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FINITENESS THEOREMS FOR THE COHOMOLOGY OF AN OVERCONVERGENT ISOCRYSTAL ON A CURVE

BY RICHARD CREW*

ABSTRACT. – Let M be an overconvergent isocrystal on a smooth affine curve X/k over a perfect field of characteristic $p > 0$, realized as a module on a suitable lifting of X with connection. We give a topological condition on the connection which guarantees that the rigid cohomology of M is finite-dimensional. As a result, one sees that M has finite-dimensional cohomology if it satisfies an analogue of Grothendieck's local monodromy theorem. Some arithmetic applications are given. © Elsevier, Paris

RÉSUMÉ. – Soit M un isocristal surconvergent sur une courbe X/k affine et lisse sur un corps parfait de caractéristique $p > 0$, réalisé comme module à connexion sur un relèvement convenable de X. Nous donnons une condition de nature topologique sur la connexion pour que la cohomologie rigide de M soit de dimension finie. Il en résulte que la cohomologie de M sera de dimension finie si M vérifie une analogue du théorème de monodromie locale de Grothendieck. On donne aussi des applications arithmétiques de ce résultat. © Elsevier, Paris

0. Introduction

Let X/k be a smooth scheme over a perfect field k of characteristic $p > 0$. In [5, 8] Berthelot constructed a category of p-adic local coefficients on X, the category of *overconvergent isocrystals* on X, and defined the *rigid cohomology* of an overconvergent isocrystal on X. This theory generalizes previous constructions of Dwork, Washnitzer, and Monsky [18, 30, 31]; there is also a definition of "rigid cohomology with supports" which generalizes Dwork's "dual theory"; (as explained in [4 §3]. Now fairly simple examples show that the cohomology of an isocrystal can be infinite-dimensional even when X is a smooth curve, and the main result of this paper is to give a fairly general sufficient condition for an overconvergent isocrystal on a smooth curve to have finite-dimensional cohomology. When this condition is satisfied, we also give a proof of Poincaré duality.

The condition we give (in §9.1 below) is an analytic condition on the behavior of the isocrystal inside the tube of a singular point. It seems a difficult one to verify in general, but it is automatic in one special case of interest, namely when the isocrystal satisfies an analogue of Grothendieck's local monodromy theorem; we call such isocrystals "quasi-unipotent" and the last few sections of this paper are devoted to some of their properties. In particular, we are able to show that much of the first two chapters of Weil II is applicable to any quasi-unipotent F-isocrystal; this includes the important theorem

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on the purity of the monodromy weight filtration [17 1.8.4–5], and the equidistribution results for the Frobenius classes.

It seems reasonable to expect that any overconvergent isocrystal "of geometric origin" is quasi-unipotent in the above sense. In fact, it might not be unreasonable to expect that any overconvergent F-isocrystal is quasi-unipotent, though evidence for this is rather fragmentary. In any case, there certainly do exist "strict isocrystals" (*i.e.* isocrystals satisfying the finiteness condition) that have no Frobenius structure and thus do not come from geometry, and one would like to know what their significance is. In this connection, we must consider another construction of Berthelot, that of the category of *arithmetic $\mathcal{D}$-modules* [7]. If X is a smooth curve, $X \hookrightarrow \overline{X}$ a smooth compactification of X, and M is an overconvergent isocrystal on X, then the direct image sp_*M of M under specialization is a $\mathcal{D}^\dagger$-module on $\overline{X}$ ([7] §4). It is not, however, necessarily a coherent $\mathcal{D}^\dagger$-module on $\overline{X}$, and one is led to ask when sp_*M is coherent. Is it sufficient that M be strict? This is not unreasonable, since both strictness of M and coherence of sp_*M guarantee that M has finite-dimensional cohomology; furthermore, in the case of the Kummer isocrystal ([6 5.12] and 6.10 below) these conditions turn out to be the same. On the other hand, strictness of M is a condition only on the behaviour of M on the tube of $\overline{X} - X$, and the same cannot be said a priori for the coherence of sp_*M. We hope that clarifying this issue would shed light on both theories.

The first part of this paper is just a review of non-archimedean functional analysis; in writing it I had in mind primarily the needs of algebraic geometers for whom this may not be their favorite subject. It contains no new results, though it does collect a few facts that can be difficult to dig out of the literature. The second and third parts are devoted to the local and global parts of the theory. The motivating ideas are all from the classical geometry of numbers (in "function field" form): if $X \hookrightarrow \overline{X}$ is a smooth projective embedding, and $\overline{\mathfrak{X}}$ is a formally smooth lifting of $\overline{X}$, then to every point of $\overline{X} - X$ we attach a "local algebra," and show that the global dagger algebra $A^\dagger$ associated to $\overline{\mathfrak{X}}$ embeds, as a closed topological subspace, into the direct sum A^{loc} of the local algebras attached to the points of $\overline{X} - X$. Furthermore A^{loc} is topologically self-dual, while $A^\dagger$ and $A^{loc}/A^\dagger$ have dual topological types.

Since the original version of this paper was written, G. Christol and Z. Mebkhout [13, 28] have also obtained finiteness results for a isocrystals on a curve. Their methods are completely different from the ones used here, but it is not unlikely that their hypotheses are closely related to the one used here (*cf.* 9.1 below).

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Notation

We will always denote by K a complete *discretely valued* field of characteristic 0, with integer ring R and residue field k of characteristic $p > 0$. The value group of $K^\times$ is $|K^\times|$, and $\sqrt{|K^\times|}$ is the group of "roots" of the value group, i.e the set of positive real numbers r such that $r^n \in |K^\times|$ for some integer n.

For any affinoid algebra A, we denote by $|\ \ |_A$ the spectral seminorm; if A is reduced and $\mathrm{Max}(A)$ is the corresponding rigid-analytic space, we will also write $|\ \ |_X$ for $|\ \ |_A$.

Part I

Functional analysis over a discretely valued field

In this section we will collect some basic results of non-archimedean functional analysis that we will need later, or that the reader will probably want to be reminded of. There is practically nothing new in this section, which should be read only as needed. Our basic references are the papers of van Tiel [39], Serre [34], and Gruson [23], the book of Monna [29], and, when all else fails, Bourbaki [EVT, AC].

We will assume throughout that K is discretely valued, although many of the results in this section are valid in the more general setting of a locally convex space over a maximally complete field. I have restricted the discussion to the case of a discretely valued field, since this seems to be the case of geometric interest, and it allows a number of technical simplifications.

1. Basic definitions

1.1. Let V be a K-vector space. A subset $C \subseteq V$ is *convex* if for every x, y, $z \in C$ and all a, b, $c \in R$ such that $a+b+c=1$, we have $ax+by+cz \in C$. A convex set C is *balanced* if $0 \in C$. One checks immediately that C is convex and balanced if and only if C is an R-module; more generally, C is convex if and only if it is a translate of a sub-R-module of V. For any subset $S \subseteq U$ we denote by $\Gamma(S)$ the convex hull of S; *i.e.* the intersection of the convex subsets of V containing S.

If V is a topological K-vector space, then V is *locally convex* if it has a neighborhood basis consisting of convex sets (in [29, 39] these are called "locally K-convex"). By the above remarks, this is the same as saying that the topology of V is *R-linear* in the sense of Bourbaki [AC III §2 Ex. 15]. The category of locally convex K-spaces is an additive category possessing arbitrary direct and inverse limits. Remember, however, that the topology of a direct limit $\varinjlim V_i$ is not necessarily separated, and is not necessarily the same as the direct limit topology in the category of topological spaces (for this reason the topology of $\varinjlim V_i$ is sometimes called the "convex direct limit"). If W is locally convex, then $\varinjlim_i V_i \to W$ is continuous if and only if each $V_i \to W$ is continuous. If V_i, W_i are locally convex and $V_i \to W_i$ is continuous for all i, then $\varinjlim V_i \to \varinjlim W_i$ is continuous.

The definitions of *absorbing* and *bounded* are the same as in the archimedean case: a subset $S \subseteq V$ is absorbing if for all $v \in V$ we have $v \in \lambda S$ for some $\lambda \in K$; $B \subset V$ is bounded if for any open neighborhood U of 0 we have $B \subset \lambda U$ for some $\lambda \in K$. Evidently a finite union of bounded sets is bounded, as is any subset of a bounded set. The closure of a bounded set is bounded, for if B is bounded and $\overline{B}$ is the closure of B, then for any open neighborhood U of 0, we have $B \subset \lambda U$ for some λ, and since $\overline{B} + \lambda U = B + \lambda U$, we have $\overline{B} \subset \lambda U$. If B is bounded, then so is the convex hull $\Gamma(B)$ [39 I Th. 2.7]. If $V = \varprojlim_i V_i$, then $B \subset V$ is bounded if and only if its image in each V_i is bounded.

Norms, seminorms, and metrics are defined in the usual way, except of course that the triangle inequality is replaced by the ultrametric inequality. There is a one-to-one correspondence between absorbing convex sets and seminorms with value group $|K^\times|$, similar to the one which one obtains in the non-archimedean case: if S is absorbing and convex, then we define $p_S := \inf_{x \in \lambda S} |\lambda|$; if p is a seminorm with the same value group as the valuation of K, we define $S_p = \{v \mid p(v) \leq 1\}$; then $S = S_{p_S}$ and $p = p_{S_p}$ [37 I Th. 2.9].

Recall that if $f : V \to W$ is a continuous linear map, then f is *strict* if $\operatorname{Coim} f \simeq \operatorname{Im} f$; *i.e.* if the subspace and quotient topologies on the image of f coincide.

1.2. Compactness notions play an important role in the duality theory of locally convex spaces over a locally compact field. If K is not locally compact we need to use the notion of R-linear compactness (referred to, henceforth, as linear compactness). If M is any R-module, then a filter $\mathcal{F}$ on M is *convex* if it has a base consisting of convex sets; *i.e.* translates of sub-R-modules. We say that M is *linearly compact* if every convex filter in M has an accumulation point (*cf.* [AC III §2 Ex. 15]). Suppose now that C is a convex set in a locally convex K-space V; then C is a translate of a sub-R-module M of V, and we say that C is linearly compact if M is. This is of course the same as saying that any convex filter on C has an accumulation point; in particular, linear compactness is the same as what van Tiel and Monna call c-compactness [29, 39].

1.3. Let V be a locally convex K-space. A linearly compact subset of V is closed, and a closed convex subset of a linearly compact set is linearly compact. A linearly compact subset of V is not necessarily bounded; for example K itself, being discretely valued, is linearly compact (in fact, K is linearly compact if and only if it is maximally complete, *cf.* [29 Ch. III §4]). The image of a convex linearly compact set under a continuous map is linearly compact. A finite union of convex balanced linearly compact sets is linearly compact. If $\{V_i\}$ is an inverse system of locally convex K-spaces and V is the inverse limit, then a convex closed $C \subset V$ is linearly compact if and only if its image in each V_i is.

Recall that in a complete metric space, the compact subsets are the same as the closed, totally bounded sets. A similar description of the convex linearly compact sets is true for Banach spaces over a maximally complete field, as was first shown by Gruson [23]. If we assume that the field is discretely valued, then this is true in greater generality. Recall that a topological vector space is *quasi-complete* if every closed bounded set is relatively linearly compact (a complete space is quasi-complete, but not conversely).

1.4. Proposition – *Let C be a closed locally convex bounded subset of a* complete *convex K-space V. Then C is linearly compact if and only if for any open neighborhood U of* 0, *there is a finite set $F \subseteq C$ such that $C \subseteq \Gamma(F) + U$.*

Proof. – By translation we can assume that C is balanced; *i.e.* that C is a sub-R-module of V. Furthermore in 1.4 we can restrict our attention to the U that are convex open neighborhoods of $0 \in V$, so that the U are sub-R-modules of V as well. Since V is complete and C is closed, C is complete and is the inverse limit of the $C/(C \cap U)$ where U runs through a cofinal set of convex open neighborhoods of 0 (*cf.* [TG III 7.3 Cor.

2]). Furthermore C is linearly compact if and only if each of the $C/(C \cap U)$ are linearly compact R-modules in the discrete topology [AC III §2 Ex. 16(a)]. Since C is bounded, the R-module $C/(C \cap U)$ is annihilated by a power of p. Since K is discretely valued, the condition of 1.4 is equivalent to saying that $C/(C \cap U)$ is artinian. Thus, it is enough to show that a discrete R-module M annihilated by a power of p is linearly compact if and only if it is artinian. But M has a filtration whose quotients are annihilated by a uniformizer of K, so by *loc. cit.* Ex. 15(c) it is enough to show that a discrete k-vector space is linearly compact if and only if it is finite-dimensional. Now by *loc. cit.* Ex. 20(d), a vector space is linearly compact if and only if it is a product k^I, and this is discrete if and only if the index set I is finite.

□

1.5. Let $\mathcal{B}$ be a bornology on V, *i.e.* a nonempty collection of subsets of V with the property that a finite union of elements of $\mathcal{B}$ is in $\mathcal{B}$, and a subset of an element of $\mathcal{B}$ is in $\mathcal{B}$. For any locally convex space W, we denote by $\mathrm{Hom}_{\mathcal{B}}(V, W)$ the space of continuous linear maps $V \to W$, with the topology defined by taking as a basis of the neighborhoods of 0 the sets $\{f : V \to W \mid F(B) \subseteq U\}$ for $B \in \mathcal{B}$ and U open in W. When $W = K$, we put $\mathrm{Hom}_{\mathcal{B}}(V, K) = V'_{\mathcal{B}}$, and denote by V' the underlying vector space. The most important cases of the latter construction are when $\mathcal{B}$ is the set of bounded sets, yielding the *strong dual* V'_s of V; the set of finite sets, yielding the *weak dual* V'_w of V (the more usual term for this is the weak* topology of V'), and the set of convex linearly compact sets, yielding the *(convex linearly)-compact dual* V'_c. We will also use on occasion the so-called weak topology of V itself, which is the weakest topology on V such that all of the linear functionals in V' are continuous (*i.e.* the topology $\sigma(V, V')$ in the notation of [EVT]).

1.6. For any locally convex V, the natural map $V \to (V'_w)'$ is an isomorphism of K-vector spaces [39 III Th. 4.10]. For any $S \subseteq V$, we denote by S° the *polar* of S, defined by

$$S^\circ = \{f \in V' \mid |f(S)| \leq 1\}.$$

When S is a linear subspace of V, then S° is of course just the annihilator $S^\perp$ of S in V'.

The *bipolar* $S^{\circ\circ} \subseteq V$ of S is the polar of $S^\circ \subseteq V'_w$, where we identify $(V'_w)' \simeq V$; it also has the description

$$S^{\circ\circ} = \{v \in V \mid |S^\circ(v)| \leq 1\}.$$

If S is K-convex and closed we have then $S^{\circ\circ} = S$ since we have assumed that the valuation of K is discrete; more generally, $S^{\circ\circ}$ is the closure of $\Gamma(S)$ [39 III 4.14]. When K is not discretely valued, the situation is a little more complicated.

Recall that a set $B \subset \mathrm{Hom}(V, W)$ is equicontinuous if for every open subset $U \subset W$ there is an open set $U_1 \subset V$ such that $f(U_1) \subset U$ for all $f \in B$. As in the archimedean case, a subset of V' is equicontinuous if and only if it is contained in the polar of a neighborhood of $0 \in V$, and if and only if its polar contains a neighborhood of $0 \in V$. From this one sees easily that the topology of V is the same as that of uniform convergence in the equicontinuous subsets of V' (via the identification $V \simeq (V'_w)'$).

1.7. A locally convex space V is *barreled* if a subset T of V that is closed, convex, balanced, and absorbing is a neighborhood of 0 (such sets are called *barrels*). The Banach-Steinhaus theorem is true for barreled spaces: if V is barreled and W is locally convex, then a weakly bounded subset of $\mathrm{Hom}(V, W)$ is equicontinuous (in fact, this property characterizes the barreled spaces). For any barreled space V and its strong dual V'_s, a subset B of V (resp. V'_s) is bounded if and only if it is contained in the polar of an open neighborhood U of 0 in V'_w (resp. V), and a subset U of V is a neighborhood of 0 in V (resp. V'_s) if and only if it contains the polar of a bounded set of V'_s (resp. V). Finally, any quotient, inductive limit, and direct sum of barreled spaces is barreled. Closed subspaces of barreled spaces are not necessarily barreled.

1.8. A locally convex space is *bornological* if every subset $S \subseteq V$ which absorbs the bounded sets of V is a neighborhood of 0 (*i.e.* if for any bounded set B, there is a λ such that $B \subset \lambda S$; such subsets are sometimes called *bornivorous*). Bornological spaces are precisely the spaces V for which continuous maps $V \to W$ to a locally convex space are the same as locally bounded maps (*i.e.* $f : V \to W$ is continuous if and only if for every bounded set $B \subset V$, $f(B)$ is bounded, *cf.* [39 III 4.30]). Any inductive limit of bornological spaces is bornological.

1.9. A locally convex K-vector space is *semi-reflexive* (resp. *reflexive*) if the natural map $V \to (V'_s)'_s$ is an isomorphism of K-vector spaces (resp. of topological K-vector spaces). Since $(V \oplus W)'_s \simeq V'_s \oplus W'_s$, a locally convex space $V = M \oplus N$ is semi-reflexive (resp. reflexive) if and only if both M and N are semi-reflexive (resp. reflexive).

1.10. The most important category of spaces for us will be the category of *Montel* spaces. A locally convex space is a Montel space if it is barreled, and if every closed convex bounded subset is linearly compact (in [29] and [39] these are called c-Montel spaces). A Montel space V is reflexive. If $V = M \oplus N$, then V is Montel if and only if both M and N are Montel.

2. Fréchet and Banach spaces

2.1. For any set I, we define $c(I)$ to be the Banach space of all sequences $\{a_i\}_{i\in I}$ where $a_i \in K$ and $a_i \to 0$ for the Frechet filter, and with norm given by $|\{a_i\}| = \max_i |a_i|$. If V is a Banach space over K whose norm is such that $|V| = |K|$, then V is isometric to a space $c(I)$ for some I. More generally, one can always (since K is discretely valued) find a norm on V equivalent to the original one with the above property, so that any Banach space over K is isomorphic (but not necessarily isometric) to some $c(I)$. For such spaces, the cardinality of I is an isometry invariant of K; in fact if V_0 is the R-module of $v \in V$ such that $|v| \leq 1$, then $V_0 \otimes_R k \simeq k^I$ as algebraic vector spaces. In fact $|I|$ is an isomorphism invariant, since norms defining the same topology are comparable.

If V is a Banach space over K, then an *orthonormal basis* of V is by definition a set of vectors in V which correspond, under some isomorphism $V \simeq c(I)$, to the set of "standard" basis vectors $\{\delta_{ij}\}_{j\in I}$ of $c(I)$.

2.2. If $V = c(I)$, then one checks immediately that the strong dual V'_s of V can be identified with that space of bounded sequences $\{a_i\}_{i\in I}$; it is a Banach space under the norm $|\{a_i\}| = \max_i |a_i|$. We have $V'_s \simeq c(J)$ for some index set J, and if I is infinite then $|J| = |k|^{|I|}$. From this it follows that if V is a *reflexive* Banach space over a discretely valued field K, then V is finite-dimensional. In particular there are no infinite-dimensional "Hilbert spaces" over K. This result holds under the weaker hypothesis that K is maximally complete.

2.3. For any Banach space V over K, and any closed subspace $W \subset V$, the natural projection map $V \to V/W$ has a splitting; i.e a closed subspace of a Banach space is a direct summand. The idea is that an orthonormal basis of V/W can be lifted to a basis of a complement of W in V, *cf.* [34 Prop. 2].

2.4. Kolmogoroff's characterization of normed spaces is valid in the non-archimedean case: a locally K-convex space is normable if and only if the origin has a bounded neighborhood [29 III §3]. If $U \ni 0$ is bounded, open and convex, then the the convex functional p_U associated to U (see 1.2.1) is a norm.

