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INTRINSIC MICROLOCAL ANALYSIS AND INVERSION FORMULAE FOR THE HEAT EQUATION ON COMPACT REAL-ANALYTIC RIEMANNIAN MANIFOLDS

BY FRANÇOIS GOLSE, ERIC LEICHTNAM AND MATTHEW STENZEL

ABSTRACT. – This paper is devoted to a new intrinsic description of microlocal analytic singularities on a connected compact C^ω Riemannian manifold (X, g). In this approach, the microlocal singularities of a distribution u on X are described in terms of the growth, as $t \to 0^+$, of the analytic extension of $e^{-t\Delta}u$ to a suitable complexification X' of X, identified with a tubular neighborhood of the zero section in T^*X. First we show that the analytic extension of the heat kernel of (X, g) to X' is an F.B.I. transform in the sense of Sjöstrand. Then we establish various inversion formulae for the heat semigroup $e^{-t\Delta}$ analogous to Lebeau's inversion formula for the Euclidean Fourier-Bros-Iagolnitzer transform.

0. Introduction

The purpose of this article is to use the complexification of a compact, real analytic Riemannian manifold to give a new, intrinsic description of the analytic wave front set of a distribution (*), and to prove an inversion formula for the heat equation analogous to Lebeau's formula in the case of Euclidean space [L]. Our substitute for the Fourier transform methods traditionally used to analyze microlocal singularities is the Fourier–Bros–Iagolnitzer (F.B.I.) transform. The kernel of this transform is defined by the analytic continuation of the heat kernel in the manifold variables. In order to extract useful information from this transform, and prove the inversion formula, the main difficulty is to understand precisely the singularity of its kernel as $t \to 0^+$.

NOTATION 0.0. – *If P is a differential operator on a manifold Z and f is a smooth function defined on $Z \times ... \times Z$, we denote by $P_k f$ the action of P on the k-th variable in f.*

In the following (X, g) will be a compact, connected, orientable, real analytic, n-dimensional manifold and p the leading symbol of the (non-negative) Laplace-Beltrami

(*) We recall that the C^∞ (resp. real analytic) wave front set of a distribution u is, roughly speaking, the points at which u fails to be locally equal to a C^∞ (resp. real analytic) function together with the codirections contributing to the singularity.

operator. There is canonically associated to (X,g) an integrable complex structure on a sufficiently small tubular neighborhood U of the zero section in T^*X (*see* [G-S1], [L-S]). We complexify X by identifying it with the zero section in such a tube. To pass from estimates on Riemannian objects in the real domain given in term of exponential coordinates to estimates in the complex domain, it will be convenient to parameterize this tube by the analytic continuation of the exponential map:

$$(x,v)\in T^{\epsilon}X\rightarrow \mathrm{Exp}_x\sqrt{-1}\,v\in U,$$

where $T^{\epsilon}X$ is the set of tangent vectors of length less than ϵ. For ϵ sufficiently small this is a diffeomorphism, and we will denote its image by M_ϵ.

Let $E(t,x,y)$ denote the heat kernel of (X,g). The F.B.I. transform is the map $u\rightarrow e^{-t\Delta}u$. It is well known that for every distribution u on X, $e^{-t\Delta}u$ is a real analytic function. We will show that there is an $\epsilon>0$, independent of u, such that $e^{-t\Delta}u$ (resp. $E(t,x,y)$) can be analytically continued to M_ϵ (resp. $(0,\infty)\times M_\epsilon\times M_\epsilon$). We denote by d^2 the square of the Riemannian distance function and its analytic continuation to a neighborhood of the diagonal Δ_X in $M_\epsilon\times M_\epsilon$. Our first main result is Theorem 0.1 (*see* below). Part (i) gives the asymptotic expansion of $E(t,x,y)$ as t tends to 0^+ modulo an exponentially decreasing term for x and y complex near the diagonal, and part ii) characterizes the analytic wave front set of u in terms of the growth of its F.B.I. transform in the complex domain as $t\rightarrow 0^+$.

