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THE HOMOLOGY OF SPECIAL LINEAR GROUPS
OVER POLYNOMIAL RINGS (1)

BY KEVIN P. KNUDSON (2)

ABSTRACT. - We study the homology of $SL_n(F[t,t^{-1}])$ by examining the action of the group on a suitable simplicial complex. The $E^1$-term of the resulting spectral sequence is computed and the differential, $d^1$, is calculated in some special cases to yield information about the low-dimensional homology groups of $SL_n(F[t,t^{-1}])$. In particular, we show that if $F$ is an infinite field, then $H_2(SL_n(F[t,t^{-1}]),\mathbb{Z}) = K_2(F[t,t^{-1}])$ for $n \geq 3$. We also prove an unstable analogue of homotopy invariance in algebraic $K$-theory; namely, if $F$ is an infinite field, then the natural map $SL_n(F) \rightarrow SL_n(F[t])$ induces an isomorphism on integral homology for all $n \geq 2$.

RÉSUMÉ. – Nous étudions l’homologie de $SL_n(F[t,t^{-1}])$ en examinant l’action de ce groupe sur un complexe simplicial adéquat. Le terme $E^1$ de la suite spectrale associée est déterminé et la différentielle $d^1$ est calculée dans certains cas, ce qui permet alors de comprendre l’homologie du groupe $SL_n(F[t,t^{-1}])$ en bas degré. En particulier, nous montrons que si $F$ est un corps infini alors $H_2(SL_n(F[t,t^{-1}]),\mathbb{Z}) = K_2(F[t,t^{-1}])$ pour $n \geq 3$. Nous prouvons aussi un analogue instable de l’invariance homotopique en $K$-théorie algébrique : si $F$ est un corps infini alors la flèche naturelle $SL_n(F) \rightarrow SL_n(F[t])$ induit un isomorphisme en homologie entière pour $n \geq 2$.

Since Quillen’s definition of the higher algebraic $K$-groups of a ring [15], much attention has been focused upon studying the (co)homology of linear groups. There have been some successes –Quillen’s computation [14] of the mod $l$ cohomology of $GL_n(F_q)$, Soulé’s results [18] on the cohomology of $SL_3(\mathbb{Z})$– but few explicit calculations have been completed. Most known results concern the stabilization of the homology of linear groups. For example, van der Kallen [11], Charney [7], and others have proved quite general stability theorems for $GL_n$ of a ring. Also, Suslin [19] proved that if $F$ is an infinite field, then the natural map

$$H_i(GL_m(F)) \rightarrow H_i(GL_n(F))$$

is an isomorphism for $i \leq m$. Other noteworthy results include Borel’s computation of the stable cohomology of arithmetic groups [1], [2], the computation of $H^*(SL_n(F),\mathbb{R})$ for $F$ a number field by Borel and Yang [3], and Suslin’s isomorphism [20] of $H_3(SL_2(F))$ with the indecomposable part of $K_3(F)$.

This paper is concerned with studying the homology of linear groups defined over the polynomial rings $F[t]$ and $F[t,t^{-1}]$. One motivation for this is an attempt to find unstable analogues of the fundamental theorem of algebraic $K$-theory [15]: If $R$ is a regular ring,

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then there are natural isomorphisms

\[ K_i(R[t]) \cong K_i(R) \]

and

\[ K_i(R[t, t^{-1}]) \cong K_i(R) \oplus K_{i-1}(R). \]

In this paper, we study the homology of \( SL_n(F[t, t^{-1}]) \). Before stating our main result, we first establish some notation.

The group \( SL_n(F[t, t^{-1}]) \) acts on a contractible \((n - 1)\)-dimensional building \( X \) with fundamental domain an \((n - 1)\)-simplex \( C \). This yields a spectral sequence converging to the homology of \( SL_n(F[t, t^{-1}]) \) with \( E^1 \)-term satisfying

\[ E^1_{p,q} = \bigoplus_{\text{dim } \sigma = p} H_q(\Gamma_\sigma) \]

where \( \Gamma_\sigma \) denotes the stabilizer of the \( p \)-simplex \( \sigma \) in \( SL_n(F[t, t^{-1}]) \), and \( \sigma \) is contained in \( C \). The vertex stabilizers are isomorphic to \( SL_n(F[t]) \), and the other stabilizers break up into isomorphism classes in such a way that in each class, there is a group \( \Gamma_\sigma \) which fits into a split short exact sequence

\[ 1 \longrightarrow K \longrightarrow \Gamma_\sigma \longrightarrow \longrightarrow P_\sigma \longrightarrow 1 \]

where \( P_\sigma \) is a parabolic subgroup of \( SL_n(F) \) and \( K \) consists of the matrices in \( SL_n(F[t]) \) which are congruent to the identity modulo \( t \). Our main result is the following.

**Theorem (cf. Theorem 5.1).** – If \( F \) is an infinite field, then the inclusion \( P_\sigma \longrightarrow \Gamma_\sigma \) induces an isomorphism

\[ H_\bullet(P_\sigma, \mathbb{Z}) \longrightarrow H_\bullet(\Gamma_\sigma, \mathbb{Z}). \]

If \( \sigma \) is a vertex, we have \( \Gamma_\sigma = SL_n(F[t]) \) and \( P_\sigma = SL_n(F) \). In this case the theorem reduces to the following unstable analogue of (1).

**Theorem (cf. Theorem 3.4).** – If \( F \) is an infinite field, then the inclusion \( SL_n(F) \longrightarrow SL_n(F[t]) \) induces an isomorphism

\[ H_\bullet(SL_n(F), \mathbb{Z}) \longrightarrow H_\bullet(SL_n(F[t]), \mathbb{Z}). \]

This theorem improves on a result of Soulé [17].

Theorem 5.1 completes the computation of the \( E^1 \)-term of the spectral sequence (3). However, the differential \( d^1 \) is difficult to calculate in general. In Section 6 we compute the map in a few special cases and obtain information about the low dimensional homology groups of \( SL_n(F[t, t^{-1}]) \). In particular, we show that if \( F \) is an infinite field, then for \( n \geq 3 \), there is an isomorphism

\[ H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}) \cong K_2(F[t, t^{-1}]). \]

The homology of \( SL_2(F[t, t^{-1}]) \) was studied by the author in [12] using slightly different techniques than those used here. The main result of [12] is the following.
THEOREM (cf. [12, Theorem 5.1]). - Let $F$ be a number field and denote by $r_1$ (resp. $r_2$) the number of real (resp. conjugate pairs of complex) embeddings of $F$. Then for $k \geq 2r_1 + 3r_2 + 2$ there is a natural isomorphism

$$H_k(SL_2(F[t, t^{-1}]), Q) \cong H_{k-1}(F^\times, Q).$$

The results of this paper reprove and generalize the results of [12]. In particular, Theorems 3.1 and 4.3 of [12] hold for infinite fields of arbitrary characteristic, not just fields of characteristic zero.

This paper is organized as follows:

In Section 1 we present the necessary background material on the Bruhat-Tits building $X$. We also introduce a complex $Y$ which will be used in subsequent sections.

In Section 2 we study the action of $SL_n(F[t, t^{-1}])$ on $X$ and examine the structure of the various stabilizers.

In Section 3 we prove Theorem 3.4, the unstable version of (1). Even though this is a special case of Theorem 5.1, we prove it separately for two reasons. First, it is a striking result which deserves to be called a theorem in its own right, and second, the proof sets the stage for the proof of Theorem 5.1.

In Section 4 we find fundamental domains for the actions of the various stabilizers on the complex $Y$ introduced in Section 1.

In Section 5 we prove Theorem 5.1.

Finally, in Section 6 we compute the $d^1$-map in the spectral sequence (3) in some special cases.

**Notation.** - If $G$ is a group acting on a simplicial complex $X$ and if $\sigma$ is a simplex in $X$, we denote the stabilizer of $\sigma$ in $G$ by $G_\sigma$. If $R$ is a ring, we denote the group of units by $R^\times$. The set of $n \times n$ matrices over $R$ will be denoted by $M_n(R)$. Unless otherwise stated, $F$ will be an infinite field of arbitrary characteristic.

**1. Preliminaries on buildings**

In this section, we summarize the basic facts about the Bruhat-Tits building associated to a vector space over a field with discrete valuation. The building was constructed in [6]; more detailed information may be found there (or see Brown [4, Ch. VI]).

Let $K$ be a field with discrete valuation, $v$. Denote by $O$ the valuation ring of $v$; that is,

$$O = \{x \in K : v(x) \geq 0\}.$$

Choose a field element $\pi$ satisfying $v(\pi) = 1$, and denote by $k$ the residue field $O/\pi O$. By a lattice in $K^n$, we mean a finitely generated $O$-submodule which spans $K^n$; such a submodule is free of rank $n$. Two lattices $L, L'$ are called equivalent if there is some nonzero field element $x$ such that $L' = xL$. Denote the equivalence class of the lattice $L$ by $[L]$. If $v_1, \ldots, v_n$ are linearly independent elements of $K^n$, denote the equivalence class of the lattice they span by $[v_1, \ldots, v_n]$. 
Assign a type to a lattice class as follows. If \([v_1, \ldots, v_n]\) is a lattice class, we define its type to be the element

\[ v(\det(v_1, \ldots, v_n)) \mod n, \]

where \(\det(v_1, \ldots, v_n)\) denotes the determinant of the matrix having \(v_1, \ldots, v_n\) as columns.

Construct a simplicial complex \(X\) in the following manner. The vertices of \(X\) are equivalence classes of lattices in \(K^n\). A collection of vertices \(\Lambda_0, \Lambda_1, \ldots, \Lambda_m\) forms an \(m\)-simplex if there exist representatives \(L_0, L_1, \ldots, L_m\) satisfying

\[ \pi L_m \subseteq L_0 \subseteq L_1 \subseteq \cdots \subseteq L_m. \]

Since \(L_i/\pi L_m\) is a subspace of the \(n\)-dimensional \(k\)-vector space \(L_m/\pi L_m\), the maximal simplices of \(X\) have \(n\) vertices; that is, \(\dim X = n - 1\). Moreover, the complex \(X\) is contractible [4, p. 137]. There is an obvious action of \(GL_n(K)\) on \(X\). Note that this action is transitive on the vertices of \(X\).

We now find a fundamental domain for the action of \(SL_n(K)\) on \(X\). Let \(C\) be the \((n - 1)\)-simplex with vertices \([e_1, \ldots, e_i, \pi e_{i+1}, \ldots, \pi e_n]\), \(i = 1, \ldots, n\), where \(e_1, \ldots, e_n\) is the standard basis of \(K^n\). Then we have the following result (see [4, p. 137]).