2.5. If V and W are Banach spaces, then a linear continuous map $f : V \to W$ is *completely continuous* if it is contained in the closure in $\mathrm{Hom}_s(V, W)$ of the subspace of linear maps of finite rank. A continuous map $f : V \to W$ is completely continous if and only if for any bounded set $B \subset V$, $f(B)$ is relatively linearly compact. In fact, if f is completely continuous, let B be a bounded set in V and choose, for any convex open neighborhood U of $0 \in W$, a linear map f_U of finite rank such that $f(b) - f_U(b) \in U$ for any $b \in B$. Since $f_U(B)$ is a bounded subset of a finite-dimensional space, it is a finite R-module, and thus there is a finite set $F \subseteq f(B)$ such that $f(B) \subseteq \Gamma(F) + U$. By 1.4, the closure of $f(B)$ is linearly compact. Conversely, if the closure of $f(B)$ is linearly compact, then for any open neighborhood U of $0 \in W$, we choose a finite $F \subset f(B)$ such that $f(B) \subseteq \Gamma(F) + U$. Since the subspace spanned by F is closed ([EVT I §2 Cor. 2]), there is by 2.3 a projection map $p : W \to \langle F \rangle$, and if $f_U = p \circ f$, then $f(b) - f_U(b) \in U$ for all $b \in B$; thus f is in the closure in $\mathrm{Hom}_s(V, W)$ of the space of maps of finite rank.

For local fields this was first shown by Serre [34 Prop. 5], with "linearly compact" replaced by compact. The case when K is maximally complete was treated by Gruson [23]. Note that the archimedean case of this is true for Hilbert spaces, but not, in general, for Banach spaces, even reflexive ones [19].

2.6. There is a characterization of Frechet spaces similar to the result of Kolmolgoroff mentioned above: a locally convex space is Frechet if and only if it is complete, and the filter of neighborhoods of 0 has a countable basis.

2.7. A Frechet space is barreled. The proof is basically the same as in the archimedean case: let $T \subseteq V$ be a barrel; then there is a countable sequence $a_n \in K$ such that $\cup_n a_n T = V$. By Baire's theorem, T must then be a neighborhood of some $v \in V$. Since $-v \in T$ as well, $-v + T = T$ is a neighborhood of the origin. One can also show that a Frechet space is bornological, *cf.* [EVT III §2 Prop. 2].

We will (rather abusively!) say that a locally convex K-space is *dual-of-Frechet* if it is isomorphic to the strong dual of a *reflexive* Frechet space (it is thus a DF-space in the usual sense of the term, though not every DF-space is of this type). If $V = M \oplus N$, then V is dual-of-Frechet if and only if both M and N are dual-of-Frechet; sufficiency is clear, and to prove necessity, we note that if $V = W'_s$ for some reflexive Frechet space W, then $W \simeq M'_s \oplus N'_s$. Then M'_s and N'_s are Frechet, and are reflexive by 1.9; it follows that M and N are dual-of-Frechet.

2.8. Proposition – *If a locally convex space V is both Frechet and dual-of-Frechet, then it is finite-dimensional.*

Proof. – The hypotheses imply that V is Banach. This is (or should be) a well known fact in archimedean functional analysis; since it will be important in what follows, it is probably worthwhile to recall the proof. By the criterion of 2.6, there is a countable convex basis $\{U_i\}_i$ of the neighborhoods of 0 in V. Since V is the strong dual of a Frechet space, there is a countable family $\{B_i\}_i$ of convex bounded sets of V such that any bounded subset of V is contained in some B_i (*cf.* 1.5). For each n, choose a $\lambda_n \in K$ such that $\lambda_n B_n \subseteq U_n$. It is easily checked that $\sum_n \lambda_n B_n$ is bounded, and is of course convex since the B_n are. Thus the closure U of $\sum_n \lambda_n B_n$ is bounded, closed, convex, and it is easily seen to be absorbing; *i.e.* U is a barrel. Since V is barreled, U is a neighborhood of the origin, and since U is bounded, V is normable by Kolmogoroff's theorem. Since V is complete, it is Banach.

Since V is the dual of a reflexive Frechet space, it is itself reflexive, and being a Banach space, it is finite-dimensional by 2.2.

□

The following refinement of 1.4 is a non-archimedean version of [EVT IV §3 no. 5 Cor. 1]:

2.9. Proposition – *Let V be a Frechet space and $C \subset V$ a balanced convex closed bounded subset. Then the following are equivalent:*

(i) C *is linearly compact;*

(ii) $C \subseteq \Gamma(F)$ *for some compact subset* $F \subset C$;

(iii) $C \subseteq \Gamma(F)$, *where F is the closure of a sequence of points of C converging to* 0.

Proof. – Suppose $C \subset \Gamma(F)$ for some compact subset of $F \subset C$, and let U be any open neighorhood of 0. Since F is totally bounded, we have $F \subset F_U + U$ for some *finite* subset $F_U \subset F$ and thus $C \subset \Gamma(F) + U \subset \Gamma(F_U) + U$; the linear compactness of C follows from 1.4, since U was arbitrary. Thus (ii) implies (i), and (iii) implies (ii), since a sequence of points tending to 0 is relatively compact. To show that (i) implies (iii), we suppose that C is balanced, convex (and thus a sub-R-module of V), bounded and linearly compact. Then from 1.4 (or from its proof) we see that for any neighborhood of 0, the R-module $C/(C \cap U)$ is artinian. Since V is Frechet, it has a countable fundamental system $\{U_i\}_{i \geq 0}$ of neighborhoods of 0 in V. We now choose a sequence of finite sets $F_i \subset C$ as follows: let F_0 be any finite subset of C mapping to a set of generators of $C/(C \cap U_0)$, and having chosen F_{n-1}, we let F_n be the union of F_{n-1} and a finite subset of $C \cap U_{n-1}$ mapping to a set of generators of the artinian module $(C \cap U_{n-1})/(C \cap U_n)$. Then F_n generates $C/(C \cap U_n)$ for all $n \geq 0$ and thus the union of the F_n generates C. Let F be the closure of this union; then $F \subset C \subseteq \Gamma(F)$, and F is the closure of a sequence of points tending to 0.

□

Remark. – In the notation of the preceding proof, we have $\overline{\Gamma(F)} + U_n = \Gamma(F) + U_n = \Gamma(F_n) + U_n$ for all n. It then follows from 1.4 that $\overline{\Gamma(F)}$ is linearly compact, *i.e.*, that $\Gamma(F)$ is relatively linearly compact.

Remark. – By translation, we can omit the hypothesis that C is balanced, at the cost of replacing the phrase "converging to 0" in (iii) by "convergent."

3. LF-spaces, strict maps, and duality

3.1. A separated locally convex K-space is an *LF-space* if there is a countable inductive system $\{V_i\}_i$ of Frechet spaces such that $V = \varinjlim_i V_i$.

What is here defined as an LF-space is called a "generalized LF-space" by most writers, who take an LF-space to be one for which the inductive limit is strict. We will not find this last concept so useful. Since a separated quotient of a Frechet space is Frechet, we can assume that the V_i are (not necessarily closed) subspaces of V.

3.2. Since a separated quotient of a Frechet space is Frechet, a separated quotient of an LF-space is an LF-space. A closed subspace of an LF-space is not necessarily an LF-space; if $V = \varinjlim V_i$ and $W \subseteq V$ is closed, then $W = \varinjlim W \cap V_i$ as vector spaces, which gives W an LF-space topology which may, however, be distinct from its original (subspace) topology.

3.3. If $V = M \oplus N$, then V is an LF-space if and only if M and N are LF-spaces. The "only if" part follows from 3.2, since M and N are separated quotients of $M \oplus N$. On the other hand, if $M = \varinjlim M_i$, $N = \varinjlim N_i$ with M_i, N_i Frechet, then we have $M \oplus N \simeq \varinjlim M_i \oplus \varinjlim N_i \simeq \varinjlim (M_i \oplus N_i)$, since filtered inductive limits commute.

Since a Frechet space is bornological and barreled, and since these properties are passed on to inductive limits, we see that an LF-space is bornological and barreled. In particular, the Banach-Steinhaus theorem holds for LF-spaces, and a map from an LF-space to a locally convex K-space is continuous if and only if it is bounded.

3.4. The open mapping theorem is valid for LF-spaces: a continuous surjective map $f : V \to W$ of LF-spaces is open [EVT II.36 Prop.10], and therefore strict. A continuous map $f : V \to W$ of LF-spaces with closed image is not necessarily strict, however, since the induced topology on a closed subspace of an LF-space is not necessarily an LF-space topology. This is the case, however, if the image is a topological direct summand:

3.5. Proposition – *If $f : V \to W$ is a continuous map of LF-spaces such that* $\operatorname{Im} f$ *is a direct summand of W, then f is strict.*

Proof. – Since finite direct sums in the category of locally convex K-spaces are the same as products, $\operatorname{Im} f$ is a quotient of W, and therefore an LF-space. The result then follows from the open mapping theorem.

□

3.6. Corollary – *If $f : V \to W$ is a continuous map of LF-spaces such that* $\operatorname{Coker} f$ *is separated and finite-dimensional, then f is strict.*

Proof. – The hypothesis says that $\operatorname{Im} f$ is closed and of finite codimension, so by [EVT I.15 Prop. 3] $\operatorname{Im} f$ is a summand of W, and the corollary follows from 3.4.

□

3.7. Strictness of a map can be difficult to show, and we will need a number of criteria for it. Here is a simple one that is suprisingly useful: if $f_i : V_i \to W_i$ is a *finite* collection of continuous linear maps between locally convex K-spaces, then the direct sum $f = \oplus_i f_i$ is strict if and only if each of the f_i is strict. Since $\operatorname{Im} f = \oplus \operatorname{Im} f_i$, the question reduces immediately to the cases where f is either injective or surjective. When f is injective, this follows from [EVT] II §5 n^o 5 Prop. 8, since a finite sum is the same as a finite product. When f is surjective, this follows from the fact that a projection map of a product onto one of its factors is open.

3.8. Proposition – *Let V, W be separated, locally convex K-spaces, and $f : V \to W$ a continuous linear map. Then f is strict if and only if*

(i) *the image of ${}^t f$ is closed in V'_w;*

(ii) *any equicontinuous subset of $\operatorname{Im}{}^t f$ is the image of an equicontinuous subset of W'_s.*

If f is strict, then

$$\operatorname{Ker}{}^t f = (\operatorname{Im} f)^\perp \qquad \operatorname{Im}{}^t f = (\operatorname{Ker} f)^\perp \tag{3.8.1}$$

and there are canonical isomorphisms

$$\operatorname{Coker}{}^t f \xrightarrow{\sim} (\operatorname{Ker} f)' \qquad \operatorname{Ker}{}^t f \xrightarrow{\sim} (\operatorname{Coker} f)' \tag{3.8.2}$$

of K-vector spaces.

See [EVT IV §4 n^o 1 Prop. 2]; note that the isomorphisms in 3.8.2 are not necessarily topological isomorphisms for the strong dual topology. We will apply 3.8 in the case when V and W are LF-spaces, in which case "equicontinuous" in the above proposition can be replaced by "bounded."

We will say that an exact sequence of locally convex K-spaces is *strict exact* if the maps in it are strict (this is sometimes called "topologically exact").

3.9. Proposition – *The transpose of a strict exact sequence of separated locally convex spaces is exact.*

Proof. – If $U \to V \to W$ is strict exact, then it is strict exact for the weak topologies on U, V, and W [EVT II §6 no. 5 Cor. 3] and then the exactness of $W' \to V' \to U'$ follows from [EVT II §6 no. 5 Rem. (1)].

□

Note that in 3.9, the case $V \to W \to 0$ is obvious, and the case $0 \to W \to V$ is just the Hahn-Banach theorem. To be sure, the Hahn-Banach theorem is itself an important ingredient in the proofs of 3.8 and 3.9; For a proof of Hahn-Banach in the case of a maximally complete field, see [29 Ch. 5 §1].

The transpose of a strict map is not necessarily strict for the strong topology on the dual spaces. For the linearly-compact dual topology, we have the following result (the argument is basically that of [EVT IV §4 no. 2 lemme 1]):

3.10. Proposition – *Let $f : V \to W$ be a continuous linear map of semi-complete locally convex K-vector spaces. Then ${}^t f : W'_c \to V'_c$ is strict if and only if $f(V)$ is closed in W, and for every convex bounded linearly compact subset $C \subset W$, there is a convex bounded linearly compact subset $D \subset V$ such that $f(D) = C$.*

Proof. – It follows from Mackey's theorem [39 III 4.18.a] that we can identify $V = (V'_c)'$ and $W = (W'_c)'$; then f can be identified with the transpose of ${}^t f$, and the convex equicontinuous subsets of V and W are exactly the convex relatively linearly compact subsets of V resp. W. Condition (ii) of 3.8 follows directly from this. As to condition (i), we need only note that a subspace of W is closed if and only if it is closed in the weak topology of W (*cf.* [39 III 4.20.b]).

□

3.11. Proposition – *If V and W are Frechet spaces, then a continuous linear map $f : V \to W$ is strict if and only if ${}^t f : W'_c \to V'_c$ is strict.*

Proof. – If ${}^t f$ is strict, then $f(V)$ is closed in W by 3.10, and so f is strict. To show the converse, it is enough to show that for any balanced convex linearly compact $C \subset W$, there is a convex linearly compact $D \subset V$ such that $f(D) = C$. By 2.9 we can write $C \subset \Gamma(F)$, Where F is the closure of a sequence of points converging to 0. Since f is strict and V, W are metrizable, we can find a set $G \subset V$ that is the closure of a sequence converging to 0, such that $f(G) = F$. We then have $f(\Gamma(G)) = \Gamma(F)$, and by the remark after Proposition 2.9, $\overline{\Gamma(G)}$ is linearly compact. Then $D = \overline{\Gamma(G)} \cap f^{-1}(C)$ is linearly compact as well, and $f(D) = C$.

□

Of course, if V, W are Montel spaces (the case of interest to us) then the linearly compact dual is the same as the strong dual:

3.12. Corollary – *If V and W are Frechet-Montel spaces, then $f : V \to W$ is strict if and only if ${}^t f : W'_s \to V'_s$ is strict.*

Finally we will need the following result, whose proof is left the reader:

3.13. Proposition – *Suppose that the rows of the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

are strict exact. Then in all of the maps in the six-term exact sequence

$$0 \to \operatorname{Ker} f' \to \operatorname{Ker} f \to \operatorname{Ker} f'' \to \operatorname{Coker} f' \to \operatorname{Coker} f \to \operatorname{Coker} f'' \to 0$$

are continuous, and the maps

$$\begin{array}{l} \operatorname{Ker} f' \to \operatorname{Ker} f \\ \operatorname{Coker} f \to \operatorname{Coker} f'' \end{array}$$

are strict.

Part II

Local Duality

The "local algebras" introduced in §4 play a role here similar to that of local fields in the classical geometry of numbers. The main results are the various duality theorems, either "quasi-coherent" (Theorem 5.4) or "de Rham" (Theorem 6.3).

4. Local algebras

4.1. Let I be an interval (closed, open, or half-open) in the set $[0,\infty]$ of nonnegative extended real numbers. We denote by A_I the K-algebra of formal Laurent series in the variable x convergent when $|x| \in I$ (when $\infty \in I$, this just means that the Laurent series is a power series in x^{-1}). When $I \subset J$, then there is a natural ring homomorphism $A_J \hookrightarrow A_I$, and we have

$$A_I = \bigcap_{J \subseteq I} A_J \tag{4.1.1}$$

where J runs through the set of closed intervals contained in I. We define, finally,

$$A = \varinjlim_{r<1} A_{(r,1)}$$

so that A is the algebra of Laurent series convergent in some annulus $r < |x| < 1$. We will call a topological K-algebra isomorphic to A a *local algebra*; in later sections such algebras will be attached to points on a smooth curve over k.

Note that A is also the direct limit of the $A_{[r,1)}$ for $r < 1$; one could further restrict the r to belong to a dense subset of $\mathbb{R}$ (for example $\sqrt{|K^\times|}$. For later use we record the observation that a Laurent series $\sum_{n\in\mathbb{Z}} a_n x^n$ defines an element of A if and only if its coefficients satisfy the condition

$$|a_{-n}| < Cr^n \text{ for some } C > 0,\ r < 1 \text{ and all } n \geq 0; \tag{4.1.2}$$

$$\text{for every } s < 1, \text{ there is a } C_s > 0 \text{ such that } |a_{-n}| < C_s s^n \text{ for all } n \leq 0 \tag{4.1.3}$$

and these conditions imply that

$$f \in A_{[r,1)} \tag{4.1.4}$$

$$\begin{cases} \text{for every } s < 1, \text{ there is a } C_s > 0 \text{ such that } |f|_{[r,s]} \leq \max(C, C_s) \text{ whenever} \\ r < s < 1. \end{cases} \tag{4.1.5}$$

Obviously 4.1.3 is equivalent to the condition that for all positive $s < 1$, the set of $|a_n|s^n$ for $n \geq 0$ is bounded.

4.2. The rings A_I just introduced all have obvious topologies. If I is a closed interval, then A_I is a Banach space; if furthermore the endpoints of I belong to $\sqrt{|K^\times|}$, then A_I is a reduced affinoid algebra, and the Banach norm is the supremum norm on the corresponding affinoid space. *We will assume from now on that the endpoints of all intervals belong to* $\sqrt{|K^\times|}$. If I is open or half-open, then we give A_I the inverse limit topology arising from 4.1.1. Note that only a countable set of intervals J appear in 4.1.1, since the endpoints belong to $\sqrt{|K^\times|}$; thus the topology of A_I is Frechet. A basis of the open neighborhoods of 0 is given by the sets

$$U_{\epsilon,J} = \{f \in A_I \mid |f(x)| \leq \epsilon \quad \text{when } |x| \in J\}$$

where $\epsilon > 0$ and J is a closed interval contained in I. A subset $B \subset A_I$ is bounded if and only if it is bounded in each of the A_J for $J \subset I$; *i.e.* if and only if it is uniformly bounded on each closed annulus contained in I. For $I = [r, 1)$, this is just the condition that the estimates 4.1.2 and 4.1.3 hold for all elements of B with a fixed choice of C, C_s.

Finally, we give $A = \varinjlim A_{(r,1)}$ the inductive limit topology. When we define the local pairing (§5.1) we will see that A is separated; the reader can check that there is no vicious circle. Since the $A_{(r,1)}$ are Frechet, A is an LF-space, since again only a countable set of r is involved. In particular, A is bornological and barreled, but not metrizable, as we shall see later. Note, finally, that A is the (topological) inductive limit of the $A_{[r,1)}$ for $r < 1$.

It is clear that the rings A, A_I with the topologies just defined are topological rings; *i.e.* that multiplication is continuous.

For any finite free A-module M, the open mapping theorem shows that the topology on M arising from an identification $M \simeq A^n$ is independent of the chosen isomorphism. One checks immediately that any A-linear map $M \to N$ of finite free A-modules is continuous; in particular the topology on a finite free A-module M is the quotient topology for any surjective A-linear map $A^n \twoheadrightarrow M$. One could try to use the same procedure to topologize any A-module of finite type, but we will see later that such topologies are not necessarily separated; consequently it is *not*, in general, true that the image of a continuous map $f : A^n \to A^m$ is closed. Of course if the image of such an f is a direct summand of A^m, then it is closed, being the kernel of some continuous map $A^m \to A^{m'}$.

For any interval I, let I^0 denote the interior of I.

4.3. Lemma – *If I, J are closed intervals such that $J \subset I^0 \subset I$, then the inclusion $A_I \hookrightarrow A_J$ is completely continuous.*

Proof. – The assertion is that the inclusion $i : A_I \hookrightarrow A_J$ is a limit of maps of finite rank. This is clear: if $J = [r, s]$, $I = [t, u]$, then $t < r$, $s < u$, and we define $i_N : A_I \to A_J$ by

$$i_N : \sum_{n \in \mathbb{Z}} a_n x^n \longmapsto \sum_{|n| < N} a_n x^n.$$

We have $|i - i_N| \leq \max((t/r)^N, (s/u)^N)$, whence $i = \lim_N i_N$.

□

4.4. Corollary – *For any* open *interval I, a bounded set in A_I is relatively linearly compact.*

Proof. – Suppose that $B \subset A_I$ is bounded; we must show that for any closed $J \subset I$, the image of B in A_J is relatively linearly compact. Now we can choose a closed interval J' such that $J \subset (J')^0 \subset J' \subset I$, and as the image of B in $A_{J'}$ is bounded, its image in A_J is relatively linearly compact by 4.3 and 2.5.

