ANNALES SCIENTIFIQUES DE L'É.N.S.

ADAM EPSTEIN
MICHAEL YAMPOLSKY

Geography of the cubic connectedness locus : intertwining surgery

Annales scientifiques de l'É.N.S. 4e série, tome 32, nº 2 (1999), p. 151-185
<http://www.numdam.org/item?id=ASENS_1999_4_32_2_151_0>

© Gauthier-Villars (Éditions scientifiques et médicales Elsevier), 1999, tous droits réservés.

L'accès aux archives de la revue « Annales scientifiques de l'É.N.S. » (http://www.elsevier.com/locate/ansens) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. scient. Éc. Norm. Sup.,
4e série, t. 32, 1999, p. 151 à 185.

GEOGRAPHY OF THE CUBIC CONNECTEDNESS LOCUS: INTERTWINING SURGERY

BY ADAM EPSTEIN AND MICHAEL YAMPOLSKY

ABSTRACT. – We exhibit products of Mandelbrot sets in the two-dimensional complex parameter space of cubic polynomials. Cubic polynomials in such a product may be renormalized to produce a pair of quadratic maps. The inverse construction intertwining two quadratics is realized by means of quasiconformal surgery. The associated asymptotic geography of the cubic connectedness locus is discussed in the Appendix. © Elsevier, Paris

RÉSUMÉ. – Nous trouvons des produits de l'ensemble de Mandelbrot dans l'espace à deux variables complexes des polynômes cubiques. La renormalisation d'un polynôme cubique appartenant à un tel produit donne deux polynômes quadratiques. Le procédé inverse qui entrelace deux polynômes quadratiques est obtenu par chirurgie quasiconforme. La géométrie asymptotique du lieu de connexité cubique associée est décrit dans l'appendice. © Elsevier, Paris

Contents

1. Introduction 152

2. Preliminaries 152
- 2.1. Polynomial dynamics . . . 152
- 2.2. Polynomial-like maps . . . 154
- 2.3. Quadratic polynomials . . . 155
- 2.4. Cubic polynomials . . . 157
- 2.5. Surgical tools . . . 159

3. Outline of the Results 162

4. Intertwining surgery 164
- 4.1. History . . . 164
- 4.2. Construction of a cubic polynomial . . . 164
- 4.3. Quasiconformal interpolation . . . 167

5. Renormalization 169
- 5.1. Birenormalizable cubics . . . 169
- 5.2. Properness . . . 171
- 5.3. Injectivity . . . 172
- 5.4. Conclusions . . . 174

6. Measure of the Residual Julia Set 175

Appendix A. Discontinuity at the corner point 178
Appendix B. Non local-connectivity of the real connectedness locus 179
References 184

1. Introduction

The prevalence of Mandelbrot sets in one-parameter complex analytic families is a well-studied phenomenon in conformal dynamics. Its explanation in [DH2] has given rise to the theory of *renormalization*, and has inspired many efforts, starting with the seminal work [BD], to invert this procedure by means of *surgery* on quadratic polynomials.

In this paper we exhibit products of Mandelbrot sets in the two-dimensional complex parameter space of cubic polynomials. These products were observed by J. Milnor in computer experiments which inspired Lavaurs' proof of non local-connectivity for the cubic connectedness locus [La]. Cubic polynomials in such a product may be renormalized to produce a pair of quadratic maps. The inverse construction is an *intertwining surgery* on two quadratics. The idea of intertwining first appeared in a collection of problems edited by Bielefeld [Bi2]. Using quasiconformal surgery techniques of Branner and Douady [BD], we show that any two quadratics may be intertwined to obtain a cubic polynomial. The proof of continuity in our two-parameter setting requires further considerations involving ray combinatorics and a pullback argument.

After this project was finished, we were informed by P. Haissinsky that he is independently working on related problems [Haï].

2. Preliminaries

In this section we discuss the relevant facts and tools of holomorphic dynamics. We assume that the reader is familiar with the basic notions and principles of the theory of quasiconformal maps (see [LV] for a comprehensive account). The knowledgeable reader is invited to proceed directly to §3.

2.1. Polynomial dynamics. Julia sets, external rays, landing theorems, combinatorial rotation number, Yoccoz inequality

We recall the basic definitions and results in the theory of polynomial dynamics. Supporting details may be found in [Mil1].

Let $P : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree $d \geq 2$. The *filled Julia set* of P is defined as

$$K(P) = \{z \in \mathbb{C} | \{P^{\circ n}(z)\} \text{ is bounded}\}$$

and the *Julia set* as $J(P) = \partial K(P)$. Both of these are nonempty compact sets which are connected if and only if all critical points of P have bounded orbits.

Recall that if P is a monic polynomial with connected Julia set then there exists a unique analytic homeomorphism (the *Böttcher map*)

$$B_P : \mathbb{C} \setminus K(P) \to \mathbb{C} \setminus \bar{\mathbb{D}}$$

which is tangent to the identity at infinity, that is $B_P(z)/z \to 1$ as $z \to \infty$. The Böttcher map conjugates P to $z \mapsto z^d$,

$$B_P(P(z)) = (B_P(z))^d,$$

thereby determining a dynamically natural polar coordinate system on $\mathbb{C} \setminus K(P)$. For $\rho > 1$ the *equipotential* E_ρ is the inverse image under B_P of the circle $\{\rho e^{2\pi i\theta} | \theta \in \mathbb{R}\}$. The *external ray at angle* θ is similarly defined as the inverse image r_θ of the radial line $\{\rho e^{2\pi i\theta} | \rho > 1\}$. Since P maps r_θ to $r_{d\cdot\theta}$, the ray r_θ is *periodic* if and only if the angle θ is periodic (mod 1) under multiplication by d. An external ray r_θ is said to *land* at a point $\zeta \in J(P)$ when

$$\lim_{\rho \to 1} B_P^{-1}(\rho e^{2\pi i\theta}) = \zeta.$$

We note that if the Julia set of P is locally connected then all rays r_θ land, and their endpoints depend continuously on the angle θ (see the discussion in [Mil1]). We refer to [Mil1] for the proofs of the following results:

THEOREM 2.1 (Douady and Hubbard, Sullivan). – *If K(P) is connected, then every periodic external ray lands at a periodic point which is either repelling or parabolic.*

THEOREM 2.2 (Douady, Milnor, Yoccoz). – *If $K(P)$ is connected, every repelling or parabolic periodic point is the landing point of at least one external ray which is necessarily periodic.*

The landing points of such rays depend continuously on parameters:

LEMMA 2.3 ([GM]). – *Let P_t be a continuous family of monic degree-d polynomials with continuously chosen repelling periodic points ζ_t. If the ray of angle θ for P_{t_0} lands at ζ_{t_0}, then for all t close to t_0 the ray of angle θ for P_t lands at ζ_t.*

Kiwi has proved the following useful separation principle which directly illustrates why a degree-d polynomial admits at most $d-1$ non-repelling periodic orbits; the latter result was earlier shown by Douady and Hubbard and appropriately generalized to rational maps by Shishikura.

THEOREM 2.4. – *Let P be a polynomial with connected Julia set, n a common multiple of the periods of non-repelling periodic points, $\mathcal{R}$ the union of all external rays fixed under $P^{\circ n}$ together with their landing points, and $U_1, \ldots, U_m$ be the connected components of $\mathbb{C} \setminus \bigcup_{j=0}^{n} P^{-\circ j}(\mathcal{R})$. Then:*

- *Each component U_i contains at most one non-repelling periodic point;*
- *Given any non-repelling periodic orbit $\zeta_1, \ldots, \zeta_\ell$ passing through $U_{i_1}, \ldots, U_{i_\ell}$, at least one of the components U_{i_k} also contains some critical point.*

We assume henceforth that $K(P)$ is connected. Let $r = r_\theta$ be a periodic external ray landing at the periodic point $\zeta \in K(P)$, whose orbit we enumerate

$$\zeta = \zeta_0 \mapsto \zeta_1 \mapsto \ldots \mapsto \zeta_n = \zeta.$$

Denote by $A_i \subset \mathbb{Q}/\mathbb{Z}$ the set of angles of the rays in the orbit of r landing at ζ_i. The iterate $P^{\circ n}$ fixes each point ζ_i, permuting the various rays landing there while

preserving their cyclic order. Equivalently, multiplication by d^n carries the set A_i onto itself by an order-preserving bijection. For each i we may label the angles in A_i as $0 \leq \theta^1 < \theta^2 < \ldots < \theta^q < 1$; then

$$d^n \theta^i \equiv \theta^{i+p} \pmod 1$$

for some integer p, and we refer to the ratio p/q as the *combinatorial rotation number* of r. The following theorem of Yoccoz (see [Hub]) relates the combinatorial rotation number of a ray landing at a period n point ζ to the multiplier $\lambda = (P^{\circ n})'(\zeta)$.

YOCCOZ INEQUALITY. – *Let P be a monic polynomial with connected Julia set, and $\zeta \in K(P)$ a repelling fixed point with multiplier λ. If ζ is the landing point of m distinct cycles of external rays with combinatorial rotation number p/q then*

$$\frac{\mathrm{Re}\rho}{|\rho - 2\pi i p/q|^2} \geq \frac{mq}{2\log d}. \tag{2.1}$$

where ρ is the suitable choice of $\log \lambda$.

More geometrically, the inequality asserts that ρ lies in the closed disc of radius $\log d/(mq)$ tangent to the imaginary axis at $2\pi i p/q$.

2.2. Polynomial-like maps. Hybrid equivalence, Straightening Theorem, continuity of straightening

Polynomial-like mappings, introduced by Douady and Hubbard in [DH2], are a key tool in holomorphic dynamics. A *polynomial-like mapping of degree d* is a proper degree-d holomorphic map $f : U \to V$ between topological discs, where U is compactly contained in V. One defines the filled Julia set

$$K(f) = \{z \in U | f^{\circ n}(z) \in U, \ \forall n \geq 1\}$$

and the Julia set $J(f) = \partial K(f)$. We say that the map f is *quadratic-like* if of $d = 2$, and *cubic-like* if $d = 3$.

Polynomial-like maps $f : U \to V$ and $\tilde{f} : \tilde{U} \to \tilde{V}$ are *hybrid equivalent*

$$f \underset{\mathrm{hb}}{\sim} \tilde{f}$$

if there exists a quasiconformal homeomorphism h from a neighborhood of $K(f)$ to a neighborhood of $K(\tilde{f})$, such that $h \circ f = \tilde{f} \circ h$ near $K(f)$ and $\bar{\partial} h = 0$ almost everywhere on $K(f)$. We remark that h can be chosen to be a conjugacy between $f|_U$ and $\tilde{f}|_{\tilde{U}}$. Notice that h is conformal on the interior of $K(f)$ and therefore preserves the multipliers of attracting periodic orbits. In view of the well-known quasiconformal invariance of indifferent multipliers, we observe:

REMARK 2.5. – *A hybrid equivalence between polynomial-like maps sends repelling orbits to repelling orbits, and preserves the multipliers of attracting and indifferent orbits.*

The following is fundamental:

THEOREM 2.6 (Straightening Theorem, [DH2]). – *Every polynomial-like mapping $f : U \to V$ of degree d is hybrid equivalent to a polynomial P of degree d. If $K(f)$ is connected then P is unique up to conjugation by an affine map.*

For a quadratic-like f with connected Julia set, we write $\chi(f) = c$ where

$$f_c(z) = z^2 + c$$

is the unique hybrid equivalent polynomial. The following theorem is due to Douady and Hubbard; we employ the formulation of [McM2, Prop. 4.7]:

THEOREM 2.7. – *Let $f_k : U_k \to V_k$ be a sequence of quadratic-like maps with connected Julia sets, which converges uniformly to a quadratic-like map $f : U \to V$ on a neighborhood of $K(f)$. Then $\chi(f_k) \to \chi(f)$.*

The proof of the uniqueness assertion in Theorem 2.6 relies essentially on the following general lemma due to Bers [LV]:

LEMMA 2.8. – *Let $U \subset \mathbb{C}$ be open, $K \subset U$ be compact, and ϕ and Φ be two mappings $U \to \mathbb{C}$ which are homeomorphisms onto their images. Suppose that ϕ is quasiconformal, that Φ is quasiconformal on $U \setminus K$, and that $\phi = \Phi$ on K. Then Φ is quasiconformal, and $\bar{\partial}\phi = \bar{\partial}\Phi$ almost everywhere on K.*

2.3. Quadratic polynomials. Mandelbrot set, renormalizable maps and tuning

Basic facts on the structure of the Mandelbrot set are found in [DH1]. Our account of renormalization and the Yoccoz construction follows [Lyu3] (see also [Mil5] and [McM1]).

The *connectedness locus* of the quadratic family $f_c(z) = z^2 + c$ is the ever-popular Mandelbrot set

$$\mathcal{M} = \{c \in \mathbb{C} |\ J(f_c) \text{ is connected}\}$$

depicted in Fig. 1. The following results are shown in [DH1].

THEOREM 2.9 (Douady and Hubbard). – *The Mandelbrot set is compact and connected, with connected complement.*

By definition, the *hyperbolic components* of $\mathcal{M}$ are the connected components H of $\mathring{\mathcal{M}}$ such that f_c has an attracting periodic orbit for $c \in H$. Recalling that there can be at most one such orbit, we denote its multiplier $\lambda_H(c)$.

THEOREM 2.10 (Douady and Hubbard). – *Let H be a hyperbolic component. The multiplier map*

$$\lambda_H : H \to \mathbb{D}$$

is a conformal isomorphism. This map extends to a homeomorphism between $\bar{H}$ and the closed disc $\bar{\mathbb{D}}$.

Let f_c be a quadratic polynomial with connected Julia set. By Theorem 2.1 the external ray of external argument 0 lands at a fixed point of f_c, necessarily repelling or parabolic with multiplier 1, henceforth denoted β_{f_c}. The main hyperbolic component H_0 is the set

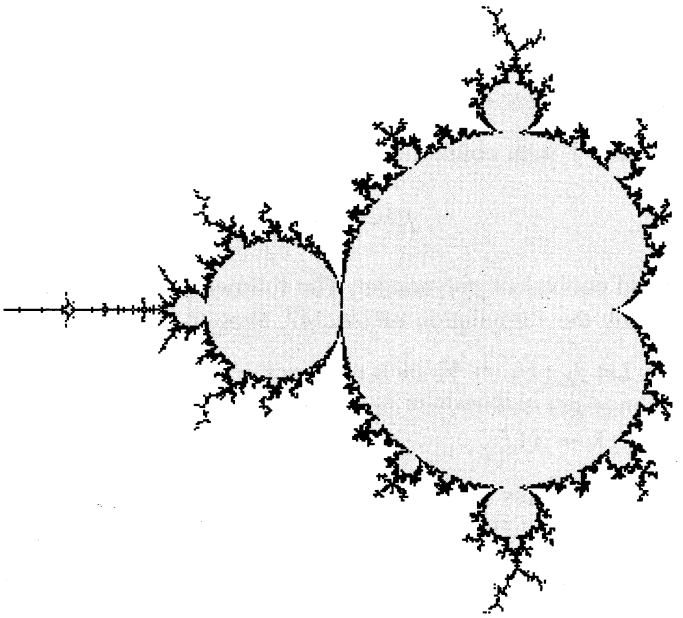

Fig. 1. – The Mandelbrot set $\mathcal{M}$

of all c for which the other fixed point α_{f_c} is attracting; the boundary point $c = 1/4$ is hereafter referred to as the *root* of $\mathcal{M}$.

