{\xi}_j| \geqslant (C_0\varepsilon|\tau,\gamma|)^{1/3}$, for $j \neq k$. Thus,

$$|Q(\xi,Y)| = \prod_{j=1}^{3} |\xi - \tilde{\xi}_j| \geqslant \left(C_0\varepsilon|\tau,\gamma|\right)^{2/3} \cdot M\varepsilon^{1/3}\gamma^{1/3+\delta} \geqslant C_0^{2/3} M\varepsilon\gamma^{1+\delta}. \tag{12.13}$$

For $M$ such that $C_0^{2/3}M > 3C$ ($C$ as in (12.12)), (12.12), (12.13) and Rouché's Theorem imply

$$\xi_j(Y) = \tilde{\xi}_j + \mathrm{O}\left(\varepsilon^{1/3}\gamma^{1/3+\delta}\right) = \xi_0 + \left[C_0\varepsilon(i\gamma-\tau)\right]_j^{1/3} + \mathrm{O}\left(\varepsilon^{1/3}\gamma^{1/3+\delta}\right). \tag{12.14}$$

Thus,

$$\xi_j(X) = \frac{\xi_0}{\varepsilon} + \varepsilon^{-2/3}\left[C_0(i\gamma-\tau)\right]_j^{1/3} + \varepsilon^{-2/3}\mathrm{O}\left(\gamma^{1/3+\delta}\right) \tag{12.15}$$

for

$$\gamma = \gamma(M), \quad \varepsilon < \varepsilon_0(\gamma), \quad |\eta| \leqslant \gamma^{1+\delta}, \quad |\tau| \leqslant \varepsilon^{-a}, \quad 0 < a < 1/3. \tag{12.16}$$

**2.** We now determine the magnitude and direction of $\varepsilon Q(X)$ (12.7) when $X = ((\tau_0,\eta_0)/\varepsilon) + (\tau - i\gamma, \eta)$ is in the range defined by (12.16) and

$$x = \lambda\varepsilon^{(2+a)/3} \quad \text{for } 0 < \delta_1 < \lambda < \delta_2,\ \delta_1,\delta_2 \text{ small}. \tag{12.17}$$

(12.15) implies the main contribution to $\varepsilon Q(X)$ is given by

$$\varepsilon \cdot \frac{\xi_0^2}{\varepsilon^2} e^{i\lambda\xi_0\varepsilon^{(a-1)/3}} \left\{ \frac{e^{i\lambda C_0^{1/3}\varepsilon^{a/3}(i\gamma-\tau)_1^{1/3}} - e^{i\lambda C_0^{1/3}\varepsilon^{a/3}(i\gamma-\tau)_2^{1/3}}}{\varepsilon^{-2/3}C_0^{1/3}[(i\gamma-\tau)_1^{1/3} - (i\gamma-\tau)_2^{1/3}]} \right\}. \tag{12.18}$$

The remainder is smaller by a factor of $c(\varepsilon,\gamma) \ll 1$.

Since $|\tau| \leqslant \varepsilon^{-a}$, for $\delta_2$ small enough the numerator inside the braces in (12.18) equals

$$i\lambda C_0^{1/3}\varepsilon^{a/3}\left[(i\gamma-\tau)_1^{1/3} - (i\gamma-\tau)_2^{1/3}\right] \cdot F(\varepsilon,\gamma,\tau), \tag{12.19}$$

where $|F-1| < 1/10$ for $(\varepsilon,\gamma,\tau)$ as in (12.16). Thus,

$$(12.18) = \left(i\lambda\xi_0^2 e^{i\lambda\xi_0\varepsilon^{(a-1)/3}}\right) \cdot \left(\varepsilon^{(a-1)/3}\right) \cdot F. \tag{12.20}$$

The important property of (12.20) is that it blows up like $\varepsilon^{(a-1)/3}$ and maintains a nearly *fixed direction* (for fixed $\varepsilon,\lambda$) for all $|\tau| \leqslant \varepsilon^{-a}$. For $\varepsilon_1 < \varepsilon_0(\gamma)$ small enough, $\varepsilon Q(X)$ has the same property for $\varepsilon < \varepsilon_1$.