**Proposition 1.1.** – The \((n - 1)\)-simplex \(C\) is a fundamental domain for the action of \(SL_n(K)\) on \(X\).

**Proof.** – Let \(C'\) be an arbitrary \((n - 1)\)-simplex with vertices \(\Lambda_0, \ldots, \Lambda_{n-1}\), with \(\Lambda_i\) of type \(n - i\). By the Invariant Factor Theorem, there is a basis \(f_1, \ldots, f_n\) of \(K^n\) such that

\[ \Lambda_0 = [f_1, \ldots, f_n], \quad \Lambda_1 = [f_1, \pi f_2, \ldots, \pi f_n], \ldots, \quad \Lambda_{n-1} = [f_1, \ldots, \pi f_n], \]

and \(\det(f_1, \ldots, f_n) = \pi^{n-r} u\) for some integer \(r\) and \(u \in \mathcal{O}^\times\). Replacing \(f_1\) by \(\pi^{-r} u^{-1} f_1\), and \(f_i\) by \(\pi^{-r} f_i, i = 2, \ldots, n\), we still have

\[ \Lambda_0 = [f_1, \ldots, f_n], \ldots, \quad \Lambda_{n-1} = [f_1, \ldots, \pi f_n], \]

but now \(\det(f_1, \ldots, f_n) = 1\). Let \(g\) be the matrix having \(f_1, \ldots, f_n\) as columns. Then \(g\) takes \(C\) to \(C'\). Since the action of \(SL_n(K)\) preserves type, it follows that \(C\) is a fundamental domain.

The stabilizer of \([e_1, \ldots, e_n]\) in \(SL_n(K)\) is the subgroup \(SL_n(\mathcal{O})\). Thus, the stabilizer of \([e_1, \ldots, e_i, \pi e_{i+1}, \ldots, \pi e_n]\) is

\[ g_i SL_n(\mathcal{O}) g_i^{-1}, \]

where

\[ g_i = \text{diag}(1, 1, \ldots, 1, \pi, \ldots, \pi), \]

the first \(\pi\) appearing in the \((i + 1)\)st column. The stabilizer of an edge is the intersection of the stabilizers of its vertices; the stabilizer of a 2-simplex is the intersection of the stabilizers of its edges, and so on.
In this paper, we shall be interested in studying various group actions on two Bruhat-Tits buildings associated to two different fields associated to a field $F$.

**Example 1.2.** Denote by $\mathcal{L}$ the field of formal Laurent series over $F$. Define a valuation $v$ on $\mathcal{L}$ by

$$v\left(\sum_{i \geq n_0} a_i t^i\right) = n_0, \quad a_{n_0} \neq 0.$$  

Here, we choose $\pi = t$. Observe that the ring $F[t, t^{-1}]$ is dense in $\mathcal{L}$. Denote by $X$ the Bruhat-Tits building associated to $\mathcal{L}^n$.

**Example 1.3.** Denote by $F(t)$ the field of fractions of $F[t]$. Define a valuation $v_\infty$ on $F(t)$ by

$$v_\infty(a/b) = \deg b - \deg a, \quad b \neq 0.$$  

In this case, we choose $\pi = 1/t$. Denote by $Y$ the Bruhat-Tits building associated to $F(t)^n$.

**Remark.** Denote by $\hat{K}$ the completion of $K$ with respect to the valuation $v$. Then the Bruhat-Tits buildings of $K$ and $\hat{K}$ are isomorphic. In particular, the completion $\hat{F}(t)$ of $F(t)$ is isomorphic to $\mathcal{L}$ via the map $t \mapsto t^{-1}$. It follows that the complexes $X$ and $Y$ are isomorphic. Although these complexes are isomorphic, it will be convenient to distinguish them when doing homological computations.

### 2. The action of $SL_n(F[t, t^{-1}])$ on $X$

We now investigate the action of the group $SL_n(F[t, t^{-1}])$ on the complex $X$ of Example 1.2. Since $F[t, t^{-1}]$ is a dense subring of the field $\mathcal{L}$, we have the following result.

**Lemma 2.1.** The subgroup $SL_n(F[t, t^{-1}])$ is dense in $SL_n(\mathcal{L})$.

**Proof.** The closure of $SL_n(F[t, t^{-1}])$ in $SL_n(\mathcal{L})$ contains the subgroup of elementary matrices over $\mathcal{L}$. Since these matrices generate $SL_n(\mathcal{L})$, the result follows.

Denote by $V$ the vector space $\mathcal{L}^n$ and let $GL(V)^\circ$ denote the kernel of the homomorphism

$$v \circ \det : GL(V) \to \mathbb{Z}.$$  

Then we have the following (cf. [16, Thm. 2, p. 78]).

**Proposition 2.2.** If $G$ is a subgroup of $GL(V)^\circ$ whose closure contains $SL(V)$, then the $(n - 1)$-simplex $C$ (see Proposition 1.1) is a fundamental domain for the action of $G$ on $X$.

**Proof.** We know that $C$ is a fundamental domain for the action of $SL(V)$ on $X$. Let $C'$ be an $(n - 1)$-simplex in $X$. There is an element $s$ of $SL(V)$ with

$$sC = C'.$$
Let $U$ be the subgroup of $GL_n(\mathcal{O})$ consisting of the matrices which are congruent to the identity mod $t$; this is an open subgroup of $GL(V)$. By hypothesis, there is an element $u$ of $U$ and an element $g$ of $G$ with $g = su$. Observe that $u$ fixes each vertex of $C$. Hence, we have the chain of equalities
\[ gC = suC = sC = C', \]
and since $G$ preserves type, it follows that $C$ is a fundamental domain for the action of $G$ on $\mathcal{X}$.

The preceding two results imply that the $(n - 1)$-simplex $C$ is a fundamental domain for the action of $SL_n(F[t, t^{-1}])$ on $\mathcal{X}$.

We now identify the stabilizers in $SL_n(F[t, t^{-1}])$ of the simplices of $C$. Label the vertices of $C$ as
\[ p_i = [e_1, \ldots, e_{i-1}, te_i, \ldots, te_n], \quad i = 1, 2, \ldots, n. \]
Note that $p_1 = [te_1, \ldots, te_n] = [e_1, \ldots, e_n]$. Evidently, the stabilizer of $p_1$ in $SL_n(F[t, t^{-1}])$ is the subgroup
\[ SL_n(F[t]) = SL_n(\mathcal{O}) \cap SL_n(F[t, t^{-1}]). \]
Denote by $g_i$ the matrix
\[ g_i = \text{diag}(1, \ldots, 1, t, \ldots, t), \quad i = 2, \ldots, n \]
where the first $i - 1$ entries are equal to 1. Then the stabilizer of $p_i$ in $SL_n(F[t, t^{-1}])$ is
\[ g_iSL_n(F[t])g_i^{-1}. \]
Denote by $\Gamma_{i_1, \ldots, i_k}$ the stabilizer of the $(k - 1)$-simplex having vertices $p_{i_1}, \ldots, p_{i_k}$. The group $\Gamma_{i_1, \ldots, i_k}$ is the intersection of the stabilizers $\Gamma_{i_1}, \ldots, \Gamma_{i_k}$ of the vertices of the simplex. Elements of $\Gamma_{i_1, \ldots, i_k}$ have the form
\[
\begin{pmatrix}
L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\
& tV_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\
& & tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\
& & & \vdots & \ddots & \ddots & \ddots & \vdots \\
& & & & & tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1}
\end{pmatrix}
\]
where we have
\[
L_r \in M_{i_r-i_{r-1}}(F[t]), \quad 1 \leq r \leq k + 1 \\
V_{r,s} \in M_{i_r-i_{r-1},i_s-i_{s-1}}(F[t]), \quad 1 \leq r, s \leq k + 1
\]
(here, we set $i_0 = 1$ and $i_{k+1} = n + 1$).
Consider the stabilizers $\Gamma_{1, j_1, \ldots, j_k}$. These are subgroups of $\Gamma_1 = SL_n(F[t])$. Elements of the group $\Gamma_{1, j_1, \ldots, j_k}$ have the form

$$
\begin{pmatrix}
L_1 & V_{12} & V_{13} & \cdots & V_{1, k-1} & V_{1, k} \\
V_{21} & L_2 & V_{23} & \cdots & V_{2, k-1} & V_{2, k} \\
V_{31} & V_{32} & L_3 & \cdots & V_{3, k-1} & V_{3, k} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
V_{k, 1} & V_{k, 2} & V_{k, 3} & \cdots & V_{k, k-1} & L_k \\
V_{k+1, 1} & V_{k+1, 2} & V_{k+1, 3} & \cdots & V_{k+1, k} & L_{k+1}
\end{pmatrix}
$$

where we have

- $L_r \in M_{j_r + 1 - j_r}(F[t])$, $1 \leq r \leq k$,
- $V_{r, s} \in M_{j_r + 1 - j_r - j_s}(F[t])$, $1 \leq r, s \leq k$.

(here, we set $j_1 = 1$ and $j_{k+1} = n + 1$).

These groups are related as follows.

**Proposition 2.3.**  The group $\Gamma_{i_1, \ldots, i_k}$ is conjugate to $\Gamma_{1, (i_2 - i_1 + 1), \ldots, (i_k - i_1 + 1)}$ inside $GL_n(F[t, t^{-1}])$.

**Proof.**  First conjugate $\Gamma_{i_1, \ldots, i_k}$ by the element

$$
g = \text{diag}(t, t, \ldots, t, 1, \ldots, 1)
$$

where the first $i_1 - 1$ entries are equal to $t$. The resulting group has elements of the form

$$
\begin{pmatrix}
L_1 & tV_{12} & tV_{13} & \cdots & tV_{1, k} & V_{1, k+1} \\
V_{21} & L_2 & V_{23} & \cdots & V_{2, k} & V_{2, k+1} \\
V_{31} & V_{32} & L_3 & \cdots & V_{3, k} & V_{3, k+1} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
V_{k, 1} & V_{k, 2} & V_{k, 3} & \cdots & V_{k, k} & V_{k, k+1} \\
V_{k+1, 1} & V_{k+1, 2} & V_{k+1, 3} & \cdots & V_{k+1, k} & L_{k+1}
\end{pmatrix}
$$

where the $L_r$ and $V_{r, s}$ are as above. Now conjugate by the permutation matrix corresponding to the permutation

$$
1 \mapsto n - i_1 + 2
$$

$$
2 \mapsto n - i_1 + 3
$$

$$
\vdots
$$

$$
i_1 - 1 \mapsto n
$$

$$
i_1 \mapsto 1
$$

$$
\vdots
$$

$$
i_2 - 1 \mapsto i_2 - i_1
$$

$$
i_2 \mapsto i_2 - i_1 + 1
$$

$$
i_2 + 1 \mapsto i_2 - i_1 + 2
$$

$$
\vdots
$$

$$
n \mapsto n - i_1 + 1.
$$
Note that if $\tau$ denotes the $n$-cycle $(12\cdots n)$, then this permutation is simply $\tau^{-1}$. The resulting group has the form

$$
\begin{pmatrix}
L_2 & V_{23} & V_{24} & \cdots & V_{2,k+1} & V_2 \\
V_{32} & L_3 & V_{34} & \cdots & V_{3,k+1} & V_3 \\
tV_{42} & tV_{43} & L_4 & \cdots & V_{4,k+1} & V_{41} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
tV_{k,2} & tV_{k,3} & tV_{k,4} & \cdots & L_k & V_{k,k+1} \\
tV_{k+1,2} & tV_{k+1,3} & tV_{k+1,4} & \cdots & tV_{k+1,k} & L_{k+1} \\
tV_{12} & tV_{13} & tV_{14} & \cdots & tV_{1,k} & V_{1,k+1}
\end{pmatrix}
$$

which is precisely the group $\Gamma_{1,(i_2-i_1+1),\ldots,(i_k-i_1+1)}$.