For nonzero $p/q \in \mathbb{Q}/\mathbb{Z}$ with $(p,q) = 1$, we define the *p/q-limb* $\mathcal{L}_{p/q}$ to be the connected component of $\mathcal{M} \setminus H_0$ whose boundary contains

$$\mathrm{root}_{p/q} = \lambda_{H_0}^{-1}(e^{2\pi i p/q}),$$

and denote $H_{p/q}$ the hyperbolic component attached to H_0 at this point; by convention, $\mathcal{L}_{0/1} = \mathcal{M}$ and $H_{0/1} = H_0$. In view of the following, we may refer to α_{f_c} as the *dividing fixed point.*

LEMMA 2.11. – *For $q \geq 2$, a parameter value $c \in \mathcal{M}$ lies in $\mathcal{L}_{p/q}$ if and only if α_{f_c} is the landing point of an external ray with combinatorial rotation number p/q.*

Consider a polynomial f_c with connected Julia set. Let $\zeta_0 \mapsto \zeta_1 \mapsto \ldots \mapsto \zeta_m = \zeta_0$ be a repelling cycle of f_c, such that each ζ_i is the landing point of at least two external rays. Let $\mathcal{R}$ be the collection of all external rays landing at these points, and let $\mathcal{R}' = -\mathcal{R}$ be the symmetric collection. Let us also choose an arbitrary equipotential E. Denote by Ω the component of $\mathbb{C} \setminus (\mathcal{R} \cup \mathcal{R}' \cup E)$ containing 0. This region is bounded by four pieces of external rays and two pieces of E. Let n be the period of these rays, $\zeta = \zeta_i$ the element of the cycle contained in $\partial\Omega$, and $\Omega' \subset \Omega$ the component of $f_c^{\circ -n}(\Omega)$ attached to ζ. If $0 \in \Omega'$ then $f_c^{\circ n} : \Omega' \to \Omega$ is a branched cover of degree 2.

Following Douady and Hubbard, we say that a polynomial f_c is *renormalizable* if there exists a repelling cycle $\{\zeta_i\}$ as above, such that $0 \in \Omega'$ and 0 does not escape Ω' under iteration of $f_c^{\circ n}$. In this case $f_c^{\circ n}|_{\Omega'}$ can be extended to a quadratic-like map $f_c^{\circ n} : U \to V$ with connected Julia set by a *thickening procedure* (a version of this procedure is employed in §5). To emphasize the dependence of this construction on the choice of periodic orbit, we shall say that this *renormalization* of f_c is associated to ζ.

Recall that the *ω-limit set* of a point z under a map f is defined as

$$\omega_f(z) = \{w | f^{\circ n_k}(z) \to w \text{ for some } n_k \to \infty\}.$$

When $f = f_c$ we simply write $\omega_c(z)$ and pay special attention to the ω-limit set of the critical point 0. The following observation will be useful along the way:

REMARK 2.12. – *For a renormalizable quadratic polynomial f_c with n as above,*

$$\omega_c(0) \subset \bigcup_{i=0}^{n-1} f_c^{\circ i}(\overline{\Omega'}) \cap J(f_c).$$

In particular, $\beta_{f_c} \notin \omega_c(0)$.

THEOREM 2.13 (Douady and Hubbard, [DH2]). – *Let f_{c_0} be a renormalizable quadratic polynomial with associated periodic point ζ. Then there exists a canonical embedding of the Mandelbrot set $\mathcal{M}$ onto a subset $\mathcal{M}' \ni c_0$ such that every map f_c with $c \in \mathcal{M}' \setminus \{$ one point$\}$ is renormalizable with associated repelling periodic point ζ_c, where $c \mapsto \zeta_c$ is continuous and $\zeta_{c_0} = \zeta$.*

These subsets $\mathcal{M}'$ are customarily referred to as the *small copies* of the Mandelbrot set. The inverse homeomorphism $\kappa : \mathcal{M}' \to \mathcal{M}$ is defined in terms of the straightening map χ:

$$\mathcal{M}' \ni c \;\mapsto\; f_c^{\circ n} : U_c \to V_c \;\overset{\chi}{\mapsto}\; \kappa(c) \in \mathcal{M}.$$

The periodic point ζ_c becomes parabolic with multiplier 1 at the excluded parameter value, hereafter referred to as the *root* of $\mathcal{M}'$. We write $\mathcal{M}_{p/q}$ for the small copy "growing" from the hyperbolic component $H_{p/q}$, its root being the point $\text{root}_{p/q}$.

2.4. Cubic polynomials. Connectedness locus, types of hyperbolic components, $\text{Per}_n(\lambda)$-curves, real cubic family

We now turn our attention to cubic polynomials. Our presentation follows the detailed discussion in [Mil2].

Observe that every cubic polynomial is affine conjugate to a map of the form

$$F_{a,b}(z) = z^3 - 3a^2 z + b, \tag{2.2}$$

with critical points a and $-a$. This normal form is unique up to conjugation by $z \mapsto -z$, which interchanges $F_{a,b}$ and $F_{a,-b}$. The pair of complex numbers $A = a^2$ and $B = b^2$ parametrizes the space of cubic polynomials modulo affine conjugacy.

The *cubic connectedness locus* is the set $\mathscr{C} \subset \mathbb{C}^2$ of all pairs (A, B) for which the corresponding polynomial $F_{a,b}$ has connected Julia set. As in the quadratic case, the connectedness locus is compact and connected with connected complement. These results were obtained by Branner and Hubbard [BH] who showed moreover that this set is *cellular*, the intersection of a sequence of strictly nested closed discs. On the other hand, Lavaurs [La] proved that $\mathscr{C}$ is not locally connected (compare with Appendix B).

Milnor distinguishes four different types of hyperbolic components, according to the behavior of the critical points: *adjacent, bitransitive, capture,* and *disjoint* [Mil2].

We are exclusively interested in the last possibility: a component $\mathcal{H} \subset \overset{\circ}{\mathscr{C}}$ is of disjoint type $D_{m,n}$ if $F_{a,b}$ has distinct attracting periodic orbits with periods m and n for every $(a^2, b^2) \in \mathcal{H}$. By definition, the $\mathrm{Per}_n(\lambda)$-curve consists of all parameter values for which the cubic polynomial $F_{a,b}$ has a periodic point of period n and multiplier λ. The geography of $\mathrm{Per}_1(0)$ was studied in [Mil3] and [Fa].

Notice that any cubic polynomial with real coefficients is affine conjugate to a map of the form 2.2 with $A, B \in \mathbb{R}$, that is $a, b \in \mathbb{R} \cup i\mathbb{R}$. Thus we may consider the connectedness locus of real cubic maps, the set of pairs $(A, B) \in \mathbb{R}^2$ such that $J(F_{a,b})$ is connected. This locus $\mathscr{C}_{\mathbb{R}}$ is also compact, connected and cellular [Mil2]. We refer the reader to Fig. 2 which was generated by a computer program of Milnor. The real slices of various hyperbolic components are rendered in different shades of gray. Certain disjoint type components are indicated, as are the curves $\mathrm{Per}_1(1) \cap \mathscr{C}_{\mathbb{R}}$ and $\mathrm{Per}_2(1) \cap \mathscr{C}_{\mathbb{R}}$.

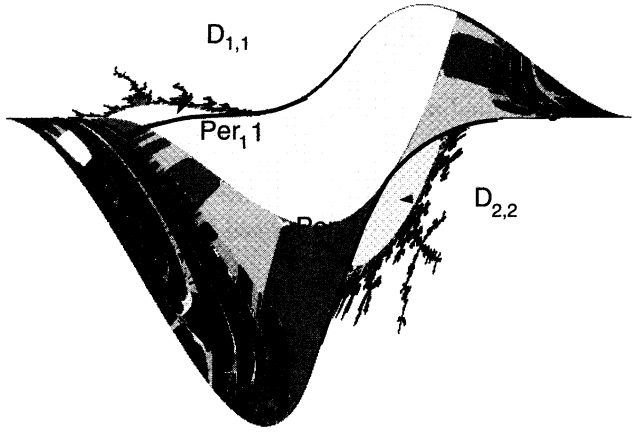

Fig. 2. – Connectedness locus $\mathscr{C}_{\mathbb{R}}$ in the real cubic family

To avoid ambiguities arising from the choice of normalization, we will actually work in the family of cubics

$$P_{A,D} = A(w^3 - 3w) + D, \quad A \neq 0$$

with *marked critical points* -1 and $+1$. The reparametrization

$$\mathbb{C}^* \times \mathbb{C} \ni (A, D) \mapsto (A, AD^2) = (A, B) \in \mathbb{C}^* \times \mathbb{C}$$

is branched over the *symmetry locus* $B = 0$ consisting of normalized cubics which commute with $z \mapsto -z$ (see Fig. 3). In particular,

$$\mathscr{C}^{\#} = \{(A, D) \subset \mathbb{C}^* \times \mathbb{C} |\ J(P_{A,D}) \text{ is connected}\}$$

is a branched double cover of $\mathscr{C} \cap (\mathbb{C}^* \times \mathbb{C})$. The marking of critical points allows us to label the attracting cycles of maps in disjoint type components $\mathcal{H} \subset \mathscr{C}^{\#}$, and we denote the corresponding multipliers $\lambda^{\pm}_{\mathcal{H}}(A, D)$. It is shown in [Mil4] that the maps $\Lambda_{\mathcal{H}} : \mathcal{H} \to \mathbb{D} \times \mathbb{D}$ given by

$$\Lambda_{\mathcal{H}}(A, D) = (\lambda^-_{\mathcal{H}}(A, D), \lambda^+_{\mathcal{H}}(A, D))$$

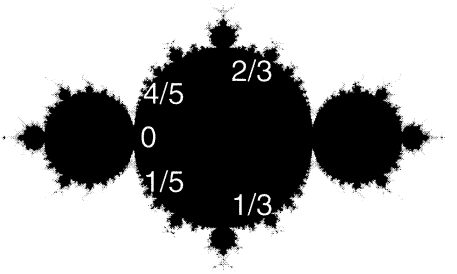

Fig. 3. – Symmetry locus in the family $P_{A,D}$

are biholomorphisms. The omitted curve $A = 0$, consisting of maps with a single degenerate critical point, is irrelevant to the discussion of disjoint type components.

This useful change of variable has the unfortunate side-effect that the values $(A, D) \in \mathbb{R}^* \times \mathbb{R}$ only account for the first and third quadrants of the real (A, B)-plane, the second and fourth quadrants being parametrized by $\mathbb{R}^* \times i\mathbb{R}$. We are therefore unable to furnish a faithful illustration of the entire locus

$$\mathscr{C}_{\mathbb{R}}^{\#} = \{(A, D) | \, (A, AD^2) \in \mathscr{C}_{\mathbb{R}}\}.$$

2.5. Surgical tools

For the reader's convenience let us review the notion of an almost complex structure. Let $\sigma = \{E_z\}_{z \in G}$ be a measurable field of ellipses on a planar domain G with the ratio of major to minor axes at the point z denoted by $K(z)$. The *complex dilatation* is a complex valued function $\mu : G \to \mathbb{D}$, where $|\mu(z)| = (K(z) - 1)/(K(z) + 1)$, and the argument of $\mu(z)$ is twice the argument of the major axis of E_z. A *bounded measurable almost complex structure* is a field of ellipses σ with $\|\mu\|_\infty < 1$. The *standard almost complex structure* σ_0 is a field of circles, thus having identically vanishing complex dilatation.

Given an ellipse field σ on G and an almost everywhere differentiable homeomorphism $h : W \to G$ the *pullback of* σ is an ellipse field $h^*\sigma$ on W obtained as follows. For almost every $z \in W$, there is a linear tangent map

$$T_z h : T_z W \to T_{h(z)} G.$$

Let $\sigma = \{E_\zeta \subset T_\zeta G\}_{\zeta \in G}$; then $h^*\sigma$ is given by $\{T_z h^{-1}(E_{h(z)}) \subset T_z W\}_{z \in W}$. We note that when the map h is quasiconformal the pullback of the standard structure $\sigma = h^*\sigma_0$ is a bounded almost complex structure.

The proofs of the following general principles can be found in [LV]:

THEOREM 2.14. – *Let h be a quasiconformal map such that $h^*\sigma_0 = \sigma_0$. Then h is conformal.*

THEOREM 2.15 (Measurable Riemann Mapping Theorem). – *If σ is a bounded almost complex structure on a domain $G \subset \mathbb{C}$, then there exists a quasiconformal homeomorphism $h : G \to h(G)$, such that*

$$\sigma = h^*\sigma_0.$$

Let $f : W' \to W$ be a quadratic-like map with connected Julia set, ζ a repelling fixed point with combinatorial rotation number p/q and associated quotient torus $T_\zeta = (D \setminus \{\zeta\})/f^{\circ q}$, where D is a fixed but otherwise arbitrary linearizing neighborhood $D \ni \zeta$. Given $S \subset W \setminus \{\zeta\}$ with $f^{\circ q}(S \cap D) = S \cap f^{\circ q}(D)$, we denote $\hat{S}$ its projection to T_ζ; in particular, $\hat{K}_1(f), \ldots, \hat{K}_q(f) \subset T_\zeta$ are the quotients of the various components of $K(f) \setminus \{\zeta\}$. As any two annuli $A_1 \supset \hat{K}_{i_1}(f)$ and $A_2 \supset \hat{K}_{i_2}(f)$ are isotopic we may speak of a distinguished isotopy class of open annuli $A \subset T_\zeta$, namely $A \sim \hat{K}(f)$ if and only if A is isotopic to an annulus containing some $\hat{K}_i(f)$. Moreover, it is easy to see if $A \subset \hat{K}_i(f)$ does not separate T_ζ then $A \sim \hat{K}(f)$; it follows then that ζ is on the boundary of an immediate attracting basin. Consider

$$\underline{\mathrm{mod}}_\zeta\, \hat{K}_i(f) = \sup\{\mathrm{mod} A \mid A \subset \hat{K}_i(f)\}$$

and

$$\overline{\mathrm{mod}}_\zeta\, \hat{K}_i(f) = \inf\{\mathrm{mod} A \mid A \supset \hat{K}_i(f)\}$$

over open annuli $A \sim \hat{K}(f)$. Notice that these quantities are independent of i. In view of the following we may simply write $\mathrm{mod}_\zeta\, \hat{K}(f)$:

LEMMA 2.16. – *In this setting,* $\underline{\mathrm{mod}}_\zeta\, \hat{K}_i(f) = \overline{\mathrm{mod}}_\zeta\, \hat{K}_i(f)$.

Proof. – It is obvious that $\underline{\mathrm{mod}}_\zeta\, \hat{K} \leq \overline{\mathrm{mod}}_\zeta\, \hat{K}$ for $\hat{K} = \hat{K}_i(f)$. Conversely, given $R_n \searrow R_\infty = e^{\overline{\mathrm{mod}}_\zeta K}$ there exist conformal embeddings $h_n : \mathbb{A}_{R_n} \to T_\zeta$ such that $h_n(\mathbb{A}_{R_n}) \supset \hat{K}_i(f)$, where

$$\mathbb{A}_R = \{z : 1 < |z| < R\}.$$

It follows from standard estimates in geometric function theory that the h_n form a normal family on $\mathbb{A}_{R_\infty}$; moreover, as all of these embeddings are isotopic, every limit $h_\infty = \lim_{k\to\infty} h_{n_k}$ is univalent. Clearly, $h_\infty(\mathbb{A}_{R_\infty}) \subset \hat{K}$ and therefore $\overline{\mathrm{mod}}_\zeta\, \hat{K} \leq \underline{\mathrm{mod}}_\zeta\, \hat{K}$. $\square$

As $\underline{\mathrm{mod}}\, \hat{K}(f)$ is defined in terms of the interior of $K(f)$, we observe:

REMARK 2.17. – *The quantity* $\mathrm{mod}_\zeta\, \hat{K}(f)$ *depends only on the hybrid equivalence class; furthermore,* $\mathrm{mod}_\zeta\, \hat{K}(f) > 0$ *if and only if* ζ *lies on the boundary of an immediate attracting basin.*

Let $f : W' \to W$ be a quadratic-like map with connected Julia set, and ζ a repelling fixed point with combinatorial rotation number p/q. An *invariant sector* with vertex ζ is a simply connected domain $S \subset W$ bounded by an arc of ∂W and two additional arcs γ_1 and γ_2 with $\gamma_j \subset f^{\circ q}(\gamma_j)$ and a common endpoint at ζ. We write $S = \backslash\gamma_1, \gamma_2/$ for the sector between γ_1 and γ_2 as listed in counterclockwise order. The quotient $\hat{S} \subset T_\zeta$ is an open annulus whose modulus will be referred to as the *opening modulus* $\mathrm{mod} S$ of the sector S.

