ALAN ROCHE

Types and Hecke algebras for principal series representations of split reductive $p$-adic groups


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TYPES AND HECKE ALGEBRAS
FOR PRINCIPAL SERIES REPRESENTATIONS
OF SPLIT REDUCTIVE P-ADIC GROUPS

BY ALAN ROCHE

ABSTRACT. - We construct types in the sense of Bushnell and Kutzko for principal series representations of split connected reductive p-adic groups (with mild restrictions on the residual characteristic) and describe the resulting Hecke algebras. We discuss their interpretation as Iwahori Hecke algebras of related reductive groups (in general disconnected). In addition, we describe how (parabolic) induction and (Jacquet) restriction functors and questions about square-integrability can be transferred to this context. © Elsevier, Paris

RÉSUMÉ. – On construit des types au sens de Bushnell et Kutzko pour les représentations des séries principales des groupes déployés connexes réductifs p-adiques (avec de légères restrictions sur la caractéristique résiduelle) et on décrit les algèbres de Hecke qui en résultent. On discute comment ces algèbres peuvent être interprétées comme des algèbres d’Iwahori-Hecke relatives à des groupes réductifs reliés (en général non-connexes). De plus, on décrit comment les foncteurs d’induction et de restriction et les questions relatives aux représentations de carré intégrable peuvent être considérés dans ce contexte. © Elsevier, Paris

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ANNALES SCIENTIFIQUES DE L’ÉCOLE NORMALE SUPÉRIEURE. – 0012-9593/98/03© Elsevier, Paris
1. Introduction

Let \( F \) be a non-Archimedean local field and \( G \) the group of \( F \)-rational points of a connected reductive algebraic group \( \mathbb{G} \) defined over \( F \). Let \( \mathfrak{R}(G) \) denote the category of smooth complex representations of \( G \). The Bernstein decomposition expresses the abelian category \( \mathfrak{R}(G) \) as a direct product

\[
\mathfrak{R}(G) = \prod_{s \in \mathfrak{B}(G)} \mathfrak{R}_s(G)
\]

of full subcategories \( \mathfrak{R}_s(G) \). The objects of \( \mathfrak{R}_s(G) \) are all \((\pi, V) \in \mathfrak{R}(G)\) such that all irreducible subquotients of \( \pi \) have fixed supercuspidal support modulo unramified twists. Thus the indexing set \( \mathfrak{B}(G) \) consists of irreducible supercuspidal representations of Levi subgroups of \( G \) (up to equivalence) modulo \( G \)-conjugation and twisting by unramified characters. Following [13], we call \( \mathfrak{B}(G) \) the Bernstein spectrum of \( G \) and refer to the subcategories \( \mathfrak{R}_s(G) \) for \( s \in \mathfrak{B}(G) \) as the components of the Bernstein decomposition.

If \( G \) is quasi-split (over \( F \)), then a minimal Levi subgroup of \( G \) is (the group of \( F \)-rational points of) a torus. Thus an irreducible supercuspidal representation of a minimal Levi subgroup is in this case a smooth character. We refer to the resulting subcategories \( \mathfrak{R}_s(G) \) of \( \mathfrak{R}(G) \) as the principal series components of the Bernstein decomposition (and the corresponding elements of the Bernstein spectrum as principal series elements).

In this paper we study the principal series components of \( \mathfrak{R}(G) \) via compact open data under the assumption that \( G \) is split (over \( F \)). Fix such a component \( \mathfrak{R}_s(G) \). When \( F \) has characteristic zero and certain mild restrictions are placed on the residual characteristic, we explicitly describe an \( s \)-type in \( G \) in the sense of Bushnell and Kutzko [13]. We also describe the structure of the resulting Hecke algebras. (See the list preceding Theorem 4.15 and Remark 4.14 for a precise statement of the restrictions on residual characteristic and related comments. If each irreducible factor of \( G \) is of type \( B, C \) or \( D \), the only excluded residual characteristic is two.) By very slightly modifying the methods of section 4, the results of this paper may also be seen to hold when \( F \) has positive characteristic while keeping the same restrictions on residual characteristic (see [2]).

These results may be rephrased as follows. Fix a split maximal torus \( T \) and a Borel subgroup \( \mathbb{B} \) containing \( T \) and write \( T \) and \( \mathbb{B} \) for the corresponding groups of \( F \)-rational points. Let \( 0^T \) be the unique maximal compact subgroup of \( T \) and \( \chi : 0^T \to \mathbb{C}^\times \) be a smooth character. We construct a pair \((J, \rho) = (J_\chi, \rho_\chi)\) where \( J \) is a compact open subgroup of \( G \) and \( \rho \) is a smooth character of \( J \) (satisfying, in particular, \( J \cap T = 0^T \) and \( \rho \mid J \cap T = \chi \)). We show that a smooth irreducible representation \((\pi, V) \) of \( G \) contains \( \rho \) (on restriction to \( J \) if and only if \( \pi \) is equivalent to a \( G \)-subquotient of \( \text{Ind}_H^G(\tilde{\chi}) \) where \( \tilde{\chi} \) is a character of \( T \) extending \( \chi \). This is precisely the statement that \((J, \rho)\) is an \( s_\chi \)-type where \( s_\chi \) is the principal series element of \( \mathfrak{R}(G) \) canonically determined by \( \chi \). We write the resulting component of the Bernstein decomposition as \( \mathfrak{R}_{s_\chi}(G) \) (i.e. \( \mathfrak{R}_{s_\chi}(G) = \mathfrak{R}_{s_\chi}(G) \)).

The pair \((J, \rho)\) gives rise to an idempotent \( e_{\rho} \) in \( \mathcal{H}(G) \), the convolution algebra of compactly supported, locally constant functions \( \Phi : G \to \mathbb{C} \). If \((\pi, V) \) is a smooth representation of \( G \), then \( \pi(e_{\rho})V = e_{\rho}V \) is the space \( V_\rho \) of \( \rho \)-isotypic vectors in \( V \) (i.e. the sum of all \( J \)-subspaces of \( V \) isomorphic to \( \rho \)). It is clear that \( V_\rho \) is a module over the subalgebra \( e_{\rho} \mathcal{H}(G)e_{\rho} \) of \( \mathcal{H}(G) \). The process \( \mathbb{V} \mapsto \mathbb{V}_\rho \) defines a functor from \( \mathfrak{R}(G) \) to \( e_{\rho} \mathcal{H}(G)e_{\rho} \) (the category of left \( e_{\rho} \mathcal{H}(G)e_{\rho} \)-modules). From the general
theory developed in Bushnell and Kutzko [13], the restriction of this functor to $\mathcal{R}_\chi(G)$ is an equivalence of categories. Further let $\mathcal{R}_\rho(G)$ denote the full subcategory of $\mathcal{R}(G)$ consisting of all smooth representations of $G$ generated by their $\rho$-isotypic subspaces. Then (again from [13]) the categories $\mathcal{R}_\chi(G)$ and $\mathcal{R}_\rho(G)$ are equal (as subcategories of $\mathcal{R}(G)$). (In fact we rederive these results here by observing that $(J,\rho)$ is an $s_\chi$-type in $G$ if and only if the smooth compactly induced representation $\text{ind}_J^G \rho$ is a (finitely generated, projective) generator of the category $\mathcal{R}_\chi(G)$. We then invoke a general criterion ([3] chap. 2, thm. 1.5) giving conditions under which a category admitting a projective generator is equivalent to a module category (over the endomorphism ring of the projective generator)). When $\chi$ is trivial, $J$ is an Iwahori subgroup of $G$ and $\rho$ is the trivial character. These results therefore generalise the well-known results of Borel [6] and Casselman [15] relating the unramified principal series to representations generated by their Iwahori-fixed vectors.

Since $\rho$ is a character, the algebra $e_\rho \mathcal{H}(G)e_\rho$ is simply $\mathcal{H}(G,\rho)$ (the convolution algebra of compactly supported, $\rho^{-1}$-spherical functions on $G$). We show there exists a family of isomorphisms from $\mathcal{H}(G,\rho)$ to an algebra $\mathcal{H}_\chi$ where $\mathcal{H}_\chi$ is a certain affine Hecke algebra twisted by a certain complex group algebra (both defined in terms of $\chi$). When $\chi$ is trivial, this reduces to the Iwahori-Matsumoto description of the Iwahori Hecke algebra. More generally when $\chi$ factors through $\mathbb{C} \to \mathbb{T}(k_F)$ (where $k_F$ is the residue field of $F$), $J$ is again an Iwahori subgroup and a description of the Hecke algebra $\mathcal{H}(G,\rho)$ in these terms has already been given by Goldstein in [17] and (as a very special case of the work of) Morris in [27].

We also construct a split reductive group $\tilde{H}$ (in general disconnected) such that the Hecke algebras $\mathcal{H}(G,\rho)$ and $\mathcal{H}(\tilde{H},1_\chi)$ are isomorphic via a family of support-preserving isomorphisms. Here $1_\chi$ is the trivial character of an Iwahori subgroup of $\tilde{H}$ (the identity component of $\tilde{H}$). The groups $H$ and $\tilde{H}$ admit natural interpretations in terms of Langlands dual groups. (I am grateful to Neil Chriss and Allen Moy for suggesting versions of this.) In particular, $H$ is an endoscopic group of $G$. We discuss this in Section 8 and show that the quotient $\tilde{H}/H$ is always abelian and is trivial when $G$ (more properly $G_0$) has connected centre. Combining Kazhdan-Lusztig [24] with some Mackey theory arguments, we see that the simple $\mathcal{H}(G,\rho)$-modules (and thus the irreducible objects in $\mathcal{R}_\chi(G)$) are essentially classified. (The group $H$ however need not have connected centre even if $G$ has.) The dual group interpretation of $\mathcal{H}(G,\rho)$ may be used to naturally attach Langlands parameters to principal series representations via the corresponding objects for representations of $H$ having a non-trivial Iwahori-fixed vector (which are known [24, 26] at least when $H$ has connected centre). We will write up the details and some consequences elsewhere.

The character $\chi$ also canonically determines a component $\mathcal{R}_\chi(T)$ of the Bernstein decomposition of $T$. This is the full subcategory of $\mathcal{R}(T)$ consisting of all smooth representations of $T$ whose restriction to $0^T$ is a multiple of $\chi$. The pair $(0^T,\chi)$ is clearly a type for $\mathcal{R}_\chi(T)$. In particular, the functor $\mathcal{W} \to \mathcal{W}^\chi : \mathcal{R}_\chi(T) \to \mathcal{H}(T,\chi) - \text{Mod}$ is an equivalence of categories. The functors of (normalised and unnormalised) parabolic induction from $\mathcal{R}(T)$ to $\mathcal{R}(G)$ (with respect to $B$) restrict to $\mathcal{R}_\chi(T)$ to yield corresponding functors from $\mathcal{R}_\chi(T)$ to $\mathcal{R}_\chi(G)$. We show in section 9 that these functors correspond via the equivalences of categories $\mathcal{R}_\chi(T) \simeq \mathcal{H}(T,\chi) - \text{Mod}$ and $\mathcal{R}_\chi(G) \simeq \mathcal{H}(G,\rho) - \text{Mod}$ to 'algebraic' induction functors between $\mathcal{H}(T,\chi) - \text{Mod}$ and $\mathcal{H}(G,\rho) - \text{Mod}$ induced by appropriate twists of an algebra embedding $t_B : \mathcal{H}(T,\chi) \to \mathcal{H}(G,\rho)$. Given our
knowledge of the structure of the Hecke algebra $\mathcal{H}(G, \rho)$, these results are special cases of (or immediate consequences of) a general result of Bushnell and Kutzko [13].

In addition, we show that the isomorphism of Hecke algebras between $\mathcal{H}(G, \rho)$ and $\mathcal{H}(\widetilde{H}, 1\mathcal{T})$ can be chosen to respect this extra structure. In particular, if $\varpi$ is a fixed uniformiser in $F$ and $Y$ denotes the cocharacter lattice of $T$ then the homomorphism $y \mapsto y(\varpi) : Y \rightarrow T$ splits the short exact sequence

$$1 \rightarrow 0T \rightarrow T \rightarrow Y \rightarrow 0.$$  

Via this splitting we may view $\chi$ as a character of $T$ (trivial on $y(\varpi)$ for $y \in Y$). The group $H$ also has $T$ as a maximal torus and the Borel subgroup $B$ in $G$ determines a Borel subgroup $B(H)$ of $H$ (containing $T$). We show there is an equivalence of categories between $\mathfrak{B}_X(G) = \mathfrak{B}_\rho(G)$ and $\mathfrak{B}_{1\mathcal{T}}(\widetilde{H})$ (induced by an appropriately chosen algebra isomorphism between $\mathcal{H}(G, \rho)$ and $\mathcal{H}(\widetilde{H}, 1\mathcal{T})$) under which the normalised induced representation $i_G^H(\chi \nu)$ corresponds to $i_{B(H)}^H(\nu)$. Here $\nu$ is any unramified character of $T$. This reduces questions concerning reducibility and composition series of general principal series representations to the unramified case plus some simple Mackey theory (taking account of the abelian component group). In the unramified case much information is available (see Reeder [30] and the references therein).

The Hecke algebras we consider carry canonical inner products which are preserved by our Hecke algebra isomorphisms. This has consequences for square-integrability and formal degrees (more generally Plancherel measure). Following a standard abuse of terminology, square-integrable here actually means square-integrable-mod-centre. We borrow an argument from [10] to show that the equivalence of categories between $\mathfrak{B}_\rho(G)$ and $\mathfrak{B}_{1\mathcal{T}}(\widetilde{H})$ (induced by a norm-preserving algebra isomorphism between $\mathcal{H}(G, \rho)$ and $\mathcal{H}(\widetilde{H}, 1\mathcal{T})$) preserves square-integrability and formal degrees (given natural choices of Haar measures on $G/Z_G$ and $\widetilde{H}/Z_{\widetilde{H}}$). As a trivial consequence of these calculations, we note that the category $\mathfrak{B}_X(G) = \mathfrak{B}_\rho(G)$ contains (non-zero) square-integrable representations if and only if the endoscopic group $H$ of $G$ is elliptic.

The idea of studying the entire smooth dual of a $p$-adic group by the method of restriction to compact open subgroups was pioneered by Howe. In particular, an early paper of his [20] discusses the principal series of $GL_N(F)$ in terms of compact open data (via a generalisation of the Satake isomorphism). Bushnell and Kutzko have described the entire smooth dual of $GL_N(F)$ in terms of types [10] [11]. For general reductive groups a similar description of the level-zero situation is given in Morris [28] (cf. also Moy and Prasad [29]). Some special cases (of principal series types) for $SL_N(F)$ were also treated in Sanje-Mpacko [31].

Our presentation is arranged as follows. In section 3, we construct the compact open subgroup $J = J_X$ and the smooth character $\rho = \rho_X$. We also prove that $J$ has an Iwahori decomposition with respect to any Borel subgroup containing $T$ and factors as $0T$ times the product of the affine root groups it contains (in any order). In section 4, we compute the $G$-intertwining of $\rho$ (equivalently the set of $J$ double cosets which support a non-zero element in $\mathcal{H}(G, \rho)$). The key step adapts an argument from Howe and Moy ([19] Lemma 4.4) to our situation. Section 5 contains a slight modification of a theorem in [13] (which we use in showing that various Hecke algebra isomorphisms preserve inner products). In section 6 we use this (along with known results in the level-zero situation)
to describe the structure of $\mathcal{H}(G, \rho)$. In section 7 we recall the definition of a type and some consequences and show that $(J_\chi, \rho_\chi)$ is an $\mathfrak{s}_{\chi}$-type. We define the groups $H$ and $\bar{H}$ in section 8 and show that $\mathcal{H}(G, \rho)$ is isomorphic to the Iwahori Hecke algebra of $\bar{H}$. We also discuss their interpretation in terms of Langlands parameters for $T$ and give some examples. Section 9 discusses the connection with parabolic induction. Finally, section 10 contains the calculations on square-integrability described above.

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2. Notation and some preliminaries

We continue with notation already introduced. Thus $G$ is a split connected reductive algebraic group defined over $F$ with $F$-split maximal torus $T$ where $F$ is a non-Archimedean local field. We write $O_F$ for the valuation ring in $F$, $P_F$ for the unique prime ideal in $O_F$ and $k_F$ for the residue field of $F$. Replacing $G$ if necessary by an $F$-isomorphic group, we may (and do) assume that $G$ and $T$ are defined and split over $Z$. Then $\mathcal{N} = N_G(T)$ (the normaliser of $T$ in $G$) is also defined over $Z$. Put $^0 T = T(O_F)$. It is the unique maximal compact subgroup of $T$. If $X = \text{Hom}(T, G^\alpha)$ is the lattice of rational characters of $T$, then $\Phi = \{x \in X | \text{val}(x(t)) = 0 \}$ for $X = \text{Hom}(T, G^\alpha)$. We write $\Phi = \Phi(G, T)$ for the roots of $G$ with respect to $T$ and $\Phi^\vee = \Phi^\vee(G, T)$ for the corresponding coroots. We fix a positive system $\Phi^+$ in $\Phi$ and write $\text{II}$ for the unique simple system contained in $\Phi^+$. We often write $\mathcal{B} = TU$ for the corresponding Borel subgroup (with unipotent radical $U$) and $\mathcal{B} = \overline{TU}$ for the opposite Borel subgroup (with unipotent radical $\overline{U}$). Write $G, T, N, B, U$ etc., for the corresponding groups of $F$-rational points. We may (and do) fix root group isomorphisms $x_\alpha : \mathcal{G}_\alpha \to \mathcal{U}_\alpha$ for $\alpha \in \Phi$ such that the following hold:

i) There exists a $Z$-homomorphism $\phi_\alpha : SL_2 \to \mathcal{G}$ satisfying $x_\alpha(u) = \phi_\alpha \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $x_{-\alpha}(u) = \phi_{-\alpha} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$. Furthermore, $\alpha^\vee(t) = \phi_\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

ii) There exist integers $C_{\alpha, \beta; i, j}$ ($\alpha, \beta \in \Phi, \beta \neq -\alpha, i \alpha + j \beta \in \Phi$) such that the commutator of $x_\alpha(u)$ and $x_\beta(v)$ is given by:

$$[x_\alpha(u), x_\beta(v)] = \prod_{i, j > 0, i \alpha + j \beta \in \Phi} x_{i \alpha + j \beta}(C_{\alpha, \beta; i, j} u^i v^j)$$

where the roots in the product are listed in some fixed order.

iii) If $X_\alpha = dx_\alpha(1) \in \text{Lie}(G)$ and $H_\alpha = [X_\alpha, X_{-\alpha}] \in \text{Lie}(T)$ for $\alpha \in \Phi$, then the following equations hold:

$$(2.1) \quad [X_\alpha, X_\beta] = C_{\alpha, \beta; 1, 1} X_{\alpha + \beta} \quad \text{if } \alpha + \beta \in \Phi$$

$0 \quad \text{if } \alpha + \beta \notin \Phi \cup \{0\},$$
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(2.2) \[ \text{Ad} x_\alpha(u).X_\beta = \sum_{j \geq 0, \beta + j\alpha \in \Phi} C_{\alpha,\beta,j} u^j X_{\beta+j\alpha} \quad (\alpha, \beta \in \Phi, \beta \neq -\alpha), \]

(2.3) \[ \text{Ad} x_\alpha(u).X_{-\alpha} = X_{-\alpha} + uH_\alpha - u^2 X_\alpha, \]

(2.4) \[ \text{Ad} x_\alpha(u).H = H - \alpha(H)uX_\alpha \quad (H \in \text{Lie}(T)). \]

We write \( g = \text{Lie}(G)(F) \) and \( t = \text{Lie}(T)(F) \). Finally, we write \( U_{\alpha,k} = x_\alpha(P_F^k) \) for \( k \in \mathbb{Z} \).

3. The pair \( (J_X, \rho_X) \)

Let \( \chi : ^0T \to \mathbb{C}^* \) be a smooth complex character. In this section we construct a compact open subgroup \( J_X \) of \( G \) and a smooth irreducible representation (in fact, a smooth character) \( \rho_X \) of \( J_X \). The subgroup \( J_X \) contains \( ^0T \) and the character \( \rho_X \) extends \( \chi \). We also establish some properties of \( J_X \) (in particular Iwahori decompositions and root group factorizations) for use in later sections. Our proof of these properties forces some restrictions on the characteristic of the residue field \( k_F \) of \( F \). More precisely, we assume \( \text{char} k_F \) is prime to any integers which occur as ratios of squares of root lengths for pairs of roots in the same irreducible component of \( \Phi \) (i.e. \( \text{char} k_F \) is prime to any positive number of bonds connecting nodes in the Dynkin diagram of \( \Phi \)). Thus \( \text{char} k_F \neq 2 \) (resp. \( \text{char} k_F \neq 3 \)) if \( \Phi \) has factors of type \( B_n, C_n \) or \( F_4 \) (resp. \( G_2 \)). We begin by recalling, in a very special case, some results of Bruhat and Tits.

Suppose a function \( f : \Phi \to \mathbb{Z} \) satisfies:

\[
\begin{align*}
  f(\alpha + \beta) &\leq f(\alpha) + f(\beta) & & \text{if } \alpha, \beta, \alpha + \beta \in \Phi, \\
  f(\alpha) + f(-\alpha) &\geq 1 & & \text{for } \alpha \in \Phi.
\end{align*}
\]

In particular, \( f \) is concave in the sense of Bruhat and Tits (see [8], 6.4.3 and 6.4.5).

Let \( U_f = \langle U_{\alpha,f(\alpha)} : \alpha \in \Phi \rangle \) and \( J_f = \langle ^0T, U_f \rangle \). Note \( J_f = \langle ^0T U_f = U_f \rangle ^0T \).

Further, let \( U_f^+ = U \cap U_f \), \( U_f^- = U \cap U_f \), \( U_{f,\alpha} = U_f \cap U_\alpha \) (for \( \alpha \in \Phi \)) and \( U_{f,\alpha}' = \langle U_{\alpha,f(\alpha)}, U_{-\alpha,f(-\alpha)} \rangle \) (for \( \alpha \in \Phi \)).

**Lemma 3.1.** With notation as above:

\[ U_{f,\alpha}' = U_{-\alpha,f(-\alpha)} \alpha'(1 + \langle \alpha' \rangle + f(-\alpha)) U_{\alpha,f(\alpha)} \]

for each \( \alpha \in \Phi \).

**Proof.** The homomorphism \( \phi_\alpha : SL_2(F) \to \langle U_\alpha, U_{-\alpha} \rangle \) satisfies \( \phi_\alpha \begin{pmatrix} 1 & P_F^k \\ 0 & 1 \end{pmatrix} = U_{\alpha,k} \)

and \( \phi_\alpha \begin{pmatrix} 1 & P_F^f(\alpha) \\ 0 & 1 \end{pmatrix} = U_{-\alpha,k} \) (for \( k \in \mathbb{Z} \)). By direct calculation,

\[
\left( \begin{pmatrix} 1 & P_F^{f(\alpha)} \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} T_{f(\alpha)+f(-\alpha)} \begin{pmatrix} 1 & P_F^{f(\alpha)} \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & P_F^{f(\alpha)} \\ 0 & 1 \end{pmatrix} T_{f(\alpha)+f(-\alpha)} \begin{pmatrix} 1 & P_F^{f(\alpha)} \\ 0 & 1 \end{pmatrix} \]

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where \( T_{f(a)+f(-a)} = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in 1 + \mathcal{P}_F^{f(a)+f(-a)} \right\} \). Applying \( \phi_a \), we obtain the

Lemma (since \( \phi_a \left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) = \alpha^\vee(t) \)). □

**Lemma 3.2.** With notation as above:

i) \( U_{f,a} = U_{a,f(a)} \) for \( a \in \Phi \).

ii) The product map \( \prod_{a \in \Phi^\pm} U_{f,a} \to U_f^\pm \) is bijective for any ordering of the factors in the product.

iii) \( U_f = U_f^{-} U_f^{\pm} \) where \( 0T_f = \prod_{a \in \Phi} \alpha^\vee(1 + \mathcal{P}_F^{f(a)+f(-a)}) \).

iv) \( J_f = U_f^{-} 0T_f U_f^{\pm} \).

**Proof.** Since \( f \) is concave i), ii) and iii) follow from the corresponding parts of [8] prop. 6.4.9 provided we show \( N_f = N \cap U_f \) coincides with \( 0T_f \). From [8] prop. 6.4.9 iv), \( N_f = \langle N_f(a) : \alpha \in \Phi \rangle \) where \( N_f(a) = N \cap U_f(a) \). Lemma 3.1 implies \( N \cap U_f(a) = \alpha^\vee(1 + \mathcal{P}_F^{f(a)+f(-a)}) \). Hence \( N_f = 0T_f \).

Part iv) is clear. □

If \( \lambda : \mathcal{O}_F^\times \to \mathbb{C}^\times \) is a smooth character, we define the conductor of \( \lambda \), denoted \( \text{cond}(\lambda) \), to be the least integer \( n \geq 1 \) such that \( 1 + \mathcal{P}_F^n \subseteq \ker(\lambda) \). (This is slightly different than the usual notion (in that the trivial character has conductor one) but more convenient for our purposes.) For each \( a \in \Phi \), we may view \( \chi \circ \alpha^\vee \) as a smooth character of \( \mathcal{O}_F^\times \). We write \( c_a \) for \( \text{cond}(\chi \circ \alpha^\vee) \).

**Definition 3.3.** Define a function \( f_\chi : \Phi \to \mathbb{Z} \) as follows:

\[
\begin{align*}
f_\chi(\alpha) &= \left[ c_\alpha / 2 \right] & \text{for } \alpha \in \Phi^+; \\
                 &= \left[ (c_\alpha + 1) / 2 \right] & \text{for } \alpha \in \Phi^-.
\end{align*}
\]

Here \([x]\) denotes the largest integer \( \leq x \).

**Lemma 3.4.** Suppose \( \text{char } k_F \neq 2 \) (resp. \( \text{char } k_F \neq 3 \)) if \( \Phi \) has irreducible factors of type \( B_n, C_n \) or \( F_4 \) (resp. \( G_2 \)). Then:

i) \( f_\chi(\alpha + \beta) \leq f_\chi(\alpha) + f_\chi(\beta) \) if \( \alpha, \beta, \alpha + \beta \in \Phi \).

ii) \( f_\chi(\alpha) + f_\chi(-\alpha) \geq 1 \) for \( \alpha \in \Phi \).

**Proof.** It is clear that ii) holds. To see i), note that by hypothesis \( \text{char } k_F \) is prime to any integers which occur as ratios of squares of root lengths for pairs of roots in the same irreducible factor of \( \Phi \). Hence:

\[
p(\alpha + \beta)^\vee = q\alpha^\vee + r\beta^\vee
\]

where \( p, q, r \in \mathbb{Z} \) are each prime to \( \text{char } k_F \). Since \( x \mapsto x^s : 1 + \mathcal{P}_F^s \to 1 + \mathcal{P}_F^s \) (\( i \geq 1 \)) is bijective for \( s \) prime to \( \text{char } k_F \), it follows that \( c_{\alpha + \beta} \leq \max(c_\alpha, c_\beta) \). Using this observation, a short calculation (which we omit) establishes i). □

We assume from now on that the residual characteristic of \( F \) is restricted as in Lemma 3.4. We define \( J_\chi \) to be \( J_{f_\chi} \). Similarly, let \( U_\chi = U_{f_\chi} \) and \( 0T_\chi = 0T_{f_\chi} \). From Lemma 3.2, \( J_\chi/U_\chi \cong 0T/0T_\chi \). By construction, \( 0T_\chi \subseteq \ker \chi \). Hence \( \chi \) defines a character of \( 0T/0T_\chi \).
and so can be lifted to a character $\rho_\chi$ of $J_\chi$. As $\chi$ is fixed, we often drop the subscripts and write the pair $(J_\chi, \rho_\chi)$ as $(J, \rho)$.

**Example 3.5.** Suppose $G = SL_2$. Since $^0T = O_F^*$, we may view $\chi : T \to C^\times$ as a character of $O_F^*$. Let $n = \text{cond}(\chi)$. Then

$$J = \left( O_F^{[n/2]} \cap SL_2(F) \right).$$

Note that $\chi$ extends to a character of $J$ which is trivial on $U_{\alpha,[(n+1)/2]}$ and on $U_{-\alpha,[(n+1)/2]}$ (where $\alpha$ is the unique positive root). By definition, $\rho$ is this extension.