□

Since A_I is Frechet (and in particular barreled), we see that A_I is a Montel space whenever I is open.

4.5. We will need some algebraic results on modules over A_I and A, which are apparently well known, but for which I do not know of a convenient reference. For the case of $A_{[0,r)}$, most of these can be found in [26]; for later use we will treat more generally the case of an algebra of rigid-analytic functions on an connected admissible open subset $X \subset \mathbb{P}^1_K$. Let $\mathcal{A} = \Gamma(X, \mathcal{O}_X)$, and for any affinoid $I \subseteq X$ set $\mathcal{A}_I = \Gamma(I, \mathcal{O}_I)$; of course we have in mind primarily the cases $\mathcal{A} = A_{(r,1)}$, $\mathcal{A} = A_{[s,t]}$ for $[s,t] \subset (r,1)$.

Let us now choose a countable set $\mathcal{S}$ of open affinoids $I \subseteq X$ such that $X = \cup_{\mathcal{S}} I$; then $\mathcal{A} = \cap_I \mathcal{A}_I$. By [21 I 1.7, 1.2] we may suppose that the I are ordered by inclusion. Let $\mathrm{Div}(\mathcal{A}_I)$ (resp. $\mathrm{Div}^+(\mathcal{A}_I)$) denote the group of divisors (resp. positive divisors) of A_I, and define $\mathrm{Div}(\mathcal{A}) = \varprojlim_I \mathrm{Div}(\mathcal{A}_I)$, $\mathrm{Div}^+(\mathcal{A}) = \varprojlim_I \mathrm{Div}^+(A_I)$. For $f \in \mathcal{A}_I$ or $\mathcal{A}$ we denote by $[f]$ the corresponding divisor. By [21 I 8.7 Cor] we know that the class group of $\mathcal{A}$ is trivial, *i.e.* for any $D \in \mathrm{Div}^+(\mathcal{A})$ (resp. $\mathrm{Div}^+(\mathcal{A}_I)$) there is an f in $\mathcal{A}$ (resp. $\mathcal{A}_I$) such that $[f] = D$. Since $\mathcal{A}_I$ is noetherian, it then follows that $\mathcal{A}_I$ is a PID.

Recall that an integral domain is a *Prüfer ring* (resp. a *Bezout ring*) if any finitely generated ideal in it is projective (resp. principal). The first thing to observe is that $\mathcal{A}$ is a Bezout ring:

4.6. Proposition – *Any finitely generated ideal of $\mathcal{A}$ is principal.*

Proof. – It is enough to see that any ideal (f, g) is principal. Since any element of $\mathrm{Div}^+(\mathcal{A})$ is the divisor of some element of $\mathcal{A}$, there is an $h \in \mathcal{A}$ such that $[h]$ is the greatest common divisor of $[f]$, $[g]$. By division, it is enough to show that if the divisors of f and g are relatively prime, then $(f, g) = \mathcal{A}$. But if $[f]$, $[g]$ are relatively prime, then for any $I \subseteq X$, the sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}_I & \to & \mathcal{A}_I \oplus \mathcal{A}_I & \to & \mathcal{A}_I & \to 0 \\ & & h & \mapsto & (-gh, fh) & & & \\ & & & & (x,y) & \mapsto & xf + yg & \end{array}$$

is exact, since $\mathcal{A}_I$ is a PID. Since for each I, the image of $\mathcal{A}$ in each $\mathcal{A}_I$ is dense, the Mittag-Leffler criterion of [EGA 0_{III} 13.2.4] shows that the inverse limit of the above set of exact seqences is exact. Since $\mathcal{A} = \varprojlim_I \mathcal{A}_I$, we have $(f, g) = \mathcal{A}$.

□

4.7. Corollary – *Any torsion-free $\mathcal{A}$-module is flat.*

Proof. – From 4.6 we see that $\mathcal{A}$ is a Prüfer ring, and any torsion-free module over such a ring is flat [CE VII Prop. 4.2].

□

4.8. Proposition – *Let M be an $\mathcal{A}$-module of finite presentation. Then*

(i) *M is coherent;*

(ii) *M is the direct sum of a finite free $\mathcal{A}$-module and a torsion $\mathcal{A}$-module of finite presentation;*

(iii) *M has a presentation*

$$0 \to \mathcal{A}^n \to \mathcal{A}^m \to M \to 0; \qquad (4.8.1)$$

(iv) $M = \varprojlim_I M \otimes \mathcal{A}_I$.

Proof. – Since $\mathcal{A}$ is a domain, 4.6 shows that any ideal of finite type in $\mathcal{A}$ is of finite presentation. It then follows that $\mathcal{A}$ is coherent, and thus that any $\mathcal{A}$-module of finite presentation is coherent. If M is finitely generated and torsion-free, then it is free (in fact for any Prüfer ring, any finitely generated torsion-free module is a sum of finitely generated locally free ideals; here the ideals are actually free). For any M of finite presentation, let M_{tor} denote the torsion submodule; then since $N = M/M_{tor}$ is finitely generated and torsion-free, it is free. Thus we can write $M = N \oplus M_{tor}$, from which it follows that M_{tor} is coherent, and is thus of finite presentation. This proves (ii); to show (iii), we note that for any Prüfer ring, a finitely generated submodule of a finitely generated free module is a finite sum of finitely generated ideals [CE I Prop 6.1]. Thus for a Bezout ring, a finitely generated submodule of a finite free module is finite free. Now since M is coherent, there is some surjective map $A^m \to M$ whose kernel is coherent; the kernel is then finitely generated, and therefore free, which proves (iii). Finally, since $\mathcal{A}_I$ is a torsion-free $\mathcal{A}$-module, it is flat, and so tensoring 4.8.1 with each of the $\mathcal{A}_I$ yields an inverse system of exact sequences

$$0 \to \mathcal{A}_I^n \to \mathcal{A}_I^m \to M \otimes \mathcal{A}_I \to 0$$

and the same Mittag-Leffler argument as in 4.6 shows that $M \simeq \varprojlim_I M \otimes \mathcal{A}_I$.

□

Since the ring A is the direct limit of the $A_{(r,1)}$, an A-module M of finite presentation is the extension of scalars $M = M_r \otimes A$ for some $A_{(r,1)}$-module M_r, $r < 1$. Furthermore if we set $M_s = M_r \otimes A_{(s,1)}$ for $r < s < 1$ then $M = \varinjlim M_s$. If $f : M \to N$ is a morphism of A-modules of finite presentation, then there is an $r < 1$, $A_{(r,1)}$-modules M_r, N_r of finite presentation, and a morphism $f_r : M_r \to N_r$ such that $M = M_r \otimes A$, $N = N_r \otimes A$, and $f = f_r \otimes A$.

4.9. Proposition – *A is a Bezout ring. In particular, a torsion-free A-module is flat, and an A-module of finite presentation is the direct sum of a finite free module and a torsion A-module of finite presentation.*

Proof. – Since any finitely generated ideal of A is induced from a finitely generated ideal of some $A_{(r,1)}$, 4.6 implies that a finitely generated ideal of A is principal. The assertions of 4.9 follow from this, using the same argument as in the proof of 4.8.

□

5. Local duality I

5.1. We define Ω^1_A to be the free A-module of rank one with basis dx/x, endowed with the obvious topology. The main object of study in this section is the pairing

$$\begin{aligned} A \times \Omega^1_A &\to K \\ (f, \omega) &\mapsto \langle f, \omega \rangle = \mathrm{Res} f\omega \end{aligned} \tag{5.1.1}$$

where Res denotes the usual residue at $x = 0$; *i.e.* the coefficient of dx/x. We will often use the basis element dx/x to identify A and Ω^1_A, in which case 5.1.1 becomes a pairing $A \times A \to K$, and $\langle f, g \rangle$ is the constant term of the product fg. This allows us to make a number of arguments by symmetry. We let $d : A \to \Omega^1_A$ be the usual exterior derivative, so that we have $\mathrm{Res}\, df = 0$, and consequently $\langle f, dg \rangle = -\langle g, df \rangle$.

For any finite free A-module M, the pairing 5.1.1 extends in the obvious way:

$$(5.1.2)\qquad \begin{array}{l} M\times (M^\vee\otimes\Omega^1_A)\to K \\ (m,m^\vee\otimes dx/x)\mapsto \mathrm{Res}\ m^\vee(m)\otimes dx/x \end{array}$$

Our first task is to show that 5.1.1 and 5.1.2 induce perfect topological dualities of the spaces involved. We begin with some simple observations. The first is that if $I=[r,s]$ and $f=\sum_{n\in\mathbb{Z}} a_n x^n\in A_I$, then

$$\begin{aligned} |f|_I &= \max |f|_{[r,r]}, |f|_{[s,s]}\quad \text{by the maximum principle}\\ &= \max_n\{|a_n|r^n,\ |a_n|s^n\}\\ &= \max\{|a_n|r^n,\ n<0;\ |a_0|;\ |a_n|s^n,\ n>0\}\end{aligned}$$

and therefore

$$|a_0|\le |f|_I.$$

From this it follows that for any $r<1$ and any closed $I\subset (r,1)$ we have

$$(5.1.3)\qquad |\langle f,g\otimes dx/x\rangle|\le |fg|_I\le |f|_I|g|_I$$

if f, g are defined on $(r,1)$. If $f\in A_{(r,1)}$, the linear functional $\omega\mapsto\langle f,\omega\rangle$ is continuous on $A_{(s,1)}\otimes dx/x$ for every s such that $s<1$ (just pick an $I\subset (r,1)\cap(s,1)$). Thus $\omega\mapsto\langle f,\omega\rangle$ is continuous on Ω^1_A, and by symmetry we conclude from this that the pairing 5.1.1 is continuous in each argument, *i.e.* induces continuous maps

$$(5.1.4)\qquad \begin{array}{l} A\to(\Omega^1_A)'_w \\ \Omega^1_A\to A'_w \end{array}$$

Since $f=\sum_{n\in\mathbb{Z}}\langle f,x^{-n-1}dx\rangle x^n$, we see that $0\in A$ is the intersection of the kernels of the continuous functionals $\langle\ \ ,x^{n-1}dx\rangle$ for $n\in\mathbb{Z}$. Thus $A\to(\Omega^1_A)'$ is injective, and A is separated. In the same way, one sees using 5.1.2 that any finite free A-module M is separated in its natural topology, and the natural map $M\to(M^\vee\otimes\Omega^1_A)'_w$ is continuous.

The next observation is the following: suppose $\ell: A_{[s,1)}\to K$ is a linear functional on $A_{[s,1)}$ such that $|\ell(U_{\epsilon,[s,t]})|\le 1$ for some $s<t<1$. Then if π is a uniformizer of K and $|\pi|=q^{-1}$, we have

$$(5.1.5)\qquad |\ell(x^n)|\le\begin{cases} q\epsilon^{-1}t^n & n\ge 0\\ q\epsilon^{-1}s^n & n\le 0.\end{cases}$$

In fact we can find elements $\sigma_n\in K$ such that

$$(5.1.6)\qquad q^{-1}\epsilon\le|\sigma_n x^n|_{[s,t]}\le\epsilon$$

Then since

$$|x^n|_{[s,t]}=\begin{cases} t^n & n\ge 0\\ s^n & n\le 0\end{cases}$$

5.1.6 yields

$$|\sigma_n|^{-1}\le\begin{cases} q\epsilon^{-1}t^n & n\ge 0\\ q\epsilon^{-1}s^n & n\le 0\end{cases}$$

On the other hand, 5.1.6 and the hypothesis on ℓ yield

$$|\ell(\sigma_n x^n)| \leq 1$$

and thus

$$|\ell(x^n)| \leq |\sigma_n|^{-1}$$

from which 5.1.5 follows.

We can now show that the maps 5.1.4 are surjective; it is enough to treat the case of $\Omega^1_A \to A'$. If $\ell : A \to K$ is a continuous linear functional, then for every $s < 1$ the induced map $\ell : A_{[s,1)} \to K$ is continuous, and thus for every $s < 1$ there is a $s < t < 1$ and an $\epsilon > 0$ depending on s such that $|\ell(B_{\epsilon,[s,t]})| \leq 1$. Then 5.1.5 shows that the numbers $a_{-n} = \ell(x^n)$ satisfy the estimates 4.1.2 and 4.1.3, with r equal to any of the t's obtained above. We conclude that the Laurent series $f = \sum_{n\in\mathbb{Z}} a_n x^n$ belongs to A, and evidently $\ell(g) = \langle g, f \otimes dx/x\rangle$ for any $g \in A$.

5.2. Lemma – *For $B \subset A$, the following are equivalent:*

(i) *B is bounded in A;*
(ii) *$B \subset A_{[r,1)}$ for some $r < 1$, and is bounded in $A_{[r,1)}$;*
(iii) *the image of B under the map $A \to (\Omega^1_A)'_s$ is bounded.*

Proof. – Evidently (ii) implies (i). Since $A \to (\Omega^1_A)'_w$ is continuous, the image of a bounded set in A is weakly bounded. Since Ω^1_A is barreled, the weakly and strongly bounded sets in $(\Omega^1_A)'$ coincide. Thus (i) implies (iii), and it remains to show that (iii) implies (ii). If the image of B in $(\Omega^1_A)'$ is weakly bounded, then by the Banach-Steinhaus theorem, this image is equicontinuous, so there is an open set $U \subset A$ such that $|\langle f, U \otimes dx/x\rangle| \leq 1$ for all $f \in B$. Thus for every $s < 1$ there is a $s < t < 1$ and an $\epsilon > 0$ depending on s such that $|\langle f, U_{\epsilon,[s,t]} \otimes dx/x\rangle| \leq 1$, and as before this shows that the estimates 4.1.2, 4.1.3 hold uniformly for all $f \in B$; once again r can be any of the t just obtained. We conclude that $B \subset A_{[r,1)}$ and that B is bounded in $A_{[r,1)}$.

□

5.3. Corollary – *For any free A-module M of finite type, M is a Montel space.*

Proof. – Since $M \simeq A^n$ topologically, it is enough to check the case $M = A$. We know that A is barreled, and by 5.2 and 4.4, a bounded set in A is relatively linearly compact.

5.4. Theorem – *For any free A-module M of finite type, the map*

$$M \to (M^\vee \otimes \Omega^1_A)'_s \tag{5.4.1}$$

induced by the pairing 5.1.2 is a topological isomorphism.

Proof. – It is enough to check this for $M = A$. We first show that 5.4.1 is continuous, and since A is bornological, it suffices to show that the image of a bounded set B in A is bounded in $(\Omega^1_A)'_s$. Now since Ω^1_A is barreled, the weakly and strongly bounded sets in $(\Omega^1_A)'$ coincide. Thus it is enough to see that the image of a bounded set in A is weakly bounded in $(\Omega^1_A)'$, and this is true because $A \to (\Omega^1_A)'_w$ is continuous.

We have already shown that $A \to (\Omega^1_A)'_s$ is surjective. To show that it is strict, we will identify $A \simeq \Omega^1_A$. Since A is Montel, it is reflexive, and thus the map $A \to A'_s$ coincides with its own strong dual. Again since A is Montel, A'_s is barreled, and so by 3.8 it suffices to show that a subset of A is bounded if and only if its image in A'_s is bounded, but this is true by lemma 5.2 (in place of 3.8, one could use the description of the open subsets of a barreled space in 1.7).

□

We now define A^+ (resp. A^-) to be the subspace of A consisting of formal Laurent series in x for which the coefficient of x^n vanishes for $n < 0$ (resp. $n \geq 0$), and we set $\Omega^+ = A^+ dx$, $\Omega^- = A^- dx$. Evidently $(A^\pm)^\perp = \Omega^\pm$ for the local pairing, from which it follows that A^+, A^- are closed subspaces of A, and thus that $A = A^+ \oplus A^-$ topologically. The same of course goes for $\Omega^\pm$.

5.5. Proposition – *The induced topology of A^+ is Frechet, that of A^- is dual-of-Frechet, and with respect to the local pairing we have*

$$(A^\pm)'_s \simeq \Omega^\mp$$

Proof. – We have $A^+ = A_{[0,1)}$ which is Frechet in its natural topology; this coincides with the topology induced from A by 3.5. Since $A = A^+ \oplus A^-$, we have $A'_s = (A^+ \oplus A^-)'_s = (A^+)'_s \oplus (A^-)'_s$, and as $(A^\pm)^\perp = \Omega^\pm$, 5.4 implies that $(A^\pm)'_s \simeq \Omega^\mp$. It then follows that A^- is the dual of the reflexive Frechet space Ω^+.

□

Remark. – Using results of Komatsu [25, 32], one can actually show directly that A^+ (resp. A^-) is Montel and Frechet (resp. dual-of-Frechet), and so give another proof of 5.3 and 5.4.

5.6. We now consider homomorphisms of local algebras, *i.e.* of topological K-algebras isomorphic to A. Let $A(a)$, $A(b)$ be two such algebras, viewed as algebras of formal Laurent series in the variables t_a resp. t_b. Suppose that $A(a) \to A(b)$ is an injective algebra homorphism such that

$$t_a = \sum_{n \geq 0} a_n t_b^n \qquad a_n \in R \tag{5.6.1}$$

and we assume that a_0 is a non-unit in R, but that one of the a_n is a unit. Then $A(a) \to A(b)$ induces a homorphism $A(a)^+ \to A(b)^+$, and if N is the smallest integer such that $a_N \in R^\times$, then $A(b)$ (resp. $A(b)^+$) is free of rank N over $A(a)$ (resp. $A(a)^+$), with basis $1, t_b, \ldots, t_b^{N-1}$. We will call this integer the *degree* of a over b, and denote it by $\deg(b/a)$. Note that $A(b)^+$ being of finite rank over $A(a)^+$ does *not* follow simply from $A(b)$ being of finite rank over $A(a)$. We shall express all of these assumptions simply by saying that the homomorphism $A(a) \to A(b)$ is *adapted* to the parametrizations t_a, t_b of $A(a)$ and $A(b)$.

6. Local duality II

As always, a *connection* on an A-module M is a K-linear map $\nabla : M \to M \otimes \Omega^1_A$ satisfying Leibnitz's rule; the same goes for connections on A_I-modules for any I. If (M, ∇) is an A-module of finite presentation with connection, then there is an

$r < 1$ and an $A_{[r,1)}$-module (M_r, ∇_r) of finite presentation with connection such that $(M, \nabla) = (M_r, \nabla_r) \otimes A$. If $M \to N$ is a horizontal map of A-modules of finite presentation with connection, then for some $r < 1$ there are A_r-modules M_r, N_r of finite presentation with connections and a horizontal map $M_r \to N_r$ which induces $M \to N$.

Since the transpose of $d : A \to \Omega_A^1$ is $-d$, it follows that for any connection $\nabla : M \to M \otimes \Omega_A^1$ on a locally free A-module M, the transpose of ∇ can, via 5.4, be identified with the negative of the dual connection: ${}^t\nabla = -\nabla^\vee$.

6.1. Proposition – *If (M, ∇) is an A-module of finite presentation with connection, then M is free.*

Proof. – We will show that this is also the case for a module of finite presentation over any A_I. In any case, it is enough, by 4.8 and 4.9, to show that the module is torsion-free. Consider first the case of an A_I-module M of finite presentation, where I is closed. Then A_I is a principal ideal domain, so let (f) be the annihilator of M_{tor}, and suppose m is a nonzero element of M_{tor}. From $fm = 0$ we get $f'm + f\nabla(m) = 0$, whence $f^2\nabla(m) = 0$. Thus $\nabla(m) \in M_{tor}$, so $f\nabla(m) = 0$ and then $f'm = 0$. Therefore $f' \in (f)$, and since A_I is a principal ideal domain containing the rational numbers, (f) must be the unit ideal, whence $M_{tor} = 0$.

In general A_I is an inverse limit of A_J with J closed, and for any A_I-module of finite presentation we have $M = \varprojlim_J M \otimes A_J$ by 4.8. A connection on M induces one on each of the $M \otimes A_J$; since these are of finite presentation, they are torsion free, and so M is too. Finally, suppose that (M, ∇) is an A-module of finite presentation with a connection, and suppose that m is a torsion element of M. Then for some $r < 1$, we have that $M = M_r \otimes A$ for some $A_{[r,1)}$-module M_r with connection. For r sufficiently close to 1 we have $m \in M_r$, whence $m = 0$.