Consider a restriction of a quadratic polynomial f_c with a connected Julia set to the domain $W \supset K(f_c)$ bounded by an equipotential E_ρ. Invariant sectors for this map may be constructed as follows [BD]: given a ray r_θ landing at a fixed point ζ with combinatorial rotation number p/q, consider

$$S_t(r_\theta) = B_{f_c}^{-1}(\{e^{r+2\pi i\gamma} \mid 0 < r < \rho \text{ and } |\gamma - \theta| < tr\}).$$

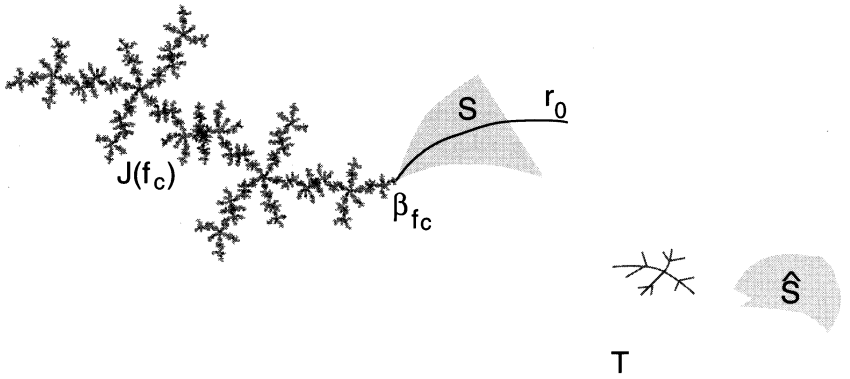

Fig. 4. – An invariant sector $S_t(r_0)$ with vertex at β_{f_c}

Fig. 4 depicts an invariant sector $S = S_t(r_0)$ and its quotient $\hat{S}$.

LEMMA 2.18 ([BF], Prop. 4.1). – *Given $\rho > 0$ there exists $\tau > 0$ such that for any $t < \tau$ the domains $S_t(r_{2^{i-1}\theta}) \subset W$ for $i = 1, \ldots, q$ are disjoint invariant sectors containing the rays $r_{2^{i-1}\theta}$ landing at ζ.*

Let $\lambda = f'(\zeta)$ denote the multiplier of the repelling fixed point ζ. Let $\varphi : D \to \mathbb{C}$ be a linearizing map conjugating the action of f in the neighborhood $D \ni \zeta$ to multiplication by λ in a neighborhood of the origin. For an invariant sector $S \subset D$ select a branch of the mapping

$$z \mapsto w = \frac{1}{q\mathrm{Log}\lambda}\mathrm{Log}(\varphi(z))$$

defined in S. In the *log-linear coordinate* w the map $f^{\circ q}$ becomes the unit translation $\tau : w \to w + 1$; the image of the sector S is a horizontal strip H with $\tau^{-1}(H) \subset H$. Following the terminology of [Bi1] we make the following definition:

DEFINITION. – Let $\Gamma \subset \mathbb{C}$ be a simple rectifiable curve with the invariance property $\tau^{-1}(\Gamma) \subset \Gamma$. Let $t \mapsto \gamma(t) \in \Gamma$ be a smooth parametrization. A differentiable map $\phi(\Gamma) \to \mathbb{C}$ is a *near translation* if there exists $C > 1$ such that

$$|\phi(\gamma(t)) - \gamma(t)| < C \text{ and } 1/C < d\phi(\gamma(t))/d\gamma(t) < C.$$

The definition clearly does not depend on the particular parametrization. An example to keep in mind is given by the following proposition with an obvious proof:

PROPOSITION 2.19. – *Let $f : W' \to W$ be a quadratic-like map with connected Julia set, ζ a repelling fixed point with combinatorial rotation number p/q and $S = \backslash l, r/ \subset W$ an invariant sector with vertex ζ bounded by smooth arcs. Let $\tilde{f} : \tilde{W}' \to \tilde{W}$ be another quadratic-like map having a repelling fixed point $\tilde{\zeta}$ with the same combinatorial rotation number, and $\tilde{S} = \backslash \tilde{l}, \tilde{r}/$ similarly defined. Consider a smooth map $\psi : l \to \tilde{l}$ conjugating the dynamics:*

$$\psi \circ f^{\circ q} = \tilde{f}^{\circ q} \circ \psi.$$

In the log-linear coordinates the map ψ becomes a near translation.

The following lemma of Bielefeld [Bi1] will be instrumental in our quasiconformal surgery construction:

LEMMA 2.20. – *Let H be a strip bounded by simple rectifiable curves Γ_1 and Γ_2 with $\tau^{-1}(\Gamma_i) \subset \Gamma_i$, and $\tilde{H}$ another strip whose boundary curves $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ have the same invariance property. Let maps $\phi_i : \Gamma_i \to \tilde{\Gamma}_i$ be near translations. Then there exists a quasiconformal homeomorphism Φ mapping a neighborhood of $-\infty$ in H to $\tilde{H}$ with boundary values ϕ_i.*

3. Outline of the Results

In the picture of the real cubic connectedness locus (Fig. 2) one observes several shapes reminiscent of the Mandelbrot set (Fig. 1). We quote Milnor ([Mil2]): "... these embedded copies tend to be discontinuously distorted at one particular point, namely the period one saddle node point $c = 1/4$, also known as the root of the Mandelbrot set. The phenomenon is particularly evident in the lower right quadrant, which exhibits a very fat copy of the Mandelbrot set with the root point stretched out to cover a substantial segment of the saddle-node curve $\mathrm{Per}_2(1)$. ... As a result of this stretching, the cubic connectedness locus fails to be locally connected along this curve."

The original goal of our investigation was to explain the appearance of these distorted copies of the Mandelbrot set embedded in $\mathscr{C}_{\mathbb{R}}$. This has lead us to the following results:

For $p/q \in \mathbb{Q}/\mathbb{Z}$ we consider the set $C_{p/q} \subset \mathscr{C}^{\#}$ consisting of cubic polynomials for which $2q$ distinct external rays with combinatorial rotation number p/q land at some fixed point ζ. As there can be at most one such point, the various $C_{p/q}$ are disjoint. Each $C_{p/q}$ is in turn the disjoint union of subsets $C_{p/q,m}$ indexed by an odd integer $1 \leq m \leq 2q-1$ specifying how many of these rays are encountered in passing counterclockwise from the critical point -1 to the critical point $+1$. In particular, C_0 consists of those cubics in $\mathscr{C}^{\#}$ whose fixed rays r_0 and $r_{1/2}$ land at the same fixed point.

THEOREM 3.1 (Main Theorem). – *Given p/q and m as above, there exists a homeomorphic embedding*

$$h_{p/q,m} : \mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\} \times \mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\} \longrightarrow C_{p/q,m}$$

mapping the product of hyperbolic components $H_{p/q} \times H_{p/q}$ onto a component $\mathcal{H}_{p/q,m}$ of type $D_{q,q}$.

We note that $\mathcal{H}_{p/q,m}$ is the unique $D_{q,q}$ component contained in $C_{p/q,m}$ as will follow from Theorem 5.6. The restriction of $h_{p/q,m}$ to $H_{p/q} \times H_{p/q}$ is easily expressed in terms of the multiplier maps defined in §2:

$$h_{p/q,m}(\rho, \tilde{\rho}) = \Lambda^{-1}_{\mathcal{H}_{p/q,m}}(\, \lambda_{H_{p/q}}(\rho), \lambda_{H_{p/q}}(\tilde{\rho})\,).$$

Discontinuity of $h_{p/q,m}$ at the corner point $(1,1)$ is a special case of a phenomenon studied by one of the authors:

THEOREM 3.2 [Ep]. – *Each algebraic homeomorphism*

$$\Lambda_{\mathcal{H}_{p/q,m}} : \mathcal{H}_{p/q,m} \to \mathbb{D} \times \mathbb{D}$$

extends to a continuous surjection $\bar{\mathcal{H}}_{p/q,m} \to \bar{\mathbb{D}} \times \bar{\mathbb{D}}$. *The fiber over* $(1,1)$ *is the union of two closed discs whose boundaries are real-algebraic curves with a single point in common, and all other fibers are points.*

The following reasonable conjecture appears to be inaccessible by purely quasiconformal techniques:

CONJECTURE 3.1. – *Each* $h_{p/q,m}$ *extends to a continuous embedding*

$$\mathcal{L}_{p/q} \times \mathcal{L}_{p/q} \setminus \{(\mathrm{root}_{p/q}, \mathrm{root}_{p/q})\} \longrightarrow C_{p/q,m}\,.$$

We draw additional conclusions from the natural symmetries of our construction. The central disk in Fig. 3 is parametrized by the eigenvalue $-3A$ of the attracting fixed point at 0; this region corresponds to symmetric cubics whose Julia sets are quasicircles. Each value $A = -\frac{1}{3}e^{2\pi i p/q}$ yields a map with a parabolic fixed point at 0. These parameters are evidently the roots of small embedded copies of $\mathcal{M}$, and our results confirm this observation for odd-denominator rationals. More specifically, it will follow that the latter copies are the images of $h_0 \circ \Delta$ and $h_{p/q,q} \circ \Delta$ for odd $q > 1$, where

$$\mathbb{C} \ni c \overset{\Delta}{\mapsto} (c,c) \in \mathbb{C}^2$$

is the diagonal embedding. As $P^{\circ 2}_{A,0}$ and $P^{\circ 2}_{-A,0}$ are conjugate by $z \mapsto -z$, our construction also accounts for the copies with $q \equiv 2 \pmod 4$. Every map in the symmetry locus is semiconjugate, via the quotient determined by the involution, to a cubic polynomial with a fixed critical value. Such maps were studied by Branner and Douady [BD] who effectively prove that the entire limb attached at the parameter value $A = -1/3$ is a homeomorphic copy of the limb $\mathcal{L}_{1/2} \subset \mathcal{M}$.

Similar considerations applied to the antidiagonal embedding yield results for the real connectedness locus. In view of the fact that real polynomials commute with complex conjugation, $\mathscr{C}^{\#}_{\mathbb{R}} \cap C_{p/q} = \emptyset$ unless $p/q \equiv -p/q \pmod 1$, and it therefore suffices to consider the real slices of $C_{1/2}$ and C_0.

THEOREM 3.3. – *There exist homeomorphic embeddings*

$$\Psi_{1/2,1} : \mathcal{L}_{1/2} \setminus \{\mathrm{root}_{1/2}\} \to \mathscr{C}^{\#}_{\mathbb{R}} \cap C_{1/2,1}$$

$$\Psi_{1/2,3} : \mathcal{L}_{1/2} \setminus \{\mathrm{root}_{1/2}\} \to \mathscr{C}^{\#}_{\mathbb{R}} \cap C_{1/2,3}$$

and

$$\Psi_0 : \mathcal{M} \setminus \{\mathrm{root}\} \to \mathscr{C}^{\#}_{\mathbb{R}} \cap C_0.$$

It follows from recent work of Buff [Bu] that these maps are compatible with the standard embeddings in the plane (see the discussion in §5.4). Their projections in $\mathscr{C}_{\mathbb{R}}$ are indicated in Fig. 2. Notice that the two images of $\mathcal{L}_{1/2} \setminus \{\mathrm{root}_{1/2}\}$ have been identified while the image of $\mathcal{M} \setminus \{\mathrm{root}\}$ has been folded in half. The latter defect is overcome through passage to the $(A, \sqrt{B})$-plane, at the cost of both copies of $\mathcal{L}_{1/2} \setminus \{\mathrm{root}_{1/2}\}$; we thank J. Milnor for enabling us to include Fig. 9 where the comb on the $D_{1,1}$ component is better resolved. The existence of this comb is verified with the aid of techniques developed by Lavaurs [La].

THEOREM 3.4. – *The real cubic connectedness locus is not locally connected.*

The remainder of this paper is structured as follows. In §4 we construct cubic polynomials by means of *quasiconformal surgery* on pairs of quadratics. The issues of uniqueness and continuity are addressed in §5 through the use of the renormalization operators R and Я defined for *birenormalizable* cubics; together they essentially invert the surgery. We show that R×Я is a homeomorphism and then complete the proofs of Theorems 3.1 and 3.3 in §5.4. The measurable dynamics of birenormalizable maps is discussed in §6. In Appendix A we comment further on the discontinuity described in Theorem 3.2, and we conclude by proving Theorem 3.4 in Appendix B.

It is worth noting that quasiconformal surgery is only employed in the proof of surjectivity for R×Я. More generally, we might associate a pair of renormalization operators to any disjoint type hyperbolic component $\mathcal{H} \subset \mathscr{C}^{\#}$ in the hope of finding an embedded product of the limbs "growing" from $\mathcal{H}$, but we are unable to adapt our surgery construction to this broader setting. A forthcoming paper will present a different approach to proving surjectivity of birenormalization, culminating in a more general version of Theorem 3.1.

4. Intertwining surgery

4.1. History

The intertwining construction was described in the 1990 Conformal Dynamics Problem List [Bi2]: "Let P_1 be a monic polynomial with connected Julia set having a repelling fixed point x_0 which has ray landing on it with rotation number p/q. Look at a cycle of q rays which are the forward images of the first. Cut along these rays and get q disjoint wedges. Now let P_2 be a monic polynomial with a ray of the same rotation number landing on a repelling periodic point of some period dividing q (such as 1 or q). Slit this dynamical plane along the same rays making holes for the wedges. Fill the holes in by the corresponding wedges above making a new sphere. The new map is given by P_1 and P_2, except on a neighborhood of the inverse images of the cut rays where it will have to be adjusted to make it continuous."

4.2. Construction of a cubic polynomial

Fix p/q written in lowest terms and an odd integer $m = 2k+1$ between 1 and $2q-1$. Our aim is to construct a map

$$h_{p/q,m} : \mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\} \times \mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\} \longrightarrow C_{p/q,m}.$$

Fixing parameter values c and $\tilde{c}$ in $\mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\}$, consider quadratic-like maps $f : W' \to W$ and $\tilde{f} : \tilde{W}' \to \tilde{W}$ hybrid equivalent to f_c and $f_{\tilde{c}}$ respectively, the choice of the hybrid equivalences to be made below. In what follows we will identify W and $\tilde{W}$ to obtain a new surface. The reader is invited to follow the construction in the particular case $p/q = 1/2$ with $m = 3$, as illustrated in Fig. 5.