If $\sigma$ is a $p$-simplex in $C$, denote by $\Gamma_\sigma$ the stabilizer of $\sigma$ in $SL_n(F[t,t^{-1}])$. Since the complex $\mathcal{X}$ is contractible, we have a spectral sequence converging to the homology of $SL_n(F[t,t^{-1}])$ with $E^1$-term

$$
E^1_{p,q} = \bigoplus_{\dim \sigma = p} H_q(\Gamma_\sigma)
$$

where $\sigma$ ranges over the $p$-simplices of $C$. By Proposition 2.3, we need only compute the homology of each $\Gamma_{1,j_2,\ldots,j_k}$; we do this in Section 5.

In the next section we single out the $\Gamma_i$, $i = 1,\ldots,n$ and compute their homology.

3. The vertex stabilizers.

The homology of $SL_n(F[t])$

Notation. – For $G$ a subgroup of $GL_n(R)$, $R$ a commutative ring with unit, denote by $\bar{G}$ the subgroup $G \cap SL_n(R)$.

Consider the stabilizers $\Gamma_1,\ldots,\Gamma_n$ of the vertices of $C$. Each of these is isomorphic to $SL_n(F[t])$. To compute homology we use the Bruhat-Tits building $\mathcal{Y}$ of Example 1.3. Recall that this is the building associated to the $n$-dimensional vector space $V = F(t)^n$.

There is an obvious left action of $SL_n(F[t])$ on $\mathcal{Y}$. Let $e_1,\ldots,e_n$ be the standard basis of $V$. Then the subcomplex $T$ having vertices

$$
[e_1t^{r_1},e_2t^{r_2},\ldots,e_{n-1}t^{r_{n-1}},e_n], \quad \text{where} \quad r_1 \geq r_2 \geq \cdots \geq r_{n-1} \geq 0
$$

is a fundamental domain for the action of $SL_n(F[t])$ on $\mathcal{Y}$ [17].

The complex $T$ is an infinite wedge. Denote by $v_0$ the vertex $[e_1,\ldots,e_n]$ and by $v_i$ the vertex $[e_1t,e_2t,\ldots,e_it,e_{i+1},\ldots,e_n], i = 1,2,\ldots,n-1$. For a $k$ element subset $I = \{i_1,\ldots,i_k\}$ of $\{1,2,\ldots,n-1\}$, define $E^{(k)}_I$ to be the subcomplex of $T$ which is the union of all rays with origin $v_0$ passing through the $(k-1)$-simplex $(v_{i_1},\ldots,v_{i_k})$. There are $\binom{n-1}{k}$ such $E^{(k)}_I$. Observe that if $I = \{1,2,\ldots,n-1\}$, then $E^{(n-1)}_{\{\}} = T$. When we write $E^{(l)}_I$, the superscript $l$ denotes the cardinality of the set $I$. 

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Define a filtration $V^*$ of $T$ by setting $V^{(0)} = v_0$ and

$$V^{(k)} = \bigcup_I E^{(k)}_I, \quad 1 \leq k \leq n - 1$$

where $I$ ranges over all $k$-element subsets of $\{1, 2, \ldots, n - 1\}$. Note that $V^{(n-1)} = T$.

Evidently, the stabilizer of $v_0$ in $SL_n(F[t])$ is the subgroup $SL_n(F)$. For any other vertex $v = [e_1 t^{r_1}, e_2 t^{r_2}, \ldots, e_{n-1} t^{r_{n-1}}, e_n] \in T$, let $\Gamma_v$ denote the stabilizer of $v$ in $SL_n(F[t])$. The subgroup $\Gamma_v$ is the semidirect product of a reductive group $L_v$ contained in $SL_n(F)$ and a unipotent group $U_v$ contained in $SL_n(F[t])$. If $p_{kl}$ denotes the polynomial in the $k$th row and $l$th column of an element of $\Gamma_v$, then we have $\deg p_{kl} \leq r_k - r_l$. It follows that the subgroup $\Gamma_v$ has a block form

$$\begin{pmatrix}
L_1 & V_{12} & V_{13} & \cdots & V_{1m} \\
L_2 & V_{23} & \cdots & V_{2m} \\
& \ddots & \ddots & \ddots \\
0 & \cdots & L_{m-1} & V_{m-1,m} \\
& & & & L_m
\end{pmatrix}$$

where the $L_k$ and $V_{kl}$ satisfy

$$L_k \in GL_{i_k-i_{k-1}}(F), \quad \text{where } \ r_{i_k-1+1}=r_{i_k-1+2}=\cdots=r_{i_k}$$

$$V_{kl} \in M_{i_k-i_{k-1},i_i-i_{i-1}}(F[t]), \quad \text{where } \ r_{i_{i-1}+1}=r_{i_{i-1}+2}=\cdots=r_{i_i}$$

(we set $i_0 = 0$). Observe that the stabilizers $\Gamma_{v_i}, i = 1, 2, \ldots, n - 1$, have the block form of the $n - 1$ maximal parabolic subgroups in $SL_n$. If $I = \{i_1, \ldots, i_k\}$ and if $v$ is a vertex in $E^{(k)}_l$ which does not lie in any $E^{(k-1)}_J$, where $J \subset I$, then $\Gamma_v$ has the block form of the intersection $\Gamma_{v_{i_1}} \cap \cdots \cap \Gamma_{v_{i_k}}$. Observe that if $v$ is a vertex of $T$ not lying in any $E^{(n-2)}_J$, then the $r_i$ are positive and distinct and hence the group $\Gamma_v$ is upper triangular.

If $e$ is an edge with vertices $v, w$, then the stabilizer $\Gamma_e$ is simply the intersection $\Gamma_v \cap \Gamma_w$. Similarly, the stabilizer of a 2-simplex is the intersection of the edge stabilizers, and so on. It follows that if $l \leq k$ and if $\sigma$ is an $l$-simplex in $E^{(k)}_I$, where $I = \{i_1, \ldots, i_k\}$, not lying entirely in any $E^{(k-1)}_J$, where $J \subset I$, then $\Gamma_{\sigma}$ has the block form of the intersection $\Gamma_{v_{i_1}} \cap \cdots \cap \Gamma_{v_{i_k}}$.

The case $n = 3$ is shown in Figure 1.

Since the complex $\mathcal{Y}$ is contractible, we have a spectral sequence converging to $H_*(SL_n(F[t]), \mathbb{Z})$ with $E^1$-term satisfying

$$E^1_{p,q} = \bigoplus_{\dim \sigma = p} H_q(\Gamma_{\sigma})$$

where $\sigma$ ranges over the simplices of $T$. 

\textit{Annales scientifiques de l'École normale supérieure}
3.1. The homology of the stabilizers

We now compute the homology of the groups $\Gamma_\sigma$. Suppose that $A$ is an $F$-algebra. Let $P$ be a subgroup of $GL_{n+m}(A)$ having block form

$$P = \begin{pmatrix} L_1 & M \\ 0 & L_2 \end{pmatrix}$$

where $L_1 \subseteq GL_n(A)$, $L_2 \subseteq GL_m(A)$, and $M$ is a vector subspace of $M_{n,m}(A)$ such that $L_1 M = M = M L_2$. Suppose that each $L_i$ contains the group of diagonal matrices over $F$. Denote by $L$ the subgroup of $P$ defined by

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}.$$

A proof of the following is deduced easily from [10, Lemma 9] by observing that the argument used works with $F$ replaced by $A$. Recall that $\overline{G}$ denotes the intersection $G \cap SL_n(R)$.

**Proposition 3.1.** If $F$ is an infinite field, then the inclusion $\overline{L} \hookrightarrow \overline{P}$ induces an isomorphism

$$H_\bullet(\overline{L}, \mathbb{Z}) \longrightarrow H_\bullet(\overline{P}, \mathbb{Z}).$$
**Corollary 3.2.** - Suppose that $P$ is a subgroup of $GL_n(A)$ having block form

$$
\begin{pmatrix}
L_1 & V_{12} & V_{13} & \cdots & V_{1m} \\
L_2 & V_{23} & \cdots & V_{2m} \\
\vdots & & \ddots & \vdots \\
0 & & \cdots & L_{m-1} & V_{m-1,m} \\
& & & 0 & L_m
\end{pmatrix}
$$

where each $L_i \subseteq GL_{n_i}(A)$ and each $V_{ij}$ is a vector subspace of $M_{n_i,n_j}(A)$ such that $L_iV_{ij} = V_{ij}L_j$. Assume that each $L_i$ contains the group of diagonal matrices over $F$. Denote by $L$ the subgroup

$$
L = \begin{pmatrix}
L_1 & \cdots & 0 \\
\ddots & \ddots & \ddots \\
0 & \cdots & L_m
\end{pmatrix}
$$

of $P$. Then the inclusion $L \rightarrow P$ induces an isomorphism

$$H_\bullet(L, \mathbb{Z}) \rightarrow H_\bullet(P, \mathbb{Z}).$$

**Proof.** - Consider the sequence of inclusions

$$
L \rightarrow \begin{pmatrix}
L_1 & 0 & 0 \\
\ddots & \ddots & \ddots \\
0 & L_{m-1} & V_{m-1,m} \\
0 & \cdots & L_m
\end{pmatrix} \rightarrow
$$

$$
\begin{pmatrix}
L_1 & 0 & 0 & 0 \\
\ddots & \ddots & \ddots & \ddots \\
0 & L_{m-2} & V_{m-2,m-1} & V_{m-2,m} \\
0 & \cdots & L_{m-1} & V_{m-1,m} \\
0 & \cdots & 0 & L_m
\end{pmatrix} \rightarrow
$$

$$
\begin{pmatrix}
L_1 & 0 & \cdots & \cdots & 0 \\
0 & L_2 & V_{23} & \cdots & V_{2m} \\
\vdots & & \ddots & \ddots & \ddots \\
\vdots & & & L_{m-1} & V_{m-1,m} \\
0 & 0 & \cdots & \cdots & L_m
\end{pmatrix} \rightarrow P.
$$

By Proposition 3.1, each of these maps induces a homology isomorphism. It follows that the inclusion $L \rightarrow P$ induces an isomorphism

$$H_\bullet(L, \mathbb{Z}) \rightarrow H_\bullet(P, \mathbb{Z}).$$
If $\sigma$ is a simplex in $T$, then the subgroup $\Gamma_{\sigma}$ has a block form as in the corollary. We have an extension

$$1 \longrightarrow U_{\sigma} \longrightarrow \Gamma_{\sigma} \longrightarrow L_{\sigma} \longrightarrow 1$$

where $U_{\sigma}$ is a unipotent group and $L_{\sigma}$ is a reductive subgroup of $SL_n(F)$. The corollary implies that the inclusion $L_{\sigma} \rightarrow \Gamma_{\sigma}$ induces an isomorphism

$$H_\bullet(L_{\sigma}, \mathbb{Z}) \longrightarrow H_\bullet(\Gamma_{\sigma}, \mathbb{Z}).$$

Let $I = \{i_1, \ldots, i_k\}$ be a subset of $\{1, 2, \ldots, n-1\}$. If $\sigma$ is a simplex in $E_{W(I)}$ then $r^{(k)}$ has the block form of the intersection $\Gamma_{v_{i_1}} \cap \cdots \cap \Gamma_{v_{i_k}}$. If $\tau$ is another such simplex, then $\Gamma_\tau$ has the same block form. Thus, $L_{\sigma} = L_{\tau}$ and it follows that $\Gamma_{\sigma}$ and $\Gamma_{\tau}$ have the same homology. Moreover, if $\sigma$ is a face of $\tau$, then the map $\Gamma_\tau \rightarrow \Gamma_{\sigma}$ induces an isomorphism on homology.