**3.** Consider the integral (12.6) over the region $|\tau| \leqslant \varepsilon^{-a}$, $|\eta| \leqslant \gamma^{1+\delta}$ for $\varepsilon < \varepsilon_1$, $|y| < b$, $t \sim \varepsilon^{(2+a)/3}$, and $x$ as in (12.17). (12.4) and the fact that $e^{it\tau+iy\eta} \sim 1$ imply there is essentially no cancellation of the contributions from (12.20) for $|\tau| \leqslant b$. Thus, the modulus of the integral over $|\tau| \leqslant \varepsilon^{-a}$, $|\eta| \leqslant \gamma^{1+\delta}$ is bounded below by

$$C(\lambda,\gamma,b)\varepsilon^{(a-1)/3}, \quad 0<a<1/3. \tag{12.21}$$

**4.** Since $\hat{g}_1(\tau)$ is rapidly decreasing in $\tau$ and $|e^{ix\xi_j(X)}| \leqslant 1$, any polynomial (in $|\tau,\varepsilon^{-1}|$) upper bound on $|\xi_1(X)-\xi_2(X)|^{-1}$ for $|\tau| \geqslant \varepsilon^{-a}$ will imply that the contribution to (12.6) from the region $|\tau| \geqslant \varepsilon^{-a}$, $|\eta| \leqslant \gamma^{1+\delta}$ is negligible.

Let

$$S = \{Z=(\tau,\gamma,\eta)\colon |\tau,\gamma,\eta| = 1,\ \gamma \geqslant 0\}.$$

For $\delta_3$ small let

$$S_1 = \{Z \in S\colon |(\gamma,\eta)-(0,\eta_0)| \leqslant |\tau|^{4/3} \leqslant \delta_3^{4/3}\}$$
$$S_2 = \{Z \in S\colon |(\gamma,\eta)-(0,\eta_0)| \leqslant \delta_3^{4/3},\ |\tau| \geqslant \delta_3\}.$$

Since $p(\xi,\tau,\eta)$ has triple zeros in $\xi$ only when $\tau=0$, we have

$$|\xi_1(Z)-\xi_2(Z)| \geqslant C > 0 \quad \text{for } Z \in S_2. \tag{12.22}$$

For $Z \in S_1$ an argument using Rouché's Theorem as in part **1** of this proof shows

$$|\xi_1(Z)-\xi_2(Z)| \geqslant C|\tau|^{1/3} \quad \text{for } Z \in S_1. \tag{12.23}$$

Using the homogeneity of $\xi_j(X)$, we see that if $X = ((\tau_0,\eta_0)/\varepsilon) + (\tau - i\gamma,\eta)$ projects to $S_2$ (respectively $S_1$) under $X \to X/|X|$, (12.22) and (12.23) imply

$$\begin{gathered} |\xi_1(X)-\xi_2(X)| \geqslant C|X| \\ \big(\text{respectively } |\xi_1(X)-\xi_2(X)| \geqslant C|X|^{2/3}|\tau|^{1/3}\big). \end{gathered} \tag{12.24}$$

Provided $0<a<1/3$ and $(4/3)(1-a)<1$, the restrictions on $(\tau,\gamma,\eta,\varepsilon)$ imply these are the only $X$ we need to consider. From (12.21) and (12.24) we conclude for such $a$

$$|\partial_x^2 u_\varepsilon| \geqslant C(\lambda,\gamma,b)\varepsilon^{(a-1)/3}, \tag{12.25}$$

for $|t| \leqslant b_1\varepsilon^a$, $x = \lambda\varepsilon^{(2+a)/3}$, $0<\delta_1<\lambda<\delta_2$, $|y| \leqslant b_2$.

**5.** As in [8] we demonstrate blow-up in $v_\varepsilon$ using a simple fact about ODEs.

LEMMA 12.3. – *Suppose $a(t)$ is a smooth nonnegative function on $[0,\infty)$ and $\int_0^\infty a(t)\,dt > 1/y_0$. The maximal interval of existence $[0,T_0)$ for the initial value problem*

$$y' = a(t)y^2, \quad y(0) = y_0 > 0$$

*is given by $\int_0^{T_0} a(t)\,dt = 1/y_0$.*

Consider $v_\varepsilon$ on the characteristic of $\partial_t + \partial_x$ starting at $(x,y,t)=0$. (12.25) and Lemma 12.3 imply that if $\beta$ is large enough ($\beta > (2+a)/(1-a)$), the maximal time of existence on this characteristic is less than $C\varepsilon^{(2+a)/3}$.