Without loss of generality we assume that 0 is the critical point for both f and $\tilde{f}$. Let ζ be the unique repelling fixed point of f with combinatorial rotation number p/q, that is $\zeta = \beta_f$ for $p/q = 0$ and $\zeta = \alpha_f$ otherwise, and $S_i \equiv \backslash l_i, r_i/$, $i = 0, \ldots, q-1$ in $W \setminus K(f)$ a cycle of disjoint invariant sectors with vertex ζ, indexed in counterclockwise

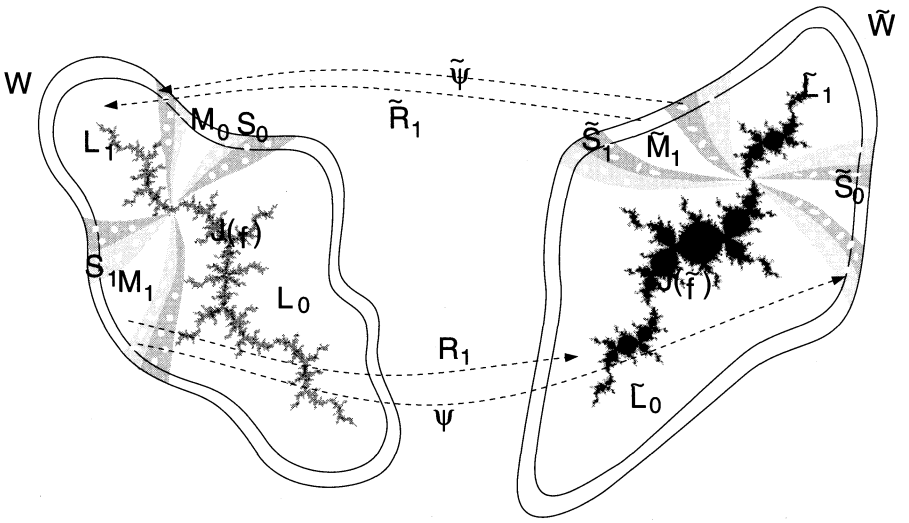

Fig. 5. – Superimposing the regions W and $\tilde{W}$

order so that the critical point 0 lies in the complementary region between S_{q-1} and S_0. We similarly specify $\tilde{\zeta}$ and a cycle of invariant sectors $\tilde{S}_i$ for $\tilde{f}$. Let

$$\phi : \bigcup_{i=0}^{q-1} l_i \cup r_i \to \bigcup_{i=0}^{q-1} \tilde{l}_i \cup \tilde{r}_i,$$

sending l_i to $\tilde{r}_{i+q-k}$ and r_i to $\tilde{l}_{i+q-k-1}$, where $k = (m-1)/2$ and indices are understood modulo q, be any smooth conjugacy,

$$\phi(f(z)) = \tilde{f}(\phi(z)).$$

The sector S_i should now correspond to the component of $K(\tilde{f}) \setminus \{\tilde{\zeta}\}$ containing $\tilde{f}^{\circ i+q-k}(0)$. An informal rule known as *Shishikura's Principle* warns against altering the conformal structure on regions of uncontrolled recurrence, and we will therefore employ invariant sectors $M_i \subset S_i \subset N_i \subset W \setminus K(f)$ and $\tilde{M}_i \subset \tilde{S}_i \subset \tilde{N}_i \subset \tilde{W} \setminus K(\tilde{f})$ to be determined below. For $i = 0, \ldots, q-1$ denote L_i the component of $W \setminus \bigcup_{j=0}^{q-1} N_j$ containing $f^{\circ i}(0)$, and $\tilde{L}_i$ the corresponding component of $\tilde{W} \setminus \bigcup_{j=0}^{q-1} \tilde{N}_j$. Let

$$R_i : M_i \to \tilde{L}_{i+q-k}$$

be the conformal Riemann map which extends continuously to the boundary, mapping $\partial M_i \cap W$ to $\partial \tilde{L}_{i+q-k} \cap \tilde{W}$ so that ζ maps to $\tilde{\zeta}$, and let

$$\tilde{R}_i : \tilde{M}_i \to L_{i-q+k+1}$$

be the Riemann map sending $\partial \tilde{M}_i \cap \tilde{W}$ to $\partial L_{i-q+k+1} \cap W$ so that $\tilde{\zeta}$ maps to ζ. It remains to fill in the gaps:

PROPOSITION 4.1 (Quasiconformal interpolation). – *For any pair f_c and $f_{\tilde{c}}$ as above the hybrid equivalent quadratic-like maps f and $\tilde{f}$ and the invariant sectors*

$$N_i \supset S_i \supset M_i \ \textit{and} \ \tilde{N}_i \supset \tilde{S}_i \supset \tilde{M}_i$$

may be chosen so that there exist quasiconformal maps

$$\psi : \bigcup_{i=0}^{q-1}(S_i \setminus M_i) \to \bigcup_{i=0}^{q-1}(\tilde{N}_i \setminus \tilde{S}_i)$$

$$\tilde{\psi} : \bigcup_{i=0}^{q-1}(\tilde{S}_i \setminus \tilde{M}_i) \to \bigcup_{i=0}^{q-1}(N_i \setminus S_i)$$

with

$$\begin{array}{l} \psi|_{\partial M_i} = R_i|_{\partial M_i} \ \textit{and} \ \psi|_{\partial S_i} = \phi|_{\partial S_i} \\ \psi|_{\partial \widetilde{M}_i} = \widetilde{R}_i|_{\partial \widetilde{M}_i} \ \textit{and} \ \widetilde{\psi}|_{\partial \widetilde{S}_i} = \phi^{-1}|_{\partial \widetilde{S}_i} \end{array}.$$

Let us complete the construction assuming the truth of Proposition 4.1. We choose sectors N_i, M_i, $\tilde{N}_i$, $\tilde{M}_i$, and maps ψ, $\tilde{\psi}$, R_i and $\tilde{R}_i$ as specified above. Consider the almost complex structure σ on W, given by $\psi^*\sigma_0$ on $\bigcup_{i=0}^{q-1} S_i \setminus M_i$ and by σ_0 elsewhere; similarly, let $\tilde{\sigma}$ be the almost complex structure on $\tilde{W}$ given by $\tilde{\psi}^*\sigma_0$ on $\bigcup_{i=0}^{q-1} \tilde{S}_i \setminus \tilde{M}_i$, and σ_0 elsewhere. In view of Theorem 2.15 there exist quasiconformal homeomorphisms $h : W \to X$ and $\tilde{h} : \tilde{W} \to \tilde{X}$ such that $\sigma = h^*\sigma_0$ and $\tilde{\sigma} = \tilde{h}^*\sigma_0$. Consider the Riemann surface obtained from $W \sqcup \tilde{W}$ with identifications ψ, $\tilde{\psi}$, R_i, $\tilde{R}_i$, whose atlas is given by h and $\tilde{h}$. It has the conformal type of a punctured disc, and we obtain a conformal disc Δ by replacing the puncture with a point $\star$.

Setting

$$\Delta' = \left(W' \setminus \bigcup_{i=0}^{q-1} S_i\right) \cup \left(\tilde{W}' \setminus \bigcup_{i=0}^{q-1} \tilde{S}_i\right) \cup \{\star\} \subset \Delta,$$

we define a new map $F : \Delta' \to \Delta$ by

$$F(z) = \begin{cases} f(z) & \text{for} \quad z \in W' \setminus \bigcup_{i=0}^{q-1} S_i, \\ \tilde{f} & \text{for} \quad z \in \tilde{W}' \setminus \bigcup_{i=0}^{q-1} \tilde{S}_i, \\ \star & \text{for} \quad z \in \{\star, -\zeta, -\tilde{\zeta}\}. \end{cases}$$

It is easily verified that F is a three-fold branched covering with critical points $0 \in W$ and $0 \in \tilde{W}$, and analytic except on the preimage of

$$\mathcal{S} = \bigcup_{i=0}^{q-1}(S_i \setminus M_i) \cup (\tilde{S}_i \setminus \tilde{M}_i).$$

Recalling that the sectors S_i and $\tilde{S}_i$ are invariant and disjoint, we consider the following almost complex structure on Δ:

$$\begin{cases} \hat{\sigma} = (F^{\circ n})^*\sigma_0 & \text{on } F^{\circ -n}(\mathcal{S}) \\ \hat{\sigma} = \sigma_0 & \text{elsewhere.} \end{cases}$$

By construction, the complex dilatation of $\hat{\sigma}$ has the same bound as that of $F^*\sigma_0$, and moreover

$$F^*\hat{\sigma} = \hat{\sigma}.$$

It follows from Theorem 2.15 that there is a quasiconformal homeomorphism $\varphi : \Delta \to V \subset \mathbb{C}$ with $\hat{\sigma} = \varphi^*\sigma_0$. Setting $U = \varphi(\Delta')$, we obtain a cubic-like map

$$G = \varphi \circ F \circ \varphi^{-1} : U \to V.$$

In view of Theorem 2.6 there is a unique hybrid equivalent cubic polynomial $P_{A,D}$ whose critical points -1 and $+1$ correspond to the critical point of f and $\tilde{f}$ respectively. The construction yields extensions of the natural embeddings

$$\pi : K(f_c) \to K(P_{A,D}) \text{ and } \tilde{\pi} : K(f_{\tilde{c}}) \to K(P_{A,D})$$

to neighborhoods of the filled Julia sets.

REMARK 4.2. – *By construction, the projections π and $\tilde{\pi}$ are conformal on the respective filled Julia sets:*

$$\bar{\partial}\pi = 0 \text{ a.e. on } K(f_c) \text{ and } \bar{\partial}\tilde{\pi} = 0 \text{ a.e. on } K(f_{\tilde{c}}).$$

We write $P_{A,D} \approx f_c \underset{p/q,m}{\curlyvee} f_{\tilde{c}}$ for *any* cubic polynomial so obtained. It is not yet clear that this correspondence is well-defined, let alone continuous. These issues will be addressed in §5.

4.3. Quasiconformal interpolation. Proof of Proposition 4.1

Note first that were it not for the condition of quasiconformality, the existence of the interpolating maps ψ and $\tilde{\psi}$ would follow without any additional argument. Any smooth interpolations are quasiconformal away from the points of ζ and $\tilde{\zeta}$, the issue is the compatibility of the the local behaviour of ϕ at ζ with that of R_i and $\tilde{R}_i$.

LEMMA 4.3. – *Given any $c \in \mathcal{L}_{p/q} \setminus \{\text{root}_{p/q}\}$ and $\upsilon > 0$, there exists a quadratic-like map f which is hybrid equivalent to f_c and admits disjoint invariant sectors S_i as above with $\text{mod}_\zeta\ S_i > \upsilon$.*

Proof. – We begin by fixing a quadratic-like restriction $f_c : G' \to G$ between equipotentially bounded regions, and apply Lemma 2.18 to obtain a cycle of disjoint invariant sectors $S_t(r_i) \subset G$. Let φ be a quasiconformal homeomorphism from the annulus $\hat{S}_t(r_0) \subset T_\zeta$ to some standard annulus $\mathbb{A}_\rho$ with $\rho > e^\upsilon$. The almost complex structure $\sigma = \varphi^*\sigma_0$ on $\hat{S}_t(r_0)$ lifts to an almost complex structure on the sector $S_t(r_0)$. We extend this structure by pullback to the various $S_t(r_i)$ and their preimages, and extend by σ_0 elsewhere, to obtain an invariant almost complex structure $\bar{\sigma}$ on G. In view of Theorem 2.15 there exists a quasiconformal homeomorphism $q : G \to q(G) \subset \mathbb{C}$ with $\bar{\sigma} = q^*\sigma_0$, giving a hybrid equivalence between f_c and the quadratic-like map

$$f = q \circ f_c \circ q^{-1} : q(G') \to q(G).$$

It follows from Theorem 2.14 that $\operatorname{mod}_\zeta S_i = \upsilon$ where

$$S_i = q(S_t(r_i)).$$

□

Given $c, \tilde{c} \in \mathcal{L}_{p/q} \setminus \{\text{root}_{p/q}\}$, we apply Lemma 4.3 to f_c and $f_{\tilde{c}}$ to obtain hybrid equivalent quadratic-like maps f and $\tilde{f}$ admitting invariant sectors S_i and $\tilde{S}_i$ with

$$\operatorname{mod}_\zeta S_i > \operatorname{mod}_{\tilde{\zeta}} \hat{K}(f_{\tilde{c}}) \text{ and } \operatorname{mod}_{\tilde{\zeta}} \tilde{S}_i > \operatorname{mod}_\zeta \hat{K}(f_c).$$

In view of Remark 2.17 we may then choose disjoint invariant sectors $N_i \supset S_i$ and $\tilde{N}_i \supset \tilde{S}_i$ so that

$$\operatorname{mod}_\zeta S_i > \operatorname{mod}_{\tilde{\zeta}} \tilde{L}_j \text{ and } \operatorname{mod}_{\tilde{\zeta}} \tilde{S}_i > \operatorname{mod}_\zeta L_j$$

for the complementary invariant sectors L_j,$\tilde{L}_j$ as above. Finally, we choose $M_i \subset S_i$ and $\tilde{M}_i \subset \tilde{S}_i$ with

$$\operatorname{mod}_\zeta M_i = \operatorname{mod}_{\tilde{\zeta}} \tilde{L}_j \text{ and } \operatorname{mod}_{\tilde{\zeta}} \tilde{M}_i = \operatorname{mod}_\zeta L_j.$$

We now exploit the following observation of [BD]:

LEMMA 4.4. – *With this choice of maps and invariant sectors there exist desired quasiconformal interpolations*

$$\psi : \bigcup_{i=0}^{q-1} (S_i \setminus M_i) \to \bigcup_{i=0}^{q-1} (\tilde{N}_i \setminus \tilde{S}_i)$$

$$\tilde{\psi} : \bigcup_{i=0}^{q-1} (\tilde{S}_i \setminus \tilde{M}_i) \to \bigcup_{i=0}^{q-1} (N_i \setminus S_i)$$

with

$$\psi|_{\partial M_i} = R_i|_{\partial M_i} \text{ and } \psi|_{\partial S_i} = \phi|_{\partial S_i}$$
$$\tilde{\psi}|_{\partial \widetilde{M}_i} = \widetilde{R}_i|_{\partial \widetilde{M}_i} \text{ and } \tilde{\psi}|_{\partial \widetilde{S}_i} = \phi^{-1}|_{\partial \widetilde{S}_i}.$$

Proof. – We give the argument for the existence of the interpolating map ψ following [Bi1]. The equality of the opening moduli implies the existence of conformal homeomorphisms $\hat{r}_i$ of the quotient annuli $\widehat{M_i} \subset T_\zeta$ and the corresponding $\widehat{\tilde{L}_j} \subset T_{\tilde{\zeta}}$. "Unrolling" the annuli we lift these maps to conformal maps $r_i : M_i \cap D_\zeta \to \tilde{L}_j \cap D_{\tilde{\zeta}}$, where D_ζ and $D_{\tilde{\zeta}}$ are some linearizing neighborhoods of the fixed points. It follows immediately from the definition that the maps $r_i|_{\partial M_i}$ are near translations in log-linear coordinates. By a straightforward application of Schwarz Reflection Principle (cf. [Bi1, Lemma 6.4]), the maps R_i and r_i have the same asymptotics at ζ, and thus R_i are near translations as well when viewed in log-linear coordinates. It remains to note that $\phi|_{\partial S_i}$ are near translations by Proposition 2.19. An application of the Interpolation Lemma 2.20 concludes the argument. □

5. Renormalization

5.1. Birenormalizable cubics

Throughout this section we will work with fixed values of p/q and m as specified above. Here we describe the construction which will provide the inverse to the map $h_{p/q,m}$.

We start with a cubic polynomial $P = P_{A,D}$ with $(A, D) \in C_{p/q,m}$. Let ζ be the landing point of the periodic rays with rotation number p/q. Denote $V_0, \ldots, V_{2q-1}$ the components of $\mathbb{C}$ with the point ζ and the $2q$ rays landing at ζ removed. We enumerate them in counterclockwise order so that $V_0 \ni -1$. The polynomial P is *renormalizable to the left* if the forward orbit of the critical point -1 is contained in $\{\zeta\} \cup \bigcup_{i=0}^{q-1} V_{2i}$. In this case the map P has a quadratic-like restriction to a neighborhood of the critical point -1, as seen from the *thickening construction* below.

Assuming that P is renormalizable to the left, let $K_{-1}^i \subset V_{2i}$ for $i = 0, \ldots, q-1$ denote the connected component of $K(P) \setminus \zeta$. Let $r_{\theta_1^i}$ and $r_{\theta_2^i}$ be the two external rays landing at ζ which separate K_{-1}^i from the other rays landing at the same point, where the values θ_1^i and θ_2^i are chosen so that $\theta_2^i < \theta_1^i$ and the rays landing at K_{-1}^i have angles in $[\theta_2^i, \theta_1^i]$. Choose a neighborhood $U \ni \zeta$ corresponding to a round disc in the local linearizing coordinate. Fix an equipotential E and a small $\epsilon > 0$, and consider the segments of the rays $r_{\theta_1^i+\epsilon}$ and $r_{\theta_2^i-\epsilon}$ connecting the boundary of U to E. Let $\Omega \supset \cup_{i=0}^{q-1} K_{-1}^i$ be the region bounded by these ray segments and the subtended arcs of E and ∂U, and consider the component Ω' of $P^{-1}(\Omega)$ with $\Omega' \subset \Omega$. In view of the fact that ζ is repelling, $\bar{\Omega}' \subset \Omega$ provided that ϵ is sufficiently small. Thus,

$$P : \Omega' \to \Omega$$

is a quadratic-like map which filled Julia set will be denoted K_{R}. Since $\{P^n(-1)\}_{n=0}^{\infty} \in \cup_{i=0}^{q-1} K_{-1}^i$, this set is connected, and we refer to the unique hybrid conjugate quadratic polynomial f_c as the *left renormalization* $\mathrm{R}(P)$. By construction f_c has a fixed point with combinatorial rotation number p/q, that is $c \in \mathcal{L}_{p/q}$.