Let $\Phi^{+'}$ be a system of positive roots in $\Phi$. Then $\Phi^{+'}$ determines a unique Borel subgroup $B'$ containing $T$. We have $B' = TU'$ where $U' = \prod_{\alpha \in \Phi^{+'}} U_\alpha$. Let $\bar{B}'$ be the opposite Borel so that $\bar{B}' = T\bar{U}'$ where $\bar{U}' = \prod_{\alpha \in \Phi^{-'}} U_\alpha$ (where $\Phi^{-'} = -\Phi^{+'}$). We say $J$ has an Iwahori decomposition with respect to the pair $(T, B')$ or with respect to the positive system $\Phi^{+'}$ if the product map:

$$(J \cap U') \times (J \cap T) \times (J \cap U') \to J$$

is a bijection. (It is then automatically a homeomorphism.) We record some properties of $(J, \rho)$ in the next proposition.

**Proposition 3.6.** The pair $(J, \rho)$ has the following properties:

i) $J$ has Iwahori decompositions with respect to any positive system $\Phi^{+'}$ in $\Phi$. Further, $J \cap \bar{U}' = \prod_{\alpha \in \Phi^{+'}} U_{\alpha, f_\chi(\alpha)}$, $J \cap T = T$, $J \cap U' = \prod_{\alpha \in \Phi^{+'}} U_{\alpha, f_\chi(\alpha)}$ where $U'$ and $\bar{U}'$ denote the unipotent radicals of the pair of opposite Borel subgroups, $B'$ and $\bar{B}'$, determined by $\Phi^{+'}$.

ii) The character $\rho : J \to C^\times$ satisfies $\rho(j_-j_0'j_+') = \chi(j_0')$ where $j_-$ is in $J \cap U'$, $j_0' \in J \cap T$ is $^0T$ and $j_+'$ is in $J \cap U'$ for $U'$ and $U'$ as in i).

**Proof.** Part i) follows from Lemma 3.2 (using the positive system $\Phi^{+'}$ in place of $\Phi^{+}$). Part ii) is clear from the definition of $\rho$. $\square$

### 4. Intertwining

In this section we calculate the $G$-intertwining of $\rho$. This serves as an essential first step in our approach to determining the algebra structure of the Hecke algebra $\mathcal{H}(G, \rho)$.

First we recall the relevant definitions. Let $\tau$ be a smooth character of a compact open subgroup $K$ of $G$. An element $g \in G$ is said to intertwine $\tau$ if

$$\tau \mid K \cap gK = g\tau \mid K \cap gK$$

where $gK = gKg^{-1}$ and $g\tau(x) = \tau(g^{-1}xg)$ for $x \in gK$. We write $\mathcal{I}_G(\tau)$ for the set of $g \in G$ which intertwine $\tau$. The Hecke algebra $\mathcal{H}(G, \tau)$ is the convolution algebra of compactly supported functions $\phi : G \to C$ such that $\phi(k_1xk_2) = \tau(k_1)^{-1}\phi(x)\tau(k_2)^{-1}$ for $k_1, k_2 \in K, x \in G$. It is easy to see that an element $g \in G$ intertwines $\tau$ if and only if the double coset $KgK$ supports a non-zero function in $\mathcal{H}(G, \tau)$. Such a function (if it exists)
is clearly unique up to a non-zero scalar. Thus the collection of double cosets $K \backslash I_G(\tau)/K$ parametrizes a vector space basis of $H(G, \tau)$.

We write $\bar{W}$ for the (ordinary) Weyl group of $T$ in $G$. Thus $\bar{W} \simeq N/T$. We have $N = 0^N T$ and $0^N \cap T = 0^T$ where $0^N = N(0_F)$ and $0^T = T(0_F)$. It follows that $\bar{W}$ is also isomorphic to $0^N/0^T$. We write $W$ for the extended affine Weyl group of $T$ in $G$. Thus $W \simeq W^N/T$.

We have $N_\gamma = \gamma N$ and $\gamma N \cap T = \gamma T$ where $\gamma N = N(0_p)$ and $\gamma T = T(0_r)$. It follows that $W$ is also isomorphic to $\gamma N/\gamma T$.

We write $W$ for the extended affine Weyl group of $T$ in $G$. Thus $\gamma W = Y \times \gamma W$ where $Y = \text{Hom}(G_m, \bar{T})$ is the lattice of rational cocharacters of $T$. Let $X = \text{Hom}(T, \mathbb{Z})$ be the lattice of rational characters of $T$ (as above) and $(\cdot, \cdot) : X \times Y \to \mathbb{Z}$ the usual perfect pairing defined by $x \circ y(-) = \langle x, y \rangle$ for $x \in X$, $y \in Y$. We have the following short exact sequence:

\[
1 \longrightarrow 0^T \longrightarrow T \xrightarrow{H_T} Y \longrightarrow 0
\]

where $i : 0^T \to T$ is the inclusion map and the map $H_T : T \to Y$ is defined by $(H_T(t), x) = \text{val}_F(x(t))$ for $x \in X$, $t \in T$. The group $0^N$ acts by conjugation on $T$ and $0^T$. The subgroup $0^T$ acts trivially. We then obtain the usual action of $\bar{W} \simeq 0^N/0^T$ on $T$ and $0^T$ and hence on $T/0^T$. Of course, $\bar{W}$ also acts on $Y$ and it is easy to check that the isomorphism $T/0^T \simeq Y$ induced by $H_T$ is $\bar{W}$-equivariant. Hence $0^N/0^T \times T/0^T \simeq \bar{W} \times Y$, i.e. $N/0^T \simeq \bar{W} \times Y = W$.

$N$ acts on the group of smooth complex characters $(0^T)^\gamma$ of $0^T$ by $n(x) = x(n^{-1} t n)$ for $n \in N$, $t \in 0^T$, $\mu \in (0^T)^\gamma$. $T$ and hence $0^T$ act trivially. Thus the action factors to an action of $\bar{W}$ or $W$ on $(0^T)^\gamma$. We let $N_\chi = \{ n \in N \mid n \chi = \chi \}$, $\bar{W}_\chi = \{ w \in \bar{W} \mid w \chi = \chi \}$, $W_\chi = \{ w \in W \mid w \chi = \chi \}$. Note $W_\chi = Y \times \bar{W}_\chi$, $N_\chi/0^T \simeq W_\chi$, $N_\chi/T \simeq \bar{W}_\chi$.

**Proposition 4.1.** With notation as above:

\[I_G(\rho) \cap N = N_\chi\]

**Proof.** Suppose $n \in N$ lies in $I_G(\rho)$. The condition

\[n^\rho \mid J \cap n^\rho J = \rho \mid J \cap n J\]

then implies $n \chi \mid 0^T = \chi \mid 0^T$, i.e. $n \chi = \chi$ or $n \in N_\chi$.

Suppose $n \in N_\chi$ maps to $w$ under $N/T \simeq \bar{W}$. Let $j \in J \cap n J$. We have to show $\rho(j) = \rho(n^{-1} j n)$. Write $j = j_0 j_+ j_-$ where $j_- \in J \cap U^-$, $j_0 \in 0^T$ and $j_+ \in J \cap U^+$. Let $w^{-1} \Phi^+ = \Phi^+$, $w^{-1} \Phi^- = \Phi^-$. Then by Propn. 3.6, $J$ has an Iwahori decomposition with respect to the positive system $\Phi^+$. Let $U''$ and $\bar{U}''$ be the unipotent radicals of the corresponding pair of opposite Borels. The root space decompositions of $U$, $\bar{U}$, $U'$ and $\bar{U}'$ imply $n^{-1} U n = U'$ and $n^{-1} \bar{U} n = \bar{U}'$. Hence $n^{-1} j n = n^{-1} j_- n^{-1} j_0 n^{-1} j_+ n$ expresses $n^{-1} j n \in J$ as an element of $\bar{U}' T U'$. From $J = (J \cap U') (J \cap T) (J \cap U')$ and uniqueness of expression in $\bar{U}' T U'$, we see $n^{-1} j_- n \in J \cap \bar{U}'$ and $n^{-1} j_+ n \in J \cap U'$. Since both $J \cap \bar{U}'$ and $J \cap U'$ are contained in $\ker(\rho)$, we obtain:

\[\rho(n^{-1} j n) = \chi(n^{-1} j_0 n) = n \chi(j_0) = \chi(j_0) = \rho(j)\]

This proves the proposition. \qed

**Remark 4.2.** Suppose $\chi$ is a level-zero character, i.e. $\chi$ factors through $0^T \to T(k_F)$. Since $J$ is then the standard Iwahori subgroup of $G$ (with respect to $\Phi^+$), Proposition 4.1
and the Bruhat decomposition imply $I_G(\rho) = JW^J$. If $\chi$ does not factor through $0^T \to \mathbb{T}(k_F)$, we say $\chi$ has positive level. In the remainder of this section, we show $I_G(\rho) = JW^J$ also holds when $\chi$ has positive level (given char $F = 0$ and certain restrictions on char $k_F$ — see the list preceding theorem 4.15 for a precise statement).

The map from $Y \otimes_k F^x$ to $T$ induced by $y \otimes \lambda \mapsto y(\lambda)$ for $y \in Y$, $\lambda \in F^x$ is an isomorphism of (topological) groups. For $q \in \mathbb{N}$, we write $T_q$ for the image of $Y \otimes (1 + \mathcal{P}_F^q)$ under this map. We also have the isomorphism (of topological groups) from $Y \otimes_k F$ to $t$ induced by $y \otimes \lambda \mapsto y(\lambda)$ for $y \in Y$, $\lambda \in F$. We write $t_r$ for the image of $Y \otimes \mathcal{P}_F^r$ for $r \in \mathbb{Z}$. The standard congruence (compact open) subgroups $K_q$ ($q \in \mathbb{N}$) of $G$ and the corresponding $\mathcal{O}_F$-lattices $\mathfrak{r}_r$ ($r \in \mathbb{Z}$) of $\mathfrak{g}$ are defined as follows:

$$K_q = \langle T_q, U_{\alpha, q} : \alpha \in \Phi \rangle \quad (q \geq 1)$$

$$\mathfrak{r}_r = t_r + \sum_{\alpha \in \Phi} \mathcal{P}_F^r X_\alpha \quad (r \in \mathbb{Z}).$$

Fix a $\mathbb{Z}$-basis $\{y_1, \ldots, y_n\}$ of $Y$. For $q \in \mathbb{N}$, each element of $Y \otimes \mathcal{P}_F^q$ has a unique expression of the form $y_1 \otimes \lambda_1 + \cdots + y_n \otimes \lambda_n$ where $\lambda_1, \ldots, \lambda_n \in \mathcal{P}_F^q$. Define a function $\gamma' : Y \otimes \mathcal{P}_F^q \to Y \otimes (1 + \mathcal{P}_F^q)$ by:

$$\gamma'(y_1 \otimes \lambda_1 + \cdots + y_n \otimes \lambda_n) = \prod_{i=1}^n y_i \otimes (1 + \lambda_i).$$

This induces a function from $t_q$ to $T_q$ which we again denote by $\gamma'$. For $q \in \mathbb{N}$ define $\gamma : \mathfrak{r}_q \to K_q$ by:

$$\gamma(q + \sum_{\alpha \in \Phi} u_\alpha X_\alpha) = \gamma(a) \prod_{\alpha \in \Phi} x_\alpha(u_\alpha) \quad (a \in t_q, u_\alpha \in \mathcal{P}_F^q (\alpha \in \Phi)).$$

Here the elements of $\Phi$ are listed in some fixed order. It is clear that $\gamma$ is a bijection of sets. Suppose $s \in \mathbb{N}$ is such that $q \leq s \leq 2q$. Then the bijections $\gamma : \mathfrak{r}_q \to K_q$ and $\gamma : \mathfrak{r}_s \to K_s$ induce group isomorphisms:

$$\mathfrak{r}_q / \mathfrak{r}_s \simeq K_q / K_s \quad (q \leq s \leq 2q).$$

These isomorphisms are independent of the choice of $\mathbb{Z}$-basis of $Y$ and the listing of roots in $\Phi$.

We now construct a non-degenerate, $AdG$-invariant, symmetric bilinear form $B$ on $\mathfrak{g}$ given char $F = 0$ and certain restrictions on the residual characteristic (see 4.5). The form will satisfy $\mathfrak{r}_r^\perp = \mathfrak{r}_{1-r}$ for $r \in \mathbb{Z}$ where $\mathfrak{r}_r^\perp = \{ X \in \mathfrak{g} \mid B(X, Y) \in \mathcal{P}_F (\forall Y \in \mathfrak{r}_r) \}$ (see Lemma 4.3).

We assume thus that char $F = 0$. Then $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ where $\mathfrak{g}'$ is semisimple and $\mathfrak{z} = Z(\mathfrak{g})$ (i.e. $\mathfrak{z}$ is the center of $\mathfrak{g}$). The Killing form $B_1$ of $\mathfrak{g}$, defined by $B_1(X, Y) = \text{tr}(\text{ad}X \text{ad}Y)$ for $X, Y \in \mathfrak{g}$, is an $AdG$-invariant, symmetric bilinear form on $\mathfrak{g}$ with radical $\mathfrak{z}$.

Let $Q^\vee = \mathbb{Z} \Phi^\vee$ be the coroot lattice of $G$ with respect to $T$. Let

$$X_0^\vee = \{ y \in Y \mid \langle y, \alpha \rangle = 0 \ (\forall \alpha \in \Phi) \}.$$
Then $Q^\vee + X_0^\vee$ has finite index in $Y$ (see Springer [32]). Let $V^\vee = Y \otimes \mathbb{Q}$, $V = X \otimes \mathbb{Q}$, $V_1^\vee = Q^\vee \otimes \mathbb{Q}$, $V_0^\vee = X_0^\vee \otimes \mathbb{Q}$. The perfect pairing $(\cdot , \cdot) : X \times Y \to \mathbb{Z}$ extends to a non-degenerate $\mathbb{Q}$-bilinear pairing $(\cdot , \cdot) : V \times V^\vee \to \mathbb{Q}$. We have $V^\vee = V_1^\vee \oplus V_0^\vee$. Let $p_1 : V^\vee \to V_1^\vee$ be projection onto the first factor. The coweight lattice $P^\vee$ is defined by

$$P^\vee = \{ v \in V_1^\vee \mid (v, \alpha) \in \mathbb{Z} \ (\forall \alpha \in \Phi) \}. \tag{4.3}$$

Then $Q^\vee \subset P^\vee$ and $P^\vee / Q^\vee$ is finite. From the definition of $X_0^\vee$, we see $Y \cap V_0^\vee = X_0^\vee$. It follows that $p_1 : V^\vee \to V_1^\vee$ induces an embedding:

$$Y/(Q^\vee + X_0^\vee) \hookrightarrow P^\vee / Q^\vee. \tag{4.4}$$

We impose the following condition on the characteristic of $k_F$:

$$\text{char } k_F \text{ does not divide } 2 | P^\vee : Q^\vee | \tag{4.5}$$

Thus if $\Phi$ is irreducible the excluded residual characteristics are as follows:

- for type $A_n$ primes dividing $2(n + 1)$
- for type $B_n$, $C_n$, $D_n$ 2
- for type $E_6$ 2, 3
- for type $E_7$ 2
- for type $E_8$, $F_4$, $G_2$ 2.

If $\Phi$ is not irreducible, then the excluded primes are those attached to each of its irreducible factors.

From 4.4 and 4.5, we obtain

$$t_q = Y \otimes P_F^q = (Q^\vee + X_0^\vee) \otimes P_F^q = Q^\vee \otimes P_F^q \oplus X_0^\vee \otimes P_F^q \quad (q \in \mathbb{Z}),$$

$$t = Y \otimes F = (Q^\vee + X_0^\vee) \otimes F = Q^\vee \otimes F \oplus X_0^\vee \otimes F.$$

Define a form $g(\cdot , \cdot) : Q^\vee \times Q^\vee \to \mathbb{Z}$ as follows:

$$g(q_1, q_2) = \sum_{\beta \in \Phi} (\beta, q_1)(\beta, q_2) \quad (q_1, q_2 \in Q^\vee).$$

Then $g(\cdot , \cdot)$ is symmetric, $\mathbb{Z}$-bilinear, positive-definite (hence non-degenerate) and $\overline{W}$-invariant. By extension of scalars, we obtain a (non-degenerate) $O_F$-bilinear form $g(\cdot , \cdot) : Q^\vee \otimes O_F \times Q^\vee \otimes O_F \to O_F$ (resp. a (non-degenerate) $F$-bilinear form $g(\cdot , \cdot) : Q^\vee \otimes F \times Q^\vee \otimes F \to F$).

The embedding:

$$y \otimes \lambda \mapsto y(\lambda) : Q^\vee \otimes F \hookrightarrow t \quad (y \in Q^\vee, \lambda \in F)$$

has image $t' = t \cap g'$. The element $\alpha^\vee \otimes 1$ is sent to $H_\alpha = [X_\alpha, X_{-\alpha}]$ for $\alpha \in \Phi$. Note that under this map the form $g(\cdot , \cdot) : Q^\vee \otimes F \times Q^\vee \otimes F \to F$ coincides with $B_1 | t' \times t'$. Indeed $\text{ad } H_\alpha.X_\gamma = \gamma(H_\alpha)X_\gamma = (\gamma, \alpha^\vee)X_\gamma$ for $\alpha, \beta \in \Phi$. Hence:

$$B_1(H_\alpha, H_\beta) = \sum_{\gamma \in \Phi} (\gamma, \alpha^\vee)(\gamma, \beta^\vee) \quad (\alpha, \beta \in \Phi)$$

so that $B_1(H_\alpha, H_\beta) = g(\alpha^\vee, \beta^\vee)$ for $\alpha, \beta \in \Phi$ as required.
Suppose $\Phi^\vee$ has irreducible factors $\Phi_1^\vee, \ldots, \Phi_r^\vee$. Let $g_i(,) = g_i(,) \mid Q_i^\vee$ where $Q_i^\vee = Z \Phi_i^\vee$. Then $g(,) = \oplus_{i=1}^r g_i(,) \mid Q_i^\vee$ (in the obvious notation). Let $l_i$ be the minimum of $g_i(\alpha^\vee, \alpha^\vee)$ as $\alpha^\vee$ ranges through $\Phi_i^\vee$. We define $g'(,) \mid Q' = \oplus_{l_i}^{-1} g_i(,) \mid Q_i^\vee$. Denote by $B'$ the corresponding form on $g'$. Thus $B'$ equals $g'(,) \mid t' \times t'$ and $B'$ coincides with $B^{-1}$ on the subalgebra of type $\Phi_i (i = 1, \ldots, r)$.

If $\beta \in \Phi_i$ and $\alpha^\vee \in \Phi_i^\vee$, we have $\langle \beta, \alpha^\vee \rangle = \frac{2g(\alpha^\vee, \beta^\vee)}{g_i(\alpha^\vee, \alpha^\vee)}$. Suppose $g_i(\gamma^\vee, \gamma^\vee) = l_i$ for $\gamma^\vee \in \Phi_i^\vee$. Then

$$\langle \beta, \alpha^\vee \rangle = \frac{2g_i(\gamma^\vee, \gamma^\vee)}{g_i(\beta^\vee, \beta^\vee)} l_i^{-1} g_i(\alpha^\vee, \beta^\vee)$$

$$= \frac{2g_i(\gamma^\vee, \gamma^\vee)}{g_i(\beta^\vee, \beta^\vee)} g'(\alpha^\vee, \beta^\vee).$$

The quantity $\frac{g_i(\gamma^\vee, \gamma^\vee)}{g_i(\beta^\vee, \beta^\vee)}$ is a ratio of squares of root lengths in $\Phi_i^\vee$ and thus is a unit in $O_F$ (see the first paragraph of section 3). In particular, the matrix $(g'(\alpha^\vee, \beta^\vee)) \in \Pi^\vee$ can be transformed into the Cartan matrix of the root system $\Phi^\vee$ by multiplying certain columns by elements of $O_F$. Since the determinant of the Cartan matrix of $\Phi^\vee$ equals $[P^\vee : Q^\vee]$, we have $\det(g'(\alpha^\vee, \beta^\vee)) \in O_F$. It follows that $g'(,) \mid Q' \times Q'$ induces an isomorphism of $O_F$-modules between $Q' \otimes O_F$ and $\text{Hom}_{O_F}(Q' \otimes O_F, O_F)$. We use this observation in the proof of the next Lemma.

Fix a $Z$-basis $\{l_1, \ldots, l_r\}$ of $X_0^\vee$. Define a non-degenerate, symmetric, $Z$-bilinear form $(,)' : X_0^\vee \times X_0^\vee \to Z$ by $(l_i, l_j)' = \delta_{ij}$ for $1 \leq i, j \leq r$. It extends to a non-degenerate, $F$-bilinear form $(,)' : X_0^\vee \otimes F \times X_0^\vee \otimes F \to F$.

Since $t = Q^\vee \otimes F \oplus X_0^\vee \otimes F$, $g'(,) \oplus (,)'$ defines a non-degenerate, symmetric, bilinear form on $t \times t$. This extends to the form $B = B' \oplus (,)'$ on $g \times g$ (since $g = g' \oplus \delta$ and $\delta = X_0^\vee \otimes F$). Note that $B$ is non-degenerate and $\text{Ad} G$-invariant (since $\text{Ad} G$ acts trivially on $\delta$ and $B \mid g' \times g'$ is a multiple of the Killing form on each of the simple factors of the semisimple algebra $g'$).

If $\mathcal{L}$ is an $O_F$-lattice in $g$ (resp. $t$), we set:

$$\mathcal{L} = \{X \in g \mid B(X, Y) \in P_F \ (\forall Y \in \mathcal{L})\}$$

**Lemma 4.3.** - With notation as above:

1. $t_q^\perp = t_{1-q} \ (q \in Z)$
2. $\mathcal{R}_q^\perp = \mathcal{R}_{1-q} \ (q \in Z)$

**Proof.** - i) It is clear that $t_{1-q} \subset t_q^\perp$. To see the opposite inclusion, let $a \in t_q^\perp$. Write $a = a_1 + z_1$ where $a_1 \in Q^\vee \otimes F$, $z_1 \in X_0^\vee \otimes F$. Then $i) B(a_1, Q^\vee \otimes P_F^s) \subset P_F$ and $ii) B(z_1, X_0^\vee \otimes P_F^s) \subset P_F$. Clearly ii) implies $z_1 \in X_0^\vee$ (since $B \mid \delta \times \delta = (,)'$). We have already observed that $g'(,) \mid Q' \times Q'$ induces an isomorphism of $O_F$-modules between $Q' \otimes O_F$ and $\text{Hom}_{O_F}(Q' \otimes O_F, O_F)$. Hence there exist elements $y_1, \ldots, y_r$ such that $g'(y_1, \alpha_i^\vee) = \delta_{ij}$ where $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$. It follows that $F y_1 + \cdots + F y_r = Q^\vee \otimes F$ and $P_F y_1 + \cdots + P_F y_r = Q^\vee \otimes P_F^s$ for any $s \in Z$. Write $a_1 = y_1 \otimes \lambda_1 + \cdots + y_r \otimes \lambda_r$ for $\lambda_1, \ldots, \lambda_r \in F$. Then $B(a_1, \alpha_i^\vee \otimes \omega_q^s) = \lambda_i \omega_q^s$ implies each $\lambda_i \in P_F^{1-q}$. Hence $a_1 \in Q^\vee \otimes P_F^{1-q}$.
ii) Since the restriction of $B$ to the subalgebra of $g'$ of type $\Phi_i$ ($i = 1, \ldots, r$) is a multiple of the Killing form, $B(t, g_\alpha) = 0$ for all $\alpha \in \Phi$ and $B(g_\alpha, g_\beta) = 0$ for $\alpha, \beta \in \Phi$ unless $\alpha + \beta = 0$. Further, $2B(X_\alpha, X_{-\alpha}) = B(H_\alpha, H_\alpha) = g'(\alpha^\vee, \alpha^\vee)$ is a unit in $\mathcal{O}_F$ (by the construction of $g'$). It follows easily that $\tilde{\mathfrak{r}}_q^+ = \mathfrak{r}_1 - q$. \hfill \Box

Fix a smooth character $\psi : F \to \mathbb{C}^\times$ such that $\mathcal{P}_F \subset \ker \psi$, $\mathcal{O}_F \not\subset \ker \psi$. Let $q, s \in \mathbb{N}$ such that $q \leq s \leq 2q$. Then $K_q/K_s \cong \mathfrak{r}_q/\mathfrak{r}_s$ and hence $(K_q/K_s)^\vee \cong (\mathfrak{r}_q/\mathfrak{r}_s)^\vee$. The map $X \mapsto \psi_X : \mathfrak{r}_s^+/\mathfrak{r}_s^\perp \to (\mathfrak{r}_q/\mathfrak{r}_s)^\vee$ defined by

$$
\psi_X(Y) = \psi(B(X, Y)) \quad (\text{for } Y \in \mathfrak{r}_q)
$$

is an isomorphism of groups. From Lemma 4.3, we see $\mathfrak{r}_1 - s/\mathfrak{r}_1 - q \cong (K_q/K_s)^\vee$.

Let $l$ be the least positive integer such that $\chi | T_i$ is trivial. Since $\chi$ has positive level, $l \geq 2$. Then $J \subset K_{l-1} \subset K_l$. We may view $\rho | K_{l-1}$ as an element of $(K_{l-1}/K_l)^\vee$. Thus $\rho | K_{l-1}$ is parameterized by some element of $\mathfrak{r}_{l-1}/\mathfrak{r}_{l-2}$. It is clear that this element has the form $a + \mathfrak{r}_{l-2}$ for some $a \in t_{l-1}$ (since $\rho$ is trivial on $U_{\alpha, l-1}$ for all $\alpha \in \Phi$). For an appropriate choice of $a \in t_{l-1}/t_{l-2}$, we will eventually show $\mathcal{I}_G(\rho) \subset J C_G(a) J$ where $C_G(a) = \{g \in G \mid \text{ad}_g a = a\}$. This is the key step in an inductive argument that determines $\mathcal{I}_G(\rho)$. The precise meaning of appropriate is contained in the following Lemma. The proof requires us to exclude some further residual characteristics. (This introduces additional restrictions only in the case of exceptional groups.) More precisely, we say a prime $p$ is bad for a root system $\Psi$ if $\mathbb{Z}\Psi/\mathbb{Z}\Psi_1$ has $p$-torsion for some closed subsystem $\Psi_1$ of $\Psi$ (see Springer-Steinberg [33] 4.3). For irreducible root systems, the bad primes are as follows:

- for type $A_n$ none
- for types $B_n, C_n, D_n$ 2
- for types $E_6, E_7, F_4, G_2$ 2, 3
- for type $E_8$ 2, 3, 5.

Clearly, the bad primes for a root system are the bad primes for its irreducible factors. We assume $\text{char } k_F \neq p$ if $p$ is a bad prime for $\Phi$.

**Lemma 4.4.** - Each coset $\bar{a} \in t_{l-1}/t_{l-2}$ has a representative $a$ such that $\alpha(a) \equiv 0 \pmod{\mathcal{P}_F^{l-1}}$ for $\alpha \in \Phi$ implies $\alpha(a) = 0$.

**Proof.** - Replacing $\bar{a}$ by $\omega^{l-1} \bar{a}$, we may assume $l = 1$, i.e. $\bar{a} \in t_0/t_1$. Let $\Phi_{\bar{a}} = \{\alpha \in \Phi : \alpha(a_1) \equiv 0 \pmod{\mathcal{P}}\}$ for any (for all) $a_1 \in \bar{a}$. It is easy to check that $\Phi_{\bar{a}}$ is a closed subsystem of $\Phi$ (i.e. if $\alpha, \beta \in \Phi_{\bar{a}}$ then $s_\alpha(\beta) \in \Phi_{\bar{a}}$ and $\mathbb{Z}\Phi_{\bar{a}} \cap \Phi = \Phi_{\bar{a}}$).

Let $Q = Z \Phi$ and $Q_{\bar{a}} = Z \Phi_{\bar{a}}$. Since $\text{char } k_F$ is not a bad prime for $\Phi$, the quotient $Q \otimes \mathcal{O}_F/Q_{\bar{a}} \otimes \mathcal{O}_F$ is torsion-free. Therefore $Q_{\bar{a}} \otimes \mathcal{O}_F$ has a complement in $Q \otimes \mathcal{O}_F$. In particular, any $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_u$ of $Q_{\bar{a}}$ extends to an $\mathcal{O}_F$-basis $\alpha_1, \ldots, \alpha_u, \beta_1, \ldots, \beta_v$ of $Q \otimes \mathcal{O}_F$.