□

If (M, ∇) is an A-module with connection, we define

$$\text{(6.1.1)} \qquad \begin{aligned} \mathrm{H}^0(M) &= \operatorname{Ker} \nabla \\ \mathrm{H}^1(M) &= \operatorname{Coker} \nabla. \end{aligned}$$

One sees immediately that any exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of A-modules with connection gives rise to a six-term exact sequence

$$\text{(6.1.2)} \quad 0 \to \mathrm{H}^0(M') \to \mathrm{H}^0(M) \to \mathrm{H}^0(M'') \to \mathrm{H}^1(M') \to \mathrm{H}^1(M) \to \mathrm{H}^1(M'') \to 0.$$

When M is free, we give the $\mathrm{H}^i(M)$ the topologies induced by those of M, $M \otimes \Omega_A^1$. Then $\mathrm{H}^0(M)$ is separated, and is finite-dimensional as well:

6.2. Proposition – *Let (M, ∇) be a flat A-module with connection. Then the natural map $A \otimes \mathrm{H}^0(M) \to M$ is injective. In particular, if M is free of finite type, then* $\dim_K \mathrm{H}^0(M) \leq \operatorname{rank}_A M$.

Proof. – Let $g : A \otimes \mathrm{H}^0(M) \to M$ denote the natural map, and suppose that $\sum_i f_i \otimes v_i \in \operatorname{Ker} g$ is a nonzero element with as few terms as possible; then the v_i

are linearly independent over K, and $f_i \neq 0$. Since $\sum_i f_i v_i = 0$, we have $\sum_i f_i' v_i = 0$ since $v_i \in \operatorname{Ker} \nabla$, so $\sum_i f_i' \otimes v_i \in \operatorname{Ker} g$. Then by the hypothesis on $\sum_i f_i \otimes v_i$, we have

$$f_1' \sum_i f_i \otimes v_i - f_1 \sum_i f_i' \otimes v_i = 0$$

so for all i we have $f_1' f_i = f_1 f_i'$. We now claim that for all i we have $f_i = c_i f_1$ for some set of $c_i \in K$. If f_1 is a unit in A, of course, then $(f_i/f_1)' = 0$, and the assertion is clear. In general, we can find an $s < 1$ such that the image of f_1 under the embedding $A \hookrightarrow A_{[s,s]}$ is a unit, and then do the same calculation in $A_{[s,s]}$. In any case we get $\sum_i f_i v_i = f_1 \sum_i c_i v_i = 0$, and since M is flat, we get $\sum_i c_i v_i = 0$, a contradiction.

□

The K-vector space $\mathrm{H}^1(M)$ is not in general finite-dimensional or separated, and the main result of this section is a sufficient condition for finite-dimensionality. Let (M, ∇) be any free A-module of finite type with connection; then the local pairing 5.1.2 induces a pairing

$$\mathrm{H}^i(M) \times \mathrm{H}^{1-i}(M^\vee) \to K. \tag{6.2.1}$$

We will say that (M, ∇) is *strict* if the connection $\nabla : M \to M \otimes \Omega_A^1$ is a strict map of topological vector spaces. We will often say simply that "M is strict" and suppress mention of the connection (even though it's what is being talked about).

6.3. Theorem – *Let (M, ∇) be a free A-module of finite type with connection. Then the following conditions are equivalent:*

(i) *M is strict,*
(ii) *$M^\vee$ is strict,*
(iii) *$\mathrm{H}^1(M)$ is finite-dimensional and separated.*

If M is strict, then the pairing 6.2.1 is a perfect pairing of finite-dimensional K-vector spaces.

Proof. – If M is strict, then the short exact sequence

$$0 \to \mathrm{H}^0(M) \longrightarrow M \xrightarrow{\nabla} M \otimes \Omega_A^1$$

is strict exact, so its transpose

$$M^\vee \xrightarrow{-\nabla^\vee} M^\vee \otimes \Omega_A^1 \longrightarrow \mathrm{H}^0(M)^\vee \longrightarrow 0$$

is exact. Since $\mathrm{H}^0(M)$ is finite-dimensional, the last exact sequence shows that the cokernel of $\nabla^\vee$ is finite-dimensional; it is also separated, by part (i) of 3.8. Then by the open mapping theorem 3.6, $M^\vee$ is strict. If $M^\vee$ is strict, then the above argument with $M^\vee$ in place of M shows that $\mathrm{H}^1(M)$ is finite-dimensional and separated. Finally if $\mathrm{H}^1(M)$ is finite-dimensional and separated, then the strictness of ∇ again follows from 3.6.

□

6.4. Corollary – *Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of finite free A-modules with strict connection. If M is strict, then so are M' and M''.

Proof. – Since M, M', and M'' are free, the exact sequence in 6.4 splits as an exact sequence of A-modules, and thus as an exact sequence of topological vector spaces over K. It is thus strict exact, and from this it easily follows that $\mathrm{H}^1(M) \to \mathrm{H}^1(M'')$ is strict and surjective. Since M is strict, $\mathrm{H}^1(M)$ is separated and finite-dimensional, so the same must be true of $\mathrm{H}^1(M'')$. Then by 6.3, M'' is strict. Since $M^\vee$ is also strict, the same argument shows that the quotient $M'^\vee$ of $M^\vee$ is strict. We conclude again by 6.3 that M' is strict.

□

I do not know if the converse of 6.4 is true. If M' and M'' are strict, then it follows from 6.3 that $\mathrm{H}^1(M)$ is finite-dimensional, but I do not know how to prove that it is separated.

6.5. Proposition – *Suppose that (M, ∇) is a finite free A-module with connection, and that L/K is a finite extension. Then M is strict if and only if the induced connection on $M \otimes L$ is.*

Proof. – If n is the degree of L/K, then $M \otimes L \simeq M^n$ and $f \otimes L \simeq f^n$, and the assertion follows from 3.7.

□

6.6. If M, N are two finite free A-modules with connection, we denote by $\mathrm{Ext}^1(M, N)$ the first Yoneda Ext group in the category of finite free A-modules with connection. One checks immediately, using the usual arguments, that

$$\mathrm{Hom}(M, N) \simeq \mathrm{H}^0(M^\vee \otimes N) \qquad \mathrm{Ext}^1(M, N) \simeq \mathrm{H}^1(M^\vee \otimes N) \tag{6.6.1}$$

functorially in M, N. With the identifications 6.6.1, the exact sequence 6.1.2 is the long exact sequence of Yoneda Ext groups

$$\begin{aligned} 0 \to \mathrm{Hom}(A, N) \to \mathrm{Hom}(A, M) \to \mathrm{Hom}(A, A) \to \\ \mathrm{Ext}^1(A, N) \to \mathrm{Ext}^1(A, M) \to \mathrm{Ext}^1(A, A) \to 0. \end{aligned} \tag{6.6.2}$$

We will say that a connection ∇ on a finite free A-module M is *unipotent* if (M, ∇) is a successive extension of trivial rank one objects (A, d).

If V is a finite-dimensional vector space and $C \in \mathrm{End}\, V$, then we denote by $(V \otimes_K A, \nabla_C)$ the A-module with connection given by

$$\nabla_C = C \otimes \frac{dx}{x} \tag{6.6.3}$$

If N is a nilpotent matrix, then the corresponding module with connection $(V \otimes_K, \nabla_N)$ is unipotent. Conversely, every module with unipotent connection has this form (*cf.* [24 2.4.3]):

6.7. Proposition – *The functor*

$$F : (V, N) \mapsto (V \otimes_K A, \nabla_N) \tag{6.7.1}$$

induces an equivalence of the category of finite-dimensional K-vector spaces with a nilpotent endomorphism, and the category of finite free A-modules with unipotent connection.

Proof. – Denote by $\mathcal{C}$ the category of (V, N), $\mathcal{D}$ the category of finite free A-modules with unipotent connection, and $F : \mathcal{C} \to \mathcal{D}$ the functor 6.7.1. We show, first, that for any two objects (V, N), (V', N') of $\mathcal{C}$, F induces isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}((V,N),(V',N')) &\xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(F((V,N)),F((V',N'))) \\ \operatorname{Ext}^1_{\mathcal{C}}((V,N),(V',N')) &\xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{D}}(F((V,N)),F((V',N'))) \end{aligned} \tag{6.7.2}$$

The Ext groups in $\mathcal{D}$ are the same as the Ext groups in the category of finite free A-modules with connection, so that we can use 6.6.1. The Ext groups in $\mathcal{C}$ have the following description: if (V, N) and (V', N') are objects of $\mathcal{C}$, then $\operatorname{Hom}((V, N), (V', N'))$ (resp. $\operatorname{Ext}^1((V, N), (V', N'))$) is the kernel (resp. cokernel) of the induced nilpotent endomorphism of $V^\vee \otimes V'$. In particular, the exact sequence of Ext groups in $\mathcal{D}$ can be truncated after the sixth term. Since any object of $\mathcal{C}$ or $\mathcal{D}$ is a successive extension of trivial ones, and since $F(K, 0) = (A, d)$, an easy induction on the length of an object reduces the proof of 6.7.2 to the case $(V, N) = (V', N) = (K, 0)$, in which case it is immediate.

From the first isomorphism in 6.7.2 we see that F is fully faithful. To show that it is essentially surjective, we again argue by induction. If (M, ∇) is an object of $\mathcal{D}$ of positive rank, then by induction there is an exact sequence

$$0 \to F(V, N) \to M \to F(K, 0) \to 0 \tag{6.7.3}$$

and the second isomorphism in 6.7.2 shows that there is an exact sequence

$$0 \to (V, N) \to (V', N') \to (K, 0) \to 0$$

in $\mathcal{C}$ whose image is 6.7.3. We conclude that $F(V, N) = M$.

□

Remark. – Recalling again the descriptions of the Ext groups, we see that for any (V, N), $F(V, N) = (M, \nabla)$, we see that the natural maps $\operatorname{Ker} N \hookrightarrow \operatorname{Ker} \nabla$, $(\operatorname{Coker} N) \otimes dx/x \to \operatorname{Coker} \nabla$ induce topological isomorphisms

$$\begin{aligned} \mathrm{H}^0(M) &\simeq \operatorname{Ker} N \\ \mathrm{H}^1(M) &\simeq \operatorname{Coker} N. \end{aligned} \tag{6.7.4}$$

Furthermore, the "normal form" provided by 6.7 shows that if (M, ∇) is unipotent and $(M, \nabla) = F(V, N)$, then

$$V = \bigcup_{n>0} \operatorname{Ker} \nabla\left(x\frac{\mathrm{d}}{dx}\right)^n.$$

Then the functor 6.7.1 has a quasi-inverse given by

$$(M, \nabla) \mapsto \left(V = \bigcup_{n>0} \operatorname{Ker} \nabla\left(x\frac{\mathrm{d}}{dx}\right)^n, N = \nabla\left(x\frac{\mathrm{d}}{dx}\right)\right) \tag{6.7.5}$$

(*cf.* [24 2.4.3.3]).

6.8. Corollary – *An A-module with unipotent connection is strict.*

Proof. – By 6.3 it is enough to show that $\mathrm{H}^1(M)$ is finite-dimensional and separated, and this follows from the previous remark.

□

We now consider the behavior of the functor 6.7.1 under base change. Let $\phi : A(a) \to A(b)$ be a homomorphism of local algebras of the type described in 5.6. Denote by $\mathcal{D}_a$, (resp. $\mathcal{D}_b$) the category of finite free $A(a)$-modules (resp. $A(b)$-modules) with unipotent connections. Since ϕ^* is exact and the pullback of a trivial connection is trivial, ϕ^* induces a functor $\phi^* : \mathcal{D}_a \to \mathcal{D}_b$.

6.9. Proposition – *With the above notation,* $\phi^* : \mathcal{D}_a \to \mathcal{D}_b$ *is an equivalence of categories.*

Proof. – Fix a pair of local parameters t_a, t_b adapted to ϕ in the sense of 5.6, and denote by $F_a : \mathcal{C} \to \mathcal{D}_a$, $F_b : \mathcal{C} \to \mathcal{D}_b$ the corresponding functors. It is enough to show that for any (V, N) in $\mathcal{C}$ there is an isomorphism

$$F_b(V,N) \xrightarrow{\sim} \phi^*(F_a(V,N)) \tag{6.9.1}$$

since this shows that ϕ^* is essentially surjective, while on the other hand the full faithfulness of ϕ^* follows from 6.9.1 and the description 6.6.1 of the Hom groups. Now an isomorphism such as 6.9.1 can be identified with an automorphism B of $V \otimes A(b)$ satisfying

$$dB \cdot B^{-1} + BNB^{-1}\frac{dt_b}{t_b} = N\frac{dt_a}{t_a}.$$

There is automorphism B_0 of V such that $B_0NB_0^{-1} = \deg(b/a)N$, so if we look for a B of the form $B = CB_0$, with C commuting with N , the above condition becomes

$$dC \cdot C^{-1} = N\left(\frac{dt_a}{t_a} - \deg(b/a)\frac{dt_b}{t_b}\right).$$

By 5.6.1 and the Weierstrass factorization theorem, we have

$$\mathrm{Res}_b dt_a/t_a = \deg(b/a)$$

which shows that there is a $g \in A(b)$ satisfying

$$dg = \frac{dt_a}{t_a} - \deg(b/a)\frac{dt_b}{t_b} \tag{6.9.2}$$

and we can take $C = \exp(gN)$.

□

Remark. – Since the choice of B_0 made above is arbitrary, we do not obtain an isomorphism of functors $F_b \simeq \phi^* \circ F_a$.

We saw earlier that $(V \otimes A, \nabla_N)$ is strict whenever N is nilpotent; we now give a criterion for strictness of $(V \otimes A, \nabla_C)$ in general. Recall that $a \in K$ is said to be a p-adic Liouville number if $|a - n| < r^{|n|}$ for some $r < 1$ and infinitely many $n \in \mathbb{Z}$. Equivalently, a is *not* p-adically Liouville if for every positive $r < 1$ there is a $C > 0$ such that $|n - a| > Cr^{|n|}$ for all n.

6.10. Proposition – *The connection ∇_C is strict if and only if the eigenvalues of C are not p-adic Liouville numbers.*

Proof. – By 3.7 and 6.5, we can assume that all the eigenvalues of C belong to K, and that C consists of a single Jordan block. If the eigenvalue of this block is an integer, then ∇_C is equivalent to ∇_N for some nilpotent matrix N; such a connection is unipotent, and therefore strict by 6.6. We will therefore assume that the eigenvalue is not an integer, and we consider first the case $n = 1$, so that the connection can be written

$$\nabla(f) = (xf' - a) \otimes \frac{dx}{x}.$$

with $a \notin \mathbb{Z}$. Since

$$\nabla(\sum_{n\in\mathbb{Z}} a_n x^n) = \sum_{n\in\mathbb{Z}} (n-a) a_n x^n \otimes \frac{dx}{x}$$

we see that if a is not p-adic Liouville, then ∇_a is actually surjective, and therefore strict by the open mapping theorem. Suppose, on the other hand, that ∇_a is strict, and that a is not an integer. Since the topological transpose of ∇_a can be identified with $-\nabla_{-a}$ and $\operatorname{Ker} \nabla_{-a} = 0$, we have $\operatorname{Coker} \nabla_a = 0$. Thus ∇_a is surjective, and since it is strict, the inverse map is continuous and must map bounded sets to bounded sets. Since $\{x^n \otimes dx/x\}_{n>0}$ is bounded, we see that the set of

$$\nabla_a^{-1}(x^n \otimes dx/x) = (n-a)^{-1} x^n \qquad n \geq 0$$

is bounded. From 4.1.3, we conclude that for all $r < 1$ there is a $C_r > 0$ such that $|n-a|^{-1} \leq C_r r^{-n}$ for all $n \geq 0$; *i.e.* $|n-a| \geq C_r^{-1} r^n$ whenever $n \geq 0$, which so to speak is half of the condition defining a p-adic non-Liouville number. To obtain the other half, we choose an $r < 1$, and can then find $a_n \in K$, $n \leq 0$ such that $q^{-1} \leq |a_n x^n|_{[r,r]} = |a_n| r^n \leq 1$, where q^{-1} is the valuation of a uniformizer of K. Since $\{a_n x^n\} \otimes dx/x$ is bounded, the same argument as above shows that

$$\nabla_a^{-1}(a_n x^n \otimes dx/x) = a_n (n-a)^{-1} x^n \qquad n \leq 0$$

is bounded. By 4.1.2 there is a $C_r > 0$ and an s such that $r < s < 1$ for which

$$|a_n (n-a)^{-1}| \leq C_r s^{-n} \leq C_r \qquad n \leq 0$$

and since $q^{-1} r^{-n} \leq |a_n|$, we have

$$|n-a| \geq C_r^{-1} |a_n| \geq (qC_r)^{-1} r^{-n} = (qC_r)^{-1} r^{|n|} \qquad n \leq 0$$

which shows that a is not p-adic Liouville.

We now consider the general case, where C is a single Jordan block with non-integral eigenvalue a. If a is not p-adic Liouville, then the above argument shows that the connection ∇_a is surjective; repeated application of the exact sequence 6.1.2 then shows that ∇_C is surjective, and is therefore strict by the open mapping theorem. If on the other hand ∇_C is known to be strict, then repeated application of 6.4 shows that ∇_a is strict, and consequently a is not p-adic Liouville.

□

Part III

Global duality and finiteness

The first step in this section is to establish a duality result of the sort familiar in the geometry of numbers (7.5, 7.7). We follow the classical procedure in reducing to the case of an open subset of the projective line; this means we need additional hypotheses on the nature of the lifting of the curve that must certainly be unnecessary (and one hopes that a better understanding of rigid-analytic duality would allow us to eliminate them). After proving the main results of the paper in §9, we give some applications in §10.

7. Dagger algebras and the global pairing

7.1. Let us first recall some ideas and results from [5,8]. Let X/k be a smooth k-scheme, and $\mathfrak{X}/k$ a formally smooth lifting. Points of the affinoid space $\mathfrak{X}^{an}$ correspond to closed formal subschemes of $\mathfrak{X}/R$ that are finite, flat, and integral over R. Then reduction modulo the maximal ideal of R yields the *specialization morphism* $sp : \mathfrak{X}^{an} \to X$. If $U \subseteq X$ is locally closed, then the *tube* $]U[_{\mathfrak{X}^{an}}$ *of* U *in* $\mathfrak{X}^{an}$ is defined to be the inverse image $sp^{-1}(U)$ of U under specialization. We will often drop the subscript and denote the tube simply by $]U[$.

Suppose $X \subset Y$ are smooth over k, with formally smooth liftings $\mathfrak{X} \subset \mathfrak{Y}$. A *strict neighborhood* V of $]X[_{\mathfrak{X}^{an}}$ in $\mathfrak{Y}^{an}$ is an admissible open subspace such that $\{V,]Y - X[\}$ is an admissible cover of $\mathfrak{Y}^{an}$ (*cf.* [8 1.2.1]). We will be mainly interested in the case when X, Y are smooth geometrically connected curves, in which case the filter of strict neighborhoods of $]X[$ in $\mathfrak{Y}$ has a countable cofinal set of *affinoid* strict neighborhoods.

We now fix our attention on the following situation: $X \hookrightarrow \overline{X}$ is an inclusion of smooth curves, X is affine, $\overline{X}$ is projective, and $X \hookrightarrow \overline{X}$ lifts to a morphism $\mathfrak{X} \hookrightarrow \overline{\mathfrak{X}}$ of formally smooth formal R-schemes. For any strict neighborhood V of $]X[$, set

$$A_V = \Gamma(V, \mathcal{O}_V)$$

and, for V running through a cofinal set of strict neighborhoods of $]X[$,

$$A_X^\dagger = \varinjlim_V A_V \qquad (7.1.1)$$

Note that $\overline{X}$ is essentially determined by X; $\mathfrak{X}$ and $\overline{\mathfrak{X}}$ are not, but we will not indicate them in the notation.