Fig. 6 illustrates this construction for a cubic polynomial in C_0. Notice that ζ becomes the β-fixed point of the new quadratic polynomial.

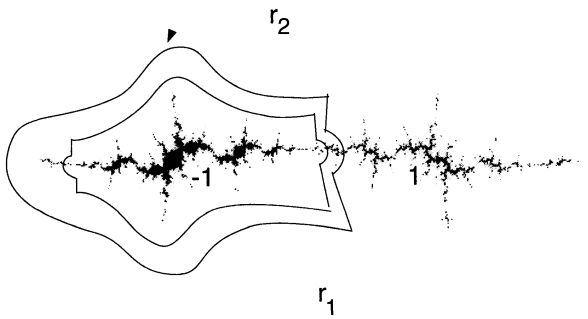

Fig. 6. – Construction of the left renormalization for a cubic in C_0

The polynomial P is *renormalizable to the right* if the forward orbit of $+1$ is contained in $\{\zeta\} \cup \bigcup_{i=0}^{q-1} V_{2i+1}$, and the set $K_{\mathfrak{R}}$ and the *right renormalization* $\mathfrak{R}(P)$ are correspondingly defined. It follows from general considerations discussed in [McM1] that the left and right renormalizations do not depend on the choice of thickened domains.

A cubic polynomial P is said to be *birenormalizable* if it is renormalizable on both left and right, in which case

$$\omega_P(-1) \cap \omega_P(+1) \subset K_{\mathrm{R}} \cap K_{\mathfrak{R}} = \{\zeta\} \tag{5.1}$$

and we set

$$\check{K}(P) = \bigcup_{i=0}^{\infty} P^{-\circ i}(K_{\mathrm{R}} \cup K_{\mathfrak{R}}).$$

The following is an easy consequence of Kiwi's Separation Theorem 2.4 and the standard classification of Fatou components:

LEMMA 5.1. – *Let P be a birenormalizable cubic polynomial. Then $\check{K}(P)$ is dense in $K(P)$, and every periodic orbit in $K(P) \setminus \check{K}(P)$ is repelling.*

We denote $B_{p/q,m}$ the set of birenormalizable cubics in $C_{p/q,m}$, writing

$$\mathrm{R}\times\mathfrak{R} : B_{p/q,m} \to (\mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\}) \times (\mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\})$$

for the map $(A, D) \mapsto (c, \tilde{c})$ where $f_c = \mathrm{R}(P_{A,D})$ and $f_{\tilde{c}} = \mathfrak{R}(P_{A,D})$. In view of Lemma 2.3 the thickening construction may be performed so that the domains of the left and right quadratic-like restrictions vary continuously for $(A, D) \in B_{p/q,m}$. Applying Theorem 2.7 we obtain:

PROPOSITION 5.2. – $\mathrm{R}\times\mathfrak{R} : B_{p/q,m} \to (\mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\}) \times (\mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\})$ *is continuous.*

The significance of intertwining rests in the following:

PROPOSITION 5.3. – $\mathrm{R}\times\mathfrak{R} : B_{p/q,m} \to (\mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\}) \times (\mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\})$ *is surjective.*

Proof. – Fix $c, \tilde{c} \in \mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\}$. We saw above that

$$f_c \underset{p/q,m}{\curlyvee} f_{\tilde{c}} \approx P$$

for some cubic polynomial $P = P_{A,D}$, and we show here that $\mathrm{R}\times\mathfrak{R}(A, D) = (c, \tilde{c})$; more precisely, we prove that $\mathrm{R}(P) = f_c$, the argument for right renormalization being completely parallel.

Let $K(f_c) \subset W' \subset W \xrightarrow{\pi} \mathbb{C}$ be as in §4. By construction, π is a quasiconformal map conjugating $f_c|_{K(f_c)}$ to $P|_{K_{\mathrm{R}}}$ and $\bar{\partial}\pi(z) = 0$ for almost every $z \in K(f_c)$. Let $\varphi_0 : W \to \Omega$ be a quasiconformal homeomorphism with

$$\varphi_0 \circ f_c|_{\partial W'} = P \circ \varphi_0|_{\partial W'}$$

which agrees with π on a small neighborhood of K. As φ_0 maps the critical value of f_c to the critical value of $P|_{\Omega'}$ there is a unique lift $\varphi_1 : W' \to \Omega'$ such that

$$\begin{array}{ccc} G' & \xrightarrow{\varphi_1} & \Omega' \\ {\scriptstyle f_c}\downarrow & & {\scriptstyle P}\downarrow \\ G & \xrightarrow{\varphi_0} & \Omega \end{array}$$

commutes and $\varphi_1|_{\partial W'} = \varphi_0|_{\partial W'}$. Setting $\varphi_1(z) = \varphi_0(z)$ for $z \in W \setminus W'$, we obtain a quasiconformal homeomorphism $\varphi_1 : W \to \Omega$ with the same dilatation bound as φ_0; moreover, $\varphi_1|_{K(f_c)} = \pi|_K$. Iteration of this procedure yields a sequence of quasiconformal homeomorphisms $\varphi_n : W \to \Omega$ with uniformly bounded dilatation. The φ_n stabilize pointwise on W, so there is a limiting quasiconformal homeomorphism $\varphi : W \to \Omega$. By construction,

$$\varphi \circ f_c|_{W'} = P \circ \varphi|_{W'}$$

and furthermore $\varphi|_{K(f_c)} = \pi|_{K(f_c)}$; it follows from Bers' Lemma 2.8 that φ is a hybrid equivalence. □

5.2. Properness

Here we deduce the properness of birenormalization from Kiwi's Separation Theorem 2.4.

PROPOSITION 5.4. – $\mathrm{R}\times\mathfrak{R}\colon B_{p/q,m} \to (\mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\}) \times (\mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\})$ *is proper.*

In view of the compactness of the connectedness loci, it suffices to prove that if $(A_k, D_k) \in B_{p/q,m}$ with

$$(A_k, D_k) \to (A_\infty, D_\infty) \in \mathscr{C}^{\#} \quad \text{and} \quad \mathrm{R}\times\mathfrak{R}(A_k, D_k) \to (c_\infty, \tilde{c}_\infty) \in \mathcal{L}_{p/q} \times \mathcal{L}_{p/q}$$

then $(A_\infty, D_\infty) \in B_{p/q,m}$ if and only if $c_\infty \neq \mathrm{root}_{p/q} \neq \tilde{c}_\infty$. If ζ_k is the unique repelling fixed point of $P_k = P_{A_k,D_k}$, where $2q$ external rays land, its multiplier is denoted μ_k. Denote by $\{\zeta_k^i\}_{i=0}^{q-1}$ the periodic orbit of period q contained in K_{R} with multiplier λ_k, and by $\{\tilde{\zeta}_k^i\}_{i=0}^{q-1}$ the similar orbit in $K_{\mathfrak{R}}$ with multiplier $\tilde{\lambda}_k$. These orbits renormalize to periodic orbits $\{\gamma_k^i\} \subset K(f_{c_k})$ and $\{\tilde{\gamma}_k^i\} \subset K(f_{\tilde{c}_k})$, whose multipliers will be denoted ρ_k and $\tilde{\rho}_k$. Passing to a subsequence if necessary, we may assume without loss of generality that the ζ_k converge to a fixed point ζ_∞ of $P_\infty = P_{A_\infty,D_\infty}$ with multiplier μ_∞, and ζ_k^i and $\tilde{\zeta}_k^i$ converge to periodic points ζ_∞^i and $\tilde{\zeta}_\infty^i$ with multipliers λ_∞ and $\tilde{\lambda}_\infty$.

LEMMA 5.5. – *In this setting, if $c_\infty \neq \mathrm{root}_{p/q} \neq \tilde{c}_\infty$ then ζ_∞, $\{\zeta_\infty^i\}$ and $\{\tilde{\zeta}_\infty^i\}$ belong to disjoint orbits. Moreover, the fixed point ζ_∞ is repelling.*

Proof. – It follows from the Implicit Function Theorem that these orbits are distinct unless one of $\{\zeta_\infty^i\}$, $\{\tilde{\zeta}_\infty^i\}$ is parabolic with multiplier 1. Without loss of generality, $\lambda_\infty = 1$, and we may further assume that either $|\lambda_k| \leq 1$ for every k or $|\lambda_k| > 1$ for every k. In the first case, $\lambda_k = \rho_k$ by Remark 2.5, and $\lambda_k \to 1$ implies $c = \mathrm{root}_{p/q}$. In the second case, it similarly follows that $|\rho_k| > 1$ for every k; in view of Yoccoz inequality, the combinatorial rotation numbers of γ_k^i are $p_k/q_k \to 0$, whence $\rho_k \to 1$ and again $c = \mathrm{root}_{p/q}$.

Because $\{\zeta_\infty^i\}$, ζ_∞ and $\{\tilde{\zeta}_\infty^i\}$ lie in distinct orbits, it follows from Theorem 2.4 that at least one of these orbits is repelling. Suppose first that $\{\zeta_\infty^i\}$ is repelling, and let $(\zeta_k^i)^* \in K(P_k)$ be the points which renormalize to $-\gamma_k^i \in K(f_{c_k})$. Then $(\zeta_k^i)^* \to (\zeta_\infty^i)^*$ where $P_\infty((\zeta_\infty^i)^*) = P_\infty(\zeta_\infty^i)$, and the rays landing at the points $(\zeta_\infty^i)^*$ separate ζ_∞ from the critical point -1. Similarly, if the orbit $\{\tilde{\zeta}_\infty^i\}$ is repelling then the rays landing at the corresponding points $(\tilde{\zeta}_\infty^i)^*$ separate ζ_∞ from $+1$. Applying Theorem 2.4 once again, we conclude that ζ_∞ is repelling. □

Continuing with the proof of Proposition 5.4, we observe by Lemma 2.3 that ζ_∞ is the common landing point of the same two cycles of rays with rotation number p/q. The thickening procedure yields a pair of quadratic-like restrictions

$$P_\infty : \Omega'_\infty \to \Omega_\infty \text{ and } P_\infty : \tilde{\Omega}'_\infty \to \tilde{\Omega}_\infty,$$

and we may arrange for Ω'_∞ and $\tilde{\Omega}'_\infty$ to be the limits of thickened domains Ω'_k and $\tilde{\Omega}'_k$ for the quadratic-like restrictions of P_k. As $P_k^{\circ n}(-1) \in \Omega'_k$ and $P_k^{\circ n}(+1) \in \tilde{\Omega}'_k$, it follows that $P_\infty^{\circ n}(-1) \in \Omega'_\infty$ and $P_\infty^{\circ n}(+1) \in \tilde{\Omega}'_\infty$. Thus, P_∞ is birenormalizable, that is, $(A_\infty, D_\infty) \in B_{p/q,m}$. $\square$

5.3. Injectivity

The time has come to show that the intertwining operations

$$(f, \tilde{f}) \mapsto f \underset{p/q,m}{\curlyvee} \tilde{f}$$

are well-defined:

PROPOSITION 5.6. – $\mathrm{R}\times\mathfrak{Я}\colon B_{p/q,m} \to (\mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\}) \times (\mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\})$ *is injective.*

The relevant *pullback argument* is formalized as:

LEMMA 5.7. – *Let $P = P_{A,D}$ and $\tilde{P} = P_{\tilde{A},\tilde{D}}$ where (A, D), $(\tilde{A}, \tilde{D}) \in B_{p/q,m}$. If*

$$\mathrm{R}\times\mathfrak{Я}(A, D) = \mathrm{R}\times\mathfrak{Я}(\tilde{A}, \tilde{D})$$

then there exists a quasiconformal homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ conjugating P to $\tilde{P}$ with $\bar{\partial}\varphi = 0$ almost everywhere on $\check{K}(P)$.

Proof. – We begin by once again restricting P and $\tilde{P}$ to domains $G \supset K(P)$ and $\tilde{G} \supset K(\tilde{P})$ bounded by equipotentials. Our first goal is the construction of a quasiconformal homeomorphism φ_0 which is illustrated in Fig. 7 for $p/q = 1/2$ and $m = 3$. Let $r_1, \ldots, r_{2q}$ be the rays landing at ζ, enumerated in counterclockwise order so that the connected component $K_{-1} \ni -1$ of $K(P) \setminus \{\zeta\}$ lies between r_1 and r_2; the component $K_{+1} \ni +1$ then lies between r_{m+1} and r_{m+2}. We label the remaining components of $K(P) \setminus \{\zeta\}$ as $K^1_{\pm 1}, \ldots, K^{q-1}_{\pm 1}$, so that K^i_{-1} lies between the rays r_{2i+1}, r_{2i+2} and similarly for K^i_{+1}. The corresponding objects associated to $\tilde{P}$ are similarly denoted with an added tilde.

It will be convenient to introduce further notation. Let $S_i \subset G$ be disjoint invariant sectors centered at r_i, and let $L^i_{\pm 1}$ be the component of $P^{\circ -1}(G) \setminus (\bigcup_{j=1}^{2q} S_j)$ containing $K^i_{\pm 1}$. The thickening procedure yields left and right quadratic-like restrictions

$$P : \Omega'_{\mathrm{R}} \to \Omega_{\mathrm{R}} \text{ and } P : \Omega'_{\mathfrak{Я}} \to \Omega_{\mathfrak{Я}}$$

and

$$\tilde{P} : \tilde{\Omega}'_{\mathrm{R}} \to \tilde{\Omega}_{\mathrm{R}} \text{ and } \tilde{P} : \tilde{\Omega}'_{\mathfrak{Я}} \to \tilde{\Omega}_{\mathfrak{Я}}.$$

By assumption, there exist hybrid equivalences h_{R} between $P|_{\Omega'_{\mathrm{R}}}$ and $\tilde{P}|_{\tilde{\Omega}'_{\mathrm{R}}}$, and $h_{\mathfrak{Я}}$ between $P|_{\Omega'_{\mathfrak{Я}}}$ and $\tilde{P}|_{\tilde{\Omega}'_{\mathfrak{Я}}}$. We now replace the domains G and $\tilde{G}$ by $G' = P^{-1}(G)$ and

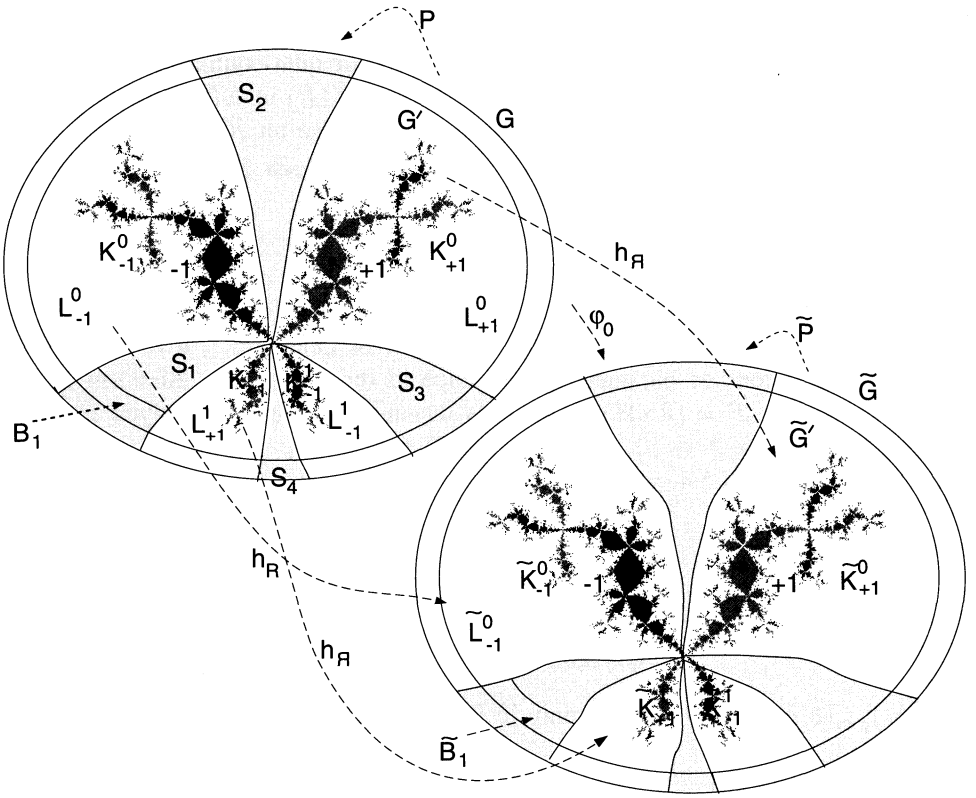

Fig. 7. – Construction of the map φ_0 in the case $p/q = 1/2$ with $m = 3$

$\tilde{G}' = \tilde{P}^{-1}(\tilde{G})$. We define the map φ_0 on $\bigcup_{i=0}^{q-1} L^i_{\pm 1} \subset G'$, setting it equal to $h_{\mathrm{R}}(z)$ for $z \in L^i_{-1}$, and $h_{\text{Я}}$ for $z \in L^i_{+1}$.