Let $\Pi = \{\gamma_1, \ldots, \gamma_r\}$ be the set of simple roots in $\Phi^+$ so that $r = u + v$. The determinant of the Cartan matrix of $\Phi$ is $[P : Q]$ where $P$ is the weight lattice (defined as in 4.3). Since there is a canonical perfect pairing

$$
P/Q \times P^\vee/Q^\vee \longrightarrow Q/\mathbb{Z}
$$
we have $[P : Q] = [P^\gamma : Q^\gamma]$. Thus char $k_F$ is prime to the determinant of the Cartan matrix of $\Phi$ so that the map

$$
t'_0 \rightarrow \mathcal{O}_F\gamma^r
\quad t \mapsto (\gamma_1(t), \ldots, \gamma_r(t))
$$

is an isomorphism of $\mathcal{O}_F$-modules. It follows that the map

$$(4.7) \quad t'_0 \rightarrow \mathcal{O}_F^r
\quad t \mapsto (\alpha_1(t), \ldots, \alpha_n(t), \beta_1(t), \ldots, \beta_v(t))$$

is also an isomorphism of $\mathcal{O}_F$-modules. Under this isomorphism, $t'_0$ is identified with $P^r_F$.

For any $a_1 \in a$, write $a_1 = a_0 + z$ where $a_0 \in t'_0$ and $z \in \mathfrak{z}$. Let $a'_0$ be the element of $t'_0$ corresponding to $(0, \ldots, 0, \beta_1(a_0), \ldots, \beta_v(a_0))$ under 4.7. Then $a = a'_0 + z \in a$ satisfies the required condition.

We assume from now on that $\rho | K_{l-1}$ corresponds to a coset $a + \mathfrak{R}_{2-l} \in \mathfrak{R}_{1-l}/\mathfrak{R}_{2-l}$ for $a$ as in the Lemma. We also assume that $l = 2l'$ is even. The slight modifications needed to deal with the case in which $l$ is odd are indicated below (see Remark 4.9).

**Definition 4.5.** - Define a compact open subgroup $L = L_a$ as follows:

$$L = \{T_{l-1}, U_{\alpha, l-1}, U_{\beta, l'} : \alpha(a) = 0, \beta(a) \neq 0\}$$

It is clear that $L$ is a subgroup of $J$. Let $\mathcal{L}$ be the $\mathcal{O}_F$-lattice in $\mathfrak{g}$ that corresponds to $L$. Thus:

$$\mathcal{L} = t_{l-1} + \sum_{\alpha(a) = 0} P_{F^-} X_\alpha + \sum_{\beta(a) \neq 0} P_{F} X_\beta$$

Now $\rho | L$ factors to $L/K_l$ and as before:

$$(L/K_l)^\gamma \simeq (\mathcal{L}/\mathfrak{R}_l)^\gamma \simeq \mathfrak{R}_{l-1}/\mathfrak{L}_l = \mathfrak{R}_{l-1}/\mathfrak{L}_l.$$

Again $\rho | L$ is parameterized by $a + \mathfrak{L}_\perp$.

The following lemma is proved in Adler [1].

**Lemma 4.6.** - An element $g \in G$ lies in $\mathcal{I}_G(\rho | L)$ if and only if

$$(4.8) \quad Ad g(a + \mathfrak{L}_\perp) \cap (a + \mathfrak{L}_\perp) \neq \emptyset$$

We sometimes write $\mathcal{I}_G(a + \mathfrak{L}_\perp)$ for the set of $g \in G$ which satisfy 4.8.

Suppose $r \in \mathbb{Z}$ and $q \in \mathbb{N}$. In the proof of the next lemma, we make use of the following identity:

$$(4.9) \quad Ad \gamma(X).Y \equiv Y + adX(Y) \pmod{\mathfrak{R}_{q+r+1}} \quad (for \ X \in \mathfrak{R}_q, \ Y \in \mathfrak{R}_r).$$

A short calculation shows that if (4.9) holds for $\gamma(X_1)$ and $\gamma(X_2)$ for $X_1, X_2 \in \mathfrak{R}_q$, then it also holds for $\gamma(X_1)\gamma(X_2)$. Hence it only necessary to check (4.9) for $X \in t_q$ and $X = uX_\alpha$ for $\alpha \in \Phi$ and $u \in P_F^-$. Further since it is $\mathcal{O}_F$-linear in $Y$, we may
assume \( Y \in t_r \) or \( Y = vX_\alpha \) for \( \alpha \in \Phi \) and \( v \in \mathcal{P}_r \). The relation therefore follows from formulas (2.1), (2.2), (2.3) and (2.4).

LEMMA 4.7. - Let \( r \in \mathbb{Z} \) and \( q \in \mathbb{N} \). Suppose \( a \in t_r \) such that \( \alpha(a) \neq 0 \) for \( \alpha \in \Phi \) implies \( \text{val}_p(\alpha(a)) = r \). Suppose \( Y \in a + \mathfrak{K}_{l+r} \) such that \( Y \in \mathfrak{Z}_g(a) \) (mod \( \mathfrak{K}_{q+r} \)) where \( \mathfrak{Z}_g(a) = \{ Y \in \mathfrak{g} \mid [a, Y] = 0 \} \). Then \( Y \) is \( \mathfrak{K}_q \)-conjugate to an element of \( \mathfrak{Z}_g(a) \).

Proof. - We show \( Y \) is \( \mathfrak{K}_1 \)-conjugate to an element of \( \mathfrak{Z}_g(a) \) (mod \( \mathfrak{K}_{q+r+1} \)). An obvious limiting argument then establishes the Lemma.

We have \( \mathfrak{Z}_g(a) = t + \sum_{\alpha(a) = 0} \mathfrak{g}_\alpha \). Put \( \mathfrak{g}(+) = \sum_{\alpha > 0} \mathfrak{g}_\alpha \), \( \mathfrak{g}(-) = \sum_{\alpha < 0} \mathfrak{g}_\alpha \). Then \( \mathfrak{g} = \mathfrak{g}(-) \oplus \mathfrak{g}_0 \oplus \mathfrak{g}(+) \) where \( \mathfrak{g}_0 = \mathfrak{Z}_g(a) \). For \( s \in \mathbb{Z} \), put \( \mathfrak{K}_s(-) = \mathfrak{g}(-) \cap \mathfrak{K}_s \) and \( \mathfrak{K}_s(+) = \mathfrak{g}(+) \cap \mathfrak{K}_s \). Write \( Y = Y_1 + Y' \) where \( Y_1 \in \mathfrak{Z}_g(a) \) and \( Y' \in \mathfrak{K}_{q+r}(-) + \mathfrak{K}_{q+r}(+) \).

For each \( s \in \mathbb{Z} \), \( \text{ad}(a) \) restricts to give isomorphisms of \( \mathcal{O}_F \)-modules:

\[
\begin{align*}
\text{ad}(a) : \mathfrak{K}_s(+) & \rightarrow \mathfrak{K}_{q+r}(+) \\
\text{ad}(a) : \mathfrak{K}_s(-) & \rightarrow \mathfrak{K}_{q+r}(-)
\end{align*}
\]

Hence we can write \( Y' = \text{ad}(a) Z' \) for some \( Z' \in \mathfrak{K}_q(+) + \mathfrak{K}_q(-) \). Then by (4.9):

\[
\begin{align*}
\text{ad} \gamma(Z'), Y & \equiv Y + [Z', Y] \pmod{\mathfrak{K}_{q+r+1}} \\
& \equiv Y + [Z', a + W] \pmod{\mathfrak{K}_{q+r+1}} \text{ where } W \in \mathfrak{K}_{q+r+1} \\
& \equiv Y - [a, Z'] \pmod{\mathfrak{K}_{q+r+1}} \\
& \equiv Y_1 + Y' - Y' \pmod{\mathfrak{K}_{q+r+1}} \\
& \equiv Y_1 \pmod{\mathfrak{K}_{q+r+1}}
\end{align*}
\]

\[\square\]

COROLLARY 4.8. - For any \( Y \in a + \mathcal{L}^\perp \), there exists an \( x \in L \) such that \( \text{Ad} x.Y \in \mathfrak{Z}_g(a) \).

Proof. - Note that \( \mathcal{L}^\perp = \mathfrak{K}_{2-l} + \mathfrak{K}_{1-l'}(+) + \mathfrak{K}_{1-l'}(-) \). Therefore \( Y \in a + \mathcal{L}^\perp \) implies \( Y \in a + \mathfrak{K}_{2-l} \) and \( Y \in \mathfrak{Z}_g(a) \) (mod \( \mathfrak{K}_{1-l'} \)). Thus we are in the situation of the previous Lemma with \( r = 1 - l \) and \( q = l' \). It is clear that the element \( \gamma(Z') \) constructed in the proof belongs to \( L \). The result follows. \( \square \)

Remark 4.9. - Suppose now that \( l = 2l' + 1 \) is odd. Define \( L \) to be the subgroup generated by \( T_{l-1} \) and the following affine root groups:

\[
\begin{align*}
U_{a,l-1} & \text{ if } \alpha(a) = 0 \\
U_{\beta,l'} & \text{ if } \beta(a) \neq 0 \text{ and } \beta > 0 \\
U_{\beta,l'+1} & \text{ if } \beta(a) \neq 0 \text{ and } \beta < 0
\end{align*}
\]

It is clear that \( L \) is again a subgroup of \( J \). Let \( \widetilde{K}_1 \) be the group generated by \( K_1 \) and the affine root groups \( U_{\beta,l-1} \) if \( \beta(a) \neq 0 \) and \( \beta > 0 \). The map \( \gamma \) (see equation 4.2) induces an isomorphism \( L/\widetilde{K}_1 \cong \mathcal{L}/\widetilde{\mathfrak{K}}_1 \) (in the obvious notation). (In particular, \( \widetilde{K}_1 \) is normal in \( L \) and the quotient is abelian.) Hence

\[(L/\widetilde{K}_1)^\sim \cong \widetilde{\mathfrak{K}}_1^\perp /\mathcal{L}^\perp.\]

The character \( \rho \mid L \) is again parametrized by \( a + \mathcal{L}^\perp \). It is straightforward to verify that Corollary 4.8 holds.
LEMMA 4.10. - Suppose \( \text{Ad} g . X_i = X_i \) where \( X_i \in \mathfrak{R}_1 \cap \mathfrak{g}(a) \) and \( X_i \equiv a \pmod{\mathfrak{R}_{2-i}} \) for \( i = 1, 2 \). Then \( \text{Ad} g . a_1 = a_1 \) for some \( a_1 \in a + \mathfrak{t}_{2-i} \).

Proof. - Fix a uniformiser \( \varpi \in F \). Replacing \( a \) by \( \varpi^{l-1} a \) and \( X_i \) by \( \varpi^{l-1} X_i \), we may assume \( a \in \mathfrak{t}_0 \) and \( X_i \in \mathfrak{R}_0 \cap \mathfrak{g}(a) \) with \( X_i \equiv a \pmod{\mathfrak{R}_1} \) \((i = 1, 2)\).

We have \( g = g' \oplus \mathfrak{z} \). Let \( \{ l_i \} \) be any basis of \( \mathfrak{z} \). Then \( X_\alpha (\alpha \in \Phi) \), \( H_\beta (\beta \in \Pi) \), \( l_i \) is a basis of \( g \) which we fix for the rest of the proof. Consider the adjoint representation \( \text{ad} : g \to \text{End}_F(\mathfrak{g}) \). Then \( \mathfrak{t} \simeq t' \) embeds in the diagonal subalgebra \( \mathfrak{d} \) of \( \text{End}_F(\mathfrak{g}) \) with respect to our fixed basis of \( g \). The trace form \( \text{Tr} \) on \( \text{End}_F(\mathfrak{g}) \) is non-degenerate. Since \( \text{char} F = 0 \) its restriction to \( \text{ad} g \simeq g' \) (the Killing form of \( g' \)) is again non-degenerate. Hence

\[
\text{End}_F(\mathfrak{g}) = g' \oplus g'_{\perp}
\]

where each summand is \( \text{Ad} G \)-invariant. Also \( \text{Tr} | \mathfrak{d} \times \mathfrak{d} \) is non-degenerate with non-degenerate restriction to \( \text{ad} t \simeq t' \) so that

\[
\mathfrak{d} = t' \oplus t'_{\perp}.
\]

Equation 2.1 and the equation

\[
\text{ad} X_\alpha . H_\beta = -(\alpha, \beta ') X_\alpha \quad (\alpha, \beta \in \Phi)
\]

imply that the matrix of \( \text{ad} X_\alpha \) with respect to our fixed basis of \( g \) has zeros on the diagonal. Hence \( \text{Tr} (d \text{ad} X_\alpha) = 0 \) for \( d \in \mathfrak{d} \) and \( \alpha \in \Phi \). It follows that \( t'_{\perp} \) (\( \perp \) in \( \mathfrak{d} \)) is contained in \( g' \perp \) (\( \perp \) in \( \text{End}_F(\mathfrak{g}) \)).

Write \( a = a_0 + z \) where \( a_0 \in \mathfrak{t}_0 \) and \( z \in \mathfrak{z} \). Note that \( \text{ad} a = \text{ad} a_0 \) viewed as an element of \( \mathfrak{d} \) may have distinct diagonal entries (eigenvalues) which are equal \( (\pmod{\mathfrak{P}_F}) \). Let \( d_0 \) be an element of \( \mathfrak{d} \) obtained by lifting such equalities \( (\pmod{\mathfrak{P}_F}) \) to equalities.

Fix an algebraic closure \( \overline{F} \) of \( F \). Write \( \overline{\mathfrak{O}} \) for the ring of integers of \( \overline{F} \) and \( \mathfrak{P} \) for the unique prime ideal in \( \overline{\mathfrak{O}} \). Let \( \overline{g} = \overline{F} \otimes_F g \). Then \( \text{End}_F(\mathfrak{g}) \) embeds canonically in \( \text{End}_{\overline{F}}(\overline{g}) \). Since each \( \text{ad} X_i \) preserves the \( \overline{\mathfrak{O}} \)-lattice \( \overline{\mathfrak{R}_0} = \overline{\mathfrak{O}} \otimes_{\mathfrak{O}_F} \mathfrak{R}_0 \) in \( \overline{g} \), their eigenvalues belong to \( \overline{\mathfrak{O}} \). Let \( \lambda \in \overline{\mathfrak{O}} \) be an eigenvalue of \( \text{ad} X_i \) and let

\[
\overline{g}_\lambda (\text{ad} X_i) = \{ v \in \overline{g} : (\text{ad} X_i - \lambda)^n v = 0 \quad \text{for some} \ n \in \mathbb{N} \}
\]

be the corresponding generalised eigenspace. Since \( \text{ad} X_i = \text{ad} a + \text{ad} Y_i \) where \( Y_i \in \mathfrak{R}_i \), we see \( \lambda \equiv \mu_0 (\pmod{\mathfrak{P}}) \) for some eigenvalue \( \mu_0 \) of \( \text{ad} a_0 \). Hence \( \lambda \equiv \mu (\pmod{\mathfrak{P}}) \) for the unique eigenvalue \( \mu \) of \( d_0 \) such that \( \mu \equiv \mu_0 (\pmod{\mathfrak{P}}) \).

We claim \( \overline{g}_\lambda (\text{ad} X_i) \subseteq \overline{g}_\mu (d_0) \) - the \( \mu \)-eigenspace of \( d_0 \). Write \( v \in \overline{g}_\lambda (\text{ad} X_i) \) as \( v = v_1 + \ldots + v_l \) where \( v_j \in \overline{g}_{\mu_j} (d_0) \) with \( \mu_i = \mu \) and \( \mu_j \neq \mu_k \) if \( j \neq k \). Since \( X_i \in \mathfrak{g}(a) = \mathfrak{g}(a_0) \), each eigenspace of \( \text{ad} a = \text{ad} a_0 \) is \( \text{ad} X_i \)-stable. Further since each eigenspace of \( d_0 \) is a sum of eigenspaces of \( \text{ad} a_0 \), we see \( \text{ad} X_i \) also stabilises these eigenspaces. Hence \( (\text{ad} X_i - \lambda)^n v_j = 0 \) for each \( j \). If \( v_j \neq 0 \) we may assume \( v_j \in \mathfrak{R}_0 \setminus \mathfrak{R}_1 \). Then \( (d_0 - \lambda)^n v_j \in \mathfrak{R}_1 \) and thus \( \mu_j \equiv \lambda (\pmod{\mathfrak{P}}) \). Hence \( v_j \neq 0 \) implies \( j = 1 \), i.e. \( \overline{g}_\lambda (\text{ad} X_i) \subseteq \overline{g}_\mu (d_0) \).

Since \( \overline{g} \) is the sum of both the generalised eigenspaces of \( \text{ad} X_i \) and the eigenspaces of \( d_0 \), we must have

\[
\sum_{\lambda \equiv \mu (\pmod{\mathfrak{P}})} \overline{g}_\lambda (\text{ad} X_i) = \overline{g}_\mu (d_0).\]
The relation $\text{Ad}_g \text{Ad}_X^\lambda \text{Ad}_{g^{-1}} = \text{ad}_X^\lambda$ then implies $\text{Ad}_g \tilde{g}_\lambda(\text{ad}_X^\lambda) = \tilde{g}_\lambda(\text{ad}_X^\lambda)$ for all $\lambda$. Hence $\text{Ad}_g$ preserves the eigenspaces of $d_0$ and thus $\text{Ad}_g d_0 \text{Ad}_{g^{-1}} = d_0$. Write $d_0 = \text{ad} a'_1 + a''_1$ where $a'_1 \in t'$ and $a''_1 \in t'^\perp \subset \mathfrak{g}'$. Then

$$d_0 = \text{Ad}_g \text{ad} a'_1 \text{Ad}_{g^{-1}} + \text{Ad}_g a''_1 \text{Ad}_{g^{-1}}.$$

Since $\mathfrak{g}'$ and $\mathfrak{g}'^\perp$ are both ad-$g$-stable, we have

$$\text{Ad}_g \text{ad} a'_1 \text{Ad}_{g^{-1}} = \text{ad} a'_1.$$

This gives $\text{ad}(\text{Ad}_g a'_1) = \text{ad} a'_1$ and hence $\text{Ad}_g a'_1 = a'_1$. Letting $a_1 = a'_1 + z$, we have $\text{Ad}_g a_1 = a_1$.

Finally we show $a_1 \subset a + t_1$ by proving $x(a_1 - a) = x(a'_1 - a_0) \in \mathcal{P}_F$ for all $x \in X$. Let $p : \mathfrak{d} \to t'$ be the projection map corresponding to the decomposition $\mathfrak{d} = t' \oplus t'^\perp$. Suppose $d \in \mathfrak{d}$ satisfies $d(X_\alpha) = d_\alpha X_\alpha$ for $\alpha \in \Pi$. Then it is easy to check that $p(d)$ is the unique element $t$ of $t'$ such that $\alpha(t) = d_\alpha$ for $\alpha \in \Pi$. It follows that the element $p(d_0) = a'_1$ satisfies

$$\alpha(a'_1) \equiv \alpha(a_0) \pmod{\mathcal{P}_F}$$

for all $\alpha \in \Pi$ and hence also for all $\alpha \in \Phi$. Further if $x \in X_0$ where

$$X_0 = \{ x \in X : \langle x, \alpha^\vee \rangle = 0 \, (\forall \alpha^\vee \in \Phi^\vee) \}$$

then $x(a'_1 - a_0) = 0$. Hence $x(a'_1 - a_0) \in \mathcal{P}_F$ for any $x \in Q + X_0$. Note that $Q + X_0$ has finite index in $X$ and the quotient embeds in $P/Q$ where $P$ is the weight lattice (exactly as in as in $4.4$). Thus char $k_F$ does not divide the index $[X : Q + X_0]$ and hence $x(a'_1 - a_0) \in \mathcal{P}_F$ for all $x \in X$. \hfill \Box

We can now compute the $G$-intertwining of $\rho \mid L$.

**Proposition 4.11.** - With notation as above:

$$\mathcal{I}_G(\rho \mid L) = L C_G(a_1) L.$$

**Proof.** - Let $g \in \mathcal{I}_G(\rho \mid L)$. By Lemma 4.6, there exist elements $X$ and $X'$ in $a + \mathfrak{L}'^\perp$ such that $\text{Ad}_g X = X'$. By Corollary 4.8, we can find $x$ and $x'$ in $L$ such that $X_1 = \text{Ad} x X$ and $X_2 = \text{Ad} x' X'$ belong to $3_g(a) \cap \mathfrak{R}_{i-1}$. Further $X_i \equiv a \pmod{\mathfrak{R}_{i-1}}$ for $i = 1, 2$ (since $L$ acts as the identity via ad on $\mathfrak{R}_{i-1}/\mathfrak{R}_{i-2}$). From $\text{Ad}(x'gx^{-1}) X_1 = X_2$ and Lemma 4.10, we see $g \in L C_G(a_1) L$. Since $L C_G(a_1) L$ is trivially contained in $\mathcal{I}_G(\rho \mid L)$, we obtain

$$\mathcal{I}_G(\rho \mid L) = L C_G(a_1) L.$$

The group $C_G(a_1)$ is the group of $F$-rational points of the reductive subgroup $C_G(a_1)$ of $G$. Since char $F = 0$, $C_G(a_1)$ is connected. Hence

$$C_G(a_1) = \langle T, U_\alpha : \alpha(a_1) = 0 \, (\alpha \in \Phi) \rangle.$$

Note that $\alpha(a_1) = 0$ for $\alpha \in \Phi$ implies $\alpha(a) \equiv 0(\pmod{\mathcal{P}_F})$ since $a \in a_1 + t_1$. From Lemma 4.4, we see $\alpha(a) = 0$. Hence $C_G(a_1)$ is contained in $C_G(a)$ and thus $\mathcal{I}_G(\rho \mid L)$
is contained in $LC_G(a)L$. Since the opposite containment also holds, we conclude $I_G(p|L) = LC_G(a)L$. □

Remark 4.12. – Our method of calculating $I_G(\rho)$ is adapted from Howe and Moy [19] (using adjoint representations in place of the natural action of $GL_n(F)$ and its Lie algebra on $F^n$). The Proposition is valid in a wider context. See Adler [1] and Adler and Roche [2].

Remark 4.13. – We have restricted the residual characteristic $p$ of $F$ in several ways. More precisely, we assume

- $p$ is prime to any integers which occur as ratios of squares of root lengths for any pair of roots in the same irreducible factor of $\Phi$
- $p$ is not a bad prime for $\Phi$
- $p$ does not divide $2[P^\nu : Q^\nu] = 2[P : Q]$.

Thus if $\Phi$ is irreducible, the excluded primes are

- for type $A_n$, divisors of $2(n + 1)$
- for types $B_n, C_n, D_n$, 2
- for types $E_6, E_7, F_4, G_2$, 2, 3
- for type $E_8$, 2, 3, 5.

If $\Phi$ is not irreducible, we exclude primes attached to its irreducible factors.

Remark 4.14. – Clearly, our restrictions are most severe in the case of groups of type $A_n$. However (using the trace form on the Lie algebra in place of the form $B$) our intertwining results are easily seen to hold for $GL_N$ without restriction on the characteristic or the residual characteristic. Our determination of the structure of $\mathcal{H}(G, \rho)$ (see section 6) is then valid for $G = GL_N$ (without restriction on $F$ or $k_F$). Further, by embedding a part of the $GL_N$ Hecke algebra in the corresponding $SL_N$ Hecke algebra (and using some additional arguments), our description of $\mathcal{H}(G, \rho)$ may also be seen to hold also for $G = SL_N$ (without restriction on $F$ or $k_F$) though we do not pursue this here.

In theorem 4.15 below we use Proposition 4.11 and an inductive argument to determine the $G$-intertwining of $\rho$. In the inductive step, we need to know that the statement of Proposition 4.11 is valid not only for $G$ but also for Levi subgroups of $G$. The following argument shows that this holds given restrictions on residual characteristic only slightly stronger than those in Remark 4.13.

Suppose $\theta : G_1 \to G_2$ is a central isogeny defined over $O_F$ (more precisely the restriction to $G_1$ of such an isogeny $\theta : G_1 \to G_2$). Then $\theta$ induces an isomorphism $g_1 \cong g_2$ (since $p$ does not divide $[P : Q]$). Write $\theta(a_1 + \Sigma_1^\perp) = a_2 + \Sigma_2^\perp$ (in the obvious notation). Then $I_G_1(a_1 + \Sigma_1^\perp) = L_1C_{G_1}(a_1)L_1$ if and only if $I_G_2(a_2 + \Sigma_2^\perp) = L_2C_{G_2}(a_2)L_2$. (The ‘if’ direction is immediate. To obtain the ‘only if’ direction, use the surjective map $G_1C_1(\overline{F}) \to G_2C_2(\overline{F})$ where $C_i$ is the centre of $G_i$).

We apply this observation to the map $Z^0 \times G_{\text{der}} \to G$ (given by multiplication) where $G_{\text{der}}$ is the group of $F$-rational points of the derived group of $G$ and $Z^0$ is the group of $F$-rational points of the identity component of the centre of $G$. Thus the statement of Proposition 4.11 applies to $G$ and its Levi subgroups if and only if it applies to $G_{\text{der}}$ and its Levi subgroups. Moreover, it applies to $G_{\text{der}}$ and its Levi subgroups if and only if it applies to each of its almost simple factors and their Levi subgroups.
If \( G \) is symplectic (resp. special orthogonal) then each Levi subgroup of \( G \) is isomorphic to a product of general linear groups and a symplectic (resp. special orthogonal) group of lower rank. Hence (using Remark 4.14) Proposition 4.11 applies to all symplectic and special orthogonal groups and all their Levi subgroups provided \( p \neq 2 \). Thus the same conclusion holds for groups each of whose irreducible factors has type \( B, C \) or \( D \).

We list explicit restrictions on \( p = \text{char} \, k_F \) under which the proof of Theorem 4.15 is valid. If \( \Phi \) is irreducible, we restrict \( p \) as follows

- for type \( A_n \) \( p > n + 1 \)
- for types \( B_n, C_n, D_n \) \( p \neq 2 \)
- for type \( F_4 \) \( p \neq 2, 3 \)
- for types \( G_2, E_6 \) \( p \neq 2, 3, 5 \)
- for types \( E_7, E_8 \) \( p \neq 2, 3, 5, 7 \).

If \( \Phi \) is not irreducible, we exclude primes attached to each of its irreducible factors. (Of course, the restriction on groups of type \( A_n \) is far too stringent and can be significantly relaxed.)

**Theorem 4.15.** – The \( G \)-intertwining of \( \rho \) is given by:

\[
\mathcal{I}_G(\rho) = JW_\chi J
\]

where \( W_\chi = \{ w \in W \mid w\chi = \chi \} \).

**Proof.** – If \( \chi \) has level zero, the result follows from Prop. 4.1 (see Remark 4.2). Thus, without loss of generality, \( \chi \) has positive level. Since \( \mathcal{I}_G(\rho) \) is then contained in \( \mathcal{I}_G(\rho \mid L) \), Prop. 4.11 implies \( \mathcal{I}_G(\rho) \subset J C_G(a) J \). We first show we may assume \( a \notin \mathfrak{z} \) so that \( C_G(a) \neq G \).

Let \( T' \) be the group of \( F \)-rational points of the subtorus \( T' = T \cap G' \) of \( T \). There exists a subtorus \( \mathcal{S} \) of \( \mathcal{T} \) such that \( T = \mathcal{S} \times T' \) (see e.g. Digne and Michel [16], Prop. 0.5). Further, this splitting is defined over \( \mathcal{O}_F \) so that \( T = \mathcal{S} \times T' \) and \( T_q = \mathcal{S}_q \times T'_q \) for all \( q \in \mathbb{N} \) (in the obvious notation). Suppose now \( a \notin \mathfrak{z} \). This implies \( \chi \mid T'_{\mathfrak{z}-1} \) is trivial. Fix a smooth character \( \chi_1 : T \to \mathbb{C}^* \) such that \( \chi_1 \mid T' = 1 \) and \( \chi_1 \mid S_{\mathfrak{z}-1} = \chi^{-1} \mid S_{\mathfrak{z}-1} \). Since \( \chi \chi_1 \circ \alpha^\vee = \chi \circ \alpha^\vee \) for \( \alpha \in \Phi \), we see \( J_\chi = J_{\chi \chi_1} \). Noting that \( \chi_1 \) extends to a character of \( G \) (which we again denote by \( \chi_1 \) ), we see \( \rho_{\chi \chi_1} = \rho_\chi \otimes \chi_1 \). Write \( \rho_1 \) for \( \rho_{\chi \chi_1} \). Then \( \mathcal{I}_G(\rho_1) = \mathcal{I}_G(\rho) \). By construction \( \rho_1 \mid K_{\mathfrak{z}-1} \) is trivial. Repeating this argument if necessary with \( \chi \chi_1 \), we may assume \( a \notin \mathfrak{z} \).