### 3.2. The homology of $SL_n(F[t])$

Given a coefficient system $\mathcal{M}$ on a simplicial complex $Z$ (i.e., a covariant functor from the simplices of $Z$ to the category of abelian groups), we may define the chain complex $C_\bullet(Z, \mathcal{M})$ by setting

$$C_p(Z, \mathcal{M}) = \bigoplus_{\dim \sigma = p} \mathcal{M}(\sigma)$$

with boundary map the alternating sum of the maps induced by the face maps in $Z$.

We shall make use of the following result (compare with [18, Lemma 6]).

**Lemma 3.3.** Suppose $F(0) \subset F(1) \subset \cdots \subset F(k) = Z$ is a filtration of the simplicial complex $Z$ by subcomplexes such that each $F(i)$ and each component of $F(i) - F(i-1)$ is contractible. Suppose that $\mathcal{M}$ is a coefficient system on $Z$ such that the restriction of $\mathcal{M}$ to each component of $F(i) - F(i-1)$ is constant. Then the inclusion $F(0) \rightarrow Z$ induces an isomorphism

$$H_\bullet(F(0), \mathcal{M}) \longrightarrow H_\bullet(Z, \mathcal{M}).$$

**Proof.** The filtration of $Z$ induces a filtration of $C_\bullet(Z, \mathcal{M})$. This yields a spectral sequence converging to $H_\bullet(Z, \mathcal{M})$ with $E^1$-term having $i$th column

$$H_\bullet(F(i), F(i-1); \mathcal{M}).$$

Consider the relative chain complex $C_\bullet(F(i), F(i-1); \mathcal{M})$. By hypothesis, this chain complex is a direct sum of chain complexes with constant coefficients. Since each $F(i)$ is contractible, it follows that

$$H_\bullet(F(i), F(i-1); \mathcal{M}) = 0, \quad i \geq 1.$$
Thus, only the 0th column $H_* (F^{(0)}, \mathcal{M})$ is nonzero. This proves the lemma.

We may now compute $H_* (SL_n (F[t]), \mathbb{Z})$. The argument in the proof below is used implicitly by Soulé in the proof of Theorem 5 of [17].

**Theorem 3.4.** If $F$ is an infinite field, then the natural inclusion $SL_n (F) \to SL_n (F[t])$ induces an isomorphism

$$H_* (SL_n (F), \mathbb{Z}) \to H_* (SL_n (F[t]), \mathbb{Z}).$$

**Proof.** Recall the spectral sequence (6). The $E^1$-term satisfies

$$E^1_{p,q} = \bigoplus_{\dim \sigma = p} H_q (\Gamma_\sigma) \Rightarrow H_{p+q} (SL_n (F[t])).$$

For each $q \geq 0$, define a coefficient system $\mathcal{F}_q$ on $T$ by

$$\mathcal{F}_q (\sigma) = H_q (\Gamma_\sigma).$$

Then the $q$th row in the spectral sequence is simply $C_\bullet (T, \mathcal{F}_q)$ and the $d^1$-map is the boundary map in this chain complex.

Recall the filtration $V^*$ of $T$ (5). For each simplex in

$$E^{(k)}_I = \bigcup_{J \subset I} E^{(k-1)}_J,$$

the stabilizers have the same reductive part and hence have the same homology (see the discussion following the proof of Corollary 3.2). It follows that the restriction of $\mathcal{F}_q$ to each component of $V^{(i)} - V^{(i-1)}$ is constant. By Lemma 3.3, the inclusion $v_0 \to T$ induces an isomorphism

$$H_* (v_0, \mathcal{F}_q) \to H_* (T, \mathcal{F}_q).$$

Observe that

$$H_p (v_0, \mathcal{F}_q) = \begin{cases} 
H_q (SL_n (F)) & p = 0 \\
0 & p > 0.
\end{cases}$$

It follows that the $E^2$-term of the spectral sequence (6) satisfies

$$E^2_{p,q} = \begin{cases} 
H_q (SL_n (F)) & p = 0 \\
0 & p > 0.
\end{cases} \quad \square$$

**Remark.** Theorem 3.4 may be viewed as an unstable version of Quillen's homotopy invariance in algebraic $K$-theory [15].

**Remark.** The $n = 2$ case of Theorem 3.4 was proved for fields of characteristic zero in [12] by considering the Mayer-Vietoris sequence associated to the amalgamated free product decomposition (due to Nagao [13])

$$SL_2 (F[t]) \cong SL_2 (F) *_{B(F)} B(F[t]).$$

(7)
where $B(R)$ denotes the upper triangular group over $R$. Proposition 3.2 of [12] shows that $B(F)$ and $B(F[t])$ are the same homologically. This implies that the Mayer-Vietoris sequence associated to (7) breaks into short exact sequences

$$0 \longrightarrow H_k(B(F)) \longrightarrow H_k(B(F[t])) \oplus H_k(SL_2(F)) \longrightarrow H_k(SL_2(F[t])) \longrightarrow 0,$$

from which it follows that $H_\bullet(SL_2(F), \mathbb{Z}) \cong H_\bullet(SL_2(F[t]), \mathbb{Z})$.

As an immediate consequence of Theorem 3.4 we have the following result.

**Corollary 3.5.** - The natural inclusion $GL_n(F) \to GL_n(F[t])$ induces an isomorphism

$$H_\bullet(GL_n(F), \mathbb{Z}) \to H_\bullet(GL_n(F[t]), \mathbb{Z}).$$

**Proof.** - Consider the commutative diagram

$$
\begin{array}{cccccc}
1 & \to & SL_n(F) & \to & GL_n(F) & \to & F^\times & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & & \\
1 & \to & SL_n(F[t]) & \to & GL_n(F[t]) & \to & F^\times & \to & 1.
\end{array}
$$

This yields a map of spectral sequences which by Theorem 3.4 is an isomorphism at the $E^2$-level.

By applying a theorem of Suslin, we have the following stability result.

**Corollary 3.6.** - If $n \leq m$, then the natural map

$$H_i(GL_n(F[t]), \mathbb{Z}) \to H_i(GL_m(F[t]), \mathbb{Z})$$

is an isomorphism for $i \leq n$.

**Proof.** - Consider the commutative diagram

$$
\begin{array}{cccc}
H_i(GL_n(F), \mathbb{Z}) & \to & H_i(GL_m(F), \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_i(GL_n(F[t]), \mathbb{Z}) & \to & H_i(GL_m(F[t]), \mathbb{Z}).
\end{array}
$$

By [19, 3.4], the top horizontal map is an isomorphism for $i \leq n$ and by Corollary 3.5, so is each of the two vertical maps.

4. The level $t$ congruence subgroup and a fundamental domain

for the action of $\Gamma_{1,j_2,...,j_k}$ on $\mathcal{Y}$

Consider the exact sequence

$$1 \longrightarrow K \longrightarrow SL_n(F[t]) \xrightarrow{t=0} SL_n(F) \longrightarrow 1$$

where $K$ consists of those matrices which are congruent to the identity modulo $t$. In the preceding section we described a fundamental domain, $\mathcal{T}$, for the action of $SL_n(F[t])$ on the complex $\mathcal{Y}$ of Example 1.2. In order to find a fundamental domain for the action of $GL_n(F[t])$, we use the congruence subgroup $K$. We shall do this by a method similar to that of Section 3. The congruence subgroup $K$ is slightly different, as it is defined by congruence modulo $t$, rather than congruence modulo $d$. However, this does not affect the fundamental domain, as the fundamental domain for the action of $SL_n(F[t])$ on $\mathcal{Y}$ includes all the necessary congruences.
\(\Gamma_{1,j_2,...,j_k}\) on \(\mathcal{Y}\), we proceed in steps. First, we find a fundamental domain for the action of \(K\), then a fundamental domain for the action of \(\Gamma_{1,2,...,n}\), and finally, a fundamental domain for the action of \(\Gamma_{1,j_2,...,j_k}\).

Denote by \(B_n(F)\) the upper triangular subgroup of \(SL_n(F)\) and choose a set \(S\) of coset representatives for \(SL_n(F)/B_n(F)\). Set

\[T' = \bigcup_{s \in S} sT.\]

**Proposition 4.1.** The complex \(T'\) is a fundamental domain for the action of \(K\) on \(\mathcal{Y}\).

**Proof.** Let \(\sigma\) be an \((n-1)\)-simplex of \(\mathcal{Y}\). There exists some \(x\) in \(SL_n(F[t])\) and a unique simplex \(\sigma_0\) of \(T\) such that \(\sigma = x\sigma_0\). Write

\[x = k y, \quad k \in K, \quad y \in SL_n(F)\]

and

\[y = s u, \quad s \in S, \quad u \in B_n(F).\]

Then

\[\sigma = k s u \sigma_0.\]

Note that \(u\) acts trivially on \(T\); i.e., \(u \sigma_0 = \sigma_0\). Hence, \(\sigma = k s \sigma_0\), and thus

\[\sigma \equiv s \sigma_0 \mod K.\]

It remains to show that no two vertices of \(T'\) are identified by \(K\).