**6.** The example is not yet completely satisfactory since the data in (12.2) is supported in $t \geqslant -T$, and we have not ruled out $[-T,-T/2]$, for example, as an interval of existence for $v_\varepsilon$ independent of $\varepsilon$. To rectify this defect replace $G(t,y)$ and $H(t,y)$ in (12.2) by

$$G_k(t,y)=\frac{1}{k^M}G(kt,y),\qquad H_k(t,y)=\frac{1}{k^M}H(kt,y).$$

This data is supported in $t \geqslant -T/k$ and for any $s$, the $H^s$ norm of $(G_k,H_k)$ is uniformly bounded with respect to $k\in\{1,2,3,\ldots\}$, provided $M=M(s)$ is big enough. Let $u_{\varepsilon,k}$, $v_{\varepsilon,k}$ be the solution to $(12.2)_k$ defined by this new data. The preceding argument shows

$$|\partial_x^2 u_{\varepsilon,k}| \geqslant C(k,\lambda,\gamma,b)\varepsilon^{(a-1)/3}, \tag{12.26}$$

and the maximal time of existence of $v_{\varepsilon,k}$ on the characteristic of $\partial_t+\partial_x$ through 0 is less than $C(k)\varepsilon^{(2+a)/3}$. Theorem 4.2 of [17] shows that if $p(\xi,\tau,\eta)=0$ had real $\xi$ roots of multiplicity at most two, a uniform (in $k$) bound $|G_k,H_k|_{H^s}\leqslant C_s$ for $s>3/2$ would imply a time of existence $T(C_s)$ for solutions to $(12.2)_k$ independent of $(k,\varepsilon)$.

*Example* 2. – Consider (12.2) again where the boundary data for $D_x u_\varepsilon$ is replaced by

$$\varepsilon\sum_{k=M}^{\infty}\frac{1}{k^{1+\mu}}G(t,y)e^{i(\tau_k,\eta_k)\cdot(t,y)/\varepsilon}, \tag{12.27}$$

where $(\tau_k,\eta_k)=(1/k,\eta_0+(1/(k^{1+\alpha})))\to(\tau_0,\eta_0)=(0,1)$ as $k\to\infty$ and $M,\mu,\alpha$ are positive constants to be chosen. In the notation of (3.11) we have for $k$ large

$$E^+(\tau_k,\eta_k)=E^+\bigl(\beta_1(\tau_k,\eta_k)\bigr)\oplus E^+\bigl(\beta_2(\tau_k,\eta_k)\bigr), \tag{12.28}$$

where $\beta_1(\tau_k,\eta_k)$ is an outgoing hyperbolic mode and $\beta_2(\tau_k,\eta_k)$ an elliptic mode with $\operatorname{Im}\beta_2>0$. Observe that the projections $P(\beta_i(\tau_k,\eta_k))$ of $E^+(\tau_k,\eta_k)$ onto its components in (12.28) blow up as $k\to\infty$ (recall Remark 3.2).

**1.** The solution to (12.2) with this new data is $u_\varepsilon=\sum_{k=M}^{\infty}u_{\varepsilon,k}$, where

$$\partial_x^2 u_{\varepsilon,k}=\frac{1}{k^{1+\mu}}e^{it\frac{\tau_k}{\varepsilon}+iy\frac{\eta_k}{\varepsilon}+\gamma t}\int e^{it\tau+iy\eta}\varepsilon\hat g_1(\tau)\hat g_2(\eta)Q(X_k)\,d\tau\,d\eta, \tag{12.29}$$

$X_k=((\tau_k,\eta_k)/\varepsilon)+(\tau-i\gamma,\eta)$, and $Q(X)$ is defined by (12.7). For $0<a<1/3$ as in Example 1, $b_1>0$, $b_2>0$ small, and $|t|\leqslant b_1\varepsilon^a$, $x\sim\varepsilon^{(2+a)/3}$, $|y|\leqslant b_2$, we proceed to estimate

$$\sum_{k=M}^{\infty}\partial_x^2 u_{\varepsilon,k} \tag{12.30}$$

from below. Rewrite (12.30) as

$$\sum_{M<k\leqslant(4\varepsilon^{1-a})^{-1}}\partial_x^2 u_{\varepsilon,k}+\sum_{k>(4\varepsilon^{1-a})^{-1}}\partial_x^2 u_{\varepsilon,k}=S_1+S_2. \tag{12.31}$$