As already observed (in the proof of Proposition 4.11) \( C_G(a) \) is (the group of \( F \)-rational points of) a connected reductive subgroup of \( G \) containing \( T \). It follows that \( C_G(a) \) is a semistandard Levi subgroup of \( G \), i.e. a Levi subgroup of \( G \) containing \( T \). Indeed \( C_G(a) = C_G(T_\mathfrak{z}) \) where \( T_\mathfrak{z} \) is the group of \( F \)-rational points of the subtorus \( T_\mathfrak{z} \) of \( T \) corresponding to the co-torsion-free submodule \( X_\mathfrak{z} \) of \( X \) where

\[
X_\mathfrak{z} = \{ x \in X \mid x(a) = 0 \}.
\]

In particular, \( C_G(a) \) is a connected reductive subgroup of \( G \) containing \( T \) of strictly smaller semisimple rank. Further \( J \cap C_G(a) \) is \( J \) for \( C_G(a) \) with respect to the positive system for \( \Phi(C_G(a), T) \) induced by \( \Phi^+ \). Also the filtration subgroups \( K_q (q \in \mathbb{N}) \) (resp. filtration
lattices \( R_r (r \in \mathbb{Z}) \) intersect with \( C_G(a) \) (resp. \( \mathfrak{g}_a(a) \)), the Lie algebra of \( C_G(a) \) to give the corresponding objects in \( C_G(a) \) (resp. \( \mathfrak{g}_a(a) \)).

Since the theorem is clear if \( G \) has semisimple rank zero (i.e. \( G = T \)) we conclude (by induction on the semisimple rank) that \( \mathcal{I}_G(\rho) \) is given by \( JW_X J \). \( \square \)

5. An isomorphism of Hecke algebras

In this section, we prove a slight modification of a result of Bushnell and Kutzko. We need this modification to show that certain Hecke algebra isomorphisms preserve canonical involutions and inner products.

Let \( P = MN \) and \( \overline{P} = M\overline{N} \) be opposite parabolic subgroups of \( G \) having Levi factor \( M \) (and unipotent radicals \( N \) and \( \overline{N} \)). Let \((K, \tau)\) be a pair consisting of a compact open subgroup \( K \) of \( G \) and a smooth character \( \tau \) of \( K \). Write \( K_M = K \cap M \), \( K_N = K \cap N \) and \( K_{\overline{N}} = K \cap \overline{N} \). We assume \((K, \tau)\) satisfies the following conditions:

i) \( K = K_{\overline{N}}K_MK_N \)

ii) \( K_{\overline{N}} \) and \( K_N \) are contained in the kernel of \( \tau \).

Let \( \tau_M = \tau | K_M \). We write \( \mathcal{I}_G(\tau) \) for the \( G \)-intertwining of \( \tau \) (as before) and assume the pair \((K, \tau)\) satisfies the further condition:

iii) \( \mathcal{I}_G(\tau) \) is contained in \( JM_J \).

From i) this is equivalent to \( \mathcal{I}_G(\tau) = K\mathcal{I}_M(\tau_M)K \) (using the obvious notation). Under these hypotheses, it is a special case of a result of Bushnell and Kutzko ([13] 7.2 (ii)) that there exists an algebra isomorphism

\[
(5.1) \quad t : \mathcal{H}(M, \tau_M) \xrightarrow{\sim} \mathcal{H}(G, \tau)
\]

which is support-preserving in the sense that \( \text{supp}(t(\phi)) = K \text{supp}(\phi)K \) for \( \phi \in \mathcal{H}(M, \tau_M) \).

The algebras \( \mathcal{H}(M, \tau_M) \) and \( \mathcal{H}(G, \tau) \) each carry a canonical involution \(*\) defined by \( f^*(x) = \overline{f(x^{-1})} \) (\( f \in \mathcal{H}(M, \tau_M) \) or \( \mathcal{H}(G, \tau) \)). However the isomorphism \( t \) is not \(*\)-preserving. We show in this section that \( t \) may be modified (by twisting by the square-root of the modulus character of \( P \)) to obtain a \(*\)-preserving isomorphism \( t_u \) (which is clearly still support-preserving). More precisely, if \( \delta_p(m) = |\det(\text{Ad}m; \text{Lie}N)| \) for \( m \in M \), the map

\[
f \mapsto f.\delta_p^{1/2} : \mathcal{H}(M, \tau_M) \longrightarrow \mathcal{H}(M, \tau_M)
\]

defines an algebra isomorphism where \( f.\delta_p^{1/2}(m) = f(m)\delta_p^{1/2}(m) \) (\( m \in M \) and \( f \in \mathcal{H}(M, \tau_M) \)). The isomorphism \( t_u : \mathcal{H}(M, \tau_M) \xrightarrow{\sim} \mathcal{H}(G, \tau) \) is obtained by precomposing with \( t \), i.e. \( t_u(f) = t(f.\delta_p^{1/2}) \) for \( f \in \mathcal{H}(M, \tau_M) \).

The existence of a support-preserving, \(*\)-preserving isomorphism between \( \mathcal{H}(M, \tau_M) \) and \( \mathcal{H}(G, \tau) \) will be used repeatedly in later sections. The results of this section may also be extended (without difficulty) to the case where \( \tau \) is a smooth irreducible representation of \( K \) (not necessarily one-dimensional).

We first recall briefly the construction of the isomorphism 5.1. See section 6 of Bushnell and Kutzko [13] for full details. Write \( \mathcal{I}^+ \) for the set of elements \( x \) in \( M \) such that
$K_N x^{-1} \subset K_N$ and $x^{-1} K_N x \subset K_N$. Such elements are called positive. The collection of functions in $\mathcal{H}(M, \tau_M)$ with support contained in $K_M T^+ K_M$ forms a subalgebra which we denote by $\mathcal{H}^+(M, \tau_M)$. Suppose $\phi \in \mathcal{H}(M, \tau_M)$ is supported on $K_M x K_M$ ($x \in T^+$). Write $\Phi$ for the unique element of $\mathcal{H}(G, \tau)$ supported on $K x K$ such that $\phi(x) = \Phi(x)$. The assignment $\phi \mapsto \Phi$ extends to an algebra embedding $t : \mathcal{H}^+(M, \tau_M) \rightarrow \mathcal{H}(G, \tau)$.

Now fix a strongly $(P, K)$ positive element $\zeta$ in the center of $M$ (see Bushnell and Kutzko [13] 6.16). In particular, $\zeta$ is positive and for each $x \in I_M(\tau_M)$ there exists a positive integer $n$ such that $\zeta^n x$ is positive. For $m \in \mathbb{Z}$, write $\phi_m$ for the unique element of $\mathcal{H}(M, \tau_M)$ supported on $K_M \zeta^m$ such that $\phi_m(\zeta^m) = 1$. Clearly

$$\phi_m \phi_n = \phi_{m+n} \quad (m, n \in \mathbb{Z}).$$

For $m \in \mathbb{Z}$, we write $\Phi_m$ for the unique element of $\mathcal{H}(G, \tau)$ supported on $K \zeta^m K$ such that $\Phi_m(\zeta^m) = 1$. Thus $t(\phi_m) = \Phi_m$ for $m \in \mathbb{N}$. Our hypotheses on $I_G(\tau)$ imply that $\Phi_1$ (and hence each $\Phi_m$ for $m \in \mathbb{N}$) is invertible in $\mathcal{H}(G, \tau)$. Then the map $t : \mathcal{H}^+(M, \tau_M) \rightarrow \mathcal{H}(G, \tau)$ extends to a (well-defined) algebra embedding of $\mathcal{H}(M, \tau_M)$ as follows. (It is independent of the choice of strongly $(P, K)$ positive element $\zeta$ in the center of $M$.) Let $\phi \in \mathcal{H}(M, \tau_M)$. There exists a positive integer $m$ such that $\phi_m \phi \in \mathcal{H}^+(M, \tau_M)$. By definition,

$$t(\phi) = \Phi_m^{-1} t(\phi_m \phi).$$

Our hypotheses on $I_G(\tau)$ imply that $t$ is a support-preserving isomorphism of $\mathbb{C}$-algebras.

The isomorphism $t_u : \mathcal{H}(M, \tau_M) \sim \mathcal{H}(G, \tau)$ is $*$-preserving if and only if it is unitary with respect to the standard inner products $(\cdot, \cdot)_M$ and $(\cdot, \cdot)_G$ on $\mathcal{H}(M, \tau_M)$ and $\mathcal{H}(G, \tau)$ (equivalently if and only if $\|\phi\|_M = \|t_u(\phi)\|_G$ for $\phi \in \mathcal{H}(M, \tau_M)$). Here $(\phi_1, \phi_2) = \int_M \phi_1(m) \overline{\phi_2(m)} \, dm$ and $\|\phi\|^2_M = (\phi, \phi)_M$ for $\phi, \phi_1, \phi_2 \in \mathcal{H}(M, \tau_M)$ where the Haar measure $dm$ on $M$ gives $K_M$ measure one (and similarly for the corresponding objects on $G$). To see this, first use the relation $(\phi_1, \phi_2)_M = (e_{\tau_M}, \phi_1 \phi_2)_M$ for $\phi_1, \phi_2$ in $\mathcal{H}(M, \tau_M)$, where $e_{\tau_M}$ is the identity element of $\mathcal{H}(M, \tau_M)$. Combining this with the corresponding relation on $G$ shows that $t_u$ is unitary implies $t_u$ is $*$-preserving. The converse follows easily from the relation

$$\phi_1 \phi_2^* (1) = (\phi_1, \phi_2)_M$$

for $\phi_1, \phi_2 \in \mathcal{H}(M, \tau_M)$ (and similarly on $G$) combined with the fact that $t_u$ is support-preserving.

**Proposition 5.1.** Suppose $(K, \tau)$ and $t : \mathcal{H}(M, \tau_M) \rightarrow \mathcal{H}(G, \tau)$ are as above. Then $t_u : \mathcal{H}(M, \tau_M) \rightarrow \mathcal{H}(G, \tau)$ defined by

$$t_u(\phi) = t(\phi, \phi_p^{1/2}) \quad (\phi \in \mathcal{H}(M, \tau_M))$$

is a $*$-preserving, support-preserving isomorphism of $\mathbb{C}$-algebras.

**Proof.** It is clear that $t$ is a support-preserving isomorphism of $\mathbb{C}$-algebras. By the discussion preceding the statement of the proposition, it is $*$-preserving if and only if

$$\|t_u(\phi)\|_G = \|\phi\|_M \quad (\phi \in \mathcal{H}(M, \tau_M)).$$
We first show this for $\phi \in \mathcal{H}^+(M, \tau_M)$. It is enough to consider the case where $\phi$ is supported on a single double coset $K_M x K_M (x \in \mathcal{I}^+)$. Then $t_u(\phi) = \delta^{1/2}_p(x) \Phi$ where $\Phi$ has support $K x K$ and $\Phi(x) = \phi(x)$. Thus $\|t_u(\phi)\|_G = \|\phi\|_M$ if and only if

$$\text{vol} (K_M x K_M) = \delta_p(x) \text{vol} (K x K).$$

To see this, note

$$\text{vol} (K x K) = [K : xKx^{-1} \cap K]$$

$$= [K_N : xK_Nx^{-1} \cap K_N][K_M : xK_Mx^{-1} \cap K_M][K_{N-} : xK_{N-}x^{-1} \cap K_{N-}]$$

$$= [K_N : xK_Nx^{-1}][K_M : xK_Mx^{-1} \cap K_M] \quad (x \in \mathcal{I}^+)$$

$$= \delta_p(x) \text{vol} (K_M x K_M).$$

Now suppose $\phi \in \mathcal{H}(M, \tau_M)$ is arbitrary. There exists a positive integer $m$ such that $\phi_m \phi \in \mathcal{H}^+(M, \tau_M)$. Clearly $\|\phi_m \phi\|_M = \|\phi\|_M$. To prove the proposition it is therefore sufficient to show

$$\|t_u(\phi_1)\|_G = \|\phi\|_G \quad (\Phi \in \mathcal{H}(G, \tau)).$$

Since $\|t_u(\phi_1)\|_G = 1$ and $\|\Phi_1 \Phi_2\|_G \leq \|\Phi_1\|_G \|\Phi_2\|_G$ (for $\Phi_1, \Phi_2 \in \mathcal{H}(G, \tau)$), this holds provided we can show

$$\|t_u(\phi_1)^{-1}\|_G = 1.$$

However

$$t_u(\phi_1)^{-1} = t_u(\phi_{-1})$$

$$= c \Phi_{-1}$$

for some constant $c$ (since $t_u$ is support-preserving). Further, $t_u(\phi_1 \phi_{-1})$ equals the identity element of $\mathcal{H}(G, \tau)$ and hence $c \Phi_1 \Phi_{-1}(1) = 1$. It follows that $c \text{vol} (K \zeta^{-1} K) = 1$ which implies $\|t_u(\phi_{-1})\|_G = 1$ as required. \qed

Remark 5.2. – Suppose $\phi \in \mathcal{H}(M, \tau_M)$ is supported on $K_M x K_M$ for some $x \in M$. As above, we denote by $\Phi$ the unique element of $\mathcal{H}(G, \tau)$ supported on $K x K$ such that $\Phi(x) = \phi(x)$. From the proof of 7.2 ii) in [13], we have

$$t(\phi) = [K_{N-} : x^{-1}K_{N-}x \cap K_{N-}] \Phi.$$

In particular, $t(\phi) = c \Phi$ where $c$ is a positive constant. Hence $t_u(\phi)$ is also a positive constant times $\Phi$. Using $\|t_u(\phi)\|_G = \|\phi\|_M$, we now see

$$t_u(\phi) = \frac{\text{vol} (K_M x K_M)^{1/2}}{\text{vol} (K x K)^{1/2}} \Phi.$$

We use this observation in the proof of Lemma 9.3.
6. The structure of $\mathcal{H}(G, \rho)$

We can now describe the structure of the Hecke algebra $\mathcal{H}(G, \rho)$. First we recall some standard notation. Let $V = Y \otimes \mathbb{Z} \mathbb{R}$. For each $\alpha \in \Phi$, $s_\alpha \in \text{Aut}(Y)$ is defined by

$$s_\alpha(y) = y - \langle y, \alpha \rangle \alpha^\vee \quad (y \in Y)$$

where $\langle , \rangle$ is the canonical perfect pairing $\langle , \rangle : X \times Y \to \mathbb{Z}$ and $\alpha^\vee \in \Phi^\vee$ is the coroot associated to $\alpha \in \Phi$. Since there is a canonical injection from $\text{Aut}(Y)$ into $\text{Aut}(V)$ ($= \text{Aut}_\mathbb{R}(V)$), we may also view $s_\alpha$ as an element of $\text{Aut}(V)$. Again we have

$$s_\alpha(v) = v - \langle v, \alpha \rangle \alpha^\vee \quad (v \in V)$$

where $\langle , \rangle$ now denotes the extension of $\langle , \rangle : X \times Y \to \mathbb{Z}$ to a (non-degenerate, $\mathbb{R}$-bilinear) pairing between $X \otimes \mathbb{Z} \mathbb{R}$ and $V$. Let $\Phi_{af}$ be the set of affine roots associated to $\Phi$. Thus

$$\Phi_{af} = \{\alpha + k : \alpha \in \Phi, k \in \mathbb{Z}\}$$

viewed as a set of affine functionals on $V$. If $a \in \Phi_{af}$, we write $s_a$ for the corresponding affine reflection. Thus if $a = \alpha + k$ ($\alpha \in \Phi, k \in \mathbb{Z}$)

$$s_a(v) = v - a(v) \alpha^\vee = v - \langle v, \alpha \rangle \alpha^\vee - k\alpha^\vee$$

for $v \in V$. We have $s_a \in \text{Aff}(V)$ (the set of affine-linear transformations of $V$). We now define some of the objects that will appear in our description of $\mathcal{H}(G, \rho)$ (cf. Goldstein [17] Definition 3.5).

**Definition 6.1.**

$$\Phi_\chi = \{\alpha \in \Phi : \chi \circ \alpha^\vee | O_F^\chi = 1\}$$

$$\Phi_{\chi, af} = \{a \in \Phi_{af} : a = \alpha + k \quad (\alpha \in \Phi_\chi, k \in \mathbb{Z})\}$$

$$W_\chi^0 = \{s_\alpha : a \in \Phi_{\chi, af}\} \leq \text{Aff}(V)$$

$$\bar{W}_\chi^0 = \{s_\alpha : \alpha \in \Phi_\chi\} \leq \text{Aut}(V)$$

It is immediate that $\Phi_\chi^\vee = \{\alpha^\vee \in \Phi^\vee \mid \chi \circ \alpha^\vee | O_F^\chi = 1\}$ is a closed sub-root system of $\Phi^\vee$. It follows that $\Phi_\chi$ is a sub-root system of $\Phi$ with Weyl group $\bar{W}_\chi^0$. Let $\Phi_\chi^+ = \Phi^+ \cap \Phi_\chi$ so that $\Phi_\chi^+$ is a positive system in $\Phi_\chi$. Define

$$C_\chi = \{v \in V \mid 0 < \alpha(v) < 1 \quad (\alpha \in \Phi_\chi^+)\}.$$ 

Then $C_\chi$ is a chamber in the decomposition of $V$ induced by $\Phi_{\chi, af}$, i.e. $C_\chi$ is a connected component of $V - \bigcup_{a \in \Phi_{\chi, af}} H_a$ where $H_a = \{v \in V \mid a(v) = 0\}$. Note that $C_\chi$ defines an ordering on $\Phi_{\chi, af}$ via $a > 0$ if and only if $a(v) > 0$ for all $v \in C_\chi$ ($a \in \Phi_{\chi, af}$). Let $\Pi_{\chi, af} = \{a \in \Phi_{\chi, af} : a$ is a minimal positive element $\}$. The walls of the chamber $C_\chi$ are then the various $H_a$'s as $a$ ranges through $\Pi_{\chi, af}$. Let

$$S_\chi^0 = \{s_a \mid a \in \Pi_{\chi, af}\}, \quad \Omega_\chi = \{w \in W_\chi \mid w C_\chi = C_\chi\}.$$
LEMMA 6.2. - \((W_\chi^0, S_\chi^0)\) is a Coxeter system and \(W_\chi\) decomposes as:

\[ W_\chi = W_\chi^0 \times \Omega_\chi. \]

The group \(\Omega_\chi\) preserves the set of generators \(S_\chi^0\).

Proof. - It follows from Bourbaki [NB] (V, section 3) that \((W_\chi^0, S_\chi^0)\) is a Coxeter system. Hence \(W_\chi^0\) acts simply transitively on the the chambers of \(V\) (with respect to \(W_\chi^0\)) and thus \(W_\chi = W_\chi^0 \Omega_\chi\) and \(W_\chi^0 \cap \Omega_\chi = 1\). Since \(\Omega_\chi\) fixes \(C_\chi\), each element of \(\Omega_\chi\) permutes the walls of \(C_\chi\). This implies that \(\Omega_\chi\) preserves the set of generators \(S_\chi^0\). In particular, \(W_\chi^0\) is a normal subgroup of \(W_\chi\).

The Hecke algebra \(\mathcal{H}(W_\chi^0, S_\chi^0)\) of the Coxeter system \((W_\chi^0, S_\chi^0)\) is the associative \(\mathbb{C}\)-algebra with \(\mathbb{C}\)-basis \(\{e_w \mid w \in W_\chi^0\}\) and multiplication given by:

\begin{align*}
(6.1) & \quad e_w e_{w'} = e_{ww'} & \text{if } l(ww') = l(w) + l(w'), \\
(6.2) & \quad e_s^2 = q e_1 + (q - 1) e & \text{for } s \in S_\chi^0.
\end{align*}

In (6.1) \(l\) is the length function of the Coxeter system \((W_\chi^0, S_\chi^0)\). It is well-known that such an algebra exists and is unique (e.g. see Humphreys [22], chap. 7).

The group \(\Omega_\chi\) acts by \(\mathbb{C}\)-algebra automorphisms on \(\mathcal{H}(W_\chi^0, S_\chi^0)\) via \(\omega e_w = e_{\omega w}\) where \(\omega(w) = \omega w \omega^{-1}\). This action clearly extends to an injective homomorphism of \(\mathbb{C}\)-algebras \(\mathbb{C}[\Omega_\chi] \rightarrow \text{End}_{\mathbb{C}-\text{alg}} \mathcal{H}(W_\chi^0, S_\chi^0)\). Via this homomorphism, we obtain a \(\mathbb{C}\)-algebra structure on the \(\mathbb{C}\)-vector space \(\mathcal{H}(W_\chi^0, S_\chi^0) \otimes \mathbb{C}[\Omega_\chi]\). Explicitly, the multiplication is determined by the formula:

\[ e_w \otimes e_{w'} = e_{ww'} \quad (w, w' \in W_\chi^0 \text{ and } \omega, \omega' \in \Omega_\chi). \]

We denote this algebra by \(\mathcal{H}(W_\chi^0, S_\chi^0) \otimes \mathbb{C}[\Omega_\chi]\). We sometimes abbreviate to \(\mathcal{H}_\chi\). We frequently write the basis elements \(e_w \otimes e_\omega\) as \(e_{ww}\) where \(w \in W_\chi^0\) and \(\omega \in \Omega_\chi\). Thus \(e_{ww} = e_w e_\omega\) and \(e_w e_\omega e_{w^{-1}} = e_{ww} (w \in W_\chi^0, \omega \in \Omega_\chi)\).

Define a function \(* : \mathcal{H}_\chi \rightarrow \mathcal{H}_\chi\) by extending \(\mathbb{C}\)-antilinearly the assignment \(e_w^* = e_{w^{-1}}\). Thus \((\alpha e_w)^* = \overline{\alpha} e_w^*\) for \(w \in W_\chi^0\) and \(\alpha \in \mathbb{C}\) where ‘bar’ denotes complex conjugation. The function \(*\) defines an involution of \(\mathcal{H}_\chi\). Define an involution \(*\) on \(\mathcal{H}(G, \rho)\) by \(f^*(x) = \overline{f}(x^{-1})\) for \(f \in \mathcal{H}(G, \rho)\) and \(x \in G\). We now state the main result of this section.

THEOREM 6.3. - i) With notation as above, there exists a family of \(*\)-preserving \(\mathbb{C}\)-algebra isomorphisms:

\[ \Psi : \mathcal{H}(W_\chi^0, S_\chi^0) \otimes \mathbb{C}[\Omega_\chi] \rightarrow \mathcal{H}(G, \rho). \]

ii) These isomorphisms are support-preserving in the sense that the function \(\Psi(e_w)\) has support \(J_w J\) for \(w \in W_\chi^0\).

iii) The restriction \(\Psi : \mathcal{H}(W_\chi^0, S_\chi^0)\) is uniquely determined.

Proof. - When \(\chi\) has level zero, i) and ii) follow easily from Morris [27] (or Goldstein [17]). For completeness we include details. From [27], we obtain a set of basis elements of \(\text{End}_G(\text{Ind } \rho)\) which multiply in a specified manner ([27] 7.12). It is
well-known that there is an isomorphism of \( \mathbb{C} \)-algebras \( t : \mathcal{H}(G, \rho) \xrightarrow{\sim} \text{End}_G(\text{Ind} \rho^{-1}) \) (see equation 7.2).

By Hewlett and Lehrer [18] 6.11, \( \chi \) extends to a character \( \tilde{\chi} \) of \( \Omega_x \). (This can also be proved by a more direct argument.) Note that \( N_x = \Omega_x Y \) (a semi-direct product) where \( Y \) is embedded in \( N_x \) by \( y \mapsto y(\omega) : Y \to N_x \) (here \( \omega \) is a fixed uniformiser). Setting \( \tilde{\chi}(y) = 1 \) yields an extension of \( \chi \) to \( N_x \). For \( w \in W_x \) let \( B_w = \tilde{\chi}(n) \theta_n \) for any \( n \in N_x \) projecting to \( w \) (see [27] 5.4). With this choice of \( B_w \) the cocycle \( \lambda \) in [27] 6.1 is trivial.

It is easy to see that in our situation we have \( \varepsilon_w = 1 \) for all \( w \in W_x \) (see [27] 7.10). It follows that the cocycle \( \mu \) in the final description of the algebra \( \text{End}_G(\text{Ind} \rho) \) ([27] 7.12) is also trivial. It is clear from this description that there exists an isomorphism of \( \mathbb{C} \)-algebras

\[
\Psi_1 : \mathcal{H}_x \xrightarrow{\sim} \text{End}_G(\text{Ind} \rho).
\]

(In the notation of [27] \( R(\chi) = W_x^{0} \), \( C(\chi) = \Omega_x \), \( \Gamma = \Phi_{x,\text{af}} \) and \( p_a = q \) for all \( a \in \Delta = \Pi_{x,\text{af}} \). Thus there exists an isomorphism of \( \mathbb{C} \)-algebras

\[
\Psi : \mathcal{H}_x \xrightarrow{\sim} \mathcal{H}(G, \rho)
\]

satisfying ii) (since \( \mathcal{H}_x \) and \( \mathcal{H}_{x^{-1}} \) coincide).

It remains to show that \( \Psi \) is \( \ast \)-preserving. To see this, define \( \ast \) on \( \text{End}_G(\text{Ind} \rho) \) to be the unique \( \mathbb{C} \)-antilinear extension of the assignment \( T_w^* = T_{w^{-1}} \) for \( w \in W_x \) (see [27] 7.9). (Thus \( \ast \) on \( \text{End}_G(\text{Ind} \rho) \) corresponds to \( \ast \) on \( \mathcal{H}_x \).) It suffices to show that \( t \circ \ast = \ast \circ t \) where \( t : \mathcal{H}(G, \rho^{-1}) \xrightarrow{\sim} \text{End}_G(\text{Ind} \rho) \) is the isomorphism in equation 7.2.

For \( n \in N_x \) let \( \Phi_n \) be the unique element of \( \mathcal{H}(G, \rho^{-1}) \) supported on \( JnJ \) such that \( \Phi_n(n) = 1 \). A straightforward calculation shows that

\[
t_{\Phi_n} = q^{l(n)} \theta_n
\]

where \( n \in N_x \) maps to \( w \in W_x \). Hence \( \tilde{\chi}(n) t_{\Phi_n} = q^{l(w)} B_w \). For \( w \in W_x^{0} \) or \( \Omega_x \), we have

\[
T_w = q^{\frac{1}{2}(l(x(w)) + l(w))} B_w
= q^{\frac{1}{2}(l(x(w)) - l(w))} \tilde{\chi}(n) t_{\Phi_n}.
\]

This clearly implies \( t \circ \ast = \ast \circ t \) and thus completes the proof of i) and ii) in the level-zero case.

When \( \chi \) has positive level, we recall from the proof of Theorem 4.15 that either \( \mathcal{T}_G(\rho) \) has \( J - J \) double coset representatives contained in a proper Levi subgroup of \( G \) containing \( T \) or there exists a character \( \chi_1 \) of \( G \) such that \( \chi \chi_1 \) (viewed as a character of \( G^{0} \)) has level zero. In this latter case, put \( \rho_1 = \rho \otimes \chi_1 \). The map \( f \mapsto f_{\chi_1} : \mathcal{H}(G, \rho) \to \mathcal{H}(G, \rho_1) \) is then a support-preserving, \( \ast \)-preserving isomorphism. Hence we may assume, without loss of generality, that \( \mathcal{T}_G(\rho) \) lies in \( JMJ \) where \( M \) is a proper Levi subgroup of \( G \) (containing \( T \)).

We are thus in exactly the situation treated in section 5. By proposition 5.1, there exists a \( \ast \)-preserving, support-preserving isomorphism from \( \mathcal{H}(G, \rho) \) to \( \mathcal{H}(M, \rho_M) \) where \( \rho_M = \rho | J \cap M \). Since \( J \cap M \) is \( J \) for \( M \) and the semisimple rank of \( M \) is strictly smaller than that of \( G \), we may complete our proof of the existence of a family of \( \ast \)-preserving,
support-preserving isomorphisms provided we do so in the case of semisimple rank zero, i.e. when $G = T$.

We assume therefore that $G = T$. Then $J = {}^0T$ and $\rho = \chi$. Also $W_\chi^0 = 1$ and $W_\chi = \Omega_\chi = Y$. We are thus reduced to showing there exists a $*$-preserving, support-preserving isomorphism of $C$-algebras

$$\Psi : C[Y] \overset{\cong}{\longrightarrow} \mathcal{H}(T, \chi).$$

This is obvious.