Let B_i be the strip of $G' \backslash P^{-1}(G')$ contained in S_i, and $\tilde{B}_i$ its counterpart in $\tilde{G}' \backslash \tilde{P}^{-1}(\tilde{G}')$. We smoothly extend φ_0 to map $B_i \to \tilde{B}_i$ in agreement with the previously specified values of φ_0 on $\partial(L^i_{-1} \cup L^i_{+1})$ and $\varphi_0 \circ P = \tilde{P} \circ \varphi_0$ on the inner boundary of B_i. We now extend φ_0 to the entire sector $S_i \cap X$ by setting $\varphi_0(z) = \tilde{P}^{\circ -n} \circ \varphi_0 \circ P^{\circ n}(z)$ when $P^{\circ n}(z) \in B_i$.

The quasiconformal homeomorphism $\varphi_0 : G' \to \tilde{G}'$ so defined conjugates P to $\tilde{P}$ on $K_{\mathrm{R}} \cup K_{\text{Я}}$, with $\bar{\partial}\varphi_0 = 0$ almost everywhere on this set, sending K_{R} to $\tilde{K}_{\mathrm{R}}$ and $K_{\text{Я}}$ to $\tilde{K}_{\text{Я}}$. We further extend φ_0 to a quasiconformal homeomorphism from G to $\tilde{G}$ so that

$$\varphi_0 \circ P|_{\partial G'} = \tilde{P} \circ \varphi_0|_{\partial G'}.$$

As φ_0 is a conjugacy between postcritical sets, there is a unique lift $\varphi_1 : G' \to \tilde{G}'$ agreeing with φ_0 on $K_{\mathrm{R}} \cup K_{\text{Я}}$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\varphi_1} & \tilde{X} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle \tilde{P}} \\ Y & \xrightarrow{\varphi_0} & \tilde{Y} \end{array}$$

As in the proof of Proposition 5.3, we set $\varphi_1(z) = \varphi_0(z)$ for z in the annulus $G \setminus G'$, and iterate the lifting procedure to obtain a sequence of quasiconformal maps φ_n with uniformly bounded dilatation. In view of the density of $\check{K}(P)$ in $K(P)$, the limiting map $\varphi : G \to \tilde{G}$ conjugates P to $\tilde{P}$. As φ_n stabilizes pointwise on $\check{K}(P)$ with $\varphi|_{K_{\mathrm{R}}} = h_{\mathrm{R}}$ and $\varphi|_{K_{\mathfrak{R}}} = h_{\mathfrak{R}}$ by construction, it follows from Bers' Lemma 2.8 that $\bar{\partial}\varphi = 0$ almost everywhere on $\check{K}(P)$. $\square$

To conclude the proof of Proposition 5.6, we show that the conjugacy just obtained is actually a hybrid equivalence: that any measurable invariant linefield on $K(P) \setminus \check{K}(P)$ has support in a set of Lebesgue measure 0. In view of Lemma 5.7, it follows from the standard considerations of parameter dependence in the Measurable Riemann Mapping Theorem (2.15) that $\mathcal{F} = (\mathrm{R}\times\mathfrak{R})^{-1}(A, D)$ is the injective complex-analytic image of a polydisc $\mathbb{D}^k$ for some $k \in \{0, 1, 2\}$; see [MSS] and [McS]. On the other hand, $\mathcal{F}$ is compact by Proposition 5.4, whence $k = 0$ and $\mathcal{F}$ is a single point. $\square$

5.4. Conclusions

Setting

$$h_{p/q,m}(c, \tilde{c}) = (\mathrm{R}\times\mathfrak{R})^{-1}(c, \tilde{c})$$

so that $h_{p/q,m}(c, \tilde{c}) = (A, D)$ if and only if $P_{A,D} = f_c \underset{p/q,m}{\curlyvee} f_{\tilde{c}}$ we obtain the embeddings

$$h_{p/q,m} : (\mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\}) \times (\mathcal{L}_{p/q} \setminus \{\mathrm{root}_{p/q}\}) \longrightarrow B_{p/q,m} \subset C_{p/q,m}$$

whose existence was asserted in Theorem 3.1. As observed in §3, if $P = P_{A,D}$ is birenormalizable and $(A, D) \in \mathscr{C}^{\#}_{\mathbb{R}}$ then $p/q = 0$ or $1/2$. It follows by symmetry that $\mathrm{R}\times\mathfrak{R}(A, D) = (c, \bar{c})$ for some $c \in \mathbb{C}$; conversely, if $\mathrm{R}\times\mathfrak{R}(A, D) = (c, \bar{c})$ then $(A, D) \in \mathscr{C}^{\#}_{\mathbb{R}}$. Writing $\bar{\Delta}$ for the antidiagonal embedding $\mathbb{C} \ni c \mapsto (c, \bar{c}) \in \mathbb{C}^2$, we define

$$\Psi_{1/2,1} = h_{1/2,1} \circ \bar{\Delta} : \mathcal{L}_{1/2} \setminus \{\mathrm{root}_{1/2}\} \to \mathscr{C}^{\#}_{\mathbb{R}} \cap C_{1/2,1}$$

$$\Psi_{1/2,3} = h_{1/2,3} \circ \bar{\Delta} : \mathcal{L}_{1/2} \setminus \{\mathrm{root}_{1/2}\} \to \mathscr{C}^{\#}_{\mathbb{R}} \cap C_{1/2,3}$$

and

$$\Psi_0 = h_0 \circ \bar{\Delta} : \mathcal{M} \setminus \{\mathrm{root}\} \to \mathscr{C}^{\#}_{\mathbb{R}} \cap C_0.$$

These are the embeddings whose existence was asserted in Theorem 3.3. Compatibility with the standard planar embeddings is a consequence of the following recent result of Buff [Bu]:

THEOREM 5.8. – *Let K_1 and K_2 be compact, connected, cellular sets in the plane, and $\varphi : K_1 \to K_2$ a homeomorphism. If φ admits a continuous extension to an open neighborhood of K_1 such that points outside K_1 map to points outside K_2, then φ extends to a homeomorphism between open neighborhoods of K_1 and K_2.*

Let us sketch the argument for the map Ψ_0. It is easily verified from the explicit expressions in [Mil2, p. 22] that for each $\mu \in \mathbb{C} \setminus \{1\}$ there is a unique pair $(A, B) \in \mathbb{R}^2$

such that the corresponding polynomial in the normal form (2.2) has a pair of complex conjugate fixed points with multipliers μ and $\bar{\mu}$, the remaining fixed point having eigenvalue

$$\nu = 1 - \frac{|\mu - 1|^2}{2\mathrm{Re}(\mu - 1)}.$$

We may continuously label these multipliers as $\mu(A, D)$, $\bar{\mu}(A, D)$ and $\nu(A, D)$ for parameter values $(A, D) \in \mathbb{R} \times i\mathbb{R}$ in a neighborhood of $\Psi_0(\mathcal{M} \setminus \{\text{root}\})$; in particular, $(A, D) \mapsto \mu(A, D)$ is a homeomorphism on such a neighborhood. It follows from Yoccoz inequality (2.1) that $\nu(A, D) > 1$, and therefore $\mu(A, D) \notin [1, \infty)$, for (A, D) in $\Psi_0(\mathcal{M} \setminus \{\text{root}\})$. Similarly, $\mu(\Psi_0(c)) \to 1$ as $c \to$ root, and thus $c \mapsto \mu(\Psi_0(c))$ extends to a embedding

$$\Upsilon : \mathcal{M} \to \mathbb{C} \setminus (1, \infty)$$

which clearly commutes with complex conjugation.

We claim that $\Upsilon^{-1} : \Upsilon(\mathcal{M}) \to \mathcal{M}$ admits a continuous extension meeting the condition of Theorem 5.8. The idea is to allow renormalizations with disconnected Julia sets. Recalling Lemma 2.3, we note that the rays r_0 and $r_{1/2}$ continue to land at the same fixed point for (A, D) in a neighborhood of $\Psi_0(\mathcal{M} \setminus \{\text{root}\})$. As before, we may construct left and right quadratic-like restrictions with continuously varying domains $\Omega'_{A,D}$. It is emphasized in Douady and Hubbard's original presentation [DH2] that straightening, while no longer canonical for maps with disconnected Julia set, may still be continuously defined: it is only necessary to begin with continuously varying quasiconformal homeomorphisms from the fundamental annuli $\Omega_{A,D} \setminus \Omega'_{A,D}$ to the standard annulus. We thereby obtain a continuous extension to a neighborhood of $\Upsilon(\mathcal{M} \setminus \{\text{root}\})$; it is easily arranged that this extension commutes with complex conjugation, so that it is trivial to obtain a further extension to an open set containing the point 1.

6. Measure of the Residual Julia Set

Recall that for a birenormalizable polynomial P,

$$\check{K}(P) \quad = \quad \bigcup_{i=0}^{\infty} P^{-\circ i}(K_{\mathrm{R}} \cup K_{\text{Я}}).$$

Here we synthetize various arguments of Lyubich to show that the *residual Julia set* $K(P) \setminus \check{K}(P)$ has Lebesgue measure 0, provided that neither renormalization lies in the closure of the hyperbolic component $H_{p/q}$. Subject to this restriction, we arrive at an alternative proof that the conjugacy constructed in Lemma 5.7 is a hybrid equivalence. We formalize the statement as follows:

THEOREM 6.1. – *Let P be a birenormalizable cubic polynomial. If neither renormalization is in $\overline{H}_{p/q}$ then $K(P) \setminus \check{K}(P)$ has Lebesgue measure 0.*

The main technical tool for us will be the celebrated Yoccoz puzzle construction which we briefly recall below:

Yoccoz puzzle and recurrence. Let $f = f_c$ be a quadratic polynomial with connected Julia set, and $G \supset K(f)$ be a domain bounded by some fixed equipotential curve. As observed in Lemma 2.11, if $c \in L_{p/q}$ for some $q \geq 2$ then α_f is the landing point of a cycle of q external rays. The *Yoccoz puzzle of depth zero* consists of the q pieces $Y_1^0, Y_2^0, \ldots, Y_q^0$ obtained by cutting G along these rays, and the puzzle pieces of depth $n > 0$ are the connected components Y_i^n of the various $f^{\circ -n}(Y_i^0)$. Each point $z \in K(f) \setminus f^{\circ -n}(\alpha)$ lies in a unique depth n puzzle piece $Y^n(z)$. A nonrenormalizable polynomial f has a *reluctantly recurrent* critical point if there exists $k \geq 0$ and a sequence of depths $n_i \to \infty$ such that the restriction $f^{\circ n_i - k} : Y^{n_i}(0) \to Y^k(f^{\circ n_i - k}(0))$ has degree 2. Note that, somewhat abusing the notation, we allow maps with non-recurrent critical point in this definition. In the complementary case of *persistently recurrent* critical point Lyubich has shown the following:

LEMMA 6.2 ([Lyu2, p. 6]). – *If the critical point of a non-renormalizable quadratic polynomial f_c is persistently recurrent then $f_c|_{\omega_c(0)}$ is topologically minimal, that is all orbits are dense in $\omega_c(0)$. In particular, $\alpha_{f_c} \notin \omega_c(0)$ and $\beta_{f_c} \notin \omega_c(0)$.*

For a cubic map $P = P_{A,D}$ as in the assumptions of Theorem 6.1 we adapt the puzzle construction as follows. Denote $\{\zeta_i\}_{i=0}^{q-1}$ the repelling periodic orbit of period q contained in K_{R}, and $\{\widetilde{\zeta}_i\}_{i=0}^{q-1}$ the similar orbit in $K_{\mathfrak{R}}$. The depth zero puzzle pieces W_i^0 are now obtained by cutting an equipotentially bounded domain $G \supset J(P)$ along every ray which lands at some fixed point or at one of the points $\zeta_i, \widetilde{\zeta}_i$, and the pieces of depth n are the connected components of the various $P^{\circ -n}(W_i^0)$. Each point $z \in J(P) \setminus P^{-\circ n}(\{\text{fixed points}\} \cup \{\zeta_i\} \cup \{\widetilde{\zeta}_i\})$ lies in a unique depth n puzzle piece $W^n(z)$.

By analogy with the quadratic case, we say that the critical point ± 1 of the cubic polynomial $P = P_{A,D}$ is *reluctantly recurrent* if there exist $k \geq 0$, $N > 0$ and a sequence of depths $n_i \to \infty$ such that $P^{\circ n_i - k}|_{W^{n_i}(\pm 1)}$ is a map of degree N. We readily observe that if P is birenormalizable and one of its renormalizations has a reluctantly recurrent critical point then the corresponding critical point of P is reluctantly recurrent. Indeed in this case the degree of the restriction $P^{\circ n-k}|_{W^n(1)}$ is not greater than that of the map $\mathrm{R}(P)^{\circ n-k}$ on the quadratic puzzle piece $Y^n(0)$, and similarly for the other renormalization.

Yarrington [Yar] has shown that if both critical points of P are reluctantly recurrent then $J(P)$ is locally connected; in particular nested sequences of puzzle pieces shrink to points in this case:

$$\bigcap_{n=0}^{\infty} W^n(z) = \{z\} \tag{6.1}$$

for every $z \in J(P) \setminus \cup P^{-\circ n}(\{\text{fixed points}\} \cup \{\zeta_i\} \cup \{\widetilde{\zeta}_i\})$ ([Yar, Theorem 3.5.7]).

Relative ergodicity. The proof of Theorem 6.1 is based on the following general principle of Lyubich:

THEOREM 6.3 ([Lyu1]). – *Let g be a rational map with $J(g) \neq \hat{\mathbb{C}}$. Then*

$$\omega_g(z) \subset \bigcup_{\gamma \in \Gamma} \omega_g(\gamma)$$

for almost every $z \in J(g)$, where Γ is the set of all critical points.

We divide the argument into two cases depending on the recurrence properties of the renormalizations of P.

Assume first that $\omega_P(-1) \cap \omega_P(+1) = \emptyset$. It follows from Theorem 6.3 that for almost every $z \in J(P)$ $\omega_P(z)$ lies in the disjoint union $\omega_P(-1) \cup \omega_P(+1)$. Without loss of generality $\omega_P(z) \subset \omega_P(-1)$, so every accumulation point of the sequence $P^{\circ n}(z)$ lies in K_{R}. In particular, $P^{\circ n}(z) \in \Omega'$ for sufficiently large n, where $\Omega' \ni -1$ is the domain of the left quadratic-like restriction of P. Thus, $P^{\circ n}(z) \in K_{\mathrm{R}}$ for large enough n, and therefore $z \in \bigcup_{i=0}^{\infty} P^{\circ -i}(K_{\mathrm{R}}) \subset \check{K}(P)$.