Suppose that $f : Y \to X$ is a k-morphism of smooth affine curves, sitting in a cartesian diagram

$$\begin{array}{ccc} Y & \hookrightarrow & \overline{Y} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ X & \hookrightarrow & \overline{X} \end{array}$$

where $\overline{X}$, $\overline{Y}$ are smooth and projective, and suppose finally that $\overline{f} : \overline{Y} \to \overline{X}$ lifts to a map $\overline{f} : \overline{\mathfrak{Y}} \to \overline{\mathfrak{X}}$ of formally smooth formal R-schemes. If V is a strict neighborhood of $]X[$

in $\mathfrak{X}^{an}$, then $f^{-1}(V)$ is a strict neighborhood of $]Y[$ in $\overline{\mathfrak{Y}}^{an}$, since the admissible cover $\{V,]\overline{X} - X[\}$ of $]\overline{X}[$ has as its inverse image the admissible cover $\{f^{-1}(V),]\overline{Y} - Y[\}$ of $\overline{\mathfrak{Y}}^{an}$. It follows that $\overline{f}$ induces a K-algebra homomorphism

$$A_X^\dagger \to A_Y^\dagger.$$

It is known that $A_X^\dagger$ is a noetherian ring [22]; however the simpler arguments of [8] are enough to show that $A_X^\dagger$ is a coherent ring, which is all that we will really need. Thus any $A^\dagger$-module of finite presentation is coherent. Since the A_V are coherent (and even noetherian, when V is affinoid), it follows from the general properties of coherent rings that the family of functors

$$\begin{array}{ccc} (A_V\text{-modules}) & \to & (A^\dagger\text{-modules}) \\ M & \mapsto & M \otimes_{A_V} A^\dagger \end{array}$$

induces an equivalence of the category of $A^\dagger$-modules with the inductive limit of the categories of A_V-modules, as V runs through a cofinal set of strict neighborhoods of $]X[_{\mathfrak{X}^{an}}$. In other words, for any coherent $A^\dagger$-module M there is a strict neighborhood V and a coherent A_V-module M_V such that $M = M_V \otimes_{A_V} A^\dagger$, and for any linear map $M \to N$ of $A^\dagger$-modules, there is a V and A_V-modules M_V, N_V such that $M \to N$ is induced by an A_V-linear $M_V \to N_V$.

For any coherent $A^\dagger$-module M such that $M = M_V \otimes A^\dagger$, we have

$$M = \varinjlim_{V' \subset V} \Gamma(V', \mathcal{M}_V) \tag{7.1.2}$$

where $\mathcal{M}_V$ is the sheaf on V corresponding to M_V, and V' runs through a cofinal set of strict neighborhoods of $]X[$. In fact, since M_V is coherent, we have

$$\Gamma(V', \mathcal{M}_V) = A_{V'} \otimes_{A_V} M_V$$

and thus

$$M = M_V \otimes A^\dagger = \varinjlim_{V'} M_V \otimes_{A_V} A_{V'} = \varinjlim_{V'} \Gamma(V', \mathcal{M}_V). \tag{7.1.3}$$

From 7.1.2 it follows that any coherent $A^\dagger$-module has a natural topology, arising from the direct limit in 7.1.2. We claim that it is separated. In fact, for any $V' \subset V$, the map $\Gamma(V', \mathcal{M}_V) \to \Gamma(\mathfrak{X}^{an}, \mathcal{M}_V)$ is continuous, and is injective by [8 2.1.11]. The map $M \to \Gamma(\mathfrak{X}^{an}, \mathcal{M}_V)$ is therefore continuous and injective. Since $\mathfrak{X}^{an}$ is affinoid, $\Gamma(\mathfrak{X}^{an}, \mathcal{M}_V)$ is a Banach space and in particular is separated; then M must be separated as well. Furthermore the natural topology of the $\Gamma(V', \mathcal{M}_V)$ is Frechet, (or even Banach, if V' is affinoid); then, since $\mathfrak{X}^{an}$ has a countable fundamental system of strict neighborhoods, the topology of M is that of an LF-space. Since an $A^\dagger$-linear map $M \to N$ arises from some map $M_V \to N_V$ of A_V-modules for some V, we also see from 7.1.2 that an $A^\dagger$-linear map of coherent $A^\dagger$-modules is continuous.

7.2. We now explain how to attach a local algebra of the sort studied in §5 to a point $a \in X$ of a smooth algebraic curve. By shrinking X, we can assume that X is affine, and thus has a formally smooth lifting $\mathfrak{X}/R$. Since $\mathfrak{X}$ is formally smooth, the inductive limit

$$A(a) = \varinjlim_{V} \Gamma(V \cap]a[, \mathcal{O}_{\mathfrak{X}^{an}}), \tag{7.2.1}$$

where V runs through the set of strict neighborhoods of $]X - a[$ in $\mathfrak{X}^{an}$, is a local algebra. More precisely, if t_a is a local section of $\mathcal{O}_{\mathfrak{X}}$ reducing to a local parameter of $\mathcal{O}_a$, then $A(a)$ is the K-algebra of formal Laurent series in t_a convergent in some annulus $r < |t_a| < 1$, and $A(a)^+$ is the K-algebra of analytic functions on $]a[$. It is clear from the definition that the natural map

$$\Gamma(V, \mathcal{O}_{\mathfrak{X}^{an}}) \longrightarrow A(a) \tag{7.2.2}$$

is continuous for the natural topologies.

This construction is essentially independent of the choice of $\mathfrak{X}$. In fact if $\mathfrak{X}'$ is another choice, then for any affine neighborhood U of a, the restrictions of $\mathfrak{X}$, $\mathfrak{X}'$ to U are isomorphic (since both are formally smooth liftings of X). It follows that $]a[_{\mathfrak{X}}$ can be identified with $]a[_{\mathfrak{X}'}$, and thus a strict neighborhood of $]U - a[$ in $\mathfrak{X}^{an}$ can be identified with a strict neighborhood of $]U - a[$ in $\overline{\mathfrak{X}}'^{an}$. The corresponding local algebras attached to a will therefore be non-canonically isomorphic.

Suppose $f : Y \to X$ is a morphism of smooth affine curves; then any lifting $\mathfrak{Y}$, $\mathfrak{X}$ of Y, X will be formally smooth, and thus there is a lifting $f : \mathfrak{Y} \to \mathfrak{X}$ of $Y \to X$. If $b \in Y$ and $f(b) = a$, then $\mathfrak{Y} \to \mathfrak{X}$ induces a K-algebra homorphism $A(a) \to A(b)$, and if we choose local sections t_a, t_b reducing to local parameters at a and b, then t_a, t_b are adapted to $A(a) \to A(b)$ in the sense of 5.6. The rank $\deg(b/a)$ of $A(b)$ over $A(a)$ is of course the same as the rank of $\mathcal{O}_b$ over $\mathcal{O}_a$.

7.3. Now suppose that we are given an embedding $X \hookrightarrow \overline{X}$ of smooth curves with X affine and $\overline{X}$ projective, and set $D = \overline{X} - X$. Fix a lifting $\mathfrak{X} \to \overline{\mathfrak{X}}$ of $X \to \overline{X}$ and let $A^\dagger = A_X^\dagger$ be as in 7.1.1; the construction of 7.2 is applicable (with $\overline{X}$ in place of X, to be sure), and the inductive limit of 7.2.2 as V ranges over all strict neighborhoods of $]X[$ in $\overline{\mathfrak{X}}^{an}$ is a continuous embedding $A^\dagger \hookrightarrow A(a)$, for any $a \in D$. If we put

$$A_X^{loc} = \bigoplus_{a \in D} A(a), \tag{7.3.1}$$

then the direct sum of the embeddings $A_X^\dagger \hookrightarrow A(a)$ is an embedding $A_X^\dagger \longrightarrow A_X^{loc}$. We define A_X^{qu} by the exactness of the sequence

$$0 \to A_X^\dagger \to A_X^{loc} \to A_X^{qu} \to 0$$

and we will drop the subscript X whenever possible. If M is any locally free $A^\dagger$-module, we have an exact sequence

$$0 \to M \to M \otimes_{A^\dagger} A^{loc} \to M \otimes_{A^\dagger} A^{qu} \to 0. \tag{7.3.2}$$

If M is a coherent $A^\dagger$-module M arising from a locally free sheaf $\mathcal{M}_V$ on a strict neighborhood V of $]X[$, then

$$M \otimes_{A^\dagger} A^{loc} = \varinjlim_{V' \subset V} \Gamma(V' \cap]D[, \mathcal{M}_V) \tag{7.3.3}$$

In fact, since $M_V = \Gamma(V, \mathcal{M}_V)$ is a finite A_V-module, we have

$$\begin{aligned} M \otimes_{A^\dagger} A^{loc} &= M_V \otimes_{A_V} A^{loc} \\ &= \varinjlim_{V'} M_V \otimes_{A_V} \Gamma(V' \cap]D[, \mathcal{O}_{V'}) \\ &= \varinjlim_{V'} \Gamma(V' \cap]D[, \mathcal{M}_V). \end{aligned}$$

As before, 7.3.3 allows us to topologize $M\otimes A^{loc}$ for any coherent $A^\dagger$-module M. However if $M\otimes A^{loc}$ is a locally free finitely generated A^{loc}-module, we have

$$M\otimes_{A^\dagger} A^{loc} = \bigoplus_{x\in D} M\otimes_{A^\dagger} A(x) \tag{7.3.4}$$

by 7.3.1, and if M is locally free, each term of the right hand side is a free $A(x)$-module (by proposition 4.8). Thus, when M is finite and locally free, there are two natural topologies on $M\otimes A^{loc}$, induced by 7.3.3 and 7.3.4, and we claim that they coincide. This is clear if $M = A^\dagger$ or, more generally, if M is free. In general, if M is finite and locally free, we can choose a finite free $A^\dagger$-module N of which M is a direct summand. Then for the topology defined by 7.3.3 or 7.3.4, M is a topological summand of N, as the inclusion $M\hookrightarrow N$ and projection $N\to M$ are both continuous. The topologies defined by 7.3.3 and 7.3.4 coincide for N, so they coincide for M. This argument also shows that this topology is separated.

Put

$$\Omega^1_{A^\dagger} = \varinjlim_V \Gamma(V,\Omega^1_V).$$

For any strict neighborhood V of $]X[$ we have $\Omega^1_{A^\dagger} = \Gamma(V,\Omega^1_V)\otimes A^\dagger$, so that $\Omega^1_{A^\dagger}$ is locally free, and the above considerations apply to $M=\Omega^1_{A^\dagger}$. We will use the abbreviations

$$\begin{aligned}\Omega^{loc} &= \Omega^1_{A^\dagger}\otimes_{A^\dagger} A^{loc}\\ \Omega^{qu} &= \Omega^1_{A^\dagger}\otimes_{A^\dagger} A^{qu}.\end{aligned}$$

If M is a locally free $A^\dagger$-module M, the *global pairing* is defined by

$$\begin{aligned}(M\otimes A_X^{loc})\times(M^\vee\otimes\Omega_X^{loc}) &\to K\\ ((m_a),(\omega_a)) &\mapsto \langle (m_a),(\omega_a)\rangle_X \overset{\text{defn}}{=} \sum_{x\in D}\langle m_a,\omega_a\rangle_a\end{aligned} \tag{7.3.5}$$

where the $\langle\ ,\ \rangle_a$ denotes the local pairing 5.1.1 on $A(a)$. From 5.4, we see that 7.3.5 induces an isomorphism of each of $M\otimes A^{loc}$, $M^\vee\otimes\Omega^{loc}$ with the strong dual of the other.

7.4. Lemma – *For the pairing 7.3.5 we have*

$$M^\vee\otimes\Omega^1_{A^\dagger}\subseteq M^\perp.$$

Proof. – From the definition, one sees immediately that it is enough to treat the case $M = A^\dagger$, in which case the assertion is equivalent to the following: for $\omega\in\Omega^1_{A^\dagger}$, we have

$$\langle 1,\omega\rangle = \sum_{x\in D}\langle 1,\omega\rangle_x = 0. \tag{7.4.1}$$

The analytic curve $\overline{\mathfrak{X}}^{an}$ has an (essentially unique) algebraization $\overline{\mathbb{X}}$; it is a smooth projective curve, and we can identify the points of $\overline{\mathfrak{X}}^{an}$ with the closed points of $\overline{\mathbb{X}}$. Denote by $\mathbb{A}$ the ring of meromorphic functions on $\overline{\mathbb{X}}$ whose poles lie in $]D[$, and by $\Omega^1_{\mathbb{A}}$ the corresponding module of 1-forms. If $\omega\in\Omega^1_{\mathbb{A}}$, then for any $a\in D$ we have

$$\mathrm{Res}_a\omega = \sum_{x\in]D[}\mathrm{Res}_x\omega \tag{7.4.2}$$

where the residue on the left hand side is the one defined in 5.1, *i.e.* the one which figures in the local pairing $\langle\ ,\ \rangle_a$, while the residues on the right hand side are the ordinary residues of ω at points x in the analytic space $]a[$. If t_a is local section of $\mathcal{O}_{\mathfrak{X}}$ chosen as in 7.2, then 7.4.2 follows from the identity $(t_a - b)^{-n} dt_a = t_a^{-n} dt_a \cdot (1 - b/t_a)^{-n}$.

It follows from 7.4.2 and the classical residue theorem that $\langle 1, \omega \rangle = 0$ for all $\omega \in \Omega^1_{\mathbb{A}}$. Since the pairing 7.3.5 is continuous and $\Omega^1_{\mathbb{A}}$ is dense in $\Omega^1_{A^\dagger}$, we must have $\langle 1, \omega \rangle = 0$ for all $\omega \in \Omega^1_{A^\dagger}$.

□

In the case when the $A^\dagger$-module M arises from a locally free sheaf $\mathcal{M}$ on $\overline{\mathfrak{X}}^{an}$, we have $\Gamma(]D[, \mathcal{M}) \subset M \otimes A^{loc}$, and with respect to the pairing 7.3.5 we have

$$\Gamma(]D[, \mathcal{M})^\perp = \Gamma(]D[, \mathcal{M}^\vee \otimes \Omega^1). \tag{7.4.3}$$

In fact, $\Gamma(]D[, \mathcal{M})$ is a free A^{loc}-module by 4.8, so 7.4.3 reduces immediately to the case $\mathcal{M} = \mathcal{O}_{\overline{\mathfrak{X}}^{an}}$, in which case it reduces to the evident equality $(A(a)^+)^\perp = A(a)^+$ for $a \in D$.

We can now prove the main result of this section:

7.5. Theorem – *Let X, $\overline{X}$, $\mathfrak{X}$, $\overline{\mathfrak{X}}$, $A^\dagger$ be as in 7.3. Then for any coherent locally free $A^\dagger$-module M, the natural topology of M is dual-of-Frechet, $M \otimes_{A^\dagger} A^{qu}$ is Frechet, and the exact sequence*

$$0 \to M \to M \otimes_{A^\dagger} A^{loc} \to M \otimes_{A^\dagger} A^{qu} \to 0 \tag{7.5.1}$$

is split exact (and consequently strict exact). With respect to the pairing 7.3.5 we have

$$M^\perp = M^\vee \otimes \Omega^1_{A^\dagger} \tag{7.5.2}$$

and the maps

$$\begin{aligned} M &\to (M^\vee \otimes \Omega^{qu})'_s \\ M \otimes \Omega^{qu} &\to (M^\vee)'_s \end{aligned} \tag{7.5.3}$$

induced by 7.3.5 and 7.5.1–2 are topological isomorphisms.

Proof. – First, note that 7.5.3 follows from 7.5.2 and the assertions about 7.5.1. Next, it is clear that 7.5 is true for a direct sum if and only if it is true for the summands, so if we represent M as a summand of a finite free $A^\dagger$-module, then we can reduce first to the case of a free module, and then to the case of $A^\dagger$ itself. Finally, we observe that if 7.5 is true after making a finite extension of scalars K'/K, then it is true by 3.7, so at any point we can pass to a finite extension of the base.

We claim that after passing to a finite extension of K (if necessary), there is a divisor E on $\overline{\mathfrak{X}}^{an}$ supported in $]D[$, whose associated line bundle $\mathcal{L} = \mathcal{L}(E)$ on $\overline{\mathfrak{X}}^{an}$ satisfies

$$\mathrm{H}^0(\overline{\mathfrak{X}}^{an}, \mathcal{L}) = \mathrm{H}^1(\overline{\mathfrak{X}}^{an}, \mathcal{L}) = 0. \tag{7.5.4}$$

In fact, if g is the genus of $\overline{\mathfrak{X}}^{an}$, a generic divisor E of degree $g-1$ satisfies the condition $\mathrm{H}^0(\overline{\mathfrak{X}}^{an}, \mathcal{L}(E)) = 0$ (*cf.* for example [2] Chapter 1, §2), and the equality $\mathrm{H}^1(\overline{\mathfrak{X}}^{an}, \mathcal{L}(E)) = 0$ follows from the Riemann-Roch theorem. Here "generic" means: belonging to a Zariski-open subset of the symmetric product $(\overline{\mathfrak{X}}^{an})^{(g-1)}$ of $g-1$ copies of $\overline{\mathfrak{X}}^{an}$; to define "Zariski-open," we fix, as in the proof of 7.4, an algebraization $\overline{\mathbb{X}}/K$ of $\overline{\mathfrak{X}}^{an}$ and identify the closed points of $(\overline{\mathbb{X}})^{(g-1)}$ with the points of $(\overline{\mathfrak{X}}^{an})^{(g-1)}$. To see that we can arrange to have E supported in $]D[$, it is sufficient to observe that the analytic subspace $(]D[)^{(g-1)} \subset (\overline{\mathfrak{X}}^{an})^{(g-1)}$, being of dimension $g-1$, intersects any Zariski-open subset of $(\overline{\mathfrak{X}}^{an})^{(g-1)}$.

Put, as usual,

$$L = \varinjlim_{V} \Gamma(V, \mathcal{L})$$

where V runs through the set of strict neighborhoods of $]X[$ in $\overline{\mathfrak{X}}^{an}$. By construction, $\mathcal{L}$ and $\mathcal{O}_{\overline{\mathfrak{X}}^{an}}$ are isomorphic on $\overline{\mathfrak{X}}^{an} - E$, So if we choose an isomorphism $\mathcal{L} \simeq \mathcal{O}_{\overline{\mathfrak{X}}^{an}}$ on $\overline{\mathfrak{X}}^{an} - E$, we obtain isomorphisms $L \simeq A^\dagger$ and $L \otimes A^{loc} \simeq A^{loc}$. If $L^\vee \otimes \Omega^1_{A^\dagger} \simeq \Omega^1_{A^\dagger}$, $L^\vee \otimes \Omega^{loc} \simeq \Omega^{loc}$ are the corresponding dual isomorphisms, then by construction they are compatible with the pairing 7.3.5. Thus it suffices to prove 7.5 when $M = L$.

We now choose, for each $a \in D$, a section t_a as in 7.2, so that $]a[$ is the locus of $|t_a| < 1$. For any $r \in \sqrt{|K^\times|}$ such that $0 < r < 1$, we let $[D]_r$ be the union of the disks $|t_a| \le r$, and $]D[_r =]D[- [D]_r$. If V is a affinoid strict neighborhood of $]X[$ in $\overline{\mathfrak{X}}^{an}$, then for some r, $\{V, [D]_r\}$ is an admissible affinoid cover of $\overline{\mathfrak{X}}^{an}$, and the complex

$$\Gamma(V, \mathcal{L}) \oplus \Gamma([D]_r, \mathcal{L}) \to \Gamma(V \cap [D]_r, \mathcal{L})$$

calculates the $\mathrm{H}^i(\overline{\mathfrak{X}}^{an}, \mathcal{L})$. By 7.5.4 these vanish, so that

$$\Gamma(V, \mathcal{L}) \oplus \Gamma([D]_r, \mathcal{L}) \xrightarrow{\sim} \Gamma(V \cap [D]_r, \mathcal{L})$$

This is actually a topological isomorphism: it is continuous, and since the spaces occurring here are all Frechet, the open mapping theorem is applicable, and the map is open. Passing to the inverse limit as $r \to 0$ yields an isomorphism

$$\Gamma(V, \mathcal{L}) \oplus \Gamma(]D[, \mathcal{L}) \xrightarrow{\sim} \Gamma(V \cap]D[, \mathcal{L})$$

which is in fact a topological isomorphism (we can regard direct sum as a direct product, and inverse limits commute with products). Similarly passing to the direct limit over V yields a topological isomorphism

$$L \oplus \Gamma(]D[, \mathcal{L}) \xrightarrow{\sim} L \otimes A^{loc} \tag{7.5.5}$$

(one could also observe that the isomorphism in 7.5.5 is continuous, and show that it is a topological isomorphism by appealing to the open mapping theorem for LF-spaces stated in 3.4). We conclude that 7.5.1 is split exact when $M = L$.