Finally we show $\Psi | \mathcal{H}(W_\chi^0, S_\chi^0)$ is uniquely determined. Suppose $\Psi$ and $\Psi'$ are two support-preserving isomorphisms from $\mathcal{H}_\chi$ to $\mathcal{H}(G, \rho)$. Then $\Theta = \Psi^{-1} \circ \Psi' | \mathcal{H}(W_\chi^0, S_\chi^0)$ is a $C$-algebra automorphism of $\mathcal{H}(W_\chi^0, S_\chi^0)$ such that $\Theta(e_w) = \lambda_w e_w$ where $\lambda_w \in \mathbb{C}$ for each $w \in W_\chi^0$. Now (6.1) implies $\lambda(ww') = \lambda_w \lambda_{w'}$ if $l(ww') = l(w) + l(w')$ ($w, w' \in W_\chi^0$) and (6.2) implies $\lambda_s = 1$ for $s \in S_\chi^0$. Hence $\lambda_w = 1$ for all $w \in W_\chi^0$ as required. □

**Remark 6.4.** - The algebra $\mathcal{H}_\chi = \mathcal{H}(W_\chi^0, S_\chi^0) \otimes C[\Omega_\chi]$ carries an inner product $\langle \,, \rangle$ defined by:

$$\langle e_w, e_{w'} \rangle = \delta_{w, w'} q^{l(w)} \quad (w, w' \in W_\chi = W_\chi^0 \rtimes \Omega_\chi)$$

where $l(w)$ is defined to equal $l(w_1)$ if $w = w_1 \omega$ ($w_1 \in W_\chi^0, \omega \in \Omega_\chi$). We also use $\langle \,, \rangle$ to denote the standard inner product on $\mathcal{H}(G, \rho)$:

$$\langle \phi_1, \phi_2 \rangle = \int \phi_1(x) \overline{\phi_2(x)} \, dx \quad (\phi_i \in \mathcal{H}(G, \rho))$$

Note $\langle \phi_1, \phi_2 \rangle = \phi_1 \phi_2^*(1) = \langle \phi_1 \phi_2^*, e_\rho \rangle$ where * is the canonical involution on $\mathcal{H}(G, \rho)$. It is easy to check that the corresponding relation holds on $\mathcal{H}_\chi$, i.e.

$$\langle e_{w_1}, e_{w_2} \rangle = \langle e_{w_1}, e_{w_2}^* \rangle \quad \text{for all } w_1, w_2 \in W_\chi.$$

Thus $e_{w_1} e_{w_2}^* = \langle e_{w_1}, e_{w_2} \rangle e_1 + f$ where $f \in \text{span} \{ e_w : w \neq 1 \}$. It follows that any support-preserving, $*$-preserving isomorphism $\Psi : \mathcal{H}_\chi \overset{\cong}{\sim} \mathcal{H}(G, \rho)$ is an isometry with respect to these inner products (assuming Haar measure on $G$ is normalised so that $J$ has measure one).

**7. Principal series types**

We show in this section that $(J, \rho)$ is a type for the component of the Bernstein decomposition determined by $\chi$. We begin by recalling the Bernstein decomposition of $\mathcal{H}(G)$ (the category of smooth complex representations of $G$). We then recall the definition of type and some consequences. Much of our presentation and notation is taken from Bushnell and Kutzko [13].

A pair $(M, \sigma)$, consisting of a Levi subgroup $M$ of $G$ and an irreducible supercuspidal representation $\sigma$ of $M$, is called a cuspidal pair. Two such pairs, $(M_i, \sigma_i)$ $(i = 1, 2)$, are said to be equivalent if there exists a $g \in G$ such that $M_2 = gM_1$ and $\sigma_2 = \sigma_1$. If we
denote by \( \text{Irr}(G) \) the set of equivalence classes of smooth irreducible representations of \( G \), then the following map is well-defined (Bernstein and Zelevinsky [5]):

\[
\text{Irr}(G) \to \{ \text{equivalence classes of cuspidal pairs} \}
\]

\[
\pi \mapsto \text{class of } (M, \sigma) \text{ if } \pi = \text{a } G\text{-subquotient of } \tilde{\rho}^P G \sigma
\]

where \( P \) is any parabolic subgroup of \( G \) with Levi factor \( M \) and \( \tilde{\rho}^P \) is the normalised parabolic induction functor. The image of \( \pi \) under this map is called the supercuspidal support of \( \pi \). We say two cuspidal pairs, \( (M_i, \sigma_i) \) \( (i = 1, 2) \), are inertially equivalent if there exists a \( g \in G \) and an unramified character \( \nu \) of \( M_2 \) such that:

\[
M_2 = g M_1 \quad \text{and} \quad g \sigma_1 = \sigma_2 \otimes \nu.
\]

We write \([M, \sigma]\) for the inertial equivalence class of \((M, \sigma)\) and \( \mathcal{B}(G) \) for the set of all inertial equivalence classes. If \((\pi, V)\) is an irreducible smooth representation of \( G \), we denote by \( \mathcal{I}(\pi) \) the inertial equivalence class of its supercuspidal support.

Now fix a class \( s \in \mathcal{B}(G) \). We denote by \( \mathcal{R}_s(G) \) the full subcategory of \( \mathcal{R}(G) \) defined as follows:

**Definition 7.1.** Let \((\pi, V) \in \mathcal{R}(G)\). Then \((\pi, V) \in \mathcal{R}_s(G)\) if and only if every irreducible \( G \)-subquotient \( \pi_0 \) of \( \pi \) satisfies \( \mathcal{I}(\pi_0) = s \).

The subcategories \( \mathcal{R}_s(G) \) \( (s \in \mathcal{B}(G)) \) split the category \( \mathcal{R}(G) \) (Bernstein and Deligne [4]). This is the Bernstein decomposition of \( \mathcal{R}(G) \). Precisely:

**Theorem 7.2.** i) The category \( \mathcal{R}(G) \) is the direct product

\[
\mathcal{R}(G) = \prod_{s \in \mathcal{B}(G)} \mathcal{R}_s(G)
\]

of the subcategories \( \mathcal{R}_s(G) \) as \( s \) ranges through \( \mathcal{B}(G) \).

i' Concretely, if \((\pi, V)\) is a smooth representation of \( G \), then, for each \( s \in \mathcal{B}(G) \), \( V \) has a unique maximal \( G \)-subspace \( V_s \in \mathcal{R}_s(G) \) and

\[
V = \sum_{s \in \mathcal{B}(G)} V_s \quad (\text{direct sum}).
\]

If \( s, s' \in \mathcal{B}(G) \) and \( s \neq s' \), then \( \text{Hom}_G(\mathcal{R}_s(G), \mathcal{R}_{s'}(G)) = 0 \) (in the obvious notation).

We can now say what we mean by an \( s \)-type in \( G \).

**Definition 7.3.** An \( s \)-type in \( G \) is a pair \((K, \rho)\), where \( K \) is a compact open subgroup of \( G \) and \( \rho \) is an irreducible smooth representation of \( K \), with the following property: an irreducible smooth representation \( \pi \) of \( G \) contains \( \rho \) if and only if \( \mathcal{I}(\pi) \in s \).

**Remark 7.4.** Clearly there is also a notion of \( s \)-type in \( G \) where \( s \) is a subset of \( \mathcal{B}(G) \) which is not necessarily a singleton. It follows from [4](section 3) that such an \( s \) is necessarily finite.

Suppose now that \((\rho, W)\) is a smooth irreducible representation of a compact open subgroup \( K \) of \( G \). If \((\pi, V)\) is a smooth representation of \( G \), we write \( V^\rho \) for the \( \rho \)-isotypic
subspace of \( \mathcal{V} \), i.e. the sum of all \( K \)-subspaces of \( \mathcal{V} \) isomorphic to \( \rho \). We have \( \mathcal{V}^\rho = e_\rho \mathcal{V} \) where \( e_\rho \) is the idempotent element of \( \mathcal{H}(G) \) defined by

\[
e_\rho(x) = \mu(K)^{-1} \dim(\rho) \text{tr}_W(\rho(x^{-1})) \quad (x \in K)
\]

\[
e_\rho(x) = 0 \quad (x \notin K).
\]

We write \( \mathcal{V}[\rho] \) for the \( G \)-subspace of \( \mathcal{V} \) generated by \( V^\rho \) (i.e. \( \mathcal{V}[\rho] = \mathcal{V}(0) V^\rho \)) and \( \mathcal{W}_\rho(G) \) for the full subcategory of \( \mathcal{H}(G) \) consisting of all \( (\pi, \mathcal{V}) \in \mathcal{H}(G) \) such that \( \mathcal{V} = \mathcal{V}[\rho] \) (i.e. \( \mathcal{V} \) is generated by its \( \rho \)-isotypic vectors).

The process \( \mathcal{V} \mapsto \mathcal{V}^\rho \) defines a functor from \( \mathcal{H}(G) \) to \( e_\rho \mathcal{H}(G)e_\rho - \text{Mod} \) (the category of left \( e_\rho \mathcal{H}(G)e_\rho \)-modules). It is well-known that this induces a bijection between smooth irreducible representations of \( G \) containing \( \rho \) and simple \( e_\rho \mathcal{H}(G)e_\rho \)-modules (e.g. see Bushnell and Kutzko [10] 4.2.3). However, it is not in general an equivalence of categories. Suppose however that \( (K, \rho) \) is an \( s \)-type in \( G \) (for some \( s \in \mathcal{B}(G) \)). In this case we show (in the next few paragraphs) that the categories \( \mathcal{H}_s(G) \) and \( \mathcal{H}_\rho(G) \) are equal (as subcategories of \( \mathcal{H}(G) \)) and that the functor \( \mathcal{V} \mapsto \mathcal{V}^\rho : \mathcal{H}_\rho(G) \to e_\rho \mathcal{H}(G)e_\rho - \text{Mod} \) is an equivalence of categories. In particular the category \( \mathcal{H}_s(G) \) is closed under subquotients. Conversely, Bushnell and Kutzko prove in [13] (using an argument due to Bernstein) that if \( \mathcal{H}_\rho(G) \) is closed under subquotients, then \( (K, \rho) \) is an \( s \)-type in \( G \) where \( s \) is a (necessarily) finite subset of \( \mathcal{B}(G) \).

Let \( \text{ind} \rho \) be the smooth representation of \( G \) compactly induced by \( \rho \). There is a natural isomorphism of \( \mathbb{C} \)-vector spaces:

\[
(7.1) \quad \text{Hom}_G(\text{ind} \rho, \pi) \cong \text{Hom}_K(\rho, \pi) \quad (\pi \in \mathcal{H}(G)).
\]

If we realise \( \text{ind} \rho \) in the usual way as the space of compactly supported functions \( f : G \to W \) such that \( f(kg) = \rho(k)f(g) \) \( (k \in K, g \in G) \) with \( G \) acting by right translations, then the maps are given by

\[
t \mapsto t_1 : \text{Hom}_G(\text{ind} \rho, \pi) \to \text{Hom}_K(\rho, \pi)
\]

\[
t \mapsto \tilde{t} : \text{Hom}_K(\rho, \pi) \to \text{Hom}_G(\text{ind} \rho, \pi)
\]

where

\[
t_1(w) = t(e_w) \quad (w \in W),
\]

\[
\tilde{t}(f) = \int_G \pi(g)t(f(g^{-1})) \, dg \quad (f \in \text{ind} \rho).
\]

Here \( e_w \in \text{ind} \rho \) is defined by \( e_w(k) = \rho(k)w \) for \( k \in K \), \( e_w(g) = 0 \) for \( g \notin K \) (and Haar measure on \( G \) is normalised so that \( K \) has measure one).

In particular \( \text{Hom}_G(\text{ind} \rho, \pi) \neq 0 \) if and only if \( \text{Hom}_K(\rho, \pi) \neq 0 \). Fix \( s \in \mathcal{B}(G) \) and suppose \( (\pi, \mathcal{V}) \in \mathcal{H}_s(G) \). Since every irreducible subquotient of \( (\pi, \mathcal{V}) \) is also in \( \mathcal{H}_s(G) \) and smooth representations of \( K \) are completely reducible, we see \( (K, \rho) \) is an \( s \)-type implies \( \text{Hom}_G(\text{ind} \rho, \pi) \neq 0 \). Further, \( (K, \rho) \) is an \( s \)-type implies \( \text{ind} \rho \in \mathcal{H}_s(G) \). Indeed if \( \text{ind} \rho \notin \mathcal{H}_s(G) \) there exists a \( G \)-surjection from \( \text{ind} \rho \) to a smooth representation \( \sigma \) where \( \sigma \in \mathcal{H}_t(G) \) for some \( t \in \mathcal{B}(G) \) with \( t \neq s \). Further, \( \sigma \) is finitely generated (since
Thus $\sigma$ has an irreducible quotient $\tau$. By 7.1, $\tau$ contains $\rho$ which means $(K,\rho)$ cannot be an $s$-type in $G$.

Conversely, suppose $\text{ind} \rho \in \mathfrak{R}_s(G)$ and $\text{Hom}_G(\text{ind} \rho, \pi) \neq 0$ for each non-zero $\pi \in \mathfrak{R}_s(G)$. It is then easy to see that $(K,\rho)$ is an $s$-type in $G$. Hence $(K,\rho)$ is an $s$-type in $G$ if and only if $\text{ind} \rho$ is a generator of $\mathfrak{R}_s(G)$, i.e. $\text{ind} \rho \in \mathfrak{R}_s(G)$ and $\text{Hom}_G(\text{ind} \rho, \pi) \neq 0$ for non-zero $\pi \in \mathfrak{R}_s(G)$ (equivalently, every object in $\mathfrak{R}_s(G)$ is a quotient of a direct sum of copies of $\text{ind} \rho$).

The isomorphism (7.1) implies that $\text{ind} \rho$ is a finitely generated, projective object in $\mathfrak{R}(G)$. Thus if $(K,\rho)$ is an $s$-type in $G$, $\text{ind} \rho$ is a finitely generated, projective generator of $\mathfrak{R}_s(G)$. By a well-known result (see e.g. Bass [3], chap 2 Theorem 1.5), the functor

$$\mathfrak{R}_s(G) \to \text{End}_G(\text{ind} \rho)^{\text{OPP}} \to \mathbf{Mod}$$

is an equivalence of categories (where $\text{Hom}_G(\text{ind} \rho, \pi)$ is given the $\text{End}_G(\text{ind} \rho)^{\text{OPP}}$-module structure defined by $t.f = f \circ t$ ($t \in \text{End}_G(\text{ind} \rho), f \in \text{Hom}_G(\text{ind} \rho, \pi)$)). Thus the functor

$$\mathfrak{R}_s(G) \to \text{End}_G(\text{ind} \rho)^{\text{OPP}} \otimes_{\mathbf{C}} \text{End}_C(W) \to \mathbf{Mod}$$

is also an equivalence of categories.

We write $\mathcal{H}(G, \rho^\vee)$ (where $(\rho^\vee, W^\vee)$ is the contragredient of $(\rho, W)$) for the convolution algebra of compactly supported functions $\Phi : G \to \text{End}_C(W)$ such that $\Phi(k_1 x k_2) = \rho(k_1) \Phi(x) \rho(k_2)$, for $k_1, k_2 \in K, x \in G$. Define maps between $\mathcal{H}(G, \rho^\vee)$ and $\text{End}_G(\text{ind} \rho)$ as follows:

$$\phi \mapsto t_\phi : \mathcal{H}(G, \rho^\vee) \to \text{End}_G(\text{ind} \rho)$$

$$t \mapsto \phi_t : \text{End}_G(\text{ind} \rho) \to \mathcal{H}(G, \rho^\vee)$$

where

$$t_\phi(f)(g) = \int_G \phi(x)(f(x^{-1} g)) \, dx \quad (f \in \text{Ind} \rho, \ g \in G)$$

$$\phi_t(g)(w) = t(e_w)(g) \quad (g \in G, \ w \in W)$$

(Here (as above) Haar measure on $G$ is normalised so that $K$ has measure one.) It is well-known and easy to check that these maps are inverse isomorphisms of $\mathbf{C}$-algebras. It is also easy to verify that the map $\Phi \mapsto \Phi'$ where $\Phi'(g) = \Phi(g^{-1})^\vee$ is an anti-isomorphism from $\mathcal{H}(G, \rho^\vee)$ to $\mathcal{H}(G, \rho)$ (where $\Phi(g^{-1})^\vee \in \text{End}_C(W^\vee)$ is the transpose of $\Phi(g^{-1}) \in \text{End}_C(W)$). Thus we have an isomorphism of $\mathbf{C}$-algebras:

$$t \mapsto \phi'_t : \text{End}_G(\text{ind} \rho)^{\text{OPP}} \cong \mathcal{H}(G, \rho).$$

Via this isomorphism and the isomorphism $\text{Hom}_G(\text{Ind} \rho, \pi) \cong \text{Hom}_K(\rho, \pi)$, we can transport the left $\text{End}_G(\text{ind} \rho)^{\text{OPP}}$-module structure on $\text{Hom}_G(\text{ind} \rho, \pi)$ to a left $\mathcal{H}(G, \rho)$-module structure on $\text{Hom}_K(\rho, \pi)$ ($\pi \in \mathfrak{R}(G)$). A short calculation shows that this is given by:

$$\Phi'.t(w) = \int_G \pi(g)t(\Phi(g^{-1})w) \, dg \quad (\Phi \in \mathcal{H}(G, \rho^\vee), \ t \in \text{Hom}_K(\rho, \pi), \ w \in W).$$
It is clear that the evaluation map

\[ \text{ev} : \text{Hom}_K(\rho, \pi) \otimes C \to V^\rho \]

\[ \text{ev}(t \otimes w) = t(w) \quad (t \in \text{Hom}_K(\rho, \pi), w \in W) \]

is an isomorphism of \( C \)-vector spaces. The space \( \text{Hom}_K(\rho, \pi) \otimes C \) is naturally a left \( \mathcal{H}(G, \rho) \otimes C \text{End}_C(W) \)-module. Recall ([10], chap. 4) that there is a canonical \( C \)-algebra isomorphism:

\[ \Psi : \mathcal{H}(G, \rho) \otimes C \text{End}_C(W) \cong e_\rho \mathcal{H}(G)e_\rho \]

\[ \Phi \otimes (w \otimes w^\vee) \mapsto (g \mapsto \text{dim } \rho(w, \Phi(g)w^\vee)) \quad (w \in W, w^\vee \in W^\vee). \]

Here \( \langle , \rangle \) denotes the canonical (evaluation) pairing and \( w \otimes w^\vee \in \text{End}_C(W) \) is defined by \( w \otimes w^\vee(w_1) = \langle w_1, w^\vee \rangle w \) for \( w_1 \in W \). Via the isomorphisms \( \Psi \) and \( \text{ev} \), \( V^\rho \) becomes a left \( e_\rho \mathcal{H}(G)e_\rho \)-module. We now show that this module structure coincides with the natural left \( e_\rho \mathcal{H}(G)e_\rho \)-module structure on \( V^\rho \) (induced by the left \( \mathcal{H}(G) \)-module structure on \( V \)).

This is equivalent to the commutativity of the diagram

\[ \begin{array}{ccc}
\mathcal{H}(G, \rho) \otimes C \text{End}_C(W) \times \text{Hom}_K(\rho, \pi) \otimes C W & \longrightarrow & \text{Hom}_K(\rho, \pi) \otimes C W \\
\downarrow \Psi \times \text{ev} & & \downarrow \text{ev} \\
\text{Hom}_K(\rho, \pi) \otimes C W & \longrightarrow & V^\rho
\end{array} \]

where the horizontal arrows come from the given module structures. Let

\[ (\Phi \otimes (w \otimes w^\vee), t \otimes w_1) \in \mathcal{H}(G, \rho) \otimes C \text{End}_C \times \text{Hom}_K(\rho, \pi) \otimes C W. \]

A short calculation shows that the clockwise image in \( V^\rho \) of this element is given by

\[ (7.3) \quad \langle w_1, w^\vee \rangle \int_G \pi(g)(t(\Phi(g)^\vee w)) \, dg \in V^\rho. \]

Another short calculation shows that the counter-clockwise image is given by

\[ (7.4) \quad \text{dim } \rho \int_G \pi(g)(t(w_1 \otimes w^\vee(\Phi(g)^\vee w))) \, dg. \]

In showing (7.3) and (7.4) are equal, we may assume the support of \( \Phi \) lies in a single double coset \( KxK \) for some \( x \in G \). Write \( KxK = \bigcup_{i=1}^r k_i x K \) where \( k_i x K \neq k_j x K \) for \( i \neq j \). Then (7.3) becomes

\[ \langle w_1, w^\vee \rangle \sum_{i=1}^r \int_K \pi(k_i)\pi(x)(k)(t(\rho(k_i^{-1})\Phi(x)^\vee \rho(k_i^{-1})w)) \, dk \]

\[ = \sum_{i=1}^r \pi(k_i)\pi(x)(t(\langle w_1, w^\vee \rangle \Phi(x)^\vee \rho(k_i^{-1})w)) \]

Equation (7.4) becomes

\[ \text{dim } \rho \sum_{i=1}^r \int_K \pi(k_i)\pi(x)(k)(t(w_1)\langle w, \rho^\vee(k_i)\Phi(x)\rho^\vee(k)w^\vee \rangle) \, dk \]

\[ = \sum_{i=1}^r \pi(k_i)\pi(x)(t[\text{dim } \rho \int_K \langle w, \rho^\vee(k_i)\Phi(x)\rho^\vee(k)w^\vee \rangle \rho(k)w_1 \, dk]). \]
Comparing these last two equations, we are reduced to showing

\[ \langle w_1, w' \rangle \Phi(x)^\vee \rho(k_i^{-1})w = \dim \rho \int_K \langle w, \rho^\vee(k) \Phi(x) \rho^\vee(k)w' \rangle \rho(k)w_1 \, dk. \]

This is equivalent to showing

\[ \int_K \langle \Phi(x)^\vee \rho(k_i^{-1})w, \rho^\vee(k)w' \rangle \langle \rho(k)w_1, w_1' \rangle \, dk = \frac{1}{\dim \rho} \langle w_1, w' \rangle \langle \Phi(x)^\vee \rho(k_i^{-1})w, w' \rangle \]

for all \( w_1 \in W^\vee \) which holds by a direct application of Schur orthogonality.

Summing up, we have shown that the functor

\[ \mathcal{V} \mapsto \mathcal{V}^\rho : \mathcal{R}_s(G) \to e_\rho \mathcal{H}(G)e_\rho - \text{Mod} \]

is an equivalence of categories when \((K, \rho)\) is an \(s\)-type in \(G\).

We now show the categories \(\mathcal{R}_s(G)\) and \(\mathcal{R}_p(G)\) are equal (when \((K, \rho)\) is an \(s\)-type). Let \((\pi, V) \in \mathcal{R}_s(G)\). We have \(\langle V/V[\rho] \rangle^\rho = 0\). This implies \(V/V[\rho] = 0\) (since every non-zero object in \(\mathcal{R}_s(G)\) contains \(\rho\)). Thus \(V = V[\rho]\) for all \((\pi, V) \in \mathcal{R}_s(G)\). To see that \(V = V[\rho]\) for \((\pi, V) \in \mathcal{R}(G)\) implies \((\pi, V) \in \mathcal{R}_s(G)\), we observe that \(V = V[\rho]\) implies \(\pi\) is a quotient of a direct sum of copies of \(\text{ind}_p\) and \(\text{ind}_p \in \mathcal{R}_s(G)\) (since by hypothesis \((K, \rho)\) is an \(s\)-type in \(G\)).

We have proved parts i) and iii) of the following theorem which records some of the main properties of types (see Theorem 4.3 in [13]).

**Theorem 7.5.** – Let \(s \in \mathcal{B}(G)\) and let \((K, \rho)\) be an \(s\)-type. Then:

i) The categories \(\mathcal{R}_p(G)\) and \(\mathcal{R}_s(G)\) are equal as subcategories of \(\mathcal{R}(G)\). In particular, \(\mathcal{R}_p(G)\) is closed under subquotients.

ii) Let \((\pi, V)\) be a smooth representation of \(G\). Then there exists a uniquely determined \(G\)-subspace \(U\) of \(V\) such that

\[ V = V[\rho] \oplus U. \]

iii) The functor

\[ \mathcal{V} \mapsto \mathcal{V}^\rho : \mathcal{R}_p(G) \to e_\rho \mathcal{H}(G)e_\rho - \text{Mod} \]

is an equivalence of categories.

Suppose now that \(\chi : {}^0T \to \mathbb{C}^x\) is a smooth character. Let \(\bar{\chi}\) be any character of \(T\) extending \(\chi\). The inertial equivalence class \([T, \bar{\chi}]\) depends only on \(\chi\). We denote it by \(s_\chi\). Let \(\mathcal{R}_\chi(G)\) be the resulting (principal series) component of the Bernstein decomposition, i.e. \(\mathcal{R}_\chi(G) = \mathcal{R}_{s_\chi}(G)\). We sometimes refer to it as the component of the Bernstein decomposition corresponding to \(\chi\).

**Lemma 7.6.** – For \(t \in T\), let \(\phi_t\) be a non-zero element of \(\mathcal{H}(G, \rho)\) supported on \(J_tJ\). Then \(\phi_t\) is invertible in \(\mathcal{H}(G, \rho)\).
Proof. – Suppose \( t \in T \) corresponds to \( w \) under \( N/0^0T \cong W \). Then \( w \in W_x \) (in fact \( w \in Y \)). By Lemma 6.2, we may write \( w = s_1 \ldots s_r \omega \) where \( s_1, \ldots, s_r \in S_x^0 \), \( t(s_1 \ldots s_r) = t \) and \( \omega \in \Omega_x \). Fix a support-preserving isomorphism \( \Psi : H_x \cong H(G, \rho) \). Then \( \Psi^{-1}(\phi_t) \) is a non-zero scalar times \( e_{s_1} \ldots e_{s_r} e_\omega \). Since each \( e_{s_i} \) is invertible and \( e_\omega \) is invertible, \( \Psi^{-1}(\phi_t) \) is also invertible. Hence \( \phi_t \) is invertible. \( \square \)

Theorem 7.7. – The pair \((J_x, \rho_x)\) is an \( s_x \)-type in \( G \).

Proof. – We apply Theorem 8.3 of Bushnell and Kutzko [13]. It is clear that the pair \((0^0T, \chi)\) is a type for \( \mathcal{R}_x(T) \) (the component of the Bernstein decomposition of \( T \) determined by \( \chi \)). Thus it is sufficient to verify that \((J, \rho)\) is a \( G \)-cover of \((0^0T, \chi)\) in the language of [13] (see Definition 8.1). From Proposition 3.6, we see that \((J, \rho)\) satisfies conditions i) and ii) in the definition of \( G \)-cover. That condition iii) is also satisfied follows immediately from Lemma 6.6. \( \square \)

Remark 7.8. – Let \( B = TU \) be a Borel subgroup containing \( T \) with unipotent radical \( U \). Let \((\pi, \mathcal{V})\) be a smooth representation of \( G \). The key point in proving Theorem 7.7 (which remains implicit in our proof since we invoke Thm. 8.3 of [13]) is to show that the Jacquet module functor \( \mathcal{V} \mapsto \mathcal{V}_U \) induces an isomorphism of \( \mathbb{C} \)-vector spaces:

\[
\mathcal{V}_U^G \cong \mathcal{V}_U^X.
\]

This is a consequence of Lemma 7.6. An easy argument (using Frobenius reciprocity) then shows that \((J, \rho)\) is an \( s_x \)-type.

Corollary 7.9. – i) The categories \( \mathcal{R}_{s_x}(G) \) and \( \mathcal{R}_\rho(G) \) are equal as subcategories of \( \mathcal{R}(G) \). In particular, if \((\pi, \mathcal{V}) \in \mathcal{R}_\rho(G)\) then every \( G \)-subquotient of \( \pi \) is also in \( \mathcal{R}_\rho(G) \).

ii) Let \((\pi, \mathcal{V})\) be any smooth representation of \( G \). There exists a uniquely determined \( G \)-subspace \( \mathcal{U} \) of \( \mathcal{V} \) such that

\[
\mathcal{V} = \mathcal{V}[\rho] \oplus \mathcal{U}.
\]

iii) The functor \( \mathcal{V} \mapsto \mathcal{V}_U^G : \mathcal{R}_\rho(G) \to \mathcal{H}(G, \rho) - \mathcal{M}od \) is an equivalence of categories.