In the other case, recall from (5.1) that $\omega_P(-1) \cap \omega_P(+1) = \{\zeta\}$. By Lemma 6.2 in the case when $\mathrm{R}(P)$ is nonrenormalizable its critical point is reluctantly recurrent, and similarly for $\mathfrak{R}(P)$. Combining Remark 2.12 and Lemma 6.2, we see that if $\mathrm{R}(P)$ (or $\mathfrak{R}(P)$) is a renormalizable quadratic polynomial then its renormalization has a reluctantly recurrent critical point. In either case we readily observe that both critical points of P are reluctantly recurrent. We conclude the argument by showing that under these conditions the Lebesgue measure of the Julia set of P is zero:

LEMMA 6.4. – *Let $P = P_{A,D}$ be a birenormalizable cubic whose critical points are both reluctantly recurrent. Then the Julia set of P has Lebesgue measure zero.*

Proof. – We adapt Lyubich's argument [Lyu2] for the quadratic case. As both critical points of P are reluctantly recurrent, there exist k and arbitrary large s and t such that $P^{\circ s}|_{W^{s+k}(-1)}$ and $P^{\circ t}|_{W^{t+k}(+1)}$ are maps of degree 2. By Theorem 6.3, for a full measure set of $z \in J(P)$, there exists n such that $P^{\circ n}(z)$ lies in $W^{s+k}(-1) \cup W^{t+k}(+1)$ for any s and t. Fixing t, s and z consider the least such n. Without loss of generality, $P^{\circ n}(z) \in W^{s+k}(-1)$ and we obtain a chain of univalent branches of P^{-1}

$$W^{s+k}(-1) = X_0 \leftarrow X_{-1} \leftarrow \ldots \leftarrow X_{-n} \ni z$$

by pulling this piece back along the orbit of z.

Fix a puzzle-piece W_i^k of depth k. As the boundary of W_i^k consists of preimages of external rays landing at five periodic orbits of P and equipotential curves it follows from Kœbe 1/4 theorem that there exist δ_i and δ_i' such that for any $u \in W_i^k$ with $\mathrm{dist}(\partial W_i^k, u) < \delta_i$ some neighborhood $U \subset W_i^k$ around u is univalently mapped by an iterate $P^{\circ j}$ to a disk $D_{\delta_i'}(P^{\circ j}(u))$ of radius δ_i' centered at $P^{\circ j}(u)$. Denote by δ the minumum of various δ_i, δ_i' and set $u = P^{\circ s+n}(z) \in P^{\circ s}(W^{s+k}(-1)) = W^k$. By the above, there exists a neighborhood $U \ni u$ in W^k and an iterate $P^{\circ j}$ univalently mapping U to $D_\delta(P^{\circ j}(u))$.

Assume first that $P^{\circ s}(-1)$ does not belong to U, or $|P^{\circ s+j}(-1) - P^{\circ j}(u)| > \delta/100$. The density of the Julia set in a disk of radius $\delta/200$ is bounded away from 1. Consider the univalent pullback $T_0 = D_{\delta/200}(P^{\circ j}(u)), T_{-1}, \ldots, T_{-s-n-j}$ along the orbit $z \mapsto P(z) \mapsto \ldots \mapsto P^{\circ s+n+j}(z) \in T_0$. By the Kœbe distortion theorem, the density of $J(P)$ in T_{-s-n-j} is also bounded away from 1. By the estimate (6.1), the disks T_{-s-n-j} shrink to the point z as s grows and therefore z is not a point of density for the set $J(P)$.

Consider now the case when $P^{\circ s}(-1) \in U$, and $|P^{\circ s+j}(-1) - P^{\circ j}(u)| < \delta/100$. Then we can find a disk D_1 centered at $P(-1)$, such that $P^{\circ s+j-1}(D_1)$ is contained between

$D_{\delta/10}(P^{\circ j}(u))$ and $D_{\delta/2}(P^{\circ j}(u))$. By Kœbe distortion theorem, the density of $J(P)$ in the disk D_1 is bounded away from 1. Consider the preimage D_0 of D_1 centered around -1 and contained in $W^{n+s}(-1)$. The density of the Julia set in D_0 is again bounded away from 1, and as in the previous case we conclude that z is not a point of density. Thus the set of density points of $J(P)$ has measure zero, and by Lebesgue density theorem, so does $J(P)$. $\square$

Appendix A

Discontinuity at the corner point

The systematic exclusion of root points is not merely an artifact of our reliance on quasiconformal surgery. It is conceivable that more powerful techniques might someday prove existence and uniqueness of intertwinings $f_c \underset{p/q,m}{\curlyvee} f_{\tilde{c}}$ for *any* pair $(c, \tilde{c}) \in \mathcal{L}_{p/q} \times \mathcal{L}_{p/q}$. Indeed, $f_c|_{J(f_c)}$ for any $c \in H_{p/q}$ is canonically topologically conjugate to $f_{\hat{c}}|_{J(f_{\hat{c}})}$ where $\hat{c} = \mathrm{root}_{p/q}$, and on these grounds we have put forth Conjecture 3.1. On the other hand, we assert in Theorem 3.2 that such a extension of $h_{p/q,m}$ is necessarily discontinuous at the corner point $(\mathrm{root}_{p/q}, \mathrm{root}_{p/q})$. This is an instance of a phenomenon investigated by one of the authors. It is shown in [Ep] that any disjoint type component consisting of maps with *adjacent* attracting basins must suffer such a discontinuity; for $c, \tilde{c} \in H_{p/q}$ the basins of $f_c \underset{p/q.m}{\curlyvee} f_{\tilde{c}}$ are adjacent by construction. Here we simply summarize the relevant considerations.

Let g be an analytic map fixing $\zeta \in \mathbb{C}$. The *holomorphic index* of g at ζ is the residue

$$\eta = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - g(z)}$$

where γ is a loop enclosing ζ but no other fixed point. It is easily checked that this quantity is conformally invariant; in fact,

$$\eta = \frac{1}{1 - \lambda}$$

so long as the multiplier $\lambda = g'(\zeta)$ is not equal to 1.

An elementary computation yields

$$\eta_a = \frac{1}{2\pi i} \int_\gamma \frac{dz}{-az^2 - z^3} = \frac{1}{a^2}$$

for the holomorphic index of $Q_a(z) = z + az^2 + z^3$ at the parabolic fixed point 0. In the terminology of [Ep], such a fixed point is described as *parabolic-attracting*, *parabolic-indifferent* or *parabolic-repelling* depending on whether $\mathrm{Re}\eta$ is greater than, equal to, or less than 1. The first of these alternatives applies when $|a^2 - \frac{1}{2}| < \frac{1}{2}$. The corresponding region in the a-plane is bounded by a lemniscate shaped like the symbol ∞; its position in the cubic connectedness locus is depicted in Fig. 8, which in view of the 4-dimensionality of $D_{1,1}$ is merely schematic. The intersection of the component boundary with $\mathrm{Per}_1(1)$

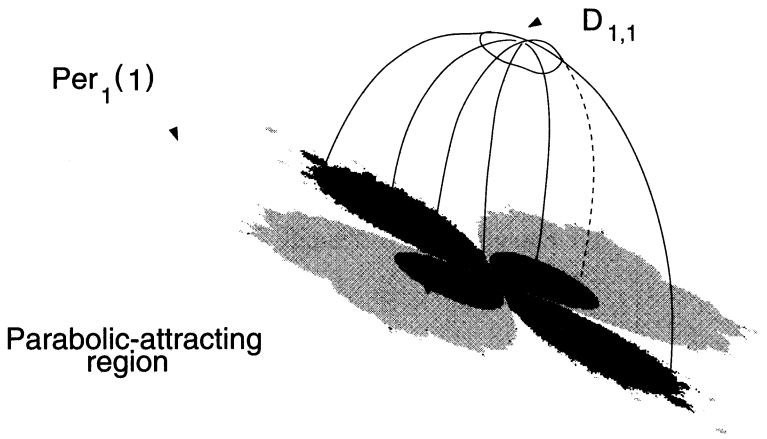

Fig. 8. – The unique component of type $D_{1,1}$ and its impression on $\mathrm{Per}_1(1)$

consists of the closure of this lemniscate (shaded in dark gray, and contained in the light gray region where both critical points lie in the parabolic basin), and a similar locus (the large lobes of the medium gray region) parameterizing maps whose other fixed point is attracting or indifferent; the latter might be described as intertwinings $f_c \underset{0}{\curlyvee} f_{\tilde{c}}$ for $c \in H_0$ and $\tilde{c} =$ root. These pieces intersect at the parameter value $a = 0$ where the parabolic fixed point becomes *degenerate*.

The crux of the matter is the following elementary observation (compare with [Mil1, Problem 9-1]):

LEMMA A.1. – *Let* $\eta \in \mathbb{C}$. *Then* $\mathrm{Re}\eta \geq 1$ *if and only if there exist continuous paths* $\lambda, \tilde{\lambda} : [0,1) \to \mathbb{D}$ *with endpoints* $\lambda(1) = 1 = \tilde{\lambda}(1)$ *such that*

$$\eta = \lim_{t \to 1} \frac{1}{1-\lambda(t)} + \frac{1}{1-\tilde{\lambda}(t)}.$$

Complex conjugate paths may be chosen when η *is real.*

Similar considerations apply to $h_{p/q,m}(H_{p/q} \times H_{p/q})$. For odd denominator p/q and $A_{p/q} = -\frac{1}{3}e^{2\pi i p/q}$, it is easy to see that $P_{A_{p/q},0}$ is the unique normalized cubic polynomial with a degenerate parabolic fixed point of multiplier $e^{2\pi i p/q}$; thus

$$\lim_{c \to \mathrm{root}_{p/q}} h_{p/q,q}(c,c) \to (A_{p/q}, 0)$$

as is evident in Fig. 3.

Appendix B

Non local-connectivity of the real connectedness locus

Here we employ a simplified version of an argument of Lavaurs [La] to conclude that the real cubic connectedness locus is not locally connected along an interval in the boundary

of $\Psi_0(\mathcal{M} \setminus \{\text{root}\})$. The existence of comb-like structures in $\Psi_{1/2,i}(\mathcal{L}_{1/2} \setminus \{\text{root}_{1/2}\})$ is similarly demonstrated; see also Nakane and Schleicher's proof of non local-connectivity for the tricorn [NS].

We begin with a brief review of the theory of parabolic bifurcations, as applied in particular to real cubic polynomials. The reader is referred to [Do] for a more comprehensive exposition; supporting technical details may be found in [Sh]. Recall that the fixed point at 0 is parabolic with multiplier 1 for every map in the family

$$Q_a(z) = z + az^2 + z^3.$$

LEMMA B.1 (Fatou coordinates). – *For $a \neq 0$ there exist topological discs $U_a^{\mathcal{A}}$ and $U_a^{\mathcal{R}}$ whose union is a punctured neighborhood of the parabolic fixed point, such that*

$$Q_a(\bar{U}_a^{\mathcal{A}}) \subset U_a^{\mathcal{A}} \cup \{0\} \text{ and } \bigcap_{k=0}^{\infty} Q_a^{\circ k}(\bar{U}_a^{\mathcal{A}}) = \{0\},$$

$$Q_a^{-1}(\bar{U}_a^{\mathcal{R}}) \subset U_a^{\mathcal{R}} \cup \{0\} \text{ and } \bigcap_{k=0}^{\infty} Q_a^{\circ -k}(\bar{U}_a^{\mathcal{R}}) = \{0\}.$$

Moreover, there exist injective analytic maps

$$\Phi_a^{\mathcal{A}} : U_a^{\mathcal{A}} \to \mathbb{C} \text{ and } \Phi_a^{\mathcal{R}} : U_a^{\mathcal{R}} \to \mathbb{C},$$

unique up to post-composition by translations, such that

$$\Phi_a^{\mathcal{A}}(Q_a(z)) = \Phi_a^{\mathcal{A}}(z) + 1 \text{ and } \Phi_a^{\mathcal{R}}(Q_a(z)) = \Phi_a^{\mathcal{R}}(z) + 1.$$

The quotients $C_a^{\mathcal{A}} = U_a^{\mathcal{A}}/Q_a$ and $C_a^{\mathcal{R}} = U_a^{\mathcal{R}}/Q_a$ are therefore Riemann surfaces conformally equivalent to the cylinder $\mathbb{C}/\mathbb{Z}$.

The quotients $C_a^{\mathcal{A}}$ and $C_a^{\mathcal{R}}$ are customarily referred to as the *Écalle-Voronin cylinders* associated to the map Q_a; we will find useful to regard these as Riemann spheres with distinguished points $\pm$ filling in the punctures. Every point in the *parabolic basin*

$$\mathcal{B}_a = \{z \in \mathbb{C} |\ Q_a^{\circ n}(z) \neq 0 \text{ for } n \geq 0 \text{ and } Q_a^{\circ n}(z) \to 0\}$$

eventually lands in $U_a^{\mathcal{A}}$, and the return map from $U_a^{\mathcal{R}} \cap \mathcal{B}_a$ to $U_a^{\mathcal{A}}$ descends to a well-defined analytic transformation

$$\mathcal{E}_a : \mathcal{W}_a \to C_a^{\mathcal{A}}$$

where $\mathcal{W}_a$ is the image of $\mathcal{B}_a$ on $C_a^{\mathcal{R}}$. It is easy to see that the ends of $C_a^{\mathcal{R}}$ belong to different components of $\mathcal{W}_a$. The choice of a conformal *transit isomorphism*

$$\Theta : C_a^{\mathcal{A}} \to C_a^{\mathcal{R}}$$

respecting these ends determines an analytic dynamical system

$$\mathcal{F}_{a:\Theta} = \Theta \circ \mathcal{E}_a : \mathcal{W}_a \to C_a^{\mathcal{R}}$$

with fixed points at $\pm$. The product of the corresponding eigenvalues $\varrho^{\pm}_{a:\Theta}$ is clearly independent of Θ; indeed

$$\varrho^{+}_{a:\Theta} \cdot \varrho^{-}_{a:\Theta} = e^{-4\pi^2(\eta_a - 1)} = e^{-4\pi^2(\frac{1}{a^2}-1)}.$$

For $a \in \mathbb{R}$ the real-axis projects to natural *equators* $R^{\mathcal{A}}_a \subset C^{\mathcal{A}}_a$ and $R^{\mathcal{R}}_a \subset C^{\mathcal{R}}_a$; the set $\mathcal{W}_a$ is disjoint from $R^{\mathcal{R}}_a$ and symmetric about it. Moreover, when $a \in (0, \sqrt{3})$ the critical points of Q_a form a complex conjugate pair in $\mathcal{B}_a$. We restrict attention to this simplest case: $J(Q_a) = \partial\mathcal{B}_a$ is a Jordan curve, as is each of the two components of $\partial\mathcal{W}_a$. It follows from the details of the construction that $\mathcal{E}_a$ has infinitely many critical points but only two critical values; these are situated symmetrically with respect to the appropriate equators, and each of the critical values $v^{\pm}_a$ has critical preimages on both sides of $R^{\mathcal{R}}_a$.