Suppose \(x : s_1 \Lambda_1 \rightarrow s_2 \Lambda_2\) where the \(s_i\) belong to \(S\) and \(x\) is some element of \(K\). Then

\[s_1 s_2^{-1}x : s_1 \Lambda_1 \rightarrow s_1 \Lambda_2.\]

Now, \(s_1 s_2^{-1}x\) belongs to \(SL_n(F[t])\) and the \(s_1 \Lambda_i\) are inequivalent modulo \(SL_n(F[t])\) (i.e., we could have taken \(s_1 T\) as a fundamental domain). Hence, \(\Lambda_1 = \Lambda_2\). Denote this common vertex by \(\Lambda\). Moreover, \(s_1 s_2^{-1}x\) stabilizes \(s_1 \Lambda\). Observe that the stabilizer of \(s_1 \Lambda\) in \(SL_n(F[t])\) is

\[s_1 (SL_n(F[t])) \Lambda s_1^{-1}.\]

It follows that

\[s_1 s_2^{-1}x = s_1 \gamma s_1^{-1}\]

where \(\gamma\) stabilizes \(\Lambda\). So,

\[(8) \quad x = s_2 \gamma s_1^{-1}.\]

We have a split exact sequence

\[1 \rightarrow (K \cap (SL_n(F[t])) \Lambda) \rightarrow (SL_n(F[t]) \Lambda) \xrightarrow{\text{id} = 0} P_\Lambda \rightarrow 1\]
where $P_{\Lambda}$ is a parabolic subgroup of $SL_n(F)$. Write $\gamma = kv$, where $k \in K$ and $v \in P_{\Lambda}$. Then
\[
x = s_2kvs_1^{-1} = s_2(vs_1^{-1})(s_1v^{-1})k(vs_1^{-1}).
\]

Since $K$ is a normal subgroup of $SL_n(F[t])$, we have
\[
(s_1v^{-1})k(vs_1^{-1}) \in K.
\]

Denote this element by $k'$. Then we may write
\[
x = s_2(vs_1^{-1})k'
\]
or
\[
(9) \quad x(k')^{-1} = s_2(vs_1^{-1}).
\]

Now, the element $x(k')^{-1}$ belongs to $K$ while the element $s_2(vs_1^{-1})$ belongs to $SL_n(F)$. Since the groups $K$ and $SL_n(F)$ intersect in the identity, both sides of equation (9) must equal 1. It follows that
\[
s_2 = s_1v^{-1}.
\]

Since $v^{-1}$ stabilizes $\Lambda$, we have
\[
s_2\Lambda = (s_1v^{-1})\Lambda = s_1\Lambda.
\]

It follows that $T'$ is a fundamental domain for the action of $K$ on $\mathcal{Y}$. \hfill \Box

**Remark.** - When $n = 2$, Proposition 4.1 allows us to deduce the free product decomposition
\[
(10) \quad K = \ast_{s \in P^1(F)} sCs^{-1}
\]

where
\[
C = \left\{ \begin{pmatrix} 1 & tp(t) \\ 0 & 1 \end{pmatrix} : p(t) \in F[t] \right\}
\]
(here, the set $S$ of coset representatives of $SL_2(F)/B_2(F)$ may be identified with $P^1(F)$). For further details see [12, 4.1].

Now consider the stabilizer $\Gamma_{1,2,\ldots,n}$ of the simplex $C$ (see Proposition 1.1). We have a split short exact sequence
\[
1 \rightarrow K \rightarrow \Gamma_{1,2,\ldots,n} \xrightarrow{t=0} B_n(F) \rightarrow 1.
\]

Choose a set of representatives for the permutation group $\Sigma_n$ in $SL_n(F)$ (e.g., we could take even permutations of the identity matrix along with odd permutations of the matrix $\text{diag}(-1, 1, \ldots, 1)$). Denote by $D_{1,2,\ldots,n}$ the subcomplex of $\mathcal{Y}$ defined by
\[
D_{1,2,\ldots,n} = \bigcup_{p \in \Sigma_n} pT.
\]
PROPOSITION 4.2. - The subcomplex $\mathcal{D}_{1,2,\ldots,n}$ is a fundamental domain for the action of $\Gamma_{1,2,\ldots,n}$ on $\mathcal{Y}$.

Proof. - We have a split extension

$$1 \longrightarrow U \longrightarrow B_n(F) \xrightarrow{\pi} T \longrightarrow 1$$

where $U$ is the unipotent radical of $B_n(F)$ and $T$ is the diagonal subgroup. The composition of $\pi$ with the map

$$\Gamma_{1,2,\ldots,n} \xrightarrow{t=0} B_n(F)$$

yields a split extension

$$1 \longrightarrow G \longrightarrow \Gamma_{1,2,\ldots,n} \longrightarrow T \longrightarrow 1.$$

Here, the group $G$ consists of matrices of the form

$$\begin{pmatrix} 1 + tp_{11} & p_{12} & \cdots & \cdots & p_{1n} \\ tp_{21} & 1 + tp_{22} & \cdots & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ tp_{n1} & \cdots & \cdots & 1 + tp_{nn} \end{pmatrix}$$

where the $p_{ij}$ lie in $F[t]$. We first show that $\mathcal{D}_{1,2,\ldots,n}$ is a fundamental domain for the action of $G$ on $\mathcal{Y}$.

Consider the extension

$$1 \longrightarrow K \longrightarrow G \xrightarrow{t=0} U \longrightarrow 1.$$

Suppose that $\sigma$ is an $(n-1)$-simplex in $\mathcal{Y}$. Then there exist $k \in K$, $s \in S$, and $\sigma_0 \in T$ such that

$$\sigma = ks\sigma_0.$$

Recall the Bruhat decomposition of $SL_n(F)$ (see e.g., [9, p. 172]):

$$SL_n(F) = \bigcup_{p \in \Sigma_n} U_pB$$

(here, $B = B_n(F)$). From this it follows that if $s$ is an element of the set $S$, then we may write $s = upv$ for some $u \in U$, $p \in \Sigma_n$, and $v \in B_n(F)$. Then we have the chain of equalities

$$\sigma = ksp\sigma_0 = kuvp\sigma_0 = kup\sigma_0.$$

The last equality follows since $B_n(F)$ acts trivially on $T$. Now, $ku$ lies in $G$. Hence,

$$\sigma \equiv p\sigma_0 \mod G.$$

It follows that $\mathcal{D}_{1,2,\ldots,n}$ is a fundamental domain for the action of $G$ on $\mathcal{Y}$. Observe that the diagonal subgroup $T$ acts trivially on $\mathcal{D}_{1,2,\ldots,n}$. 

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Lemma 4.3. – Suppose a group $H$ acts on a simplicial complex $Z$, and that there is a split extension

$$1 \rightarrow N \rightarrow H \rightarrow Q \rightarrow 1.$$ 

Suppose further that the subcomplex $A$ is a fundamental domain for the action of $N$ on $Z$ and that $Q$ acts trivially on $A$. Then $A$ is a fundamental domain for the action of $H$ on $Z$.

Proof. – It suffices to show that no two vertices of $A$ are identified by the action of $H$. Suppose that $v_1$ and $v_2$ are vertices of $A$ and that there is an element $h$ in $H$ with $hv_1 = v_2$. Write $h = nq$, where $n \in N$, and $q \in Q$. Then we have

$$v_2 = hv_1 = nqv_1 = nv_1.$$ 

Since the vertices of $A$ are inequivalent modulo $N$, we must have $v_1 = v_2$. □

The lemma implies that $D_{1,2,...,n}$ is a fundamental domain for the action of $\Gamma_{1,2,...,n}$ on $\mathcal{Y}$. This completes the proof of Proposition 4.2. □

Finally, consider the group $\Gamma_{1,2,...,j_k}$. Note that $\Gamma_{1,2,...,j_k}$ contains the subgroup $H$ of $\Sigma_n$ consisting of permutation matrices that are products of the form

$$\sigma_1 \sigma_2 \cdots \sigma_{k-1}$$

where $\sigma_i$ is a permutation of the set

$$\{j_i, j_i + 1, \ldots, j_{i+1} - 1\}$$

(we take $j_1 = 1$). Let $N$ be a set of coset representatives of $H \setminus \Sigma_n$ containing the identity. Define a subcomplex $D_{1,j_2,...,j_k}$ by

$$D_{1,j_2,...,j_k} = \bigcup_{p \in N} pT.$$ 

Proposition 4.4. – The complex $D_{1,j_2,...,j_k}$ is a fundamental domain for the action of $\Gamma_{1,j_2,...,j_k}$ on $\mathcal{Y}$.

Proof. – Observe that $\Gamma_{1,j_2,...,j_k}$ contains the group $\Gamma_{1,2,...,n}$. It follows that a fundamental domain for the action of $\Gamma_{1,j_2,...,j_k}$ on $\mathcal{Y}$ is no larger than $D_{1,2,...,n}$. If $\sigma$ is an $(n-1)$-simplex in $\mathcal{Y}$, then there exist $g \in \Gamma_{1,2,...,n}$, $p \in \Sigma_n$, and $\sigma_0 \in T$ such that

$$\sigma = g \sigma_0.$$ 

Write $p = hn$, where $h \in H$ and $n \in N$. Then we have the chain of equalities

$$\sigma = g \sigma_0 = ghn \sigma_0.$$ 

Since $gh$ lies in $\Gamma_{1,j_2,...,j_k}$, it follows that

$$\sigma \equiv n \sigma_0 \text{ mod } \Gamma_{1,j_2,...,j_k},$$

and hence, $D_{1,j_2,...,j_k}$ is a fundamental domain for the action of $\Gamma_{1,j_2,...,j_k}$ on $\mathcal{Y}$. □
5. The homology of $\Gamma_{1,j_2,\ldots,j_k}$

We now compute the homology of the various $\Gamma_{1,j_2,\ldots,j_k}$. This will complete the computation of the $E^1$-term of the spectral sequence (4) since by Proposition 2.3 each $\Gamma_{i_1,\ldots,i_k}$ is isomorphic to some $\Gamma_{1,j_2,\ldots,j_k}$.

We have a split short exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma_{1,j_2,\ldots,j_k} \longrightarrow P_{1,j_2,\ldots,j_k} \longrightarrow 1$$

where $P_{1,j_2,\ldots,j_k}$ is a parabolic subgroup of $SL_n(F)$.

**Theorem 5.1.** The natural inclusion $P_{1,j_2,\ldots,j_k} \longrightarrow \Gamma_{1,j_2,\ldots,j_k}$ induces an isomorphism

$$H_\bullet(P_{1,j_2,\ldots,j_k}, \mathbb{Z}) \longrightarrow H_\bullet(\Gamma_{1,j_2,\ldots,j_k}, \mathbb{Z}).$$

**Proof.** Since the complex $Y$ is contractible, we obtain a spectral sequence converging to the homology of $\Gamma_{1,j_2,\ldots,j_k}$ satisfying

$$(11) \quad E^1_{p,q} = \bigoplus_{\dim \sigma = p} H_q(G_\sigma)$$

where $G_\sigma$ is the stabilizer of the $p$-simplex $\sigma$ in $\Gamma_{1,j_2,\ldots,j_k}$ ($\sigma \in \mathcal{D}_{1,j_2,\ldots,j_k}$).

Recall the filtration $V^\bullet$ of $T$ (5) defined in Section 3. Define a filtration $W^\bullet$ of $\mathcal{D}_{1,j_2,\ldots,j_k}$ by setting

$$W^{(l)} = \bigcup_{p \in \mathbb{N}} pV^{(l)}, \quad 0 \leq l \leq n - 1.$$ 

Note that $W^{(0)} = v_0$ and that the group $G_{v_0}$ is precisely $P_{1,j_2,\ldots,j_k}$. Define a coefficient system $G_q$ on $\mathcal{D}_{1,j_2,\ldots,j_k}$ by

$$G_q(\sigma) = H_q(G_\sigma).$$

Then the $q$th row of the spectral sequence (11) is the chain complex

$$C_\bullet(\mathcal{D}_{1,j_2,\ldots,j_k}, G_q).$$

On each component of $W^{(i)} - W^{(i-1)}$, the coefficient system $G_q$ is constant (i.e., the stabilizers in the translate $pT$ are conjugate to the stabilizers in $T$ and hence have isomorphic homology). So we may apply Lemma 3.3 to deduce that the inclusion $v_0 \longrightarrow \mathcal{D}_{1,j_2,\ldots,j_k}$ induces an isomorphism

$$H_\bullet(v_0, G_q) \longrightarrow H_\bullet(\mathcal{D}_{1,j_2,\ldots,j_k}, G_q).$$

Now the $E^2$-term of the spectral sequence (11) satisfies

$$E^2_{p,q} = \begin{cases} H_q(P_{1,j_2,\ldots,j_k}) & p = 0 \\ 0 & p > 0. \end{cases}$$

This completes the proof of Theorem 5.1. $\square$
Remark. - Theorem 3.4 is the special case $\Gamma_1 = SL_n(F[t])$ and $P_1 = SL_n(F)$.