By Serre duality, $\mathcal{L}^\vee \otimes \Omega^1$ satisfies 7.5.4 as well, and we have $\varinjlim_V \Gamma(V, \mathcal{L}^\vee \otimes \Omega^1) \simeq L^\vee \otimes \Omega^1_{A^\dagger}$, so 7.5.5 yields

$$(L^\vee \otimes \Omega^1_{A^\dagger}) \oplus \Gamma(]D[, \mathcal{L}^\vee \otimes \Omega^1) \xrightarrow{\sim} L^\vee \otimes \Omega^{loc}. \tag{7.5.6}$$

From 7.4 we have $L^\vee \otimes \Omega^1 \subseteq L^\perp$, so 7.5.6 yields

$$L^\vee \otimes \Omega^1_{A^\dagger} = L^\perp + \Gamma(]D[, \mathcal{L}^\vee \otimes \Omega^1). \tag{7.5.7}$$

On the other hand, 7.4.3 for $L = M$ reads

$$\Gamma(]D[, \mathcal{L})^{\perp} = \Gamma(]D[, \mathcal{L}^{\vee} \otimes \Omega^1) \tag{7.5.8}$$

and since 7.3.5 is nondegenerate, 7.5.5 and 7.5.8 yield

$$L^{\perp} \cap \Gamma(]D[, \mathcal{L}^{\vee} \otimes \Omega^1) = L^{\perp} \cap \Gamma(]D[, \mathcal{L})^{\perp} = (L \otimes A^{loc})^{\perp} = 0.$$

It follows that the sum in 7.5.7 is direct:

$$L^{\vee} \otimes \Omega^{loc} = L^{\perp} \oplus \Gamma(]D[, \mathcal{L}^{\vee} \otimes \Omega^1)$$

Comparing this with 7.5.6, and recalling that $L^{\vee} \otimes \Omega^1_{A^{\dagger}} \subseteq L^{\perp}$, we see that the latter inclusion must be an equality:

$$L^{\vee} \otimes \Omega^1_{A^{\dagger}} = L^{\perp}$$

and, by symmetry,

$$L = (L^{\vee} \otimes \Omega^1_{A^{\dagger}})^{\perp} \tag{7.5.9}$$

which is 7.5.2 for $M = L$.

It remains to prove the assertion about the topologies of L and $L \otimes A^{qu}$; of course it follows from 7.5.2 and 7.5.3 that it is enough to show that $L \otimes A^{qu}$ is Frechet. By definition $L \otimes A^{qu} \simeq (L \otimes A^{loc})/L$, so 7.5.5 yields a topological isomorphism $L \otimes A^{qu} \simeq \Gamma(]D[, \mathcal{L})$. The latter space is Frechet – it is a countable inverse limit of spaces of the form $\Gamma([D]_r, \mathcal{L})$, and these are Banach. We conclude that $L \otimes A^{qu}$ is Frechet.

□

Remark. – It may seem perverse to prove, using Serre duality and the Riemann-Roch theorem, an assertion which resembles closely the ones used classically to give adelic proofs of Serre duality, Riemann-Roch, etc. But it does not seem particularly easy or enlightening to prove 7.5 in its full generality by the classical procedure, *i.e.* by reducing to the case $\overline{X} = \mathbb{P}^1$. The problem is that it is not in general possible to lift a morphism $\overline{X} \to \mathbb{P}^1$ to a morphism $\overline{\mathfrak{X}} \to \mathbb{P}^1$ over R, where $\overline{\mathfrak{X}}$ has been given in advance (*cf.* the remark after 8.3) Of course, if one doesn't care which lifting of $\overline{X}$ is being used (as is the case in §8–9) then this doesn't much matter.

Various authors [10, 38] have discussed Serre duality in the context of open rigid-analytic spaces, and 7.5 is obviously connected with this circle of ideas. For example, if $\mathcal{M}$ is a locally free sheaf on $\overline{\mathfrak{X}}^{an}$, then one can show that $(M \otimes A^{loc})/\Gamma(]D[, \mathcal{M}) \simeq \mathrm{H}^1_c(]D[, \mathcal{M})$, where the cohomology with compact support is defined in [10]; then 7.4.3 is equivalent to the isomorphism $\mathrm{H}^0(]D[, \mathcal{M})' \simeq \mathrm{H}^1_c(]D[, \mathcal{M}^{\vee} \otimes \Omega^1)$ which is a special case of the duality results proven in [10].

To interpret 7.5.3 in this vein, we suppose that the $A^{\dagger}$-module M is arises from a locally free sheaf $\mathcal{M}_V$ on some strict neighborhood V of $]X[$. If V is a Stein space (in the sense of [10 4.1]) then by [10 4.21] there is a topological isomorphism

$$\Gamma(V, (\mathcal{M}_V)^{\vee} \otimes \Omega^1_V)'_s \simeq \mathrm{H}^1_c(V, \mathcal{M}_V). \tag{7.5.10}$$

Now $]X[$ has a fundamental system of strict Stein neighborhoods, and it can be shown that there is a topological isomorphism

$$M^{qu} \simeq \varprojlim_{]X[\subset V' \subset V} \mathrm{H}^1_c(V', \mathcal{M}_V) \tag{7.5.11}$$

obtained by passing to the inverse limit in 7.5.10 and applying 7.5. On the other hand, one has

$$M^\vee \otimes \Omega^1_{A^\dagger} = \varinjlim_{]X[\subset V' \subset V} \mathrm{H}^0(V', \mathcal{M}^\vee_V \otimes \Omega^1). \tag{7.5.12}$$

If one had an *a priori proof* of 7.5.11, one could presumably deduce 7.5 from the duality of 7.5.11 and 7.5.12; however, there seems to be no simple way to relate the two side of 7.5.11, nor is it at all clear how the duality 7.5.3, which is defined by the global pairing 7.3.5, is related to the corresponding duality isomorphism of [10 4.22], which is constructed using embeddings into affine spaces, and which does not result in an explicit formula for the residue (though there is a residue map constructed (implicitly) in [10] §5).

8. Isocrystals on a curve

8.1. We now, finally, turn to the study of overconvergent isocrystals on a smooth affine curve, and of their cohomology. Since we will not consider any other kind of variety, the definitions we give will be special to the case of a curve. In what follows, "isocrystal" will mean "overconvergent isocrystal."

Let $X \subset Y$ be an open immersion of smooth curves, and set $D = Y - X$. Fix a lifting $\mathfrak{X} \subset \mathfrak{Y}$. An overconvergent isocrystal on X/K overconvergent around D is a locally free $A^\dagger_X$-module M endowed with a connection $\nabla : M \to \Omega^1_{A^\dagger}$ with the following property: if $M = M_V \otimes A^\dagger$ for some strict neighborhood of $]X[$ in $\mathfrak{Y}^{an}$, and if the connection extends to V (which it must, for some V), then the connection induces an isomorphism $p_1^* M \simeq p_2^* M$ on some strict neighborhood W of the tube $]X[_{\mathfrak{Y}^{an}\times\mathfrak{Y}^{an}}$ of the image of the diagonal $X \hookrightarrow X \times X$, where $p_i :]Y[_{\mathfrak{Y}^{an}\times\mathfrak{Y}^{an}} \to]Y[_{\mathfrak{Y}^{an}}$ are the projections; the isomorphism must satisfy the usual kind of cocycle condition. The meaning of this condition is twofold. First, for any $x \in X$, the connection must, by means of its associated Taylor series, induce a trivialization of $M|]x[$. Second, for any $x \in D$, the connection must induce a trivialization of the restriction of M to any disk in $V\cap]x[$ of sufficiently small radius, but with the condition that the radius of the disk must approach 1 as the disk itself approaches the edge of $]x[$. In [8] Berthelot shows that the category of isocrystals on X/K overconvergent around D is independent of the choice of liftings of X, Y, is functorial in X, Y, and is of local nature on Y. When $Y = \overline{X}$ is a smooth complete curve containing X, then the corresponding category is simply called the category of (overconvergent) isocrystals on X/K; as before it is functorial in X/K and is of local nature on X.

For any $A^\dagger_X$-module M with a connection, we denote by $\mathrm{H}^{\cdot}_{\mathrm{DR}}(M)$ the de Rham cohomology of M:

$$\mathrm{H}^{\cdot}_{\mathrm{DR}}(M) = \mathrm{H}^{\cdot}([M \to M \otimes \Omega^1_{A^\dagger}])$$

and similarly for any complex of $A^\dagger$-modules with connection. In particular, if M is an isocrystal on X/K represented by a finite locally free $A_X^\dagger$-module M with connection $\nabla : M \to M \otimes \Omega^1_{A^\dagger}$, then the de Rham cohomology of (M, ∇) is exactly the *rigid cohomology* of the isocrystal M:

$$\mathrm{H}^i(X, M) = \mathrm{H}^i_{\mathrm{DR}}(M) \tag{8.1.1}$$

(in general one needs sheaf hypercohomology, but since the strict neighborhoods can be taken to be quasi-Stein, we do not need to do this). To define the cohomology with compact supports, we pick a smooth compactification $X \subset \overline{X}$ and a lifting $\mathfrak{X} \subset \overline{\mathfrak{X}}$, and suppose that M is an overconvergent isocrystal on X/K, represented as a locally free sheaf with connection (M_V, ∇) on some strict neighborhood V of $]X[$ in $\overline{\mathfrak{X}}^{an}$. If $D = \overline{X} - X$ and $i_V : V \cap]D[\to V$, $j_V : V \to \overline{\mathfrak{X}}^{an}$ are the natural inclusions, then $\mathrm{H}^i_c(X, M)$ is defined by

$$\begin{aligned} \mathrm{H}^i_c(X, M) &= \mathrm{H}^i(\overline{\mathfrak{X}}^{an}, \varinjlim_V j_{V*}([M_V \to i_{V*} i_V^* M_V] \otimes^L \Omega^{\cdot}_V V)) \\ &= \varinjlim_V \mathrm{H}^i(V, [M_V \to i_{V*} i_V^* M_V] \otimes^L \Omega^{\cdot}_V V) \end{aligned} \tag{8.1.2}$$

where we now let V run through a cofinal system of strict neighborhoods of $]X[$ (since $\overline{\mathfrak{X}}^{an}$ is quasicompact, the direct limit commutes with cohomology). We can choose a set of affinoid strict neighoborhoods (quasi-Stein would suffice), in which case 7.1.3 and 7.3.4 show that

$$\mathrm{H}^i_c(X, M) = \mathrm{H}^i_{\mathrm{DR}}([M \to M \otimes_{A^\dagger} A_X^{loc}]) = \mathrm{H}^i_{\mathrm{DR}}(M \otimes A_X^{qu}[-1]) \tag{8.1.3}$$

or in other words

$$\begin{aligned} &\mathrm{H}^1_c(X, M) = \mathrm{Ker}(\nabla : M \otimes A^{qu} \to M \otimes \Omega^{qu}) \\ &\mathrm{H}^2_c(X, M) = \mathrm{Coker}(\nabla : M \otimes A^{qu} \to M \otimes \Omega^{qu}) \\ &\mathrm{H}^i_c(X, M) = 0 \qquad i \neq 1, 2. \end{aligned}$$

From 8.1.1 and 8.1.3, we see that the exact sequence of modules with connection

$$0 \to M \to M \otimes A^{loc} \to M \otimes A^{qu} \to 0$$

yields, upon taking de Rham cohomology, the exact sequence

$$\begin{aligned} &0 \to \mathrm{H}^0(X, M) \to \mathrm{H}^0_{\mathrm{DR}}(M \otimes A_X^{loc}) \to \mathrm{H}^1_c(X, M) \to \\ &\qquad \mathrm{H}^1(X, M) \to \mathrm{H}^1_{\mathrm{DR}}(M \otimes A_X^{loc}) \to \mathrm{H}^2_c(X, M) \to 0 \end{aligned} \tag{8.1.4}$$

If we think of 8.1.4 as the p-adic analogue of the long exact sequence relating the cohomology with and without supports of a lisse ℓ-adic sheaf, then we see that the $\mathrm{H}^{\cdot}_{\mathrm{DR}}(M \otimes A_X^{loc})$ are to be thought of as a sort of local cohomology (as the $\mathrm{H}^{\cdot}(D, i^* R j_* M)$, to be more precise). Pursuing the analogy leads us to define the "middle" or parabolic cohomology by

$$\mathrm{H}^1_p(X, M) = \mathrm{Im}(\mathrm{H}^1_c(X, M) \xrightarrow{\partial} \mathrm{H}^1(X, M)). \tag{8.1.5}$$

The coboundary map ∂ has the explicit description

$$\begin{aligned} \mathrm{H}^1_c(X, M) = \frac{\{m \in M \otimes A^{loc} \mid \nabla(m) \in M \otimes \Omega^1_{A^\dagger}\}}{M} &\to \mathrm{H}^1(X, M) = \frac{M \otimes \Omega^1_{A^\dagger}}{\nabla(M)} \\ m &\mapsto \nabla(m) \end{aligned}$$

which shows that

$$\mathrm{H}^1_p(X,M) \simeq \frac{(M\otimes\Omega^1_{A^\dagger})\cap\nabla(M\otimes A^{loc}_X)}{\nabla(M)} \tag{8.1.6}$$

This of course is just the classical definition of parabolic cohomology (1–cycles trivial at infinity, modulo global 1–chains).

Denote by K the "constant" isocrystal $(A^\dagger_X, d)$. By the definition given above, we have

$$\mathrm{H}^2_c(X,K) = \Omega^{loc}/(\Omega^1_{A^\dagger} + dA^{loc})$$

and since by 7.4.1 we have $\langle 1,\omega\rangle = 0$ for $\omega\in\Omega^1_{A^\dagger}$, the map

$$\begin{aligned}\Omega^{loc} &\to K\\ \omega &\mapsto \langle 1,\omega\rangle\end{aligned}$$

passes to the quotient by $\Omega^1_{A^\dagger} + dA^{loc}$. We obtain thus the *trace map*

$$\mathrm{H}^2_c(X,K)\to K \tag{8.1.7}$$

and we will see later that it is an isomorphism (this can also be deduced from the excision exact sequence (cf. [5 §3.1]) and Berthelot's comparison theorem [9 1.9] for rigid and crystalline cohomology). For any isocrystal M on X/K, we can compose 8.1.7 with the cup product

$$\mathrm{H}^i_c(X,M)\times\mathrm{H}^{2-i}(X,M^\vee)\to\mathrm{H}^2(X,K)$$

to obtain a pairing

$$\mathrm{H}^i_c(X,M)\times\mathrm{H}^{2-i}(X,M^\vee)\to K. \tag{8.1.8}$$

Since the dual of $\mathrm{H}^1_c(X,M)\to\mathrm{H}^1(X,M)$ is $\mathrm{H}^1_c(X,M^\vee)\to\mathrm{H}^1(X,M^\vee)$, 8.1.8 induces a pairing

$$\mathrm{H}^1_p(X,M)\times\mathrm{H}^1_p(X,M^\vee)\to K. \tag{8.1.9}$$

8.2. Suppose $j: U\hookrightarrow X$ is the inclusion of an open subset, and M is an isocrystal on U. For later purposes we will need some variants of the above construction, which could be thought of as providing $\mathrm{H}^1(X, Rj_*M)$ and $\mathrm{H}^1(X, j_*M)$. With $D=\overline{X}-X$ as before, and $E = X-U$, we set

$$A(E)=\bigoplus_{x\in E}A(x)\subset A^{loc}_U=\bigoplus_{x\in D\cup E}A(x)\qquad A_{X,E}=A^{loc}_U/(A(E)+A^\dagger_U).$$

Since $A(E)\cap A^\dagger_U = 0$ in A^{loc}_U, there is an exact sequence

$$0\to A(E)\to A^{qu}_U\to A_{X,E}\to 0.$$

If M is an isocrystal on U, realized as a locally free $A^\dagger_U$-module with connection, we can tensor the above sequence with M over $A^\dagger_U$

$$0\to A(E)\otimes M\to A^{loc}_U\otimes M\to A_{X,E}\otimes M\to 0. \tag{8.2.1}$$

and pursuing the same analogy as in the paragraph after 8.1.4, we define

$$\mathrm{H}^i_c(X, Rj_*M) = \mathrm{H}^{i-1}_{\mathrm{DR}}(A_{X,E} \otimes M) \tag{8.2.2}$$

Note that we don't give an independent definition of Rj_*M! The exact sequence of De Rham cohomology is

$$\begin{aligned} 0 \to \mathrm{H}^0_{\mathrm{DR}}(A(E) \otimes M) \to \mathrm{H}^1_c(U, M) \to \mathrm{H}^1_c(X, Rj_*M) \to \\ \mathrm{H}^1_{\mathrm{DR}}(A(E) \otimes M) \to \mathrm{H}^2_c(U, M) \to \mathrm{H}^2_c(X, Rj_*M) \to 0 \end{aligned} \tag{8.2.3}$$

from which one sees that the notation is justified by the ℓ-adic analogy. Define $\mathrm{H}^1_c(X, j_*M)$ to be the image of $\mathrm{H}^1_c(U, M) \to \mathrm{H}^1_c(X, Rj_*M)$, so that 8.2.3 yields an exact sequence

$$0 \to \mathrm{H}^0_{\mathrm{DR}}(A(E) \otimes M) \to \mathrm{H}^1_c(U, M) \to \mathrm{H}^1_c(X, j_*M) \to 0 \tag{8.2.4}$$

and set, finally,

$$\mathrm{H}^2_c(X, j_*M) = \mathrm{H}^2_c(U, M). \tag{8.2.5}$$

These definitions can also be justified by the ℓ-adic analogy.

Suppose now that $f : Y \to X$ is finite étale, and choose compactifications $X \hookrightarrow \overline{X}$, $Y \hookrightarrow \overline{Y}$ of X, Y; then $f : Y \to X$ extends (uniquely) to a finite, generically étale morphism $\overline{f} : \overline{Y} \to \overline{X}$. Choose, finally, a formally smooth lifting $\overline{\mathfrak{X}}$ of $\overline{X}$. Our construction of the direct image of an isocrystal under f will need:

8.3. Lemma – *With the above notation, there is a lifting $\overline{\mathfrak{Y}} \to \overline{\mathfrak{X}}$ of $\overline{Y} \to \overline{X}$, with $\overline{\mathfrak{Y}}$ proper and flat.*

Proof. – By standard arguments, it is enough to show that there is no obstruction to infinitesimal liftings. Suppose, then, that X_0, Y_0 are two curves, smooth and proper over a ring R_0 and $f_0 : Y_0 \to X_0$ is a finite, flat, generically étale R_0-morphism. Suppose also that $R_1 \to R_0$ is a surjective ring homomorphism with kernel I such that $I^2 = 0$, and X_1 is a lifting of X_0 to R_1, *i.e.* a flat R_1-scheme such that $X_0 = X_1 \otimes_{R_1} R_0$. The obstruction to finding a lifting $f_1 : Y_1 \to X_1$ of f_0 is a class in $\mathrm{Ext}^2_{Y_0}(L_{Y_0/X_0}, \mathcal{O}_{Y_0})$, where L_{Y_0/X_0} is the relative cotangent complex of f_0. Since X_0, Y_0 are smooth, we have $L_{Y_0/X_0} \simeq [f_0^*\Omega^1_{X_0/R_0} \to \Omega^1_{Y_0/R_0}]$, with the two terms in degrees -1 and 0. If we put $L^\vee_{Y_0/X_0} = \mathrm{RHom}(L_{Y_0/X_0}, \mathcal{O}_{Y_0})$, then $L^\vee_{Y_0/X_0} \simeq [(\Omega^1_{Y_0/R_0})^\vee \to (f_0^*\Omega^1_{X_0/R_0})^\vee]$, since $f_0^*\Omega^1_{X_0/R_0}$, $\Omega^1_{Y_0/R_0}$ are locally free. Since f_0 is generically étale, $f_0^*\Omega^1_{X_0/R_0} \to \Omega^1_{Y_0/R_0}$ is an isomorphism on an open subset of Y_0, so the support of $\mathcal{H}^i(L^\vee_{Y_0/X_0})$ is zero-dimensional for $i = 0, 1$ and $\mathcal{H}^i(L^\vee_{Y_0/X_0}) = 0$ for $i > 1$. Thus

$$\mathrm{Ext}^2_{Y_0}(L_{Y_0/X_0}, \mathcal{O}_{Y_0}) \simeq \mathrm{H}^2(Y_0, L^\vee_{Y_0/X_0}) = 0$$

It follows that there are no obstructions to lifting f_0 to a morphism $f_1 : Y_1 \to X_1$, with Y_1 flat over R_1.