We now consider perturbations in the family

$$Q_{a,\epsilon}(z) = \epsilon + z + az^2 + z^3$$

corresponding to

$$F(z) = z^3 - \frac{1}{3}a^2 z + \left(\frac{2}{27}a^3 + \epsilon\right) \tag{B.1}$$

in the normal form of (2.2). For small $\epsilon > 0$ the parabolic point splits into a complex conjugate pair of attracting fixed points $\zeta^{\pm}_{a,\epsilon}$ but one may still speak of attracting and repelling petals:

LEMMA B.2 (Douady coordinates). – *For small $\epsilon > 0$ there exist topological discs $U^{\mathcal{A}}_{a,\epsilon}$ and $U^{\mathcal{R}}_{a,\epsilon}$ whose union is a neighborhood of the parabolic fixed point of Q_a, and injective analytic maps*

$$\Phi^{\mathcal{A}}_{a,\epsilon} : U^{\mathcal{A}}_{a,\epsilon} \to \mathbb{C} \text{ and } \Phi^{\mathcal{R}}_{a,\epsilon} : U^{\mathcal{R}}_{a,\epsilon} \to \mathbb{C},$$

unique up to post-composition by translations, such that

$$\Phi^{\mathcal{A}}_{a,\epsilon}(Q_{a,\epsilon}(z)) = \Phi^{\mathcal{A}}_{a,\epsilon}(z) + 1 \text{ and } \Phi^{\mathcal{R}}_{a,\epsilon}(Q_{a,\epsilon}(z)) = \Phi^{\mathcal{R}}_{a,\epsilon}(z) + 1.$$

The quotients $C^{\mathcal{A}}_{a,\epsilon} = U^{\mathcal{A}}_{a,\epsilon}/Q_{a,\epsilon}$ and $C^{\mathcal{R}}_{a,\epsilon} = U^{\mathcal{R}}_{a,\epsilon}/Q_{a,\epsilon}$ are Riemann surfaces conformally equivalent to $\mathbb{C}/\mathbb{Z}$.

In view of the assumption on ϵ these cylinders come similarly equipped with equators. As in the parabolic case, the return map from the relevant portion of $U^{\mathcal{R}}_{a,\epsilon}$ to $U^{\mathcal{A}}_{a,\epsilon}$ descends to an analytic transformation $\mathcal{E}_{a,\epsilon}$ from a neighborhood of each end of $C^{\mathcal{R}}_{a,\epsilon}$ to a neighborhood of the corresponding end of $C^{\mathcal{A}}_{a,\epsilon}$. However, there is now a canonical transit isomorphism $\Theta_{a,\epsilon} : C^{\mathcal{A}}_{a,\epsilon} \to C^{\mathcal{R}}_{a,\epsilon}$, and the composition

$$\mathcal{F}_{a,\epsilon} = \Theta_{a,\epsilon} \circ \mathcal{E}_{a,\epsilon}$$

is completely specified by the dynamics of $Q_{a,\epsilon}$. In particular, the eigenvalues at $\pm$ are given by

$$\varrho^{\pm}_{a,\epsilon} = e^{\frac{-4\pi^2}{\log \lambda^{\pm}(a,\epsilon)}}, \tag{B.2}$$

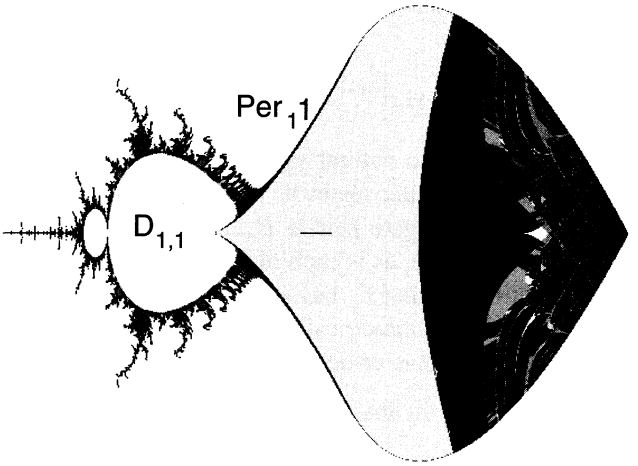

Fig. 9. – The view of the comb on the $D_{1,1}$ component in (A, b) parametrization

where $\lambda^{\pm}(a,\epsilon) = 1 \pm 2i\sqrt{a\epsilon} + O(\epsilon)$ are the complex conjugate eigenvalues of $\zeta^{\pm}_{a,\epsilon}$. A fixed but otherwise arbitrary choice of basepoints in the original petals $U^{\mathcal{A}}_{a,\epsilon}$ and $U^{\mathcal{R}}_{a,\epsilon}$ allows us to identify $C^{\mathcal{A}}_a$ with the various $C^{\mathcal{A}}_{a,\epsilon}$ and $C^{\mathcal{R}}_a$ with the various $C^{\mathcal{R}}_{a,\epsilon}$. The following fundamental theorem first appeared in [DH1] and was adapted to the case at hand in [La]:

THEOREM B.3. – *In this setting, if $a_k \to a$ and $\epsilon_k \to 0$ such that*

$$\varrho^{+}_{a_k,\epsilon_k} \to \varrho^{+}_{a:\Theta} \text{ or equivalently } \varrho^{-}_{a_k,\epsilon_k} \to \varrho^{-}_{a:\Theta}$$

then $\mathcal{F}_{a_k,\epsilon_k} \to \mathcal{F}_{a:\Theta}$ locally uniformly on $\mathcal{W}_a$.

We are now in a position to avail ourselves of an elementary but crucial observation of Lavaurs [La]:

LEMMA B.4. – *There exist $a \in (0, \sqrt{3})$ and a transit map $\Theta : C^{\mathcal{A}}_a \to C^{\mathcal{R}}_a$ respecting equators, such that both $\Theta(v^{\pm}_a)$ are superattracting fixed points for $\mathcal{F}_{a:\Theta}$.*

The relevant continuity argument is depicted in Fig. B.4. For small $a > 0$, the critical values v^{+}_a and v^{-}_a are farther apart than any pair of critical points of $\mathcal{E}_a$. All of these points move continuously as a increases towards the parameter value $\sqrt{3}$ where $v^{\pm}_a$ collide at the equator. Consequently, there exist $a \in (0, \sqrt{3})$ and a symmetric pair of critical points $c^{\pm}_a$ which are exactly as far apart as the critical values $v^{\pm}_a$. Both possibilities

$$\mathcal{E}_a(c^{\pm}_a) = v^{\pm}_a \text{ and } \mathcal{E}_a(c^{\pm}_a) = v^{\mp}_a$$

may be so arranged. Choosing the former, we see that $\Theta(v^{\pm}_a) = c^{\pm}_a$ for a suitable transit map respecting equators; in particular, each of $c^{\pm}_a$ is a superattracting fixed point for $\mathcal{F}_{a:\Theta}$.

Let a be the parameter value so obtained. In view of (B.2) there exist real ϵ_k decreasing to 0 with $\varrho^{\pm}_{a,\epsilon_k} \to \varrho^{\pm}_{a:\Theta}$. It follows from Theorem B.3 that the nearby fixed points of of $\mathcal{F}_{a,\epsilon_k}$ are attracting. Their lifts generate a complex conjugate pair of attracting

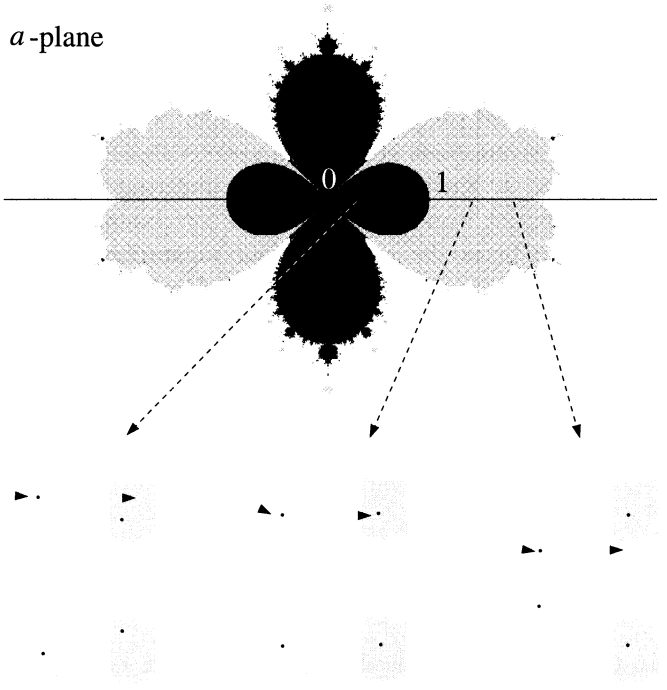

Fig. 10. – Lavaurs' continuity argument

periodic orbits in the original dynamical plane, and thus $J(Q_{a,\epsilon_k})$ is connected; moreover, Q_{a,ϵ_k} is birenormalizable as the critical orbits are separated by the real-axis. The two ways of marking the critical points of Q_a yield parameters $(A_\infty, \pm D_\infty) \in \mathrm{Per}_1(1)$ and corresponding parameters $(A_k, \pm D_k) \in \Phi_0(\mathcal{M} - \{\mathrm{root}\})$ associated to the perturbations Q_{a,ϵ_k}. It follows from (B.1) that $A_\infty < 0$, and thus $(A_\infty, \pm D_\infty)$ are the endpoints of an interval I on the simple arc

$$\mathcal{P} = \{(A, D) \in \mathrm{Per}_1(1) |\ A < 0\}.$$

The entire impression

$$\mathcal{I} = \{(A, D) \in \mathscr{C}^{\#}_{\mathbf{R}} |\ \Psi_0(c_j) \to (A, D) \text{ for some } c_j \in \mathcal{M} \setminus \{\mathrm{root}\} \text{ with } c_j \to \mathrm{root}\}$$

lies in $\mathcal{P}$ by Yoccoz Inequality (2.1); thus $I \subset \mathcal{I}$, as $\mathcal{I}$ is connected and

$$(A_\infty, \pm D_\infty) = \lim_{k \to \infty} (A_k, \pm D_k) \in \mathcal{I}$$

by construction. It follows from Lemma 2.3 and the considerations of Lemma A.1 that $\mathscr{C}^{\#}_{\mathbf{R}}$ is non-locally connected at every $(A, D) \in \mathcal{I}$ for which $P_{A,D}$ has a parabolic-repelling fixed point.

Acknowledgments

This paper was motivated by J. Milnor's Autumn 1995 Stony Brook lectures on the dynamics of cubic polynomials, and developed out of joint meditation of the two authors in front of the the full-color version of Fig. 2. We thank J. Milnor for numerous discussions of our results and many helpful suggestions as this paper progressed. We are indebted to M. Lyubich for fruitful conversations concerning various aspects of quadratic dynamics. Further thanks are due to J. Kiwi for sharing his understanding of cubic maps and discussing some of his current work, and to X. Buff for communicating his results. We would additionally like to thank the referee whose suggestions led us to the stronger formulation of our main result as presented here, and also to Tan Lei for making similar comments. This project was conducted in the congenial atmosphere of IMS at Stony Brook, and we thank our colleagues for their interest and moral support.

The computer pictures in this paper were produced using software written by J. Milnor and S. Sutherland.

REFERENCES

[Bi1] B. Bielefeld, Changing the order of critical points of polynomials using quasiconformal surgery, *Thesis*, Cornell, 1989.

[Bi2] B. Bielefeld, Questions in quasiconformal surgery, pp. 2-8, in Conformal Dynamics Problem List, ed. B. Bielefeld, Stony Brook IMS Preprint 1990/1, and in part 2 of Linear and Complex Analysis Problem Book 3, ed. V. Havin and N. Nikolskii, *Lecture Notes in Math.*, Vol. 1574, Springer-Verlag, 1994.

[BD] B. Branner and A. Douady, Surgery on complex polynomials, in Proceedings of the Symposium on Dynamical Systems, Mexico, 1986, *Lecture Notes in Math.*, Vol. 1345, Springer-Verlag, 1987.

[BF] B. Branner and N. Fagella, *Homeomorphisms between limbs of the Mandelbrot set*, MSRI Preprint 043-95.

[BH] B. Branner and J. H. Hubbard, The iteration of cubic polynomials. Part I: The global topology of parameter space, *Acta Mathematica*, Vol. 160, 1988, pp. 143-206.

[Bu] X. Buff, *Extension d'homéomorphismes de compacts de* $\mathbb{C}$, Manuscript, and personal communication.

[Do] A. Douady, Does a Julia set depend continuously on a polynomial?, *Proc. of Symp. in Applied Math.*, Vol. 49, 1994.

[DH1] A. Douady and J. H. Hubbard, Étude dynamique des polynômes complexes, I & II, *Publ. Math. Orsay*, 1984-5.

[DH2] A. Douady and J. H. Hubbard, On the dynamics of polynomial-like mappings, *Ann. scient. Éc. Norm. Sup.*, 4e *série*, Vol. 18, 1985, pp. 287-343.

[Ep] A. Epstein, *Counterexamples to the quadratic mating conjecture*, Manuscript in preparation.

[Fa] D. Faught, *Local connectivity in a family of cubic polynomials*, Thesis, Cornell 1992.

[GM] L. Goldberg and J. Milnor, Fixed points of polynomial maps II, *Ann. Scient. Éc. Norm. Sup.*, 4e série, Vol. 26, 1993, pp. 51-98.

[Haï] P. Haïssinsky, *Chirurgie croisée*, Manuscript, 1996.

[Hub] J. H. Hubbard, Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz, in *Topological methods in Modern Mathematics*, Publish or Perish, 1992, pp. 467-511 and 375-378.

[Ki] J. Kiwi, *Non-accessible critical points of Cremer polynomials*, IMS at Stony Brook Preprint 1995/2.

[La] P. Lavaurs, *Systèmes dynamiques holomorphes: Explosion de points périodiques*, Thèse, Université de Paris-Sud, 1989.

[LV] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer-Verlag, 1973.

[Lyu1] M. Lyubich, On typical behavior of the trajectories of a rational mapping of the sphere, *Soviet. Math. Dokl.*, Vol. 27, 1983, No. 1, pp. 22-25.

[Lyu2] M. LYUBICH, *On the Lebesgue measure of a quadratic polynomial*, IMS at Stony Brook Preprint 1991/10.

[Lyu3] M. LYUBICH, *Dynamics of quadratic polynomials, I. Combinatorics and geometry of the Yoccoz puzzle*, MSRI Preprint 026-95.

[MSS] R. MAÑÉ, P. SAD and D. SULLIVAN, On the dynamics of rational maps, *Ann. Scient. Éc. Norm. Sup.*, 4e *série*, Vol. 16, 1983, pp. 51-98.

[McM1] C. MCMULLEN, Complex Dynamics and Renormalization, *Annals of Math. Studies*, Princeton Univ. Press, 1993.

[McM2] C. MCMULLEN, Renormalization and 3-Manifolds which Fiber over the Circle, *Annals of Math. Studies*, Princeton Univ. Press, 1996.

[McS] C. MCMULLEN and D. SULLIVAN, *Quasiconformal homeomorphisms and dynamics III: The Teichmüller space of a rational map*, Preprint, 1996.

[Mil1] J. MILNOR, *Dynamics in one complex variable: Introductory lectures*, IMS at Stony Brook Preprint 1990/5.

[Mil2] J. MILNOR, Remarks on iterated cubic maps, *Experimental Math.*, Vol. 1 1992, pp. 5-24.

[Mil3] J. MILNOR, *On cubic polynomials with periodic critical point*, Manuscript, 1991.

[Mil4] J. MILNOR, *Hyperbolic components in spaces of polynomial maps*, with an appendix by A. Poirier, IMS at Stony Brook Preprint 1992/3.

[Mil5] J. MILNOR, *Periodic orbits, external rays and the Mandelbrot set; An expository account*, Preprint, 1995.

[NS] S. NAKANE and D. SCHLEICHER, *Non-local connectivity of the tricorn and multicorns*, in Proceedings of the International Conference on Dynamical Systems and Chaos, World Scientific, 1994.

[Sh] M. SHISHIKURA, The parabolic bifurcation of rational maps, *Colóquio Brasileiro de Matemática 19*, IMPA, 1992.

[Win] R. WINTERS, *Bifurcations in families of antiholomorphic and biquadratic maps*, Thesis, Boston University, 1989.

[Yar] B. YARRINGTON, *Local connectivity and Lebesgue measure of polynomial Julia sets*, Thesis, SUNY at Stony Brook, 1995.

(Manuscript received October 10, 1996;
revised and accepted October 19, 1998.)

A. EPSTEIN
Institute for Mathematical Sciences,
Department of Mathematics,
State University of New York at Stony Brook,
New York, 11794-3660, USA
E-mail: adame@math.sunysb.edu

M. YAMPOLSKY
Department of Mathematics,
Yale University,
New Haven CT06520-8283, USA
E-mail: yampol@math.yale.edu