Remark. - In the case of $\Gamma_{1,2,\ldots,n}$ and $P_{1,2,\ldots,n} = B_n(F)$, it is not necessary to define the filtration $W^*$ of $D_{1,2,\ldots,n}$ to prove the result. Indeed, Corollary 3.2 implies that each $G_v$ is homologically equivalent to $B_n(F)$. It follows that the $q$th row of spectral sequence (11) is the chain complex

$$C_\bullet(D_{1,2,\ldots,n}, H_q(B_n(F))).$$

Since $D_{1,2,\ldots,n}$ is contractible, the homology of the complex vanishes except in dimension zero, where we get $H_q(B_n(F))$.

Remark. - When $n = 2$, we only have the group $\Gamma_{12}$. In this case, Theorem 5.1 states that

$$H_\bullet(\Gamma_{12}) \cong H_\bullet(B_2(F)).$$

This was proved in [12] for fields of characteristic zero by examining the Lyndon-Hochschild-Serre spectral sequence associated to the extension

$$1 \longrightarrow K \longrightarrow \Gamma_{12} \longrightarrow B_2(F) \longrightarrow 1.$$  

The free product decomposition (10) for $K$ allows us to deduce that

$$H_k(K) = \bigoplus_{s \in P_1(F)} H_k(sCs^{-1}), \quad k \geq 1.$$  

Utilizing Shapiro’s Lemma and a standard center kills argument, Proposition 4.4 of [12] shows that

$$H_\bullet(B_2(F), H_k(K)) = 0, \quad k \geq 1.$$  

The $n = 2$ case of Theorem 5.1 follows easily. In [12], we used the action of $B_2(F)$ to kill the homology of $K$ rather than finding a fundamental domain for the action of $\Gamma_{12}$ on $\mathcal{Y}$. This approach works well in that case, but fails for $n \geq 3$ since we no longer have the free product decomposition for $K$.

6. The $d^1$-map

Having completed the computation of the $E^1$-term of the spectral sequence (4), we now turn our attention to the differential, $d^1$. Unfortunately, the computation of this map is rather difficult as it depends upon computing the maps induced on homology by the various inclusions $P_I \longrightarrow P_J$, where $P_I$ and $P_J$ are parabolic subgroups of $SL_n(F)$. To get a feel for the oddities which may occur, we present the following two results. Recall that for a field $F$, we denote by $B_2(F)$ the subgroup of $SL_2(F)$ consisting of upper triangular matrices.
PROPOSITION 6.1. (Dupont-Sah[8]) – The natural map
\[ H_2(B_2(C)) \rightarrow H_2(SL_2(C)) \]
is surjective.

The following result and its proof were communicated to me by J. Yang.

PROPOSITION 6.2. – If \( F \) is a number field, then the natural map
\[ j : H_2(B_2(F), \mathbb{Q}) \rightarrow H_2(SL_2(F), \mathbb{Q}) \]
is trivial.

Proof. – If \( F \) is a number field, then the group \( K_2(F) \) is torsion. Since the map
\( H_2(B_2(F), \mathbb{Z}) \rightarrow H_2(SL_2(F), \mathbb{Z}) \) factors through the map \( H_2(B_2(F), \mathbb{Z}) \rightarrow K_2(F) \), it follows that after tensoring with \( \mathbb{Q} \), the map \( j \) is trivial.

In light of these results, it seems to be a difficult question to compute the map
\[ H_k(P_1) \rightarrow H_k(P_j) \]
in general. Still, we are able to compute some special cases. In particular, we shall compute the maps \( d_{*0}^1 \) and \( d_{*1}^1 \).

6.1. The \( q = 0 \) case

Since the group \( H_0(\Gamma_\sigma) = \mathbb{Z} \) for each simplex \( \sigma \) of \( C \), the \( q = 0 \) row of the spectral sequence (4) is simply the simplicial chain complex \( S_\bullet(C) \). Since the simplex \( C \) is contractible, we have
\[ E_2^{p,0} = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p > 0. \end{cases} \]

6.2. The \( q = 1 \) case

Because we can find explicit representatives for elements of the various \( H_1(\Gamma_\sigma) \), we are able to compute the map \( d_{*1}^1 \). We begin by writing down the map explicitly.

Consider the group \( \Gamma_{1,j_2,...,j_k} \). By Theorem 5.1, we have
\[ H_1(\Gamma_{1,j_2,...,j_k}) \cong H_1(P_{1,j_2,...,j_k}). \]

By Corollary 3.2, the group \( P_{1,j_2,...,j_k} \) has the same homology as its reductive part \( L_{1,j_2,...,j_k} \). The group \( L_{1,j_2,...,j_k} \) has the form
\[
\begin{pmatrix}
B_1 & 0 \\
\vdots & \ddots \\
0 & B_k
\end{pmatrix}
\]
where each $B_i = GL_{j_{i+1} - j_i}(F)$ (see section 2). Now, for each $i$, $H_i(B_i) = F^\times$ (via the determinant map) and hence by the Künneth formula, $H_1(B_1 \times B_2 \times \cdots \times B_k) = (F^\times)^k$. It follows that

$$H_1(L_{i_1, j_2, \ldots, j_k}) \cong (F^\times)^{k-1},$$

via the map

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ 0 \\ A_k \end{pmatrix} \mapsto (\det A_1, \det A_2, \ldots, \det A_{k-1}).$$

Since each $\Gamma_{i_1, \ldots, i_k}$ is conjugate to some $\Gamma_{i_1, j_2, \ldots, j_k}$, it follows that

$$H_1(\Gamma_{i_1, \ldots, i_k}) \cong (F^\times)^{k-1}.$$

Denote the simplex with vertices $i_1, i_2, \ldots, i_k$ by $\sigma_{i_1 \ldots i_k}$. We now compute the map

$$H_1(\Gamma_{i_1, \ldots, i_k}) \to H_1(\Gamma_{\hat{i_1}, \ldots, \hat{i_k}})$$

induced by the face map $\sigma_{i_1 \ldots i_k} \to \sigma_{\hat{i_1} \ldots \hat{i_k}}$.

**Lemma 6.3.** Let $\sigma_{i_1 \ldots i_k}$ be a $(k-1)$-simplex in $C$ and suppose that $\sigma_{\hat{i_1} \ldots \hat{i_k}}$ is a face of $\sigma_{i_1 \ldots i_k}$. Then the map

$$H_1(\Gamma_{i_1, \ldots, i_k}) \to H_1(\Gamma_{\hat{i_1}, \ldots, \hat{i_k}})$$

is the map

$$(F^\times)^{k-1} \to (F^\times)^{k-2}$$

defined by

$$(\alpha_1, \ldots, \alpha_{k-1}) \mapsto \begin{cases} (\alpha_2, \alpha_3, \ldots, \alpha_{k-1}) & l = 1 \\
(\alpha_1, \ldots, \alpha_{l-1}\alpha_l, \hat{\alpha}_l, \ldots, \alpha_{k-1}) & 2 \leq l \leq k-2 \\
(\alpha_1, \alpha_2, \ldots, \alpha_k) & l = k-1. \end{cases}$$

**Proof.** To compute the map, we must chase elements around the following diagram:

```
\begin{align*}
\Gamma_{i_1, \ldots, i_k} &\to \Gamma_{1,(i_2-i_1+1),\ldots,(i_k-i_1+1)} \to L_{1,(i_2-i_1+1),\ldots,(i_k-i_1+1)} \\
\Gamma_{\hat{i_1}, \ldots, \hat{i_k}} &\to \Gamma_{1,(i_2-i_1+1),\ldots,(i_k-i_1+1)} \to L_{1,(i_2-i_1+1),\ldots,(i_k-i_1+1)} \\
\vdots &\to (F^\times)^{k-1} \\
\vdots &\to (F^\times)^{k-2}
\end{align*}
```
Consider first the case $2 \leq l \leq k - 2$. Here the first maps are the same in each row. We follow elements around the diagram. In the first row, we have

\[
\begin{pmatrix}
L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\
tV_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\
tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
L_2 & V_{23} & V_{24} & \cdots & V_{2,k+1} & V_{21} \\
tV_{32} & L_3 & V_{34} & \cdots & V_{3,k+1} & V_{31} \\
tV_{42} & tV_{43} & L_4 & \cdots & V_{4,k+1} & V_{41} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
tV_{k,2} & tV_{k,3} & tV_{k,4} & \cdots & L_k & V_{k,k+1} & V_{k,1} \\
tV_{k+1,2} & tV_{k+1,3} & tV_{k+1,4} & \cdots & tV_{k+1,k} & L_{k+1} & V_{k+1,1} \\
tV_{12} & tV_{13} & tV_{14} & \cdots & tV_{1,k} & V_{1,k+1} & L_1
\end{pmatrix}
\]

In the second row, we have

\[
\begin{pmatrix}
L_2 & L_3 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & L_{k-1} & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & L_{k+1} & V_{k+1,1} \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & L_{1}
\end{pmatrix}
\]

\[\mapsto (\det L_2, \det L_3, \ldots, \det L_k).\]
\[
\begin{pmatrix}
L_2 \\
L_3 \\
\vdots \\
L_{l-1} \\
0 \\
L_l \\
0 \\
\vdots \\
L_{k+1} \\
V_{k+1,1} \\
V_{1,k+1} \\
L_1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 \\
L_{l-1} \\
v_{l-1,l} \\
0 \\
L_l \\
0 \\
\vdots \\
L_{k+1} \\
v_{k+1,1} \\
v_{1,k+1} \\
L_1
\end{pmatrix}
\]
\[
\Rightarrow (\det L_2, \ldots, \det \begin{pmatrix} L_{l-1} & v_{l-1,l} \\ 0 & L_l \end{pmatrix}, \ldots, \det L_k)
\]
\[
= (\det L_2, \ldots, \det L_{l-1} \det L_l, \det L_{l+1}, \ldots, \det L_k).
\]

So we see that the map \((F^\times)^{k-1} \rightarrow (F^\times)^{k-2}\) is given by
\[
(\alpha_1, \ldots, \alpha_{k-1}) \mapsto (\alpha_1, \ldots, \alpha_{l-1} \alpha_l, \tilde{\alpha}_l, \ldots, \alpha_{k-1}).
\]