Proof. – This is Theorem 7.5 for the type \((J, \rho)\). \( \square \)

8. A dual group interpretation of \( \mathcal{H}(G, \rho) \)

We begin by rephrasing Theorem 6.3 on the structure of the Hecke algebra \( \mathcal{H}(G, \rho) \). We define a reductive \( \mathcal{O}_F \)-group \( \hat{H} \) in terms of \( \chi \) (in general disconnected) with group of \( F \)-rational points \( \hat{H} \) and show that there exists a \( s \)-preserving, support-preserving isomorphism from \( \mathcal{H}(H, 1) \) to \( \mathcal{H}(G, \rho) \) where \( I \) is a certain Iwahori subgroup of \( H \) (the group of \( F \)-rational points of the identity component of \( H \)). The groups \( H \) and \( \hat{H} \) have natural interpretations in terms of Langlands parameters for \( T \). (The key idea here was suggested to me by Neil Chriss and Allen Moy.) We discuss this next and give two examples. We then show the quotient \( \hat{H}/H \) is always abelian and is trivial when \( G \) has connected centre. Finally we observe that \( H \) is an endoscopic group of \( G \).

First, we define the groups \( H \) and \( \hat{H} \). The quadruple \( \Psi = (X, \Phi_X, Y, \Phi_Y) \) is a root datum. Hence there exists a connected reductive \( \mathcal{O}_F \)-group \( H \) with \( \mathcal{O}_F \)-split maximal torus
TH (unique up to $O_F$-isomorphism) such that the root datum $\Psi(H, T_H)$ equals $\Psi_X$. We may (and will) assume that $T_H = T$. As usual we denote by $H$ and $T$ the corresponding groups of $F$-rational points. Note that $\overline{W}_\chi^0 = \overline{W}(H)$ (the ordinary Weyl group of $H$). Let $\Phi_X^+ = \Phi_X \cap \Phi^+$. Then $\Phi_X^+$ is a positive system in $\Phi_X$. Put $C_X = \{w \in \overline{W}_\chi \mid w\Phi_X^+ = \Phi_X^+\}$. Write $B_H$ for the Borel subgroup of $H$ corresponding to $\Phi_X^+$ with group of $F$-rational points $B_H$. It is clear that $C_X$ acts on the based root datum $\Psi_0(H) = (X, \Pi_X, Y, \Pi_X')$ where $\Pi_X$ is the unique simple system contained in $\Phi_X^+$. Fix $O_F$-isomorphisms $x_\alpha : G_\alpha \to U_\alpha$ ($\alpha \in \Phi_X$). Put $u_\alpha = x_\alpha(1)$ ($\alpha \in \Pi_X$). The short exact sequence:

$$1 \to \text{Int} H \to \text{Aut} H \to \text{Aut} \Psi_0(H) \to 1.$$ 

is split by an isomorphism:

$$\text{Aut} \Psi_0(H) \xrightarrow{\cong} \text{Aut} (H, B_H, T, (u_\alpha)_{\alpha \in \Pi_X}).$$

Via this splitting, $C_X$ embeds in $\text{Aut} H$. By definition, $\widetilde{H} = H \rtimes C_X$. Then $\widetilde{H} = H \rtimes C_X$.

As before, let $\Phi_{X, \text{af}} = \{a \in \Phi_{\text{af}} \mid \text{gr}(a) \in \Phi_X\}$ where $\text{gr}(a)$ denotes the linear part of $a$. Note that $\Phi_{X, \text{af}}$ is the set of affine roots of $H$ with respect to $T$. Let $W(H)$ be the extended affine Weyl group of $H$ with respect to $T$. Thus $W(H) = Y \rtimes \overline{W}_\chi^0$. As above

$$C_X = \{v \in V : 0 < \alpha(v) < 1 \ (\alpha \in \Phi_X^+)\}$$

is a chamber with respect to the decomposition of $V$ induced by the affine functionals in $\Phi_{X, \text{af}}$. Let

$$\Omega(H) = \{w \in W(H) \mid wC_X = C_X\}.$$ 

**Lemma 8.1.**

i) $\overline{W}_\chi = \overline{W}_\chi^0 \rtimes C_X$.

ii) $\Omega_X = \Omega(H) \rtimes C_X$.

**Proof.**

i) The normality of $\overline{W}_\chi^0$ in $\overline{W}_\chi$ follows directly from the definition of $\overline{W}_\chi^0$. Let $w \in \overline{W}_\chi^0$. Then $w\Phi_X^+ \subset \Phi_X$ is a positive system for $\Phi_X$. Hence there exists a $w_1 \in \overline{W}_\chi^0$ (the Weyl group of $\Phi_X$) such that $w\Phi_X^+ = w_1\Phi_X^+$, i.e. $w_1^{-1}w \in C_X$. Hence $\overline{W}_\chi = \overline{W}_\chi^0C_X$. It is clear that $\overline{W}_\chi \cap C_X = 1$. This proves i).

ii) From $W(H) = Y \rtimes \overline{W}_\chi$, we see $\Omega(H) = \{(y,w) \in \Omega_X : w \in \overline{W}_\chi^0\}$. Since an element $v \in V$ lies in $C_X$ if and only if $0 < \alpha(v) < 1$ for all $\alpha \in \Phi_X^+$ and $w\Phi_X^+ = \Phi_X^+$ for all $w \in C_X$, we see $wC_X = C_X$, i.e. $C_X$ is a subgroup of $\Omega_X$. Suppose $c \in C_X$ and $(y,w) \in \Omega(H)$. Then $c(y,w)c^{-1} = (c(y),cw^{-1})$ and $cw^{-1} \in \overline{W}_\chi^0$ (by i)). Hence $C_X$ normalizes $\Omega(H)$.

Suppose now that $(y,w) \in \Omega_X$. By i) we may write $w \in \overline{W}_\chi$ as $w = w_1w_2$ where $w_1 \in \overline{W}_\chi^0$ and $w_2 \in C_X$. Then $(y,w) = (y,w_1)w_2$ is an element of $\Omega(H)C_X$ as required. $\square$

The decomposition $\Omega_X = \Omega(H) \rtimes C_X$ implies $C[\Omega_X] = C[\Omega(H)] \otimes C[C_X]$ where the twisted tensor product is defined by the conjugation action of $C_X$ on $\Omega(H)$. Hence:

$$\mathcal{H}(W_X^0, S_X^0) \otimes C[\Omega_X] = \mathcal{H}(W_X^0, S_X^0) \otimes (C[\Omega(H)] \otimes C[C_X]) = (\mathcal{H}(W_X^0, S_X^0) \otimes C[\Omega(H)]) \otimes C[C_X].$$

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The positive system $\Phi^+_x$ uniquely determines an Iwahori subgroup of $H$ which we denote by $I$. Thus

$$I = \prod_{\alpha \in \Phi^+_x} U_{\alpha,1} \prod_{\alpha \in \Phi^-_x} U_{\alpha,0}. $$

As in Iwahori and Matsumoto [23], there is a $*$-preserving, support-preserving isomorphism:

(8.2) \[ H(H,1_I) \cong H(W^0_x,S^0_x) \otimes C[\Omega(H)]. \]

It is easy to see that there is a $*$-preserving, support-preserving isomorphism:

(8.3) \[ H(\tilde{H},1_{\tilde{I}}) \cong H(H,1_I) \otimes C[C_x]. \]

Combining (8.1), (8.2), (8.3) and Theorem 6.3 we obtain the following theorem.

**Theorem 8.2.** The algebras $H(\tilde{H},1_{\tilde{I}})$ and $H(G,\rho)$ are isomorphic via a family of $*$-preserving, support-preserving (and hence inner-product preserving) isomorphisms.

Suppose now that $\tilde{\chi}$ is a character of $T$ extending $\chi$. We write $\tilde{T}$ for the complex torus dual to $T$ in the sense of Langlands (thus $X^*_{\tilde{T}} = X_*(T)$ and $X_{\tilde{T}} = X^*(T)$). Let $W_F$ be the (absolute) Weil group of $F$. The Langlands parameter of $\tilde{\chi} : T \to \mathbb{C}^*$ is the homomorphism $\tau_{\tilde{\chi}} : W_F \to \tilde{T}$ defined by

(8.4) \[ \alpha(\tau_{\tilde{\chi}}(\sigma)) = \tilde{\chi}(\alpha(\tau_F(\sigma))) \]

for all $\alpha \in X^*_{\tilde{T}} = X_*(T)$ and all $\sigma \in W_F$. Here $\tau_F : W_F \to F^\times$ induces the isomorphism $W_F^{ab} \cong F^\times$ of local class field theory. Since $\tau_F^{-1}(O_F^\times) = I_F$ (the inertia group of $F$), the restriction $\tau_{\tilde{\chi}} | I_F$ depends only on $\chi$. We denote it by $\tau_\chi$.

The centraliser in $\tilde{G}$ (the Langlands dual group of $G$) of the image of $\tau_\chi$ is the reductive subgroup of $\tilde{G}$ generated by $\tilde{T}$ and those root groups $U_\alpha$ for which $\alpha | \im \tau_\chi$ is trivial together with those Weyl group representatives $n_w$ for which $w(t) = t$ for all $t \in \im \tau_\chi$. The identity component is then generated by $\tilde{T}$ and those $U_\alpha$ for which $\alpha | \im \tau_\chi$ is trivial (see e.g. [33] 4.1). Write $\tilde{W}(\im \tau_\chi)$ for the stabiliser in $\tilde{W} = W(T,G) = W(\tilde{T},\tilde{G})$ of $\im \tau_\chi$ and $\tilde{W}(\im \tau_\chi)^0$ for the normal subgroup generated by those reflections $s_\alpha$ for which $\alpha | \im \tau_\chi = 1$. Then

$$C_{\tilde{G}}(\im \tau_\chi)/C_{\tilde{G}}(\im \tau_\chi)^0 \cong \tilde{W}(\im \tau_\chi)/\tilde{W}(\im \tau_\chi)^0. $$

Using 8.4, the condition $\alpha | \im \tau_\chi = 1$ is easily seen to be equivalent to $\chi \circ \alpha | O_F^\times = 1$, i.e. $\alpha \in \Phi^\vee_x$ (viewing $\alpha$ as an element of $\Phi(T,G)^\vee = \Phi(\tilde{T},\tilde{G})$). Again from 8.4, we see $w(\tau_\chi(\sigma)) = \tau_{\chi}(\sigma)$ for all $\sigma \in I_F$ if and only if $w(\chi) = \chi$, i.e. $w \in \tilde{W}_x$. Thus

$$\Psi(C_{\tilde{G}}(\im \tau_\chi)^0,\tilde{T}) = (X^*(\tilde{T}),\Phi^\vee_x,X_{\tilde{T}}(\tilde{T}),\Phi_\chi) = (Y,\Phi^\vee_x,X,\Phi_\chi) = \Psi(H,T)^\vee. $$

Hence $C_{\tilde{G}}(\im \tau_\chi)^0$ may be viewed as the Langlands dual group $\tilde{H}$ of $H$. The component group $\pi_0(C_{\tilde{G}}(\im \tau_\chi))$ is given by $\tilde{W}_x/\tilde{W}_x^0$ and so is isomorphic to $C_x = \pi_0(\tilde{H})$. Fix
non-trivial elements \( u_\alpha \in U_\alpha \) where \( U_\alpha \) is the \( \alpha \) root group for \( (G, T) = (G, \tau_x^0, \hat{T}) \) \( (\alpha \in \Pi^\chi) \). We have

\[
C_\chi \hookrightarrow \text{Aut}(G)/\text{Inn}(G) \simeq \text{Aut}(G, T, (u_\alpha)_{\alpha \in \Pi^\chi}).
\]

Thus we obtain an action of \( C_\chi \) on \( \Psi(H, T)^\vee \). It is clear that this action coincides (under transpose inverse) with the action of \( C_\chi \) on \( \Psi_x = \Psi(H, T) \). In rough terms \( C_G(\im \tau_x) \) might be considered ‘the dual group’ of \( \hat{H} \).

**Example 8.3** (cf. Sanje-Mpacko [31].) – Let \( G = SL(n, F) \). Then

\[
0^T \simeq \{(\lambda_1, \ldots, \lambda_n) \in (O_F^n)^n : \lambda_1 \ldots \lambda_n = 1\}.
\]

Let \( \chi_1 : O_F^n \to \mathbb{C}^* \) be a character of order \( n \) (e.g. suppose \( n \mid (q - 1) \) where \( q = |k_F| \)). Define \( \chi : 0^T \to \mathbb{C}^* \) by

\[
\chi(\lambda_1, \ldots, \lambda_n) = \chi_1(\lambda_1)\chi_1^2(\lambda_2)\ldots \chi_1^{n-1}(\lambda_{n-1}) \quad (\lambda_i \in O_F).
\]

The root datum \( \Psi_x = (X, \emptyset, Y, \emptyset) \) where \( X \simeq \mathbb{Z}^n/\mathbb{Z}(1, \ldots, 1) \) and

\[
Y \simeq \{(l_1, \ldots, l_n) \in \mathbb{Z}^n : \sum l_i = 0\}.
\]

In particular, \( \overline{W}_x^0 = \{1\} \). An easy calculation shows that \( \overline{W}_x \simeq \mathbb{Z}/n\mathbb{Z} \) with generator \( (\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_n, \lambda_1, \ldots, \lambda_{n-1}) \). Thus \( W_x \simeq Y \times \mathbb{Z}/n\mathbb{Z} \) and the Hecke algebra \( \mathcal{H}(G, \rho) \) is isomorphic to \( \mathbb{C}[W_x] \) (the group algebra of \( W_x \)). Finally, \( H = T \) and \( H = H \times \mathbb{Z}/n\mathbb{Z} \) (where e.g. \( \mathbb{I} \in \mathbb{Z}/n\mathbb{Z} \) acts by \( (\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_n, \lambda_1, \ldots, \lambda_{n-1}) \ (\lambda_i \in F^*) \)).

**Example 8.4.** – Let \( G = Sp_{2n}(F) \) \( (n \geq 2) \). Then \( 0^T \simeq (O_F^n)^n \) and thus a character \( \chi : 0^T \to \mathbb{C}^* \) is given by an \( n \)-tuple of characters of \( O_F^* \). Let \( \chi = (\chi_1, \chi_1, \ldots, \chi_1) \) where \( \chi_1^2 = 1, \chi_1 \neq 1 \). The root datum \( \Psi_x \) then equals \( (\mathbb{Z}^n, \{\pm e_i \pm e_j\}, \mathbb{Z}^n, \{\pm e_i \pm e_j\}) \) using standard notation. Thus \( \Phi_\chi = \Phi_\chi^* \) has type \( D_n \) for \( n \geq 3 \) and type \( A_1 \times A_1 \) for \( n = 2 \). Clearly, \( \overline{W}_x = \overline{W} \) and \( \overline{W}_x^0 \) is the subgroup of index two consisting of signed permutations with an even number of sign changes. The group \( H = SO_{2n}(F) \) and \( \tilde{H} = O_{2n}(F) \). In particular, \( \mathcal{H}(G, \rho) \) is isomorphic to the Iwahori Hecke algebra of \( O_{2n}(F) \).

We now show

- the group \( C_\chi \) is always abelian,
- \( G \) has connected centre implies \( C_\chi = \{1\} \) and hence \( H = \tilde{H} \).

Tensoring the split exact sequence

\[
1 \longrightarrow 1 + P_F \longrightarrow O_F^x \longrightarrow k_F^x \longrightarrow 1.
\]

with \( Y \) yields the split exact sequence

\[
1 \longrightarrow Y \otimes (1 + P_F) \longrightarrow Y \otimes O_F^x \longrightarrow Y \otimes k_F^x \longrightarrow 1.
\]

Hence the sequence

\[
1 \longrightarrow T_1 \longrightarrow 0^T \longrightarrow \text{I}(k_F) \longrightarrow 1.
\]
is again split exact and thus $\chi : \mathfrak{T} \to \mathbb{C}^x$ factors as $\chi = \chi_1 \mu$ where $\chi_1 = \chi \mid T_1$ (by abuse of notation) and $\mu$ is trivial on $T_1$. Therefore $\tau_\chi = \tau_{\chi_1} \tau_\mu$ and $\im \tau_\chi = \im \tau_{\chi_1} \im \tau_\mu$. Note $\im \tau_\mu$ is cyclic (since $k_\mu$ is cyclic) and $\im \tau_{\chi_1}$ consists of elements of $p$-power order (where $p = \text{char} k_F$).

Suppose now that $G$ has connected centre. Then $X/\mathbb{Z} \Phi$ is torsion free so that $\mathbb{Z} \Phi$ is a direct summand of $X$ (where $X = X^*(T) = X_*(\tilde{T})$). Thus the coroot lattice of $\tilde{G}$ is a direct summand of the cocharacter lattice of $\tilde{T}$. This is equivalent to $\tilde{G}$ having simply connected derived group. We use the following results of Steinberg:

- Let $G$ be a connected reductive group (over an algebraically closed field) with simply connected derived group. Let $t$ be a semisimple element of $G$. Then $C_G(t)$ (the centraliser of $t$ in $G$) is connected. (This is Steinberg’s connectedness theorem (see Steinberg [34]).)

- Suppose moreover that $t^n \in Z(G)$ for some $n$ divisible by no torsion prime for the root system of $G$. Then the derived group of (the connected group) $C_G(t)$ is again simply connected (see Steinberg [35]).

A prime $l$ is a torsion prime for the root system $\Psi$ if $\mathbb{Z} \Psi^V / \mathbb{Z} \Psi^V_1$ has $l$-torsion for some closed subsystem $\Psi_1$ of $\Psi$. (See Springer and Steinberg [33] pp. 178-179 for a listing of the possible torsion primes.) Our restrictions on $p = \text{char} k_F$ imply that $p$ is not a torsion prime for $\Phi^V$.

Applying these results, we see $\tilde{G}_1 = C_{\tilde{G}}(\im \tau_{\chi_1})$ is connected with simply connected derived group. The connectedness theorem then implies

$$C_{\tilde{G}_1}(\im \tau_\mu) = C_{\tilde{G}}(\im \tau_\chi)$$

is connected (since $\im \tau_\mu$ is cyclic). Hence $C_\chi = \{1\}$ and $H = \tilde{H}$.

If $G$ does not have connected centre then $X/\mathbb{Z} \Phi$ (the group of rational characters of the centre) has non-trivial torsion. In this case we fix a surjective homomorphism $\varphi : L \to X/\mathbb{Z} \Phi$ where $L$ is free abelian of finite rank. Consider the pullback diagram (cf. [25] pages 119-120):

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{p_1} & L \\
\downarrow p_2 & & \downarrow q \\
X & \xrightarrow{p} & X/\mathbb{Z} \Phi
\end{array}$$

Thus $\tilde{X} = \{(x, l) \in X \times L \mid p(x) = q(l)\}$ where $p$ is the obvious surjection and $p_1$, $p_2$ are the canonical projections. Let

$$\tilde{\Phi} = \{(\alpha, 0) \in X \times L : \alpha \in \Phi\}.$$ 

Note that $\tilde{\Phi}$ is contained in $\tilde{X}$ and that $p_1$ induces an isomorphism $\tilde{X}/\mathbb{Z} \tilde{\Phi} \simeq L$. In particular, $\tilde{X}/\mathbb{Z} \tilde{\Phi}$ is torsion free.

Let $\tilde{T}$ be the $F$-split $F$-torus such that $X^*(\tilde{T}) = \tilde{X}$. Let $\tilde{T} = \tilde{T}(F)$. Then

$$\tilde{T} = \text{Hom}(\tilde{X}, F^x) \supset \text{Hom}(X, F^x) = T.$$
By standard arguments (e.g. using root data) the inclusion $T \subset \widehat{T}$ extends to an inclusion $G \subset \widehat{G} = \widehat{G}(F)$ where $G$ is a connected reductive $F$-group with maximal torus $\widehat{T}$ and $\tilde{\Phi} = \Phi(\widehat{G}, \widehat{T})$. By construction $\widehat{G}$ has connected centre (since $\widehat{X}/\widehat{Z}$ is torsion free).

Let $\chi$ be any extension of $\chi$ to $\widehat{T}$. Since $\widehat{G}$ has connected centre, we have $C_{\chi} = \{1\}$ or $\widehat{W}_{\chi} = \widehat{W}_{\chi}^0$. Note also that $\widehat{W}_{\chi}^0 = \widehat{W}_{\chi}^0$ (since the image of any coroot of $\widehat{G}$ with respect to $\widehat{T}$ is contained in $T$). Thus for $c \in C_{\chi}$, we have $c\chi = \chi\mu_c$ where $\mu_c$ is trivial on $\widehat{0}T$, i.e. $\mu_c$ belongs to the Pontryagin dual $(\widehat{0}T/\widehat{0}T)^{\wedge}$ of $\widehat{0}T/\widehat{0}T$. Further, $\mu_c = 1$ if and only if $c \in C_{\chi} \cap \widehat{W}_{\chi}^0 = \{1\}$. It is easy to verify that the map

$$c \mapsto \mu_c : C_{\chi} \to (\widehat{0}T/\widehat{0}T)^{\wedge}$$

is a homomorphism (e.g. by showing that $w(t)t^{-1} \in T$ for all $t \in \widehat{T}$ and all $w \in \widehat{W}$). Thus $C_{\chi}$ embeds in $(\widehat{0}T/\widehat{0}T)^{\wedge}$. In particular, $C_{\chi}$ is abelian.

Finally we observe that $H$ is an endoscopic group of $G$. Since $H$ is $F$-split, we only have to show that $C_{\chi}(\im \tau_\chi)^0 = C_{\chi}(x)^0$ for some $x \in \widehat{T}$. We prove the following:

• Let $G$ be a connected reductive group over an algebraically closed field. Suppose $S_1$ and $S_2$ are finite subgroups of a fixed maximal torus $T$ of $G$ where $S_2$ is cyclic and the orders of $S_1$ and $S_2$ are relatively prime. Finally, suppose the order of $S_1$ is not divisible by a bad prime for $\Phi = \Phi(G, T)$. Then $C_{\chi}(S)^0 = C_{\chi}(x)^0$ for some $x \in T$ where $S = S_1S_2$.

This clearly implies that $H$ is endoscopic for $G$ (taking $(G, T) = (\widehat{G}, \widehat{T})$, $S_1 = \im \tau_\chi$ and $S_2 = \im \tau_\mu$).

We first show $C_{\chi}(S_1)^0 = C_{\chi}(t)^0$ for some $t \in T$. Let

$$\Phi_1 = \{\alpha \in \Phi : \alpha(s_1) = 1 \ \forall s_1 \in S_1\}.$$ 

Clearly $\Phi_1$ is a closed subsystem of $\Phi$. Since (by definition) $\Z\Phi/\Z\Phi_1$ can only have $l$-torsion for a prime $l$ when $l$ is a bad prime for $\Phi$ and since $s_1 \mapsto s_1^l$ is an automorphism of $S_1$, we see $\Z\Phi/\Z\Phi_1$ is torsion free. Hence $\Z\Phi_1$ is a direct summand of $\Z\Phi$. It follows that there exists an element $t \in T$ such that $\alpha(t) = 1$ for $\alpha \in \Z\Phi$ if and only if $\alpha \in \Z\Phi_1$ (see e.g. Humphreys [21] 16.2, Lemma C). Then $C_{\chi}(S_1)^0 = C_{\chi}(t)^0$.

Note $C_{\chi}(S)^0$ is the identity component of the centraliser of $s$ in $C_{\chi}(t)^0$ where $s$ generates $S_2$. This is the group generated by $T$ and those root groups $U_\alpha$ for which $\alpha(t) = 1$ and $\alpha(s) = 1$. We show this is equivalent to $\alpha(st) = 1$ by proving $\alpha(st) = 1$ implies $\alpha(t) = 1$. It follows that $C_{\chi}(S)^0 = C_{\chi}(st)^0$. Thus suppose $\alpha(st) = 1$. Then $\alpha(t)^m = 1$ where $m$ is the order of $s$ and so $m\alpha \in \Z\Phi_1$. Since $\Z\Phi/\Z\Phi_1$ is torsion free, this implies $\alpha \in \Z\Phi_1 \cap \Phi = \Phi_1$. Hence $\alpha(t) = 1$ (and $\alpha(s) = 1$).

9. Jacquet and induction functors via Hecke algebra homomorphisms

Let $B = TU$ be a Borel subgroup containing $T$ with unipotent radical $U$. Write $\Ind_B^G : \mathcal{R}(T) \to \mathcal{R}(G)$ for the unnormalised (parabolic) induction functor and $R_U : \mathcal{R}(G) \to \mathcal{R}(T)$ for the Jacquet restriction functor. As above, we write $\mathcal{R}_\chi(T)$ for the component of the Bernstein decomposition of $T$ determined by $\chi$. Thus $\mathcal{R}_\chi(T)$ is the
full subcategory of $\mathcal{R}(T)$ consisting of all smooth representations $(\sigma, \mathcal{W})$ of $T$ such that each irreducible $T$-subquotient of $\sigma$ is given by a character $\tilde{\chi} : T \to \mathbb{C}^\times$ extending $\chi$ (equivalently, $\sigma$ on restriction to $^0T$ is a multiple of $\chi$).

The functor $\text{Ind}_{G}^{\mathcal{R}(G)}$ restricts to $\mathcal{R}_\chi(T)$ to yield a functor $\text{Ind}_{G}^{\mathcal{R}(G)} : \mathcal{R}_\chi(T) \to \mathcal{R}_\chi(G)$. If $(\pi, \mathcal{V}) \in \mathcal{R}_\chi(G)$, we have

$$R_U(\mathcal{V}) \in \prod_{w \in \mathcal{W}} \mathcal{R}_{w\chi}(T).$$

Composing $R_U : \mathcal{R}_\chi(G)$ with the projection functor $p_x : \prod_{w \in \mathcal{W}} \mathcal{R}_{w\chi}(T) \to \mathcal{R}_\chi(T)$, we obtain a functor $p_x \circ R_U : \mathcal{R}_\chi(G) \to \mathcal{R}_\chi(T)$. It is clear that this functor is left adjoint to $\text{Ind}_{G}^{\mathcal{R}(T)} : \mathcal{R}_\chi(T) \to \mathcal{R}_\chi(G)$ (since $\text{Hom}_T(\mathcal{R}_{w\chi}(T), \mathcal{R}_\chi(T)) = 0$ if $w\chi \neq \chi$). By abuse of notation, we often drop the $p_x$ and simply write $R_U : \mathcal{R}_\chi(G) \to \mathcal{R}_\chi(T)$.

We write $\text{Ind}^\chi : \mathcal{R}_\chi(G) \to \mathcal{R}_\chi(T)$ and $\text{Jacquet}^\chi : \mathcal{R}_\chi(G) \to \mathcal{R}_\chi(T)$ for the normalised induction and Jacquet restriction functors. Thus if $(\sigma, \mathcal{W})$ is a smooth representation of $T$ and $\mathcal{W}$ is the modulus character for the action of $T$ on $\mathcal{W}$ by conjugation (i.e. $d(tu^{-1}) = \delta_B(t)du$ for $t \in T$, $du$ is any Haar measure on $T$), then $\text{Ind}^\chi(\sigma) = \text{Ind}_{G}^{\mathcal{R}(T)}(\sigma \otimes \delta_B^{1/2})$. If $(\pi, \mathcal{V})$ is a smooth representation of $G$, $R_U(\mathcal{V})$ is a character $\mathcal{W}_U$ of $\mathcal{R}_\chi(T)$ and $R_U : \mathcal{R}_\chi(G) \to \mathcal{R}_\chi(T)$ (more properly $p_x \circ R_U$). It is clear that $R_U$ is left adjoint to $\text{Ind}^\chi$.

From Corollary 7.9, the functor $V \mapsto V^\chi : \mathcal{R}_\chi(G) \to \mathcal{H}(G, \rho) - \text{Mod}$ is an equivalence of categories. The functor $W \mapsto W^\chi : \mathcal{R}_\chi(T) \to \mathcal{H}(T, \chi) - \text{Mod}$ is also an equivalence of categories ($^{0}T, \chi$ is a type for $\mathcal{R}_\chi(T)$). Our first aim in this section is to show that, under these equivalences, the induction and Jacquet restriction functors between $\mathcal{R}_\chi(T)$ and $\mathcal{R}_\chi(G)$ (unnormalised and normalised) correspond to induction and restriction functors between $\mathcal{H}(T, \chi) - \text{Mod}$ and $\mathcal{H}(G, \rho) - \text{Mod}$ induced by appropriate twists of a $C$-algebra embedding $t_B : \mathcal{H}(T, \chi) \to \mathcal{H}(G, \rho)$. Given our knowledge of the structure of the Hecke algebra $\mathcal{H}(G, \rho)$, these results are special cases (or immediate consequences of) a general result of Bushnell and Kutzko (see [13] 7.12, 8.4).