**2.** Choose $\alpha$ such that

$$(1-a)(1+\alpha)=1, \tag{12.32}$$

and note that for $t, y$ as above and $k > (4\varepsilon^{1-a})^{-1}$, $e^{it\frac{\tau_k}{\varepsilon}+iy\frac{\eta_k}{\varepsilon}+\gamma t}$ stays close to $e^{iy\frac{\eta_0}{\varepsilon}}$. An analysis similar to the one in parts 1–3 of Example 1 shows that each term in $S_2$ has magnitude $\geqslant (C/k^{1+\mu})\varepsilon^{(a-1)/3}$, and these contributions have a nearly fixed direction independent of $k$ (but not $\varepsilon$). In this argument $(1/k) + \varepsilon(\tau - i\gamma)$ plays the role of $\varepsilon(\tau - i\gamma)$ in Example 1. Thus,

$$|S_2| \geqslant \varepsilon^{(a-1)/3} \sum_{k>(4\varepsilon^{1-a})^{-1}} \frac{1}{k^{1+\mu}} = \frac{C}{\mu}\varepsilon^{(a-1)(1/3-\mu)}. \tag{12.33}$$

**3.** We claim that each term in $S_1$ has magnitude $\leqslant$

$$\frac{C_B}{k^{1+\mu}}\left(k^{1/3} + \varepsilon^B\right), \quad \text{for any } B > 0. \tag{12.34}$$

The $\varepsilon^B$ piece corresponds to the integral over $|\tau| \geqslant \varepsilon^{-a}$ (recall $\hat{g}_1(\tau)$ is rapidly decreasing). The $k^{1/3}$ piece appears since

$$\left|\xi_1(X_k) - \xi_2(X_k)\right| \sim \frac{1}{k^{1/3}} \quad \text{for } \frac{1}{M} > \frac{1}{k} \geqslant 4\varepsilon^{1-a},\ |\tau| \leqslant \varepsilon^{-a},$$

if $M$ is large enough. Thus,

$$|S_1| \leqslant C \sum_{M<k<(4\varepsilon^{1-a})^{-1}} k^{(1/3)-1-\mu} = C\frac{\varepsilon^{(a-1)(1/3-\mu)}}{\frac{1}{3}-\mu}. \tag{12.35}$$

**4.** $|S_2|$ blows up as $\varepsilon \to 0$ for $0 < \mu < 1/3$ and is much larger than the right side of (12.35) for $\mu$ near 0. The argument of Example 1, Part **5**, shows that $v_\varepsilon$ blows up for $t \sim \varepsilon^{(2+a)/3}$ in this example.

As in Part **6** of Example 1, we may replace $G(t,y)$ in (12.27) and $H(t,y)$ in (12.2)(b) by $G_j(t,y) = (1/j^L)G(jt,y)$, $H_j(t,y) = (1/j^L)H(jt,y)$ to obtain data supported in $t \geqslant -T/j$, $j \in \{1,2,\ldots\}$. Set

$$g_j(t,y,\theta') = \left(\sum_{k=M}^{\infty} \frac{1}{k^{1+\mu}} G_j(t,y)e^{i(\tau_k,\eta_k)\cdot\theta'}, H_j(t,y)e^{i(\tau_1,\eta_1)\cdot\theta'}\right). \tag{12.36}$$

For any $s$, $|g_j|_{\mathbb{A}_{\theta'}(H^s)} \leqslant C_s$ for all $j$, provided $L = L(s)$ is large enough. If there were no glancing modes of order $> 2$, Theorem 4.2 of [17] would yield a time of existence $T(s)$, $s > 3/2$, independent of $(j,\varepsilon)$ for solutions to $(12.2)_j$ defined by the data (12.36). In our example the maximal time of existence of $v_{\varepsilon,j}$ on the characteristic of $\partial_t + \partial_x$ through 0 is $\leqslant C(j)\varepsilon^{(2+a)/3}$.

*Remark* 12.4. – The blow-up in this example depends on a delicate balance between the size of the convergence factor $1/k^{1+\mu}$ in (12.27), the rate of blow-up of individual terms $\varepsilon^{(a-1)/3}$, and the rate of convergence $(\tau_k,\eta_k) \to (\tau_0,\eta_0)$ (recall $(1-a)(1+\alpha) = 1$). For example, if $(1-a)\mu > 1/3$, blow-up does not occur.

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Mark WILLIAMS
Department of Mathematics,
University of North Carolina,
Chapel Hill, NC 27599, USA
E-mail: williams@math.unc.edu