□

Remark. – If a lifting $\overline{\mathfrak{Y}}$ of $\overline{Y}$ is specified in advance, then one cannot in general hope to lift $\overline{f} : \overline{Y} \to \overline{X}$ to a morphism $\overline{\mathfrak{Y}} \to \overline{\mathfrak{X}}$, since in this case (with the notation of the above proof) the obstruction lies in $\mathrm{Ext}^1_{Y_0}(L_{Y_0/X_0}, \mathcal{O}_{Y_0})$, which need not vanish in this situation.

Suppose now that M is an isocrystal on Y, represented by a locally free sheaf M_V on some strict neighborhood V of $]Y[$ in $\overline{\mathfrak{Y}}^{an}$ endowed with a connection ∇ overconvergent around $\overline{Y} - Y$. By the maximum principle, there exists a strict neighborhood V_0 of $]X[$ in $\overline{\mathfrak{X}}^{an}$ such that $\overline{f}^{-1}(V_0) \subseteq V$. Since $\overline{Y} \to \overline{X}$ is finite and flat, the same is true for $\overline{\mathfrak{Y}} \to \overline{\mathfrak{X}}$, and thus for $\overline{f}^{-1}(V_0) \to V_0$. It follows that $f_* M_V$ is a locally free sheaf on V_0. The fact that the natural (Gauss-Manin) connection on $f_* M_V$ is overconvergent follows from Lemma 3.5.2 of [14][1].

For later use we will need an explicit description of the local behavior of the connection. First, for $a \in \overline{X} - X$ we have

$$(f_* M_V) \otimes A(a) = \bigoplus_{b \in \overline{Y} - Y,\ b \to a} M_V \otimes A(b) \tag{8.3.1}$$

where as in 7.2 we embed $A_X^\dagger \hookrightarrow A(a)$, $A_Y^\dagger \hookrightarrow A(b)$. Choose local coordinates t_a, t_b as in the last paragraph of 7.2. In $A(b)$ we have $d/dt_a = (dt_a/dt_b) d/dt_a$, and since t_a is a power series in t_b with bounded (in fact with integral) coefficients, dt_a/dt_b is a unit in $A(b)$. It follows that the action of $\nabla(d/dt_a)$ on $(f_* M_V \otimes A(a))$ is given by

$$\nabla(d/dt_a) | M_V \otimes A(b) = (dt_a/dt_b)^{-1} \nabla(d/dt_b). \tag{8.3.2}$$

If M is an overconvergent isocrystal on X/K, then one defines $f^* M$ in the obvious way: we represent M as a locally free sheaf with overconvergent connection on some strict neighborhood V of $]X[$ in $\overline{\mathfrak{X}}^{an}$, and the pullback by $\overline{f}$ represents $f^* M$.

Remark. – Since overconvergence is a purely local property, the use of 8.3 to define direct and inverse images is overkill, and is an artifact of our somewhat naive definition of an overconvergent isocrystal. In the situation of 8.3, the map $\overline{Y} \to \overline{X}$ is a local complete intersection, and so has local liftings. These can be used to define $f_* M$ locally, and the resulting isocrystals patch together.

9. Strict isocrystals and global duality

9.1. Suppose now that X/k is a smooth curve, and M is an (overconvergent) isocrystal on X/K. If we choose, as always, a smooth completion $X \subseteq \overline{X}$ and a lifting $\mathfrak{X} \subseteq \overline{\mathfrak{X}}$, then M can be identified with a locally free sheaf M with a connection ∇ overconvergent around $D = \overline{X} - X$. Let $A_X^\dagger$ be the dagger-algebra associated to the lifting $\overline{\mathfrak{X}}$, and for any $a \in D$ let $A(a)$ be the local algebra at a, and Ω_a^1 the corresponding module of differentials. We shall say that M is *strict at* a if the induced connection $M \otimes A(a) \to M \otimes \Omega_a^1$ is strict in the sense of §6; *i.e.* if it is strict as a map of topological K-vector spaces. From the results in 7.2, we see that this notion is independent of the choice of $X \hookrightarrow \overline{X}$ and of the lifting $\mathfrak{X} \hookrightarrow \overline{\mathfrak{X}}$, and is in fact local around a for the étale topology on $\overline{X}$. We will say that an isocrystal M on X/K is *strict* if for some smooth compactification $X \hookrightarrow \overline{X}$ it is strict around each point of $\overline{X} - X$; obviously this is independent of the choice of

[1] We should point out a misprint in [14]: the arrows in the displayed diagram of [14] 3.5.2 should go in the same direction as the corresponding arrows of [14] 3.5.1.

$\overline{X}$. An equivalent condition is that for any choice of $X \hookrightarrow \overline{X}$ and $\mathfrak{X} \hookrightarrow \overline{\mathfrak{X}}$, the induced map $M \otimes A^{loc} \to M \otimes \Omega^{loc}$ is strict.

This condition might seem like a rather difficult one to verify, and in general it is. There nonetheless seems to be plenty of examples. Suppose, for example, that k is finite and M satisfies the Robba condition with non-Liouville exponents (*cf.* [12 §4]) in each disk $]a[$, $a \in \overline{X} - X$. Then Christol and Mebkhout show [12 §6.2] that the restriction of M to each $]a[$ is isomorphic to one of the form 6.7.1, where the eigenvalues of C are non-Liouville, and it follows from 6.10 that M is strict. If the isocrystal arises from an algebraic differential equation with regular singularities, and no two singularities have the same reduction (the case studied by Adolphson [1]), then the condition is that the exponents in the usual sense are not p-adic Liouville. Another class of examples, distinct from those just described, is that of the quasi-unipotent isocrystals, which we discuss in §10.

9.2. Proposition – *Let X/k be a smooth affine curve and M an isocrystal on X/K. Then M is strict if and only if $M^\vee$ is strict.*

Proof. – This follows immediately from 6.3 and the definitions.

□

9.3. Proposition – *Let $\pi : Y \to X$ be an étale cover of smooth curves. If M is a strict isocrystal on Y/K, then $\pi_* M$ is a strict isocrystal on X/K.*

Proof. – In the notation of 8.5, it is enough to show that $\nabla(d/dt_a)$ is a strict endomorphism of $(f_* M_V) \otimes A(a)$, while by hypothesis $\nabla(d/dt_b)$ is a strict endomorphism of $M_V \otimes A(b)$ for each $b \to a$. The assertion in question follows from 8.5.1 and 8.5.2, since multiplication by the unit $(dt_a/dt_b)^{-1}$ is a topological isomorphism.

□

9.4. Proposition – *Let $\pi : Y \to X$ be a finite étale cover of smooth curves, and M an isocrystal on X/K. If $\pi^* M$ is strict, then so is M. If $\pi : Y \to X$ can be compactified to a finite étale map $\overline{Y} \to \overline{X}$ with $\overline{Y}$, $\overline{X}$ proper and smooth over k, and if M is strict, then so is $\pi^* M$.*

Proof. – The last assertion simply restates the fact the strictness of an isocrystal at a point is étale local around that point. As to the first part of 9.4, note that there is an exact sequence of isocrystals

$$0 \to M \to \pi_* \pi^* M \to N \to 0$$

on X/K. If $\pi^* M$ is strict, then so is $\pi_* \pi^* M$ by 9.3, and then the strictness of M follows from 6.4.

□

We can now prove the main result of this paper:

9.5. Theorem – *Let X/k be a smooth affine curve, and M a strict isocrystal on X/K. Then the three K-vector spaces $\mathrm{H}^i(X, M)$, $\mathrm{H}^i_c(X, M)$, and $\mathrm{H}^1_p(X, M)$ are finite-dimensional, and the pairings 8.1.8–9*

$$\begin{aligned} \mathrm{H}^i_c(X, M) \times \mathrm{H}^{2-i}(X, M^\vee) &\to K \\ \mathrm{H}^1_p(X, M) \times \mathrm{H}^1_p(X, M^\vee) &\to K \end{aligned} \tag{9.5.1}$$

are perfect pairings of finite-dimensional vector spaces.

Proof. – The map $\nabla : M \to M \otimes \Omega^1_{A^\dagger}$ induces maps $\nabla_{loc} : M \otimes A^{loc} \to M \otimes \Omega^{loc}$, $\nabla_{qu} : M \otimes A^{qu} \to M \otimes \Omega^{qu}$, which sit in a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & M \otimes A^{loc} & \longrightarrow & M \otimes A^{qu} & \longrightarrow & 0 \\
 & & \downarrow \nabla & & \downarrow \nabla_{loc} & & \downarrow \nabla_{qu} & & \\
0 & \longrightarrow & M \otimes \Omega^1_{A^\dagger} & \longrightarrow & M \otimes \Omega^{loc} & \longrightarrow & M \otimes \Omega^{qu} & \longrightarrow & 0.
\end{array} \tag{9.5.2}
$$

The proof is based on a study of this diagram, and of a portion of the six-term exact sequence

$$
\begin{aligned}
0 \to \mathrm{H}^0(M) \to \mathrm{H}^0_{\mathrm{DR}}(M \otimes A^{loc}) \to \mathrm{H}^1_c(M) \to \\
\mathrm{H}^1(M) \to \mathrm{H}^1_{\mathrm{DR}}(M \otimes A^{loc}) \to \mathrm{H}^2_c(M) \to 0
\end{aligned} \tag{9.5.3}
$$

arising from it. We endow the spaces in 9.5.3 with the topologies they have as subspaces (resp. quotients) of the spaces in the first (resp. second) row of 9.5.2. By 3.13, the maps in 9.5.3 are continuous, and $\mathrm{H}^1_{\mathrm{DR}}(M \otimes A^{loc}) \to \mathrm{H}^2_c(M)$ is actually strict. Note, finally, that by 6.7.1, the pairing 8.4.2 coincides with the pairing induced by the global pairing 6.2.6.

By 9.2, $M^\vee$ is strict. Thus if at any point we have proven a general fact about M, then it holds for $M^\vee$ too. The remainder of the proof is in a sequence of simple steps:

Step 1. ∇, ∇_{loc}, and ∇_{qu} are strict, and $\mathrm{H}^2_c(M)$ is finite-dimensional and separated: for ∇_{loc}, this is true by definition. By 6.3, $\mathrm{H}^1_{\mathrm{DR}}(M \otimes \Omega^{loc})$ is finite-dimensional and separated; then since $\mathrm{H}^1_{\mathrm{DR}}(M \otimes A^{loc}) \to \mathrm{H}^2_c(M)$ is strict and surjective, $\mathrm{H}^2_c(M)$ is finite-dimensional and separated. In particular, the image of ∇_{qu} is closed, and since $M \otimes A^{qu}$, $M \otimes \Omega^{qu}$ are Frechet spaces, ∇_{qu} is strict. Since they are Montel spaces as well, the strong dual $\nabla^\vee : M^\vee \to M^\vee \otimes \Omega^1_{A^\dagger}$ of ∇_{qu} is strict; then the same is true for ∇.

Step 2. $\mathrm{H}^1_c(M)$ is Frechet-Montel: it is certainly Frechet, being a closed subspace of a Frechet space. Being Frechet, it is barreled, so it is enough to show that any convex closed bounded subset of $\mathrm{H}^1_c(M)$ is linearly compact, which is true because $M \otimes A^{qu}$ is Montel.

Step 3. The duality pairing 8.4.2 induces a topological isomorphism $\mathrm{H}^i_c(M) \simeq \mathrm{H}^{2-i}(M^\vee)'_s$: since the sequence

$$
0 \to \mathrm{H}^1_c(M) \to M \otimes A^{qu} \to M \otimes \Omega^{qu} \to \mathrm{H}^2_c(M) \to 0 \tag{9.5.4}
$$

is strict exact, and since all the spaces in it are Frechet-Montel (this has been shown for the first three, and by step 1, $\mathrm{H}^2_c(M)$ is even finite-dimensional), the strong dual of 9.5.4 is strict exact by 3.12. By 9.2

$$
0 \to \mathrm{H}^0(M^\vee) \to M^\vee \to M^\vee \otimes \Omega^1_{A^\dagger} \to \mathrm{H}^1_c(M^\vee) \to 0 \tag{9.5.5}
$$

is also strict exact. Finally by 6.8.2, $M^\vee \to M^\vee \otimes \Omega^1_{A^\dagger}$ is the strong dual of $M \otimes A^{qu} \to M \otimes \Omega^{qu}$; thus 9.5.5 is the strong dual of 9.4.4, and the assertion follows immediately.

Step 4. $\mathrm{H}^1(M)$ is dual-of-Frechet: this follows from steps 2 and 3.

Step 5. $\mathrm{H}^1_p(M)$ is finite-dimensional: we see from 9.5.3 and step 2 that $\mathrm{H}^1_p(M)$ inherits a Frechet topology from $\mathrm{H}^1_c(M)$. On the other hand, $\mathrm{H}^1_{\mathrm{DR}}(M \otimes A^{loc})$ is finite-dimension and separated, so 9.5.3 shows that we have a direct sum decomposition

$$\mathrm{H}^1(M) \simeq \mathrm{H}^1_p(M) \oplus F$$

where F is finite-dimensional and separated. Thus $\mathrm{H}^1_p(M)$ inherits a dual-of-Frechet topology from $\mathrm{H}^1(M)$. However, since the cokernel of $\mathrm{H}^1_c(M) \to \mathrm{H}^1(M)$ is finite-dimensional and separated, $\mathrm{H}^1_c(M) \to \mathrm{H}^1(M)$ is strict by 3.6, and the finite-dimensionality of $\mathrm{H}^1_p(M)$ follows from 2.8.

Step 6. $\mathrm{H}^i(M)$, $\mathrm{H}^i_c(M)$ are all finite-dimensional: for $i = 1$, this follows from 9.5.3 and step 5; for $\mathrm{H}^2_c(M)$ this is step 1; for $\mathrm{H}^0(M)$ this is clear.

Step 7. the pairings 9.5.1 are perfect: this follows from steps 3 and 6.

□

Remark. – It would be interesting to know if the strictness of M is necessary for the conclusion of 9.5.

The isocrystal $K = (A^\dagger, d)$ is strict, and one checks easily that $\mathrm{H}^0(X, K) = K$, so we see that the trace map $\mathrm{H}^2_c(X, K) \to K$ is an isomorphism. If we assume that X has a k-rational point, then this result can be extended as follows. Pick $x_0 \in X(k)$; then for any (overconvergent) isocrystal on X/K, the $\otimes$-category $[M]$ generated by M is a neutral Tannakian category, and we denote by $DGal(M, x_0)$ the group scheme of automorphisms of the fiber functor corresponding to x_0. The group $DGal(M, x_0)$ plays the role here of a "geometric monodromy group"; it is an algebraic group with a canonical representation on the fiber M_{x_0}, and the theory of Tannakian categories gives an equivalence of $[M]$ with the category of representations of $DGal(M, x_0)$ on finite-dimensional K-vector spaces (for a summary, see [15] and the references given there). Under this equivalence, the subspace $(M_{x_0})^{DGal(M,x_0)} \subseteq M_{x_0}$ corresponds to the largest constant sub-isocrystal of M, and is thus isomorphic to $\mathrm{H}^0(X, M)$. Suppose now that M is strict; then by 9.5, $\mathrm{H}^2_c(X, M)$ is dual to the invariants of $DGal(X, x_0)$ acting on $(M^\vee)_{x_0}$, and is thus canonically isomorphic to the coinvariants of $DGal(X, x_0)$ on M_{x_0}:

9.6. Corollary – *Suppose $x_0 \in X(k)$. For any strict isocrystal M on X/K, we have*

$$\mathrm{H}^2_c(M) \simeq (M_{x_0})_{DGal(M,x_0)} \tag{9.6.1}$$

functorially in M.

Remark. – When k is finite, say $|k| = q$, the q^{th}-power Frobenius F acts linearly on the $\mathrm{H}^i(X, K)$ and the $\mathrm{H}^i_c(X, K)$. Using the excision exact sequence together with Berthelot's comparison theorem [5 Prop. 2 and 9 Prop. 1.9] for the rigid and crystalline cohomology of a proper smooth scheme, we find that

$$\mathrm{H}^2_c(X, K) \simeq K(-1) \tag{9.6.2}$$

as K-spaces with a Frobenius action; as usual, the "-1" denotes a Tate twist. Suppose now that (M, Φ) is strict F-isocrystal on X/K; *i.e.* a strict isocrystal endowed with a semi-linear isomorphism $\Phi : F^*M \xrightarrow{\sim} M$. Since the duality pairing 9.5.1 is natural in its arguments, we get

$$\mathrm{H}^2_c(M) \simeq (M_{x_0})_{DGal(M,x_0)}(-1). \tag{9.6.4}$$

10. Quasi-unipotent isocrystals

10.1. Let X/k be a smooth affine curve, and M an isocrystal on X/K. Pick a smooth compactification $X \hookrightarrow \overline{X}$, a lifting $\mathfrak{X} \hookrightarrow \overline{\mathfrak{X}}$ of it, and a locally free sheaf with connection (M, ∇) representing M. For $x \in \overline{X} - X$, we will say that M is *unipotent at* x if the local connection $M \otimes A(x)$ is unipotent in the sense of §5; *i.e.* if it is a successive extension of trivial modules with connection $(A(x), d)$. We will say that M is unipotent if it is overconvergent, and if for some choice of $X \hookrightarrow \overline{X}$ it is *unipotent* at every point of $\overline{X} - X$; the usual arguments show that if this is the case for one smooth compactification, then it is true for any other. Finally, we say that M is *quasi-unipotent* if there is a finite étale cover $\pi : Y \to X$ such that $\pi^* M$ is unipotent.

In this section we shall show that many of Deligne's results on ℓ-adic sheaves in [17] hold, with a suitable formulation, for quasi-unipotent F-isocrystals, and the reader may therefore wish for some examples. Examples will be given in a moment, but it should be pointed out that it seems reasonable that *any* overconvergent F-isocrystal on a smooth curve is quasi-unipotent. In fact, N. Tsuzuki has recently shown [36, 37] that this is indeed the case for any overconvergent *unit-root* F-isocrystal. In the general case, the assertion amounts to an analogue of Grothendieck's local monodromy theorem for F-isocrystals, so it seems reasonable to suspect that at least any F-isocrystal "of geometric origin" is quasi-unipotent.

In a number of situations studied classically, the isocrystal arises by "analytification" of an algebraic differential equation on a smooth lifting of the curve, and one can hope to establish quasi-unipotence by an explicit calculation. If the equation is regular singular, for example, then the isocrystal is quasi-unipotent if all the exponents are rational, as one sees easily using Christol's transfer theorem [11]. In the irregular singular case one must analyze the Turritin normal form of the equation at the singular points; one example, a certain generalized hypergeometric equation, is worked out in [16]. In all these examples, the isocrystals are "of geometric origin."

Suppose that X/k is a smooth curve embedded in a projective smooth curve $\overline{X}/k$, and $\overline{f} : \overline{Y} \to \overline{X}$ is proper. Suppose further that $\overline{X}$, $\overline{Y}$ can be given logarithmic structures such that $\overline{X}$, $\overline{Y}$, and $\overline{f}$ are all log-smooth. Then in [20] §2e it is shown that the relative rigid cohomology of a convergent isocrystal on $\overline{Y}$ is represented by logarithmic isocrystals on X/K, at least when $\overline{f}$ can be lifted to R étale locally on $\overline{X}$. In particular, if $f : Y \to X$ is the restriction of $\overline{f}$ to X, then the relative rigid cohomology of f should be represented by logarithmic (*i.e.* unipotent, in our sense) isocrystals on X/K. It would be interesting to know whether this kind of result could be extended to a family $\overline{f} : \overline{Y} \to \overline{X}$ which is merely "potentially logarithmic" (*i.e.* semi-stable).