We now describe the algebra embedding $t = t_B : \mathcal{H}(T, \chi) \to \mathcal{H}(G, \rho)$. This is done in a much more general context in sections 6 and 7 of [13]. Fix a uniformiser $\varpi$ in $F$. The short exact sequence (see equation 4.1)

$$1 \longrightarrow ^0T \longrightarrow T \overset{H_{\varpi}}{\longrightarrow} Y \longrightarrow 0$$

is then split by the homomorphism $y \mapsto y(\varpi) : Y \to T$. For each $y \in Y$, let $\phi_y$ be the unique element of $\mathcal{H}(T, \chi)$ with support $^0Ty$ such that $\phi_y(y(\varpi)) = 1$. It is clear that the elements $\phi_y$ for $y \in Y$ form a vector space basis of $\mathcal{H}(T, \chi)$ (and that the $C$-linear extension of the assignment $y \mapsto \phi_y$ defines an isomorphism of $C$-algebras between $C[Y]$ and $\mathcal{H}(T, \chi)$). For each $y \in Y$, let $\Phi_y$ be the unique element of $\mathcal{H}(G, \rho)$ with support $JyJ$ such that $\Phi_y(y(\varpi)) = 1$. The Borel subgroup $B$ corresponds to a system of simple roots $\Pi_B$ in $\Phi(G, T)$. Define a subsemigroup $Y^+$ of $Y$ as follows:

$$Y^+ = \{y \in Y \mid \langle y, \alpha \rangle \geq 0 \ (\alpha \in \Pi_B)\}.$$ 

Given $y \in Y$ we may write $y = y_1 - y_2$ where $y_1, y_2 \in Y^+$. By definition

$$t(\phi_y) = \Phi_{y_1} \Phi_{y_2}^{-1}.$$
(Note $\Phi_{y_2}$ is indeed invertible in $\mathcal{H}(G, \rho)$ by Lemma 7.6.) The following lemma shows that $t_B(\phi_y)$ is well-defined (i.e. independent of the choice of $y_1, y_2 \in Y^+$ such that $y_1 - y_2 = y$) and extends to a C-algebra embedding $t_B : \mathcal{H}(T, \chi) \rightarrow \mathcal{H}(G, \rho)$.

**Lemma 9.1.** - For $y_1, y_2 \in Y^+$, $\Phi_{y_1}, \Phi_{y_2} = \Phi_{y_1 + y_2}$.

**Proof.** - This is straightforward (using Iwahori decompositions). □

It is easy to check that the homomorphism $t : \mathcal{H}(T, \chi) \rightarrow \mathcal{H}(G, \rho)$ is independent of the choice of uniformiser $\omega$ (as the notation indicates).

Suppose now that $j : \mathcal{H}(T, \chi) \rightarrow \mathcal{H}(G, \rho)$ is an algebra homomorphism. Then $j$ induces restriction and induction functors $j^* : \mathcal{H}(G, \rho) - \text{Mod} \rightarrow \mathcal{H}(T, \chi) - \text{Mod}$ and $j_* : \mathcal{H}(T, \chi) - \text{Mod} \rightarrow \mathcal{H}(G, \rho) - \text{Mod}$ as follows.

If $M \in \mathcal{H}(G, \rho) - \text{Mod}$, then $j^*(M) = M$ with $\mathcal{H}(T, \chi)$-action given by $\phi, m = j(\phi)m$ for $\phi \in \mathcal{H}(T, \chi)$ and $m \in M$. If $N \in \mathcal{H}(T, \chi) - \text{Mod}$, $j_*(N) = \text{Hom}_{\mathcal{H}(T, \chi)}(\mathcal{H}(G, \rho), N)$. Here $\mathcal{H}(G, \rho)$ is viewed as an $\mathcal{H}(T, \chi)$-module via $j$ and $\mathcal{H}(G, \rho)$ acts by right translations. Thus $\Phi, \alpha(\Phi') = \alpha(\Phi' \Phi)$ for $\alpha \in \text{Hom}_{\mathcal{H}(T, \chi)}(\mathcal{H}(G, \rho), N)$ and $\Phi, \Phi' \in \mathcal{H}(G, \rho)$. It is easy to verify that $(j^*, j_*)$ is an adjoint pair, i.e. there exist natural isomorphisms (of C-vector spaces)

\[
\text{Hom}_{\mathcal{H}(T, \chi)}(j^*(M), N) \cong \text{Hom}_{\mathcal{H}(G, \rho)}(M, j_*(N))
\]

for $M \in \mathcal{H}(G, \rho) - \text{Mod}$ and $N \in \mathcal{H}(T, \chi) - \text{Mod}$.

If $\tau$ is an unramified character of $T$, we write $t_\tau : \mathcal{H}(T, \chi) \rightarrow \mathcal{H}(G, \rho)$ for the algebra embedding given by $t_\tau(\phi) = t(\phi, \tau)$ for $\phi \in \mathcal{H}(T, \chi)$ where $\phi, \tau(t) = \phi(t)\tau(t)$ ($t \in T$). For ease of notation, we write $t_B$ for the algebra embedding $t_B^{1/2}$.

**Theorem 9.2.** - Let $B = TU$ be a Borel subgroup containing $T$ with unipotent radical $U$. Let $t : \mathcal{H}(T, \chi) \rightarrow \mathcal{H}(G, \rho)$ be the associated embedding of C-algebras (as described above). Then the following diagrams commute (up to natural equivalence).

\[
\begin{align*}
\mathcal{R}_\chi(T) & \rightarrow \mathcal{H}(T, \chi) - \text{Mod} \\
\mathcal{R}_\chi(G) & \rightarrow \mathcal{H}(G, \rho) - \text{Mod}
\end{align*}
\]

\[
\begin{align*}
\mathcal{R}_\chi(T) & \rightarrow \mathcal{H}(T, \chi) - \text{Mod} \\
\mathcal{R}_\chi(G) & \rightarrow \mathcal{H}(G, \rho) - \text{Mod}
\end{align*}
\]
In each case the rows are given by the equivalences of categories
\[ V \mapsto V^\rho : \mathcal{R}_\chi(G) \xrightarrow{\cong} \mathcal{H}(G, \rho) - \text{Mod} \]
and
\[ W \mapsto W^\chi : \mathcal{R}_\chi(T) \xrightarrow{\cong} \mathcal{H}(T, \chi) - \text{Mod}. \]

Proof. – The first two diagrams are commutative by Bushnell and Kutzko [13] 8.4. In fact, if \((\pi, V) \in \mathcal{R}_\chi(G)\) then it follows from [13] 7.12 that the isomorphism \(V^\rho \cong R_U(V)\) (see Remark 7.8) satisfies
\[ (9.6) \quad R_U(t_{\delta_B}(\phi).v) = \phi.R_U(v) \]
for \(v \in V^\rho, \phi \in \mathcal{H}(T, \chi)\). The hypotheses of [13] hold in our situation by Lemma 7.6. This gives 9.2 (and also 9.3 by uniqueness of adjoints).

To prove 9.4 it is enough to show
\[ r_U(t_{\pi}(\phi_y).v) = \phi_y r_U(v) \]
for \(v \in V^\rho, y \in Y\). We may rewrite this as
\[ R_U(t(\phi_y t_B^{1/2}(y)).v) = \delta_B^{-1/2}(y)\phi_y.R_U(v) \]
for \(v \in V^\rho, y \in Y\). Equivalently
\[ R_U(t(\phi_y t_B(y)).v) = \phi_y.R_U(v) \]
for \(v \in V^\rho \) and \(y \in Y\) which we know from 9.6.

The commutativity of 9.5 follows (again by uniqueness of adjoints). \(\square\)

Thus, under the equivalences of categories \(V \mapsto V^\rho : \mathcal{R}_\chi(G) \rightarrow \mathcal{H}(G, \rho) - \text{Mod}\) and \(W \mapsto W^\chi : \mathcal{R}_\chi(T) \rightarrow \mathcal{H}(T, \chi) - \text{Mod}\), the adjoint pair \((R_U, \text{Ind}_G^H)\) corresponds to \((t^*_B, (t_{\delta_B})_*)\) and the pair \((r_U, t_B^H)\) corresponds to \((t^{*, *}_{\delta_B}, (t^*_B, t_{\delta_B})_*) = (t^*_u, (t_{\pi})_*)\).

Suppose now that the Borel subgroup \(B\) containing \(T\) corresponds to the positive system \(\Phi^+\) of \(\Phi\) (as in the definition of the subgroup \(J\)). Let \(B(H)\) be the Borel subgroup of \(H\) (with unipotent radical \(U(H)\)) corresponding to the positive system \(\Phi^+_\chi = \Phi_\chi \cap \Phi^+\). In the remainder of this section we prove a simple relation between the induction and restriction functors \(i_B^H\) : \(\mathcal{R}_\chi(T) \rightarrow \mathcal{R}_\chi(G)\) and \(r_U : \mathcal{R}_\chi(G) \rightarrow \mathcal{R}_\chi(T)\) and corresponding functors for \(H\). A precise statement is given in Theorem 9.4.

First we claim that the functor \(V \mapsto V^1_\chi : \mathcal{R}_\chi(\tilde{H}) \rightarrow \mathcal{H}(\tilde{H}, 1_\chi) - \text{Mod}\) is an equivalence of categories. This holds provided the category \(\mathcal{R}_\chi(\tilde{H})\) is closed under subquotients (see [12] 3.9 (i)). Thus suppose \(V \in \mathcal{R}_\chi(\tilde{H})\). Write \(r^H_H(V)\) for the smooth representation of \(H\) obtained by (ordinary) restriction. Since \(\tilde{H} = H \times C_\chi\) and \(C_\chi\) normalises \(I\), \(r^H_H(V) \in \mathcal{R}_\chi(\tilde{H})\). Moreover since \((I, 1_\chi)\) is a type in \(H\), each subquotient of \(r^H_H(V)\) is again in \(\mathcal{R}_\chi(H)\) and thus each subquotient of \(V\) is (a fortiori) in \(\mathcal{R}_\chi(\tilde{H})\).

It is clear that the restriction functor \(r^H_H : \mathcal{R}_\chi(\tilde{H}) \rightarrow \mathcal{R}_\chi(H)\) fits into the following commutative diagram:
\[ \begin{array}{ccc}
\mathcal{R}_\chi(\tilde{H}) & \xrightarrow{\cong} & \mathcal{H}(\tilde{H}, 1_\chi) - \text{Mod} \\
\downarrow^{r^H_H} & & \downarrow^{i'} \\
\mathcal{R}_\chi(H) & \xrightarrow{\cong} & \mathcal{H}(H, 1_\chi) - \text{Mod}
\end{array} \]
where \( i^* \) is induced by the inclusion homomorphism \( i : \mathcal{H}(H, 1_T) \to \mathcal{H}(\tilde{H}, 1_T) \) and the rows are given by the equivalences of categories \( \mathcal{V} \to \mathcal{V}^{1_T} \).

The functor \( r^H_H : \mathcal{R}_1(\tilde{H}) \to \mathcal{R}_1(H) \) has a right adjoint \( i_H^H : \mathcal{R}_1(H) \to \mathcal{R}_1(\tilde{H}) \) (ordinary induction from \( H \) to \( \tilde{H} \)). Also \( i^* : \mathcal{H}(\tilde{H}, 1_T) - \text{Mod} \to \mathcal{H}(H, 1_T) - \text{Mod} \) has a right adjoint \( i_* : \mathcal{H}(H, 1_T) - \text{Mod} \to \mathcal{H}(\tilde{H}, 1_T) - \text{Mod} \) (exactly as in 9.1). By uniqueness of adjoints, we obtain the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{R}_1(\tilde{H}) & \xrightarrow{\simeq} & \mathcal{H}(\tilde{H}, 1_T) - \text{Mod} \\
\uparrow i_H^H & & \uparrow i_* \\
\mathcal{R}_1(H) & \xrightarrow{\simeq} & \mathcal{H}(H, 1_T) - \text{Mod}.
\end{array}
\]

(9.8)

Since \((I, 1_I)\) is a type for \( \mathcal{R}_1(H) \), we have \( \mathcal{R}_1(H) = \mathcal{R}_1(\tilde{H}) \). Applying Theorem 9.2 (iii) and (iv) (in this context) and writing \( t_{u,H} \) for \( t_{1_H,2} \), we obtain further commutative diagrams:

\[
\begin{array}{cc}
\mathcal{R}_1(H) & \xrightarrow{\simeq} \mathcal{H}(H, 1_T) - \text{Mod} \\
r^H_H \downarrow & \downarrow t_{u,H} \\
\mathcal{R}_{10_T}(T) & \xrightarrow{\simeq} \mathcal{H}(T, 1_{10_T}) - \text{Mod}
\end{array}
\]

(9.9)

and

\[
\begin{array}{cc}
\mathcal{R}_1(H) & \xrightarrow{\simeq} \mathcal{H}(H, 1_T) - \text{Mod} \\
i_H^H \uparrow & \uparrow (t_{u,H})_* \\
\mathcal{R}_{10_T}(T) & \xrightarrow{\simeq} \mathcal{H}(T, 1_{10_T}) - \text{Mod}
\end{array}
\]

(9.10)

Combining 9.7 and 9.9 and also 9.8 and 9.10 (and using the obvious notation) yields the commutativity of

\[
\begin{array}{ccc}
\mathcal{R}_1(\tilde{H}) & \xrightarrow{\simeq} \mathcal{H}(\tilde{H}, 1_T) - \text{Mod} \\
r^H_H \downarrow & \downarrow t_{u,H} \\
\mathcal{R}_{10_T}(T) & \xrightarrow{\simeq} \mathcal{H}(T, 1_{10_T}) - \text{Mod}
\end{array}
\]

(9.11)

and

\[
\begin{array}{cc}
\mathcal{R}_1(\tilde{H}) & \xrightarrow{\simeq} \mathcal{H}(\tilde{H}, 1_T) - \text{Mod} \\
i_H^H \uparrow & \uparrow (t_{u,H})_* \\
\mathcal{R}_{10_T}(T) & \xrightarrow{\simeq} \mathcal{H}(T, 1_{10_T}) - \text{Mod}
\end{array}
\]

(9.12)

As above, let \( \phi_y \) for \( y \in Y \) denote the unique element of \( \mathcal{H}(T, \chi) \) supported on \( ^0Ty \) such that \( \phi_y(y(\omega)) = 1 \). Write \( \chi_y \) for the characteristic function of \( ^0Ty \). It is clear that there is a unique isomorphism of C-algebras

\[\psi = \psi_\omega : \mathcal{H}(T, \chi) \longrightarrow \mathcal{H}(T, 1_{10_T})\]

satisfying \( \psi(\phi_y) = \chi_y \). The key step in our argument is contained in the following lemma.
LEMMA 9.3. Let \( \psi (= \psi_\infty) : \mathcal{H}(T, \chi) \to \mathcal{H}(T, 1_{vT}) \) be the \( \mathbb{C} \)-algebra isomorphism described above. Then there exists a support-preserving, \(*\)-preserving \( \mathbb{C} \)-algebra isomorphism \( \Psi (= \Psi_\infty) : \mathcal{H}(G, \rho) \to \mathcal{H}(\widetilde{H}, 1_T) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{H}(G, \rho) & \xrightarrow{\Psi} & \mathcal{H}(\widetilde{H}, 1_T) \\
t_u & \uparrow & t_{u,N} \\
\mathcal{H}(T, \chi) & \xrightarrow{\psi} & \mathcal{H}(T, 1_{vT}).
\end{array}
\]

Proof. Clearly it is sufficient to show there exists an algebra isomorphism as in the statement of the lemma such that

\[ \Psi(t_u(\phi_y)) = t_{u,H}(\psi(\phi_y)) \quad (y \in Y^+). \]

If \( J^+ = J \cap U \), then \( \delta_B(y) = [J^+ : yJ^+y^{-1}]^{-1} \). Using the Iwahori decomposition of \( J \) with respect to \( B \), we have \( \delta_B(y) = \text{vol}(JyJ)^{-1} \). Similarly \( \delta_B(H)(y) = \text{vol}(IyI)^{-1} \). Thus we need to show

\[ (9.13) \quad \Psi(\text{vol}(JyJ)^{-1/2} \Phi_y) = \text{vol}(IyI)^{-1/2} \chi_{yI} \quad (y \in Y^+) \]

where \( \chi_{yI} \) is the characteristic function of \( IyI \).

Let \( M \) be minimal among Levi subgroups of \( G \) containing \( T \) such that \( \mathcal{T}_G(\rho) \subset JMJ \). From Proposition 5.1 and Remark 5.2, there exists a \(*\)-preserving, support-preserving Hecke algebra isomorphism

\[ \Psi^M : \mathcal{H}(G, \rho) \xrightarrow{\cong} \mathcal{H}(M, \rho_M) \]

such that

\[ \Psi^M(\text{vol}(JyJ)^{-1/2} \Phi_y) = \text{vol}(JM_yJM)^{-1/2} \Phi_y^M \quad (y \in Y^+) \]

where \( \rho_M = \rho | J_M \) (\( J_M = J \cap M \)) and \( \Phi_y^M \) is the unique function in \( \mathcal{H}(M, \rho_M) \) with support \( JM_yJM \) such that \( \Phi_y^M(y(\infty)) = 1 \). Moreover from the proof of theorem 4.15, there exists a character \( \chi_1 \) of \( M \) such that \( \chi_1 \) viewed as a character of \( 0^T \) has level zero. We may also assume that the character \( \chi_1 \) satisfies \( \chi_1(y(\infty)) = 1 \). Using the isomorphism \( f \mapsto f_{\chi_1} : \mathcal{H}(M, \rho_M) \to \mathcal{H}(M, \rho_M\chi_1) \), we are thus reduced to the proving 9.13 in the level-zero case.

We assume therefore that \( \chi \) is a level zero character of \( 0^T \). We use Morris [27] to show there exists a \((\ast\)-preserving, support-function) \( \mathbb{C} \)-algebra isomorphism satisfying 9.13.

As in the proof of Theorem 6.3, we fix an extension \( \tilde{\chi} \) of \( \chi \) to \( N_\chi \) satisfying \( \tilde{\chi}(y(\infty)) = 1 \) for all \( y \in Y \). We then have a basis \( B_x \) for \( x \in W_\chi = W_\chi^{-1} \) of \( \text{End}_G(\text{Ind}^{\rho_1}^{-1}) \) given by

\[ B_x = \tilde{\chi}(n)^{-1} \theta_n \]

for any \( n \in N_\chi \) projecting to \( x \in W_\chi \) (see [27] 5.4, 5.5). Write \( y = w\omega \) where \( w \in W_\chi^0 \) and \( \omega \in \Omega_\chi \). In the notation of [27], \( C(\chi) = C(\chi^{-1}) = \Omega_\chi \). Further the cocycle \( \lambda \) (in [27] 6.2) is trivial (by our choice of \( B_x \) for \( x \in W_\chi \)). Applying [27] Propn. 7.6 a,

\[ (9.14) \quad B_w B_\omega = \left( \frac{\text{Ind} w\omega}{\text{Ind} w} \right)^{1/2} B_{w\omega} = q^{\frac{1}{2}(l(y) - l(w) - l(\omega))} B_y. \]
Since $\epsilon_x = 1$ for all $x \in W_x$ and $p_w = q^{l_x(w)}$, we have

\begin{align*}
(9.15) \quad T_w &= q^{\frac{1}{2}(l(w) + l_x(w))} B_w \\
T_w &= q^{\frac{1}{2}l(w)} B_w
\end{align*}

(see [27] 7.9, 7.10). A straightforward calculation shows that the isomorphism

$$
t : \mathcal{H}(G, \rho) \xrightarrow{\cong} \text{End}_G(\text{Ind} \rho^{-1})$$

of equation 7.2 satisfies $t_{\Phi_n} = q^{l(x)} \theta_n$ for any $n \in N_x$ projecting to $x \in W_x$. Here $\Phi_n$ is the unique element of $\mathcal{H}(G, \rho)$ supported on $JnJ$ satisfying $\Phi_n(n) = 1$. Thus

\begin{align*}
q^{-\frac{1}{2}l(y)} \Phi_y &\mapsto q^{-\frac{1}{2}l(y)} t_{\Phi_y} \\
&= q^{\frac{1}{2}l(y)} \theta_{y(x)} \\
&= q^{\frac{1}{2}l(y)} B_y \quad \text{(since $\tilde{\chi}(y(w)) = 1$)} \\
&= q^{\frac{1}{2}l(w)} B_w q^{\frac{1}{2}l(w)} B_w \quad \text{(by 9.14)} \\
&= q^{-\frac{1}{2}l_x(w)} T_w T_w \quad \text{(by 9.15)}.
\end{align*}

Applying the composition of the isomorphisms between $\text{End}_G(\text{Ind} \rho^{-1})$ and $\mathcal{H}_x$ and between $\mathcal{H}_x$ and $\mathcal{H}(\tilde{H}, 1_I)$, this last element is sent to

$$
q^{-\frac{1}{2}l_x(w)} \chi_{IwI} \chi_{IwI} = q^{-\frac{1}{2}l_x(y)} \chi_{IyI}.
$$

This proves 9.13 in the level-zero case and thus completes the proof of the lemma. □

We now fix a $\mathbb{C}$-algebra isomorphism $\Psi : \mathcal{H}(G, \rho) \cong \mathcal{H}(\tilde{H}, 1_I)$ as in the Lemma. We immediately obtain the following commutative diagrams on module categories (where the top (resp. bottom) rows are induced by $\Psi$ (resp. $\psi$)):

\begin{equation}
(9.16)
\begin{array}{c}
\mathcal{H}(G, \rho) - \text{Mod} \xrightarrow{\cong} \mathcal{H}(\tilde{H}, 1_I) - \text{Mod} \\
\downarrow \psi \quad \downarrow \psi
\end{array}
\end{equation}

and

\begin{equation}
(9.17)
\begin{array}{c}
\mathcal{H}(T, \chi) - \text{Mod} \xrightarrow{\cong} \mathcal{H}(T, 1_{oT}) - \text{Mod} \\
\uparrow (t_{\psi}) \quad \uparrow (t_{\psi})
\end{array}
\end{equation}

The isomorphism $\Psi$ induces an equivalence of categories between $\mathcal{R}_x(G)$ and $\mathcal{R}_{1_I}(\tilde{H})$ (by composing the inverse of the equivalence $\mathcal{W} \mapsto \mathcal{W}^{1_I} : \mathcal{R}_{1_I}(\tilde{H}) \cong \mathcal{H}(\tilde{H}, 1_I) - \text{Mod}$ with the composition of $\Psi_* : \mathcal{H}(G, \rho) - \text{Mod} \cong \mathcal{H}(\tilde{H}, 1_I) - \text{Mod}$ and $\mathcal{V} \mapsto \mathcal{V}^\rho : \mathcal{R}_x(G) \cong \mathcal{H}(G, \rho) - \text{Mod}$). Similarly $\psi$ induces an equivalence of categories between $\mathcal{R}_x(T)$ and $\mathcal{R}_{1_{oT}}(T)$. These are compatible with induction and restriction as described in the following theorem.
**Theorem 9.4.** - The equivalences of categories $\mathcal{R}_\chi(G) \cong \mathcal{R}_{1z}(\tilde{H})$ (induced by $\Psi$) and $\mathcal{R}_\chi(T) \cong \mathcal{R}_{lo}(T)$ (induced by $\psi$) fit into the following commutative diagrams:

\[
\begin{array}{ccc}
\mathcal{R}_\chi(G) & \xrightarrow{\sim} & \mathcal{R}_{1z}(\tilde{H}) \\
r_U \downarrow & & r_U(H) \downarrow \\
\mathcal{R}_\chi(T) & \xrightarrow{\sim} & \mathcal{R}_{lo}(T).
\end{array}
\]

(9.18)

\[
\begin{array}{ccc}
\mathcal{R}_\chi(G) & \xrightarrow{\sim} & \mathcal{R}_{1z}(\tilde{H}) \\
i_B^G \uparrow & & i_B^G(H) \\
\mathcal{R}_\chi(T) & \xrightarrow{\sim} & \mathcal{R}_{lo}(T).
\end{array}
\]

(9.19)

In particular, if $\nu$ is an unramified character of $T$, then the representation $i_B^G(\chi, \nu)$ corresponds to $i_B^H(\nu)$.

**Proof.** - Combining 9.4, 9.16 and 9.11, we immediately obtain (the commutativity of) 9.18. Similarly, (the commutativity of) 9.19 follows immediately from (the commutativity of) 9.5, 9.17 and 9.12. □

**10. Square-integrability and formal degrees**

Throughout this section we fix a *-preserving, support-preserving (and hence inner-product preserving) isomorphism of $\mathbb{C}$-algebras $\Psi : \mathcal{H}(H, 1_Z) \cong \mathcal{H}(G, \rho)$. This induces an equivalence of categories between $\mathcal{H}(H, 1_Z) - \text{Mod}$ and $\mathcal{H}(G, \rho) - \text{Mod}$. Via the equivalences $W \mapsto \mathcal{W}_{1z} : \mathcal{R}_{1z}(\tilde{H}) \cong \mathcal{H}(H, 1_Z) - \text{Mod}$ and $\mathcal{V} \mapsto \mathcal{V}^\rho : \mathcal{R}_\chi(G) \cong \mathcal{H}(G, \rho) - \text{Mod}$, we also obtain an equivalence of categories between $\mathcal{R}_{1z}(\tilde{H})$ and $\mathcal{R}_\chi(G) = \mathcal{R}_\rho(G)$. Hence there is a bijection between smooth irreducible representations of $H$ containing $1_Z$ and smooth irreducible representations of $G$ containing $\rho$. In this section we show that this correspondence preserves square-integrability provided $Z_H/Z_G$ is compact. (Here $Z_H$ denotes the centre of $H$ and $Z_G$ the centre of $G$.) This condition is equivalent to the endoscopic group $H$ of $G$ being elliptic. When $Z_H/Z_G$ is non-compact, it is a trivial consequence of our calculations that there exist no (non-zero) square-integrable representations of $G$ containing $\rho$. (Following a standard abuse of terminology we use the terms square-integrable and square-integrability in place of the more accurate (but cumbersome) square-integrable-mod-centre and square-integrability-mod-centre.) We also observe that (when square-integrability is preserved) the formal degrees of corresponding square-integrable representations are equal (given certain natural choices of Haar measures). It is well-known that Hecke algebra isomorphisms may be used to make observations of this sort. This first appeared in the $p$-adic setting in Howe and Moy [19]. In section 7.7 of [10] Bushnell and Kutzko obtain analogous results for a simple type in $GL_N(F)$ (in place of $\rho$). Their treatment however is general and we have merely transposed it to our context (often almost word for word).

**Remark 10.1.** - It would not be difficult to recast the arguments of this section without explicit mention of the groups $H$ or $\tilde{H}$. Instead one would work with the 'abstract' Hecke algebra $\mathcal{H}_\chi$ or appropriate quotients (if $G$ is not semisimple) and directly define
square-integrability and formal degrees for modules over such algebras via their canonical inner products (see 6.4).

Let \((\pi, \mathcal{V})\) be a smooth irreducible representation of \(G\) with unitary central character \(\omega = \omega\) (thus \(|\omega(z)| = 1\) for \(z \in Z = Z_G\)). We say \((\pi, \mathcal{V})\) is square-integrable (by abuse of terminology) if

\[
\int_{G/Z} |(\pi(g)v, \tilde{v})|^2 \, dg < \infty \quad (v \in \mathcal{V}, \tilde{v} \in \tilde{\mathcal{V}}).
\]

Clearly we may replace \(Z\) by any cocompact subgroup \(Z_1\) of \(Z\).

Whenever \(Z_1\) is such a subgroup we fix Haar measures on \(Z_1\) and \(G/Z_1\) as follows. Let \(dz\) be the Haar measure on \(Z_1\) such that \(0Z_1 = Z_1 \cap 0T\) has measure one. Let \(dg\) be the Haar measure on \(G\) giving \(J\) measure one. We then choose the induced Haar measure \(dg\) on \(G/Z_1\) (i.e. \(dz \, dg = dg\)).