Next, consider the case \(l = k - 1\). Here the map in the second row is as follows:
\[
\begin{pmatrix}
L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\
tV_{21} & L_2 & V_{33} & \cdots & V_{2,k} & V_{2,k+1} \\
tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1}
\end{pmatrix}
\]
\[
\Rightarrow
\begin{pmatrix}
L_2 & V_{23} & V_{24} & \cdots & V_{2,k} & V_{2,k+1} & V_{21} \\
tV_{32} & L_3 & V_{34} & \cdots & V_{3,k} & V_{3,k+1} & V_{31} \\
tV_{42} & tV_{43} & L_4 & \cdots & V_{4,k} & V_{4,k+1} & V_{41} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
tV_{k,2} & tV_{k,3} & tV_{k,4} & \cdots & L_k & V_{k,k+1} & V_{k,1} \\
tV_{k+1,2} & tV_{k+1,3} & tV_{k+1,4} & \cdots & tV_{k+1,k} & L_{k+1} & V_{k+1,1} \\
tV_{12} & tV_{13} & tV_{14} & \cdots & tV_{1,k} & V_{1,k+1} & L_1
\end{pmatrix}
\]
\[
\Rightarrow
\begin{pmatrix}
L_2 \\
L_3 \\
\vdots \\
L_{k-1}
\end{pmatrix}
\mapsto
\begin{pmatrix}
L_k & V_{k,k+1} & V_{k,1} \\
0 & L_{k+1} & V_{k+1,1} \\
0 & V_{1,k+1} & L_1
\end{pmatrix}
\]
\[
\Rightarrow (\det L_2, \ldots, \det L_{k-1}).
\]

So, the map \((F^\times)^{k-1} \rightarrow (F^\times)^{k-2}\) is simply
\[
(\alpha_1, \ldots, \alpha_{k-1}) \mapsto (\alpha_1, \ldots, \alpha_{k-2}).
\]
Finally, consider the case \( l = 1 \). In this case, we are omitting the first vertex \( i_1 \). Thus, we use different conjugation maps in the isomorphisms

\[
\Gamma_{i_1, \ldots, i_k} \longrightarrow \Gamma_{1,(i_2-i_1+1),\ldots,(i_k-i_1+1)}
\]

and

\[
\Gamma_{i_2, \ldots, i_k} \longrightarrow \Gamma_{1,(i_3-i_2+1),\ldots,(i_k-i_2+1)}.
\]

Now the second row of the diagram looks like

\[
\begin{pmatrix}
L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\
tV_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\
tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
L_3 & V_{34} & \cdots & \cdots & V_{3,k+1} & V_{31} & V_{32} \\
tV_{43} & L_4 & \cdots & \cdots & V_{4,k+1} & V_{41} & V_{42} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
tV_{k+1,3} & tV_{k+1,4} & L_k & V_{k,k+1} & V_{k,1} & V_{k,2} \\
tV_{13} & tV_{14} & V_{1,k+1} & L_1 & V_{12} \\
tV_{23} & tV_{24} & V_{2,k+1} & V_{21} & L_2
\end{pmatrix}
\]

\[
\begin{pmatrix}
L_3 \\
L_4 \\
\vdots \\
L_k
\end{pmatrix}
\]

\[
\begin{pmatrix}
L_{k+1} & V_{k+1,1} & V_{k+1,2} \\
V_{1,k+1} & L_1 & V_{12} \\
V_{2,k+1} & V_{21} & L_2
\end{pmatrix}
\]

\[\mapsto (\det L_3, \ldots, \det L_k).\]

Hence, the map \((F^x)^{k-1} \longrightarrow (F^x)^{k-2}\) is given by

\[
(\alpha_1, \ldots, \alpha_{k-1}) \mapsto (\alpha_2, \ldots, \alpha_{k-1}).
\]

This completes the proof of Lemma 6.3. \(\Box\)

Denote the element \((\alpha_1, \ldots, \alpha_{k-1})\) of \(H_1(\Gamma_{i_1, \ldots, i_k})\) by \(\sigma_{i_1, \ldots, i_k} \otimes [\alpha_1, \ldots, \alpha_{k-1}]\). Then the \(d^1\)-map is given by the formula

\[
d^1 : \sigma_{i_1, \ldots, i_k} \otimes [\alpha_1, \ldots, \alpha_{k-1}] \\
\mapsto \sigma_{i_2, \ldots, i_k} \otimes [\alpha_2, \ldots, \alpha_{k-1}] \\
+ \sum_{l=2}^{k-1} (-1)^{l-1} \sigma_{i_1, \ldots, i_l, \ldots, i_k} \otimes [\alpha_1, \ldots, \alpha_{l-1}, \alpha_l, \ldots, \alpha_{k-1}] \\
+ (-1)^{k-1} \sigma_{i_1, \ldots, i_{k-1}} \otimes [\alpha_1, \ldots, \alpha_{k-2}].
\]
Let $A$ be an abelian group (written additively). Denote by $Q^{(n)}_*$ the chain complex defined as follows. To each $(k-1)$-simplex $\sigma_{i_1 \ldots i_k}$ of $C$ we assign the group $A^{k-1}$. The boundary map $d : Q^{(n)}_{k-1} \rightarrow Q^{(n)}_{k-2}$ is given by formula (12) above. We will compute the homology of $Q^{(n)}_*$ for any abelian group $A$. Taking $A = \mathbb{F}^\times$ we obtain the terms $E^2_{\ast,1}$ of the spectral sequence (4).

To compute the homology of the complex $Q^{(n)}_*$, we realize $Q^{(n)}_*$ as a quotient of another complex $C^{(n)}_*$. We shall then compute $H_*(C^{(n)}_*)$ and use this along with a long exact homology sequence to obtain $H_*(Q^{(n)}_*)$.

Construct the chain complex $C^{(n)}_*$ by assigning to each $(k-1)$-simplex $\sigma_{i_1 \ldots i_k}$ of $C$ the group $A^k$. Define the boundary map $\partial$ by

$$\partial : \sigma_{i_1 \ldots i_k} \otimes (a_1, \ldots, a_k) \mapsto \sum_{l=1}^{k} (-1)^{l-1} \sigma_{i_1 \ldots \widehat{i_l} \ldots i_k} \otimes (a_1, \ldots, \widehat{a_l}, \ldots, a_k).$$

Observe that for each $n \geq 2$, $C^{(n)}_*$ is a subcomplex of $C^{(n+1)}_*$.

Denote by $B^{(n)}_*$ the standard simplicial chain complex for $C$ with coefficients in $A$. Embed the complex $B^{(n)}_*$ into $C^{(n)}_*$ via

$$\sigma_{i_1 \ldots i_k} \otimes a \mapsto \sigma_{i_1 \ldots i_k} \otimes (a, \ldots, a).$$

Then we have the following.

**Lemma 6.4.** - The quotient complex $C^{(n)}_*/B^{(n)}_*$ is isomorphic to the complex $Q^{(n)}_*$.

**Proof.** - Denote the quotient complex by $D^{(n)}_*$. In $D^{(n)}_*$, we have assigned to each simplex $\sigma_{i_1 \ldots i_k}$ the group $A^k/A \cdot (1, \ldots, 1) \cong A^{k-1}$. We need only check that the boundary map is the same as that for $Q^{(n)}_*$. We take our isomorphism $A^k/A \cdot (1, \ldots, 1) \cong A^{k-1}$ to be the map

$$(a_1, \ldots, a_k) \mapsto (a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}).$$

To compute the boundary map in $D^{(n)}_*$, we lift elements to $C^{(n)}_*$, apply $\partial$, and then project back to $D^{(n)}_*$. Denote the projection map $C^{(n)}_* \rightarrow D^{(n)}_*$ by $\pi$. Then we have

$$\pi : \sigma_{i_1 \ldots i_k} \otimes (0, a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_{k-1})$$

$$\mapsto \sigma_{i_1 \ldots i_k} \otimes [a_1, \ldots, a_{k-1}]$$

and

$$\partial : \sigma_{i_1 \ldots i_k} \otimes (0, a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_{k-1})$$

$$\mapsto \sum_{l=1}^{k} (-1)^{l-1} \sigma_{i_1 \ldots \widehat{i_l} \ldots i_k} \otimes (0, a_1, \ldots, a_1 + \ldots + a_{l-1}, \ldots, a_1 + \ldots + a_{k-1}).$$
Applying $\pi$ to the right hand side of this equation, we see that the boundary map in $D^\cdot_0$ is the map

$$
\sigma_{i_1,...,i_k} \otimes [a_1, \ldots, a_{k-1}]
\mapsto \sigma_{i_2,...,i_k} \otimes [a_2, \ldots, a_{k-1}]
+ \sum_{l=2}^{k-1} (-1)^{l-1} \sigma_{i_1,...,i_l} \otimes [a_1, \ldots, a_{l-1} + a_l, \tilde{a}_l, \ldots, a_{k-1}]
+ (-1)^{k-1} \sigma_{i_1,...,i_{k-1}} \otimes [a_1, \ldots, a_{k-2}].
$$

It follows that $D^\cdot_1$ is isomorphic to $Q^\cdot_1$.

We now have a short exact sequence of chain complexes

$$
0 \rightarrow B^\cdot_1 \rightarrow C^\cdot_1 \rightarrow Q^\cdot_1 \rightarrow 0.
$$

The homology of $B^\cdot_1$ is easily computed (since $C$ is contractible). We now compute the homology of $C^\cdot_1$.

**Proposition 6.5.** The complex $C^\cdot_1$ is contractible. Hence, $H_i(C^\cdot_1) = 0$.

**Proof.** If $n$ is even, we define a contracting homotopy $h$ for $C^\cdot_1$ by

$$
h : \sigma_{i_1,...,i_k} \otimes (a_1, \ldots, a_k)
\mapsto \sum_{l=1}^{i_1-1} \sigma_{l_{i_1},...,i_k} \otimes (0, (-1)^{i_1+l+1}a_1, (-1)^{i_2+l+1}a_2, \ldots, (-1)^{i_k+l+1}a_k)
- \sum_{l=i_1+1}^{i_2-1} \sigma_{i_{i_1},l_{i_2},...,i_k} \otimes ((-1)^{i_1+l+1}a_1, 0, (-1)^{i_2+l+1}a_2, \ldots, (-1)^{i_k+l+1}a_k)
+ \cdots
+ (-1)^k \sum_{l=i_k+1}^n \sigma_{i_1,...,i_{k,l}} \otimes ((-1)^{i_1+l+1}a_1, \ldots, (-1)^{i_k+l+1}a_k, 0).
$$

If $n$ is odd, then $n - 1$ is even. So if $\sigma_{i_1,...,i_k}$ is a simplex in $C$ with $i_k < n$, then we may view $\sigma_{i_1,...,i_k} \otimes (a_1, \ldots, a_k)$ as belonging to the subcomplex $C^\cdot_{n-1}$. Thus, we may use the formula above. We extend $h$ to simplicies with $i_k = n$ as follows. If $i_k + 1 < n - 1$, then we define $h$ to be

$$
h : \sigma_{i_1,...,i_k} \otimes (a_1, \ldots, a_k)
\mapsto \sum_{l=1}^{i_1-1} \sigma_{l_{i_1},...,i_k} \otimes (0, (-1)^{i_1+l+1}a_1, \ldots, (-1)^{i_k+l+1}a_k)
- \sum_{l=i_1+1}^{i_2-1} \sigma_{i_{i_1}l_{i_2},...,i_k} \otimes ((-1)^{i_1+l+1}a_1, 0, (-1)^{i_2+l+1}a_2, \ldots, (-1)^{i_k+l+1}a_k)
+ \cdots
$$
\[ + (\pm 1)^{k-1} \sum_{l=i_{k-2}+1}^{n-1} \sigma_{i_1 \ldots i_{k-2},n-1,n} \otimes ((-1)^{i_1+l+1} a_1, \ldots, 0, (-1)^{n+l+1} a_k) \]
\[- \sum_{l=1}^{i_{i-1}} \sigma_{i_1 \ldots i_{k-2},n} \otimes (0, \ldots, 0, (-1)^l a_k) \]
\[+ \sum_{l=i_{i+1}}^{i_{i+1}} \sigma_{i_1 i_{k-2},n-1,n} \otimes (0, \ldots, 0, (-1)^{i_{k-2}+l+1} a_k) \]
\[+ \ldots \]
\[+ (\pm 1)^k \sum_{l=i_{k-1}+1}^{n-2} \sigma_{i_1 \ldots i_{k-2},n-1,n} \otimes (0, \ldots, 0, (-1)^l a_k). \]