10.2. Proposition – *Let X/k be a smooth affine curve. A quasi-unipotent isocrystal on X/K is strict, and any subquotient of a quasi-unipotent isocrystal is quasi-unipotent. The category of quasi-unipotent isocrystals on X/K is an abelian subcategory of the category of isocrystals on X/K, and is stable under tensor products and internal* Hom.

Proof. – The second assertion follows immediately from the definitions. As to the first, we can use 9.3 to reduce to the case of a unipotent isocrystal, in which case the assertion follows from 6.6.

□

10.3. Proposition – *Let $\pi : Y \to X$ be a finite étale map. If M is a quasi-unipotent isocrystal on Y/K, then $\pi_* M$ is quasi-unipotent on X/K. If M is an isocrystal on X/K, then M is quasi-unipotent if and only if $\pi^* M$ is quasi-unipotent.*

Proof. – This follows immediately from the definititions, and the descriptions in 8.5 of the direct and inverse images by a finite étale map.

□

10.4. From now on we suppose that k is a finite field with cardinality q, and that X has a k-rational point x_0. We denote by $F : X \to X$ the q^{th}-power Frobenius morphism on X. Suppose that (M, Φ) is an overconvergent F-isocrystal on X/K, *i.e.* an overconvergent isocrystal on X/K with a Frobenius structure $\Phi : F^* M \xrightarrow{\sim} M$. In [15] it was shown that Grothendieck's global monodromy theorem holds for the group $DGal(M, x_0)$ alluded to in the last section: the radical of $DGal(M, x_0)$ is unipotent. We also showed in [15 §5] how to construct an extension

$$0 \longrightarrow DGal(M, x_0) \longrightarrow W^M_{x_0} \longrightarrow W(\overline{k}/k) \longrightarrow 0 \tag{10.4.1}$$

of affine K-groups playing the role of a "Weil group" for M (more precisely, playing the role of the extension 1.3.7.1 of [17]). In particular, for any closed point x of X, there is a canonical conjugacy class $\mathrm{Frob}_x \in W^M_{x_0}(\overline{K})$ where $\overline{K}$ is an algebraic closure of K. If M corresponds to a representation $\rho : DGal(M) \to GL(V)$ (where $V = M_{x_0}$), then ρ extends canonically to a representation of $W^M_{x_0}$, and for any closed point $x \in |X|$, the pair $(M_{x_0}, \rho(\mathrm{Frob}_x))$ is isomorphic to (M_x, Φ_x). With the monodromy theorem and the Weil group formalism, we can use Deligne's procedure [17 §1.3] to construct a theory of determinantal weights associated to a choice of isomorphism $\iota : \overline{K} \simeq \mathbb{C}$, where $\overline{K}$ is an algebraic closure of K (*cf.* [15 5.6, 5.7]).

If ρ is the representation of $W^M_{x_0}$ corresponding to the F-isocrystal (M, Φ), then the L-function of (M, Φ) can be written

$$L(X, M, T) = \prod_{x \in |X|} \det(1 - \rho(\mathrm{Frob}_x) T^{\deg x})^{-1}. \tag{10.4.2}$$

On the other hand it has been long known that $L(M, T)$ has a cohomological formula

$$L(X, M, T) = \prod_i \det(1 - TF^* | \mathrm{H}^i_c(X, M))^{(-1)^{i+1}} \tag{10.4.3}$$

(*cf.* [18, 31, 33 and 27]). Suppose now that M is quasi-unipotent; then by 10.2, the finiteness theorem 9.5 is applicable to the objects of $[M]$. In particular, if (M, Φ) is a quasi-unipotent F-isocrystal (*i.e.* a quasi-unipotent isocrystal with a Frobenius structure), then the L-function of any tensor power of M is rational, and its denominator is controlled by the formula 9.7.4.

We are now in a position to retrace the argument of [17 §1.4–5]; we will leave this to the reader, and merely state the results. Let X/k be a smooth curve over a finite field, and (M, Φ) an overconvergent isocrystal on X/K. If $\iota : \overline{K} \simeq \mathbb{C}$ is an isomorphism, then we say that (M, Φ) is *pointwise ι-pure* of weight w if for every closed point $x \in |X|$, the eigenvalues of $\iota(\Phi_x | M_x)$ are pure of weight w; it is *ι-mixed* if it is a successive extension of pointwise ι-pure F-isocrystals; and finally it is *ι-real* if for every $x \in |X|$, the characteristic polynomial of Φ_x is real. Then following *loc. cit.* step by step yields

10.5. Theorem – *Let (M, Φ) be a quasi-unipotent F-isocrystal on a smooth curve X/k over a finite field. If M is ι-real, then the irreducible constituents of (M, Φ) are ι-pure.*

Note that for the argument to work, we need the finiteness results (and thus quasi-unipotence) not just for M but for all of the even tensor powers $\otimes^{2k} M$ of M (*cf.* [17 1.5.2.1]).

10.6. We now explain how to set up a local monodromy formalism; here again we only get useful results if we restrict our attention to quasi-unipotent isocrystals.

If A is a local algebra, then the category of finite free A-modules is a K-linear Tannakian category; it is not necessarily neutral since the obvious fiber functor (M, ∇) is A-valued and not K-valued. If we look, instead, at the category of finite free A-modules with *unipotent* connection, however, then there is a K-valued fiber functor (*cf.* [23, 2.4] where this is done in a algebraic setting). By Proposition 6.7 the category of free A-modules with a unipotent connection is equivalent to the category of K-vector spaces with a nilpotent endomorphism. If we combine the functor 6.7.5 with the obvious functor $(V, N) \mapsto V$, we obtain a K-valued fiber functor. Since the category of vector spaces with a nilpotent endomorphism is equivalent to the category of representations of the additive group $\mathbb{G}_a/K$, we see that this is also the case for the category of finite, free A-modules with unipotent connection; the equivalence, of course, depends on the choice of local parameter of A (since 6.7.5 does).

Suppose now that X/k is a smooth affine curve. Fix an embedding $X \hookrightarrow \overline{X}$, into a smooth projective curve, and let $x \in D = \overline{X} - X$. As a substitute for a generic point of the local ring of X at x, we will consider systems $(f : U \to \overline{X}, \mathcal{U}, y, t)$ where

(i) U is smooth, and $f : U \to \overline{X}$ is quasi-finite and étale outside of x,
(ii) $\mathcal{U}$ is a formal lifting of U,
(iii) y is a point of U such that $f(y) = x$, and
(iv) t is a local section of $\mathcal{O}_{\mathcal{U}}$ reducing to a local parameter of $\mathcal{O}_U$ at y.

We will say that η "lies over $x \in \overline{X}$." A morphism

$$(f' : U' \to \overline{X}, \mathcal{U}', y', t') \to (f : U \to \overline{X}, \mathcal{U}, y, t)$$

is a commutative diagram

$$\begin{array}{ccccc} \mathcal{U}' & \longleftarrow & U' & \longrightarrow & \overline{X} \\ \downarrow & & \downarrow & & \| \\ \mathcal{U} & \longleftarrow & U & \longrightarrow & \overline{X} \end{array} \tag{10.6.1}$$

such that $U' \to U$ maps $y' \to y$ (we make no condition on the parameters t, t'). The $(U, \mathcal{U}, y, t)$ form a filtered projective system.

Let $\eta = (f : U \to \overline{X}, \mathcal{U}, y, t)$ be as in the last paragraph. If M is an isocrystal on X/K, then its pullback to U can be represented as a locally free sheaf with connection

(M_V, ∇) on some strict neighborhood V of $]U - f^{-1}(x)[$ in $\mathcal{U}^{an}$, and as in 7.2 we have an embedding $\Gamma(V, \mathcal{O}_V) \hookrightarrow A(y)$. Then

$$M_{\eta,A(y)} = (M_V, \nabla) \otimes A(y)$$

is a locally free $A(y)$-module with connection. Suppose, finally, that $M_{\eta,A(y)}$ is unipotent (we will express this by saying that M is *unipotent at* η), and that y is a k-rational point of U. Then the choice (in η) of the parameter t singles out an isomorphism $M_{\eta,A(y)} = V_0 \otimes_K A(y)$ via the functor 6.7.5, and the connection has the form $\nabla = N \otimes d/t$ for some nilpotent endomorphism of V_0. We put

$$M_\eta = V_0$$

and think of M_η as the "fiber of M at η" and N as the corresponding "monodromy operator." We have

$$(M_{\eta,A(y)}, \nabla) = (M_\eta \otimes_K A(y), N \otimes dt/t). \tag{10.6.2}$$

It follows from 6.9 that this construction is essentially independent of the choice of t. To express this more precisely, let t, t' be two choices of parameters, from which we get to K-valued fiber functors on the $\otimes$-category $[M_{\eta,A(y)}]$ generated by $M_{\eta,A(y)}$; then by the general theory of Tannakian categories, these fiber functors are isomorphic, so that M_η and $M_{\eta'}$ carry (non-canonically) isomorphic representations of $\mathbb{G}_a$. From 6.9 it follows that if $\eta' \to \eta$ is a morphism and M is unipotent at η, then it is unipotent at η', and the corresponding representations of $\mathbb{G}_a$ are isomorphic.

Suppose now that M is unipotent at $\eta = (U, \mathcal{U}, y, t)$, and that M has a Frobenius structure, *i.e.* an isomorphism $\Phi : F^*M \xrightarrow{\sim} M$ such that $\Phi\nabla = \nabla\Phi$. If f is the residual degree of $k(y)/k$, and if $\phi : \mathcal{U} \to \mathcal{U}$ is a lifting of $F^f : U \to U$, then Φ^f induces an isomorphism $\phi^* M_{\eta,A(y)} \xrightarrow{\sim} M_{\eta,A(y)}$. Now we can always choose a t and a ϕ such that $\phi(t) = t^{q_y}$, q_y being the cardinality of $k(y)$, and this case one can check by a direct calculation (using the condition that $\Phi\nabla = \nabla\Phi$) that $\Phi^f | M_{\eta,A(y)}$ is induced by an endomorphism F of M_η such that

$$FN = q_y NF. \tag{10.6.3}$$

It is not so clear in this situation that the endomorphism F of M_η is independent of the choices made (in particular, that of t).

Recall now that if we are given any vector space V with a nilpotent endomorphism $N : V \to V$, then there is a unique increasing filtration $\mathcal{M}^{\cdot}$ of V, stable under N, such that $N\mathcal{M}^i \subset \mathcal{M}^{i-2}$, and N^k induces an isomorphism

$$N^k : \mathrm{gr}_k^{\mathcal{M}} V \xrightarrow{\sim} \mathrm{gr}_{-k}^{\mathcal{M}} V$$

(this is the *monodromy weight filtration; cf.* [17 1.6.1]). Consider now the case when $V = M_\eta$ and N is the monodromy operator. We obtain a filtration $\mathcal{M}^{\cdot}$ of M_η which, by 10.6.2, induces a filtration on $M_{\eta,A(y)}$ by horizontal submodules. The induced connections on the quotients $\mathrm{gr}_k^{\mathcal{M}} M_{\eta,A(y)}$ are trivial, so that the $\mathrm{gr}_k^{\mathcal{M}} M_{\eta,A(y)}$ extend canonically to isocrystals $\mathcal{G}_k$ on the point $\operatorname{Spec}(k(y))$. The $\mathcal{G}_k$ can be identified canonically with the $\operatorname{gr}_k^{\mathcal{M}} M_\eta$ which are their fibers at $t = 0$; in particular, the $\operatorname{gr}_k^{\mathcal{M}} M_\eta$ are independent, up to canonical isomorphism, of the choices of the lifting $\mathcal{U}$ and the parameter t. On the other hand, it follows from 10.6.3 that $\mathcal{M}$ is stable under F, and the induced filtration of $M_{\eta,A(y)}$ is stable under Φ^f. Now Φ^f acting on $M_{\eta,A(y)}$ is induced by F acting on M_η, and the action of Φ^f on the graded pieces $\operatorname{gr}_k^{\mathcal{M}} M_{\eta,A(y)}$ acts on the canonical extension $\mathcal{G}_k$ of the $\operatorname{gr}_k^{\mathcal{M}} M_{\eta,A(y)}$; we can then canonically identify the F-action on $\operatorname{gr}_k^{\mathcal{M}} M_\eta$ with the fiber at $t = 0$ of the Φ^f-action on the $\mathcal{G}_k$. We conclude that the action of F on the graded pieces $\operatorname{gr}_k^{\mathcal{M}} M_\eta$ of the monodromy filtration is independent, up to canonical isomorphism, of any of the choices made.

Thus if (M, Φ) is an F-isocrystal on X/K and M is unipotent at η, we can speak of the eigenvalues of Φ acting M_η, and in particular of the ι-weights of M at η. If M is quasi-unipotent, then by definition there is an η such that M is unipotent at η, and any η' such that $\eta' \to \eta$ will give rise to the same weights (the corresponding points y might have different residue fields, and so the eigenvalues need not be the same). Finally, if $x \in \overline{X} - X$, then any two η, η' lying over x are dominated by a third, so we can speak unambiguously of the monodromy filtration of M at x, and of the weights of the Frobenius on the "generic fiber" at x.

10.7. We can now explain how to prove an analogue of Theorem 1.8.4 of [17] for a quasi-unipotent F-isocrystal; the key point is to make sense of formula 1.8.1.1 of *loc. cit.*, *i.e.* the cohomological formula for the direct image of a lisse ℓ-adic sheaf on a dense open subset of a smooth curve. Suppose that X is a smooth affine curve, $j : U \hookrightarrow X$ is a dense open subset, and M is an isocrystal on U/K. As always we choose a formally smooth lifting of X, so that to any $a \in X - U$ we can associate a local algebra $A(a)$ and a map $A_X^\dagger \to A(a)$. If we represent the isocrystal M as a locally free sheaf with connection (M, ∇) on some strict neighborhood of $]U[$, then $M \otimes A(a)$ is a finite free $A(a)$-module with connection. Then $\mathrm{H}^0(M \otimes A(a))$ will play the role of the fiber of the direct image at $a \in X$. If M has a Frobenius structure Φ, then we *define*

$$L(X, j_*M, T) := L(U, M, T) \prod_{a \in X - U} \det(1 - T\Phi^{\deg a} | \mathrm{H}^0(M \otimes A(a)))^{-1}. \tag{10.7.1}$$

Now for our purposes, the essential point of [17 1.8.1.1] is that the numerator of the L-function $L(U, M, T)$ is already divisible by the product of the $\det(1 - T\Phi^{\deg a} | \mathrm{H}^0(M \otimes A(a)))$, so that $L(X, j_*M, T)$ will have no more poles than $L(U, M, T)$. To see that this is the case, we observe that by 10.4.3, 10.7.1, and 8.2.4–5 we have

$$L(X, j_*M, T) = \frac{\det(1 - TF^* | \mathrm{H}_c^1(X, j_*M))}{\det(1 - TF^* | \mathrm{H}_c^2(X, j_*M))}. \tag{10.7.2}$$

We can now follow the argument of [17 1.8.1–2] more or less word-for-word. Suppose that M is pointwise ι-pure on U of weight β (as before, X is affine, so that $\overline{X} \neq X$), and choose a point $x \in X - U$. From 10.7.2, one sees in the usual way [17 1.8.1] that $\iota \det(1 - TF^* | \mathrm{H}^0(M \otimes A(x)))$ has no pole for $|T| < q^{-(\beta+2)/2}$, so that

$$|\iota(\alpha)| \leq q_x^{(\beta+2)/2},$$

for any eigenvalue α of F^* on $\mathrm{H}^0(M \otimes A(x))$; *i.e.* the weights of F^* on $\mathrm{H}^0(M \otimes A(x))$ are at most $\beta + 2$. Applying this argument to $\otimes^k M$ in place of M, and noting that

$\otimes^k \mathrm{H}^0(M \otimes A(x)) \subset \mathrm{H}^0(\otimes^k M \otimes A(x))$, we find that the weights of F^* on $\mathrm{H}^0(M \otimes A(x))$ are at most $\beta + 2/k$. Letting $k \to \infty$, we find that the weights of F^* on $\mathrm{H}^0(M \otimes A(x))$ are at most β. From this, the argument of the proof of [17 1.8.4] yields

10.8. Theorem – *Suppose that $U \subset X$, M is ι-pure on U/K of weight β, and (M, Φ) is unipotent at some η lying above $x \in X - U$. Then for each i, $\mathrm{gr}_i^{\mathcal{M}} M_\eta$ is ι-pure of weight $\beta + i$.*

If M is the restriction of an isocrystal on X, then the monodromy filtration is trivial, and just as in [17] we get

10.9. Corollary – *If (M, Φ) is a quasi-unipotent F-isocrystal on X/K, and is ι-pure of weight β on some dense open subset $U \subset X$, then (M, Φ) is ι-pure on all of X.*

10.10. We will conclude by remarking that the results of [17 §2] on the equidistribution of Frobenius classes are valid for a quasi-unipotent F-isocrystal on a smooth curve. Let us recall how these results are formulated; the proofs are those of [17 §2], virtually without change.

Pick as before an algebraic closure $\overline{K}$ of K and an isomorphism $\iota : \overline{K} \simeq \mathbb{C}$. Let M be an F-isocrystal on X/K; then we denote by

$$0 \to G^0_{\mathbb{C}} \to G_{\mathbb{C}} \to \mathbb{Z} \to 0$$

the extension of scalars by $K \hookrightarrow \overline{K} \xrightarrow{\iota} \mathbb{C}$ of the exact sequence 10.4.1. There is a subgroup $G_{\mathbf{R}} \subset G_{\mathbb{C}}$ projecting onto $\mathbb{Z}$, such that $G^0_{\mathbb{C}} \cap G_{\mathbf{R}}$ is a maximal compact subgroup of $G^0_{\mathbb{C}}$ [17 2.2.1]. The conjugacy classes of $G_{\mathbf{R}}$ are the intersections with $G_{\mathbf{R}}$ of the conjugacy classes of $G_{\mathbb{C}}$. Suppose now that M is ι-mixed. Then as in [17 2.2.6], the semisimple components of the conjugacy classes $\iota\mathrm{Frob}_x \in G_{\mathbb{C}}$ lie in $G_{\mathbf{R}}$ (*i.e.* come from conjugacy classes of $G_{\mathbf{R}}$). Denote by $G^\natural_{\mathbf{R}}$ the space of conjugacy classes of $G_{\mathbf{R}}$, and by $\mu^\natural$ the direct image on $G^\natural_{\mathbf{R}}$ of the measure

$$\mu = \sum_{\substack{x \in |X| \\ n > 0}} \deg(x) q^{-n\deg(x)} \delta(\mathrm{Frob}^n_x)$$

where $\delta(g)$ denotes the Dirac delta-measure at $g \in G_{\mathbf{R}}$. On the other hand, denote by dg the Haar measure on $G_{\mathbf{R}}$ normalized so that $G^0_{\mathbf{R}}$ has measure one; by μ_0 the product of dg by the characteristic function of the elements of positive degree, and by $\mu^\natural_0$ the direct image of μ_0 on $G^\natural_{\mathbf{R}}$. Finally, let $G^\natural_n$ denote the subset of $G^\natural_{\mathbf{R}}$ consisting of classes of degree n. Let z be an element of the center of $G_{\mathbf{R}}$ of *positive* degree; such elements exist by the global monodromy theorem ([15 1.3.11] and 10.4 above). The equidistribution theorem is the following:

10.11. Theorem – *Suppose that (M, Φ) is ι-mixed and quasi-unipotent on X/K. Then for any i, we have*

$$z^{-n}(\mu^\natural | G^\natural_{i+n\deg(z)}) \to \mu^\natural_0 | G^\natural_i$$

weakly as $n \to \infty$.

As in [17 §2.2], the key point is to show that if M is ι-pure of weight zero, then the L-function $L(X, M, T)$ has no zero for $|T| = q^{-1}$, which is shown using the method of Hadamard and de la Vallée-Poussin [17 2.1.4 and 2.2.8–9].

Again as in [17], this assertion is the "first step" towards the Riemann hypothesis:

10.12. Theorem – *Suppose that (M, Φ) is quasi-unipotent on X/K and ι-pure of weight β. Then the weights of Frobenius on $\mathrm{H}^1_c(X, M)$ are strictly less than $\beta + 2$.*

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