Let \(\mathcal{H}(G, \omega)\) be the convolution algebra (with respect to \(dg\)) of smooth functions \(\phi : G \to \mathbb{C}\) which are compactly supported modulo \(Z_1\) and satisfy \(\phi(zg) = \omega(z)^{-1}\phi(g)\) \((z \in Z_1, g \in G)\). Further, let \(\mathcal{H}^2(G, \omega)\) be the space of (left and right) smooth functions \(\phi : G \to \mathbb{C}\) which satisfy \(\phi(zg) = \omega(z)^{-1}\phi(g)\) and are also square-integrable modulo \(Z_1\). We view \(\mathcal{H}^2(G, \omega)\) as a smooth \(G\)-representation by left translations (or equivalently as a left \(\mathcal{H}(G, \omega)\)-module). We write \(*\) for the usual involution on \(\mathcal{H}(G, \omega)\) (i.e. \(\phi^*(g) = \phi(g^{-1})\) for \(g \in G, \phi \in \mathcal{H}(G, \omega)\)). This is related to the canonical inner product \((\ , \ )\) on \(\mathcal{H}(G, \omega)\) by the standard formula

\[
(\phi, \psi) = \int_{G/Z_1} \phi(g)\overline{\psi(g)} \, dg
= \phi\psi^*(1) \quad (\phi, \psi \in \mathcal{H}(G, \omega)).
\]

It is clear that \(*\) and \((\ , \ )\) extend to \(\mathcal{H}^2(G, \omega)\).

Suppose now that \((\pi, \mathcal{V})\) contains \(\rho\). Then \(\omega | Z_1 \cap J = \rho | Z_1 \cap J\) and thus \(\rho \omega\) is a well-defined representation of \(Z_1J\). The subalgebra \(\mathcal{H}(G, \rho \omega)\) of \(\mathcal{H}(G, \omega)\) is defined in the obvious way. We have \(\mathcal{H}(G, \rho \omega) = e_{\rho \omega} \mathcal{H}(G, \omega)e_{\rho \omega}\) where

\[
e_{\rho \omega}(x) = \mu(JZ_1/Z_1)\rho \omega(x)^{-1}(x \in JZ_1)
= 0 \quad (x \notin JZ_1).
\]

The volume factor \(\mu(JZ_1/Z_1)\) is in fact one (given our choice of Haar measure on \(G/Z_1\)). The completion of \(\mathcal{H}(G, \rho \omega)\) with respect to the (restriction of the) inner product \((\ , \ )\) is \(\mathcal{H}^2(G, \rho \omega) = e_{\rho \omega} \mathcal{H}^2(G, \omega)e_{\rho \omega}\).

From [10] 7.7.4 (or 7.7.5) we have the following Proposition.

**Proposition 10.2.** Let \((\pi, \mathcal{V})\) be a smooth irreducible representation of \(G\) containing \(\rho\) with central character \(\omega\). Then \((\pi, \mathcal{V})\) is square-integrable if and only if \(\mathcal{V}^\rho = \mathcal{V}^{\rho \omega}\) is isomorphic to a left \(\mathcal{H}(G, \rho \omega)\)-submodule of \(\mathcal{H}^2(G, \rho \omega)\).

For a fixed uniformiser \(\varpi\) in \(F\), the exact sequence (see 4.1)

\[
1 \longrightarrow 0T \longrightarrow T \xrightarrow{H_T} Y \longrightarrow 0
\]
is split by the homomorphism \( y \mapsto y(w) : Y \to T \). The resulting isomorphism \( T \cong 0^* T \times Y \) restricts to \( Z \) to give \( Z \cong 0^* Z \times L \) where \( 0^* Z = Z \cap 0^* T \) and

\[
L = \{ y \in Y \mid \langle y, \alpha \rangle = 0 \ (\alpha \in \Phi) \}.
\]

We also write \( L \) for its image in \( Z \) under the embedding \( l \mapsto l(w) : L \to Z \). Thus we may view \( L \) as a cocompact subgroup of \( Z \). As above, \( \rho \omega \) defines a representation of \( JL = JZ \). There is a canonical surjective algebra isomorphism

\[
P_\omega : \mathcal{H}(G, \rho) \longrightarrow \mathcal{H}(G, \rho \omega)
\]

given by

\[
P_\omega(\phi)(g) = \int_Z \omega(z) \phi(zg) dz = \sum_{l \in L} \omega(l) \phi(lg)
\]

for \( \phi \in \mathcal{H}(G, \rho) \), \( x \in G \). For \( l \in L \), define \( \phi_l \in \mathcal{H}(G, \rho) \) by \( \text{supp} \phi_l = Jl \) and \( \phi_l(l) = \omega(l)^{-1} \).

**Lemma 10.3.** The kernel of \( P_\omega \) is the two-sided ideal of \( \mathcal{H}(G, \rho) \) generated by the elements \( \phi_i - e_\rho \) \((l \in L)\).

**Proof.** Fix a \( Z \)-basis \( l_1, \ldots, l_r \) of \( L \). Write \( \phi_1 = \phi_{l_1}, \ldots, \phi_r = \phi_{l_r} \). A simple inductive argument using the relation

\[
\phi_{l+i'} - e_\rho = (\phi_i - e_\rho)(\phi_{i'} - e_\rho) + (\phi_i - e_\rho) + (\phi_{i'} - e_\rho) \quad (l, l' \in L)
\]

shows that the two-sided ideal of \( \mathcal{H}(G, \rho) \) generated by \( \phi_1 - e_\rho, \ldots, \phi_r - e_\rho \) equals the two-sided ideal generated by \( \phi_i - e_\rho \) \((l \in L)\). It suffices therefore to show that \( \ker P_\omega \) is generated by \( \phi_i - e_\rho, \ldots, \phi_r - e_\rho \).

A straightforward calculation shows \( \phi_i - e_\rho \in \ker P_\omega \) \((1 \leq i \leq r)\). Suppose \( P_\omega(\phi) = 0 \) for \( \phi \in \mathcal{H}(G, \rho) \). We may assume \( \text{supp} \phi \subset Z J x J \) for some \( x \in G \). Since \( \phi \) has compact support, there exist integers \( m_i \leq n_i \) \((1 \leq i \leq r)\) such that

\[
\text{supp} \phi \subset \bigcup_{m_1 \leq i_1 \leq n_1} l_1^{i_1} \ldots l_r^{i_r} J x J.
\]

Note we may assume \( \sum_{i=1}^r (n_i - m_i) > 0 \). Indeed if \( n_i = m_i \) \((1 \leq i \leq r)\), then \( \text{supp} \phi \) is contained in a single \( J \)-double coset and so \( P_\omega(\phi) = 0 \) implies \( \phi = 0 \). Suppose \( m_1 \leq i_1 < n_1 \) and \( \phi(l_1^{i_1} \ldots l_r^{i_r} x) = \alpha \neq 0 \). There exists a (unique) function \( \psi \in \mathcal{H}(G, \rho) \) such that \( \psi(l_1^{m_1} \ldots l_r^{m_r} x) = \alpha \) and \( \text{supp} \psi = l_1^{m_1} \ldots l_r^{m_r} J x J \). Consider \( \phi + \psi(\phi_1 - e_\rho) \). The support of this function is contained in

\[
\bigcup_{m_1 \leq i_1 \leq n_1} l_1^{i_1} \ldots l_r^{i_r} J x J.
\]
Further,
\[
\psi \phi_1(l_1^{m_1} \ldots l_r^{m_r} x) = \int_G \psi(l_1^{m_1} \ldots l_r^{m_r} xy^{-1}) \phi_1(y) dy \\
= \int_J \psi(l_1^{m_1} \ldots l_r^{m_r} j_1^{-1}) \phi_1(jl_1) dj \\
= \psi(l_1^{m_1-1} l_2^{m_2} \ldots l_r^{m_r} x) \omega(l_1)^{-1} \\
= 0.
\]
Therefore \((\phi + \psi(\phi_1 - e_p))(l_1^{m_1} \ldots l_r^{m_r} x) = 0\) and hence
\[
\text{supp}(\phi + \psi(\phi_1 - e_p)) \subset \bigcup_{m_1 + 1 \leq i_1 \leq n_1, \ldots, m_r \leq i_r \leq n_r} l_1^{i_1} \ldots l_r^{i_r} J x J.
\]
Continuing in this way we eventually obtain a function in \(\text{Ker} P_\omega\) supported on the single double coset \(J x J\). Since such a function must be the zero function, this proves the lemma. \(\square\)

We also have the embedding \(l \mapsto l(\pi) : L \rightarrow Z_{\tilde{H}}^\ast\) (where \(Z_{\tilde{H}}\) denotes the centre of \(\tilde{H}\)). Note that \(L\) (viewed as a subgroup of \(Z_{\tilde{H}}\) via this embedding) is discrete but not necessarily cocompact. We assume the convolution algebra \(\mathcal{H}(\tilde{H}, 1_I)\) is defined with respect to the Haar measure on \(\tilde{H}\) giving \(I\) measure one. Then
\[
\Psi^{-1}(\phi_l) = \alpha_l \chi_l \quad (l \in L)
\]
where \(\chi_l\) is the characteristic function of \(I l\) and \(|\alpha_l| = 1\). We define a homomorphism \(\omega' : L \rightarrow \mathbb{C}^x\) by
\[
\omega'(l) = \alpha_l^{-1} \quad (l \in L).
\]
Then \(1_{\mathcal{I}} \omega'\) is a representation of \(I L\). Using the obvious notation, we form the convolution algebra \(\mathcal{H}(\tilde{H}, 1_{\mathcal{I}} \omega')\) (with respect to the Haar measure \(dh\) on \(\tilde{H}/L\) such that \(dl \, dh = dh\) where \(dh\) is our fixed Haar measure on \(\tilde{H}\) (giving \(I\) measure one) and \(dl\) is the counting measure on \(L\)). Again we have a surjective algebra homomorphism
\[
P_{\omega'} : \mathcal{H}(\tilde{H}, 1_{\mathcal{I}}) \longrightarrow \mathcal{H}(\tilde{H}, 1_{\mathcal{I}} \omega')
\]
given by
\[
P_{\omega'}(\phi)(x) = \sum_{l \in L} \omega'(l) \phi(l x) \quad (\phi \in \mathcal{H}(\tilde{H}, 1_{\mathcal{I}}), x \in \tilde{H}).
\]
Arguing exactly as in the proof of Lemma 10.3, we obtain the following lemma.

**Lemma 10.4.** – The kernel of \(P_{\omega'}\) is the two-sided ideal of \(\mathcal{H}(\tilde{H}, 1_{\mathcal{I}})\) generated by the elements \(\alpha_l \chi_l - e\ (l \in L)\) where \(e\) denotes the identity element of \(\mathcal{H}(\tilde{H}, 1_{\mathcal{I}})\).

**Corollary 10.5.** – There exists a unique \((\ast\)-preserving, support-preserving) isomorphism of \(\mathbb{C}\)-algebras
\[
\Psi_\omega : \mathcal{H}(\tilde{H}, 1_{\mathcal{I}} \omega') \xrightarrow{\sim} \mathcal{H}(G, \rho_\omega)
\]
such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{H}(\tilde{H},1_I) & \xrightarrow{\Psi} & \mathcal{H}(G,\rho) \\
\downarrow P_{\omega'} & & \downarrow P_{\rho} \\
\mathcal{H}(\tilde{H},1_{I\omega'}) & \xrightarrow{\Psi_\omega} & \mathcal{H}(G,\rho\omega).
\end{array}
\]

\textbf{Proof.} – This is clear from Lemma 10.3 and Lemma 10.4. □

We now verify that \( \Psi_\omega \) is unitary with respect to the standard inner products on \( \mathcal{H}(\tilde{H},1_{I\omega'}) \) and \( \mathcal{H}(G,\rho\omega) \).

\textbf{Lemma 10.6.} – The canonical inner products \((\, , \) \) on \( \mathcal{H}(\tilde{H},1_{I\omega'}) \) and \( \mathcal{H}(G,\rho\omega) \) satisfy

\[
(\Psi_\omega \phi, \Psi_\omega \theta) = (\phi, \theta) \quad (\phi, \theta \in \mathcal{H}(\tilde{H},1_{I\omega'})).
\]

\textbf{Proof.} – Fix \( \phi_1, \theta_1 \in \mathcal{H}(\tilde{H},1_I) \). Then

\[
(\Psi_\omega P_{\omega'} \phi_1, \Psi_\omega P_{\omega'} \theta_1) = \Psi_\omega P_{\omega'}(\phi_1 \theta_1^*) (1).
\]

Write \( P_{\omega'}(\phi_1 \theta_1^*) = \alpha e + \eta \) where \( \alpha \in \mathbb{C} \), \( e \) is the unit element of \( \mathcal{H}(\tilde{H},1_{I\omega'}) \) and \( \text{supp} \eta \cap JL = \emptyset \). Then

\[
\Psi_\omega P_{\omega'}(\phi_1 \theta_1^*)(1) = \alpha e + \Psi_\omega(\eta)
\]

and \( \text{supp} \Psi_\omega(\eta) \cap JL = \emptyset \). Therefore

\begin{equation}
(10.1) \quad \Psi_\omega P_{\omega'}(\phi_1 \theta_1^*)(1) = \alpha \hat{\mu}(JL/L)^{-1}
\end{equation}

where \( \hat{\mu}(JL/L) \) denotes the measure of \( JL/L \) with respect to our fixed Haar measure on \( G/L \).

We also have

\begin{equation}
(10.2) \quad (P_{\omega'} \phi_1, P_{\omega'} \theta_1) = P_{\omega'}(\phi_1 \theta_1^*)(1)
= \alpha e(1)
= \alpha \hat{\mu}(JL/L)^{-1}
\end{equation}

where \( \hat{\mu}(JL/L) \) now denotes the measure of \( JL/L \) with respect to our fixed Haar measure on \( \tilde{H}/L \).

Comparing 10.1 and 10.2 gives

\[
(P_{\omega'} \phi_1, P_{\omega'} \theta_1) = \frac{\hat{\mu}(JL/L)}{\hat{\mu}(\tilde{J}L/L)}(\Psi_\omega P_{\omega'} \phi_1, \Psi_\omega P_{\omega'} \theta_1).
\]

Since \( P_{\omega'} \) is surjective, we have

\[
(\phi, \theta) = c(\Psi_\omega \phi, \Psi_\omega \theta) \quad (\phi, \theta \in \mathcal{H}(\tilde{H},1_{I\omega'}))
\]
where \( c = \frac{\hat{\mu}(J/L)}{\hat{\mu}(J/L)} \). However, by our choices of Haar measures, we have \( \hat{\mu}(J/L) = \hat{\mu}(J/L) = 1 \). Indeed, if \( f \in L^1(G) \)

\[
\int_G f(g) dg = \int_{G/L} \sum_{l \in L} f(lg) dg.
\]

Taking \( f \) to be the characteristic function of \( J \) gives \( \hat{\mu}(J/L) = 1 \). Similarly, \( \hat{\mu}(J/L) = 1 \) and hence \( c = 1 \) as required. \( \square \)

Since \( \Psi_\omega \) preserves the canonical inner products on \( \mathcal{H}(\tilde{H}, 1_\mathcal{I} \omega') \) and \( \mathcal{H}(G, \rho \omega) \), it extends to an isomorphism

\[
\Psi_\omega : \mathcal{H}^2(\tilde{H}, 1_\mathcal{I} \omega') \rightarrow \mathcal{H}^2(G, \rho \omega)
\]
on completions and therefore induces a bijection between simple \( \mathcal{H}(G, \rho \omega) \)-submodules of \( \mathcal{H}^2(G, \rho \omega) \) and simple \( \mathcal{H}(\tilde{H}, 1_\mathcal{I} \omega') \)-submodules of \( \mathcal{H}^2(\tilde{H}, 1_\mathcal{I} \omega') \).

It is convenient to temporarily use the following terminology. We say a smooth irreducible representation \( (\sigma, \mathcal{W}) \) of \( \tilde{H} \) is \( L \)-square-integrable if its central character \( \omega_\sigma \) is unitary on \( L \) (i.e. \( |\omega_\sigma(l)| = 1 \) for \( l \in L \)) and

\[
\int_{\tilde{H}/L} |(\sigma(h)w, \bar{w})|^2 dh < \infty \quad (w \in \mathcal{W}, \bar{w} \in \bar{\mathcal{W}}).
\]

(If \( L \) is not cocompact in \( Z(\tilde{H}) \), this notion is vacuous in that only the zero representation is \( L \)-square integrable. Otherwise, \( L \)-square-integrability is equivalent to square-integrability.) From the analogue of Proposition 10.2, the process \( \mathcal{W} \mapsto \mathcal{W}^{1_\mathcal{I} \omega'} \) establishes a bijection between smooth irreducible \( L \)-square-integrable representations of \( \tilde{H} \) containing \( 1_\mathcal{I} \) and simple \( \mathcal{H}(\tilde{H}, 1_\mathcal{I} \omega') \)-submodules of \( \mathcal{H}^2(\tilde{H}, 1_\mathcal{I} \omega') \). Using Proposition 10.2 again, we obtain a bijection between (smooth, irreducible) square-integrable representations of \( G \) containing \( \rho \) and (smooth irreducible) \( L \)-square-integrable representations of \( \tilde{H} \) containing \( 1_\mathcal{I} \).

Exactly as on \( G \), we may write \( Z_H \circlearrowleft Z_H L_H \) where \( Z_H = Z_H \cap 0T \) and

\[
L_H = \{ y \in Y \mid \langle y, \alpha \rangle = 0 \ (\alpha \in \Phi_\chi) \}.
\]

Let \( L_H = L_H^C = \{ l \in L_H \mid cl = l \ (c \in C_\chi) \} \). Note that \( L_H \) embeds in \( Z_{\tilde{H}} \) (recall \( \tilde{H} = H \rtimes C_\chi \)) and \( Z_{\tilde{H}} / L_{\tilde{H}} \) is compact. Note also that we have the chain of inclusions

\[
Z_H \supset Z_{\tilde{H}} \supset Z_G
\]

(viewing each group as a subgroup of \( T \)).

It is clear that if \( L_{\tilde{H}} / L \) is infinite (equivalently if \( Z_{\tilde{H}} / Z_G \) is noncompact) there exist no (non-zero) \( L \)-square-integrable representations of \( \tilde{H} \). Therefore, in this case, there exist no (non-zero) square-integrable representations of \( G \) containing \( \rho \).

Suppose however that \( L_{\tilde{H}} / L \) is finite. Then \( Z_{\tilde{H}} / L \) is compact and hence \( L \)-square-integrability on \( \tilde{H} \) is equivalent to square-integrability. Further, it is easy to see (e.g. by using Clifford theory) that there exist (non-zero) square-integrable representations of \( \tilde{H} \).
if and only if \( \frac{L_H}{L_H^*} \) is finite (equivalently if and only if \( \frac{ZH}{Z^*} \) is compact). From the exact sequence

\[
0 \longrightarrow \frac{L^*_H}{L} \longrightarrow \frac{L_H}{L} \longrightarrow \frac{L_H}{L^*_H} \longrightarrow 0
\]

we see \( \frac{L^*_H}{L} \) and \( \frac{L_H}{L^*_H} \) are both finite if and only if \( \frac{L_H}{L} \) is finite (equivalently if and only if \( \frac{ZH}{Z^*} \) is compact).

Putting these observations together, we obtain the following theorem. Before we state it we note (for later use) that \( \frac{L_H}{L^*_H} \) is finite if and only if \( \frac{L_H}{L^*} \). Indeed, \( \frac{L_H}{L^*_H} \) is finite if and only if \( \frac{L_H}{L^*} \) is finite, and \( C_x \) is a group of automorphisms of \( L_H \). Thus it is sufficient to observe that if \( L \) is a \( Z \)-lattice and \( \alpha \in \text{Aut}_L (L) \) then \( [L : L^*] < \infty \) if and only if \( L = L^* \).

**Theorem 10.7.**

i) If \( \frac{ZH}{Z^*} \) is non-compact, there exist no (non-zero) square-integrable representations of \( G \) containing \( \rho \).

ii) If \( \frac{ZH}{Z^*} \) is compact, there exist square-integrable representations of \( G \) containing \( \rho \). Further, a smooth irreducible representation of \( G \) containing \( \rho \) is square-integrable if and only if it corresponds to a square-integrable representation of \( \hat{H} \) under the equivalence of categories between \( \mathcal{R}_p(G) \) and \( \mathcal{R}_{\chi}(H) \).

**Remark 10.8.** For \( L_1 \subset Y \), let \( L_1^* = \{ x \in X : \langle x, l_1 \rangle = 0, \forall l_1 \in L_1 \} \). Then \( \frac{Z_H}{Z_G} \) is compact if and only if \( \frac{L_H}{L} \) is finite which holds if and only if \( \frac{L^*_H}{L^*_L} \) is finite, i.e. if and only if \( \frac{Z^*_H}{Z^*_G} \) is finite. This in turn is equivalent to the statement that \( \hat{H} \) is not contained in a proper Levi subgroup of \( \hat{G} \) (see Digne and Michel [16] 14.11), i.e. that the endoscopic group \( H \) of \( G \) is elliptic.

**Remark 10.9.** If \( \frac{Z^*_H}{Z^*_G} \) is finite, our restrictions on residual characteristic (in particular, the fact that \( p = \text{char} \, k_F \) is not a bad prime for \( \Phi \) imply that \( \chi \) is essentially tamely ramified, i.e. some twist of \( \chi \) by a character of \( G \) is trivial on \( T_1 \). Thus if \( \chi \) is not essentially tamely ramified, the category \( \mathcal{R}_p(G) = \mathcal{R}_p(G) \) contains no (non-zero) square-integrable representations. The following example of Tadic [36] (Section 10) shows that the restriction on residual caracteristic (implicit in theorem 10.7) is not always necessary. (I am grateful to the referee for pointing this out.) Let \( \nu(x) = |x|_F \) for \( x \in F^x \) where \( F \) is now any non-archimedean field of characteristic not equal to two. Suppose \( \psi_1, \ldots, \psi_n \) are \( n \) non-trivial quadratic characters of \( F^x \) linearly independent over \( Z/2Z \).

Consider the character of the standard maximal torus of \( Sp(4n, F) \) corresponding to the \( 2n \)-tuple of characters \( (\nu \psi_1, \psi_1, \ldots, \nu \psi_n, \psi_n) \). The resulting induced representation of \( Sp(4n, F) \) (obtained by normalised induction from the standard Borel) has exactly \( 2^n \) square-integrable subquotients. When \( n = 2 \), one may recover Tadic’s computation of the subquotients of this induced representation by straightforward Hecke algebra computations - the relevant Hecke algebras are the Iwahori Hecke algebras of \( Sp(4, F) \times Sp(4, F) \) and \( Sp(4, F) \times O(4, F) \). When \( n > 2 \), the methods of this paper do not apply (since the existence of \( n \) quadratic characters \( \psi_i \) as above forces \( p = 2 \)).

We assume for the remainder of this section that the quotient \( \frac{ZH}{Z_G} \) is compact. Let \( (\pi, \mathcal{V}) \) be a (smooth, irreducible) square-integrable representation of \( G \) containing \( \rho \) and \( (\sigma, \mathcal{W}) \) be the corresponding (smooth, irreducible) square-integrable representation of \( H \) containing \( 1_L \).
The formal degree $d(\pi)$ of $\pi$ with respect to the Haar measure $dg$ on $G/Z$ is defined by
\begin{equation}
(10.3) \quad \int_{G/Z} (\pi(g)v_1,w_1)(\pi(g)v_2,w_2) dg = d(\pi)^{-1}(v_1,v_2)(w_1,w_2)
\end{equation}
for $v_i, w_i \in V$ where $(\cdot,\cdot)$ is a $G$-invariant inner product on $V$. It is clear (given our choices of Haar measures on $G/Z$ and $G/L$) that we may also integrate over $G/L$. Since there is an embedding of $V^\omega$ into $H^2(G,\omega)$, we can regard $V$ as a subspace of $H^2(G,\omega)$. Rewriting 10.3 using functions $\phi_i, \theta_i \in H^2(G,\omega) \cap V$ and noting $(\pi(g)\phi, \theta) = \phi^\ast\theta(g^{-1})$, we obtain
\begin{equation}
(10.4) \quad (\phi_1^\ast\theta_1^\ast, \phi_2^\ast\theta_2^\ast) = d(\pi)^{-1}(\phi_1, \phi_2)(\theta_1, \theta_2)
\end{equation}
where $(\cdot,\cdot)$ now denotes the canonical inner product on $H^2(G,\omega)$.

Let $dz$ be the Haar measure on $Z_H^\ast$ giving $Z_H^\ast = Z_H^\ast \cap 0T$ measure one. We then obtain quotient measures $dz = \frac{dz}{dl}$ on $Z_H^\ast/L$ and $dh = \frac{dh}{dl}$ on $H/Z_H^\ast$. (Here $dh$ is the Haar measure on $H$ giving $Z_H^\ast$ measure one and $dl$ is the counting measure on $L$.) There is a positive constant $c$ such that
\begin{equation}
(10.5) \quad \int_{H/L} f(\tilde{h}) \frac{dh}{dl} = c \int_{H/Z_H^\ast} \int_{Z_H^\ast/L} f(\tilde{h}) \frac{dz}{dl} \frac{dh}{dz} \quad (f \in L^1(\tilde{H}/L)).
\end{equation}
Taking $f$ to be the characteristic function of $\mathcal{I}L/L$, an easy calculation yields
\[1 = c \mu_{Z_{H^\ast}/L}(0Z_{H^\ast}L_{H^\ast}/L)\]
(using the obvious notation). Here we have used $Z_{H^\ast} = 0Z_{H^\ast}L_{H^\ast}$ (which holds since $L_{H^\ast} = L_H$). By our choice of Haar measure on $Z_{H^\ast}/L$, $\mu_{Z_{H^\ast}/L}(Z_{H^\ast}/L) = 1$ and hence $\mu_{Z_{H^\ast}/L}(0Z_{H^\ast}L_{H^\ast}/L) = [L_H:L]$ so that $c = [L_H:L]^{-1}$.

Returning to 10.4, we assume $\phi_i, \theta_i \in H^2(G,\omega) \cap V$. Write $\Psi_\omega^{-1}(\phi_i) = \xi_i, \Psi_\omega^{-1}(\theta_i) = \zeta_i$. Since $\Psi_\omega$ preserves inner products, we have
\begin{equation}
(10.6) \quad (\xi_1^*, \xi_2^*, \zeta_1^*, \zeta_2^*) = d(\pi)^{-1}(\xi_1^*, \xi_2^*)(\zeta_1^*, \zeta_2^*)
\end{equation}
where $(\cdot,\cdot)$ is now the canonical inner product on $H^2(\tilde{H},1\omega')$. We also have $\xi_i, \theta_i \in H^2(\tilde{H},1\omega')$ where $\omega'$ is the central character of $(\sigma,\mathcal{W})$ ($\omega'|L = \omega'$). Furthermore, if $f_1, f_2 \in H^2(\tilde{H},1\omega')$ then 10.5 gives
\begin{align*}
\int_{H/L} f_1(\tilde{h})f_2(\tilde{h}) \frac{dh}{dl} &= [L_H:L]^{-1} \mu_{Z_{H^\ast}/L}(Z_{H^\ast}/L) \int_{H/Z_H^\ast} f_1(\tilde{h})f_2(\tilde{h}) \frac{dh}{dz} \\
&= \int_{H/Z_H^\ast} f_1(\tilde{h})f_2(\tilde{h}) \frac{dh}{dz}
\end{align*}
using $\mu_{Z_{H^\ast}/L}(Z_{H^\ast}/L) = \mu_{Z_{H^\ast}/L}(0Z_{H^\ast}L_{H^\ast}/L) = [L_H:L]$. Therefore 10.6 also holds when $(\cdot,\cdot)$ is the canonical inner product on $H^2(\tilde{H},1\omega')$. We also have that the formal degree $d(\sigma)$ of $(\sigma,\mathcal{W})$ (with respect to our fixed Haar measure on $\tilde{H}/Z_{H^\ast}$) satisfies
\begin{equation}
(\xi_1^*, \xi_2^*, \zeta_1^*, \zeta_2^*) = d(\sigma)^{-1}(\xi_1^*, \xi_2^*)(\zeta_1^*, \zeta_2^*)
\end{equation}
where $(\cdot,\cdot)$ is still the canonical inner product on $H^2(\tilde{H},1\omega')$. Hence $d(\pi) = d(\sigma)$. Thus the equivalence of categories between $\mathfrak{M}_p(G)$ and $\mathfrak{M}_1(\tilde{H})$ preserves formal degrees (with respect to our choices of Haar measures on $G/Z$ and $\tilde{H}/Z_{H^\ast}$).
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A. ROCHE
Department of Mathematics,
Purdue University,
West Lafayette,
IN 47907-1395, U.S.A.
E-mail : aroche@math.purdue.edu