If \( i_{k-1} = n - 1 \), then
\[ h : \sigma_{i_1 \ldots i_{k-2},n-1,n} \otimes (a_1, \ldots, a_k) \]
\[ \mapsto \sum_{l=1}^{i_{i-1}} \sigma_{i_1 \ldots i_{k-2},n-1,n} \otimes (0, (-1)^{i_1+l+1} a_1, \ldots, (-1)^{n+l+1} a_k) \]
\[- \sum_{l=1}^{i_{i-1}} \sigma_{i_1 i_{k-2},n-1,n} \otimes (0, \ldots, 0, (-1)^{i_{k-2}+l+1} a_k) \]
\[+ \sum_{l=i_{i+1}}^{i_{i+1}} \sigma_{i_1 i_{k-2},n-1,n} \otimes (0, \ldots, 0, (-1)^{i_{k-2}+l+1} a_k) \]
\[+ \ldots \]
\[+ (\pm 1)^{k-2} \sum_{l=i_{k-2}+1}^{n-2} \sigma_{i_1 \ldots i_{k-2},n-1,n} \otimes ((-1)^{i_1+l+1} a_1, \ldots, 0, (-1)^{(n-1)+l+1} a_{k-1}, (-1)^{n+l+1} a_k) \]
\[- \sum_{l=1}^{i_{i-1}} \sigma_{i_1 \ldots i_{k-2},n-1,n} \otimes (0, \ldots, 0, (-1)^l a_k) \]
\[+ \sum_{l=i_{i+1}}^{i_{i+1}} \sigma_{i_1 i_{k-2},n-1,n} \otimes (0, \ldots, 0, (-1)^l a_k) \]
\[+ \ldots \]
\[+ (\pm 1)^{k-1} \sum_{l=i_{k-2}+1}^{n-2} \sigma_{i_1 \ldots i_{k-2},n-1,n} \otimes (0, \ldots, 0, (-1)^l a_k). \]

One checks that \( \partial h + h \partial = \text{identity} \). This completes the proof of the proposition. \( \square \)

**Corollary 6.6.** – The homology of the complex \( Q^{(n)}_{\bullet} \) is given by

\[ H_k(Q^{(n)}_{\bullet}) = \begin{cases} A & k = 1 \\ 0 & k \neq 1. \end{cases} \]

**Proof.** – Since \( C^{(n)}_{\bullet} \) is contractible, the long exact homology sequence implies that

\[ H_k(Q^{(n)}_{\bullet}) \cong H_{k-1}(B^{(n)}_{\bullet}). \]
The result follows since
\[ H_k(B(n)) = \begin{cases} A & k = 0 \\ 0 & k \neq 0. \end{cases} \]

Taking \( A = F^\times \), we obtain the following.

**Corollary 6.7.** - The spectral sequence (4) satisfies
\[ E^2_{p,1} = \begin{cases} F^\times & p = 1 \\ 0 & p \neq 1. \end{cases} \]

**6.3. The second homology and cohomology groups**

**Corollary 6.8.** - There is an exact sequence
\[ 0 \longrightarrow \operatorname{coker}\{d^1_{1,2} : E^1_{1,2} \to E^1_{0,2}\} \longrightarrow H_2(SL_n(F[t, t^{-1}])) \longrightarrow F^\times \longrightarrow 1. \]

**Proof.** - Since \( E^2_{p,0} = E^2_{p,1} = 0 \) for \( p > 1 \), we have \( E^2_{0,2} = E^2_{0,1} \). The group \( E^2_{0,2} \) is precisely the cokernel of \( d^1 : E^1_{1,2} \to E^1_{0,2} \). Since \( E^2_{1,1} = F^\times \), the result follows. \( \square \)

**Corollary 6.9.** - Let \( F \) be a number field and denote the number of real embeddings of \( F \) by \( r_1 \). Then
\[ H_2(SL_2(F[t, t^{-1}]), \mathbb{Q}) \cong (F^\times \otimes \mathbb{Q}) \oplus \mathbb{Q}^{2r_1}. \]

**Proof.** - By Borel-Yang [3], we have
\[ H_2(SL_2(F), \mathbb{Q}) = \mathbb{Q}^{r_1}. \]

It follows that \( E^1_{0,2} = \mathbb{Q}^{2r_1} \). By Proposition 6.2, the map \( d^1 : E^1_{1,2} \to E^1_{0,2} \) is trivial. Hence, we have an exact sequence
\[ 0 \longrightarrow \mathbb{Q}^{2r_1} \longrightarrow H_2(SL_2(F[t, t^{-1}]), \mathbb{Q}) \longrightarrow F^\times \otimes \mathbb{Q} \longrightarrow 0. \]

We now investigate the map \( d^1_{1,2} \).

**Proposition 6.10.** - If \( n \geq 3 \), then the cokernel of the map \( d^1_{1,2} : E^1_{1,2} \to E^1_{0,2} \) is isomorphic to \( H_2(SL_n(F), \mathbb{Z}) \).

**Proof.** - The term \( E^1_{0,2} \) is equal to
\[ \bigoplus_{i=1}^n H_2(\Gamma_i). \]

Since each \( \Gamma_1 \) is conjugate to \( SL_n(F[t]) \) in \( GL_n(F[t, t^{-1}]) \), by Theorem 3.4 we have
\[ E^1_{0,2} \cong H_2(SL_n(F), \mathbb{Z})^{\otimes n}. \]
Consider the map
\[ p : H_2(SL_n(F), \mathbb{Z})^\oplus_n \rightarrow H_2(SL_n(F), \mathbb{Z}) \]
defined by
\[ p(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i. \]
The map \( p \) is surjective with kernel consisting of those elements of
\[ H_2(SL_n(F), \mathbb{Z})^\oplus_n \]
whose entries sum to zero. We show that the image of \( d_{1,2} \) coincides with the kernel of \( p \).

Given a pair of integers \( i, j \) with \( 1 \leq i < j \leq n \), we have maps
\[ H_2(\Gamma_{ij}) \rightarrow H_2(\Gamma_i) \quad \text{and} \quad H_2(\Gamma_{ij}) \rightarrow H_2(\Gamma_j) \]
induced by inclusion. The map \( d_{1,2} \) is the alternating sum of these maps. To compute the image of \( d_{1,2} \) as a subgroup of \( H_2(SL_n(F), \mathbb{Z})^\oplus_n \), we make use of the diagrams
\[ \begin{array}{ccc}
H_2(\Gamma_{ij}) & \xrightarrow{\approx} & H_2(\Gamma_{1,j-i+1}) \\
\downarrow & & \downarrow \\
H_2(\Gamma_{j-i+1})
\end{array} \]
to see that the image of \( H_2(\Gamma_{ij}) \) in \( H_2(\Gamma_i) \) is isomorphic (via the identifications \( \Gamma_i \cong \Gamma_i \)) to the image of \( H_2(\Gamma_{ij}) \) in \( H_2(\Gamma_j) \). Since \( d_{1,2} \) maps \( H_2(\Gamma_{ij}) \) to \( H_2(\Gamma_i) \) with a negative sign and to \( H_2(\Gamma_j) \) with a positive sign, we see that the image of \( d_{1,2} \) in \( H_2(SL_n(F), \mathbb{Z})^\oplus_n \) lies in the kernel of \( p \).

To see that the image is all of the kernel, we use a result of Hutchinson [10, p. 200] which states that if \( F \) is an infinite field, then the map
\[ H_2(\Gamma_{12}) \rightarrow H_2(\Gamma_1) \]
is surjective for \( n \geq 3 \). It follows that the maps
\[ H_2(\Gamma_i,i+1) \rightarrow H_2(\Gamma_i) \quad \text{and} \quad H_2(\Gamma_i,i+1) \rightarrow H_2(\Gamma_i+1) \]
are surjective for \( i = 1, \ldots, n-1 \). Thus, the image of \( d_{1,2} \) contains all elements of the form
\[ (-a, a, 0, \ldots, 0), (0, -a, a, 0, \ldots, 0), \ldots, (0, \ldots, 0, -a, a) \]
and it follows that the image of \( d_{1,2} \) coincides with the kernel of \( p \). \( \square \)

Corollary 6.11. - If \( F \) is an infinite field, then for \( n \geq 3 \),
\[ H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}) = H_2(SL_n(F), \mathbb{Z}) \oplus F^\times. \]
Proof. – The spectral sequence (4) gives an exact sequence

\[ 0 \to H_2(SL_n(F), \mathbb{Z}) \xrightarrow{\phi} H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}) \to F^\times \to 0. \]

Observe that the map \( p : E^{1}_{1,2} \to E^{3}_{1,2} \) is split by inclusion onto the first factor. It follows that the map \( \phi \) is induced by the canonical inclusion \( SL_n(F) \to SL_n(F[t, t^{-1}]) \).

Observe that this map is split by the map \( SL_n(F[t, t^{-1}]) \xrightarrow{t=1} SL_n(F) \).

It follows that \( H_2(SL_n(F), \mathbb{Z}) \) is a direct summand of \( H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}) \). This proves the corollary. \( \square \)

Remark. – Since \( K_2(F[t, t^{-1}]) = K_2(F) \oplus K_1(F) \) and since

\[ K_2(F) = H_2(SL_n(F), \mathbb{Z}) \quad n \geq 3, \]

Corollary 6.11 implies that \( H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}) \) stabilizes at \( n = 3 \); i.e., for \( n \geq 3 \) we have an isomorphism

\[ H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}) \cong K_2(F[t, t^{-1}]). \]

Corollary 6.12. – If \( n \geq 3 \), then

\[ H^2(SL_n(F[t, t^{-1}]), \mathbb{Z}) \cong H^2(SL_n(F), \mathbb{Z}) \oplus \text{Hom}_\mathbb{Z}(F^\times, \mathbb{Z}). \]

Proof. – By the Universal Coefficient Theorem,

\[ H^2(SL_n(F[t, t^{-1}]), \mathbb{Z}) \cong \text{Hom}_\mathbb{Z}(H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}_\mathbb{Z}(H_1(SL_n(F[t, t^{-1}]), \mathbb{Z}), \mathbb{Z}) \cong \text{Hom}_\mathbb{Z}(H_2(SL_n(F), \mathbb{Z}), \mathbb{Z}) \oplus \text{Hom}_\mathbb{Z}(F^\times, \mathbb{Z}) \oplus 0 \cong H^2(SL_n(F), \mathbb{Z}) \oplus \text{Hom}_\mathbb{Z}(F^\times, \mathbb{Z}). \]

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