THEOREM 0.1. – *For any $x_o\in X$ there exists ϵ_2, $\rho>0$ and an open neighborhood W_1 of x_o in M_ϵ such that:*

(i) *For any $0<t<1$ and any $(x,y)\in W_1\times W_1$ we can write:*

$$E(t,x,y)=N(t,x,y)e^{-d^2(x,y)/4t}+O\big(e^{-\rho/8t}\big)$$

where the $O(\cdot)$ is uniform with respect to (x,y) as $t\rightarrow 0^+$, and $N(t,x,y)$ is an analytic symbol of order $n/2$ with respect to $1/t$ in the sense of [Sj] (see *definition 4.3). Moreover if $\xi_0\in T_{x_0}X\setminus\{0\}$ has length less than ϵ_2 then $\frac{1}{2}\sqrt{-1}\,d^2(x,y)$ is an FBI phase near $(Exp_{x_0}(\sqrt{-1}\,\xi_0),x_0)$ and the value at $Exp_{x_0}(\sqrt{-1}\,\xi_0)$ of the associated weight is $\frac{1}{2}|\xi_0|^2$.*

(ii) *Let u be any distribution on X and let $\xi_o\in T_{x_0}X\setminus\{0\}$ have length less than ϵ_2.*

If there exists C, $\delta>0$ and an open neighborhood Z of $Exp_{x_0}(\sqrt{-1}\,\xi_0)$ in M_ϵ such that for all $0<t\le 1$ and all $x\in Z$,

$$(\star)\qquad e^{-|\xi_o|^2/4t}\,|e^{-t\Delta}u(x)|\le C\,e^{-\delta/4t}$$

then the covector $(x_0,\zeta_0)\in T^*_{x_0}X$ defined by $\zeta_0:\xi_1\rightarrow g(\xi_1,-\xi_0)$ does not belong to the analytic wave front set of u. Conversely if (x_0,ζ_0) does not belong to the analytic wave front set of u then for suitable Z, C and δ the estimate $(\star)$ is satisfied.

By the definition of $N(t,x,y)$ (*see* §4), Theorem 0.1 i) means that

$$(0.1)\qquad E(t,x,y)=(4\pi\,t)^{-n/2}\,e^{-d^2(x,y)/4t}\sum_{0\le k\le\frac{1}{Ct}}u_k(x,y)t^k+O\big(e^{-\rho/8t}\big)$$

as $t \to 0^+$, where the $u_k(x,y)$ are the analytic continuation of the coefficients appearing in the formal solution of the heat equation on (X,g) (*see* [B-G-M]) and C is a constant (defined in Definition 4.2) depending on the growth (*see* prop 3.1) of the u_k in the complex domain. The formula (0.1) improves (for a real-analytic manifold) upon the result of Kannai ([K]) which says that for two *real* points x, y of X close to each other and any nonnegative integer l,

$$E(t,x,y) = (4\pi\, t)^{-n/2} e^{-d^2(x,y)/4t} \sum_{0\le k\le l} u_k(x,y)t^k + O\Big(t^{l+1-n/2} e^{-d^2(x,y)/4t}\Big).$$

Another intrinsic approach to microlocal analysis is through the following Toeplitz correspondence (*see* [G], [G-S2]). Let ω be a holomorphic form of type $(n,0)$ which is smooth up to the boundary of M_ϵ. We can integrate it along the fiber of the usual cotangent fibration to get a smooth function u on X:

$$q \in X \to u(q) = \int_{\pi^{-1}(q)} \omega. \tag{0.2}$$

Epstein and Melrose (*see* [E-M]) have proven that for ϵ small enough the correspondence $\omega \in \mathcal{O}(\overline{M}_\epsilon, \Lambda^{n,0}) \to u \in C^\infty(X)$ is an isomorphism. When (X,g) is real analytic this map extends to an isomorphism between distributions on X and the space of all holomorphic $(n,0)$ forms on M_ϵ with temperate growth near ∂M_ϵ.

It can be shown that the microlocal regularity properties of u near a boundary point α of ∂M_ϵ are equivalent to the local regularity properties of ω near α (*see* [G, §5]). The inverse of the integral transform (0.2) is microlocally equivalent to the F.B.I. transform defined by the *square root* of the Laplacian (*see* [G-S2], Theorem 5.3). Our approach seems to be different because we work with F.B.I. transform associated to a *differential* operator. We note that our method can also be applied to characterize C^∞ wave front sets.

To explain our next set of results we introduce some notation. For any $q \in X$, let Y denote the fiber $\pi^{-1}(q)$ in M_ϵ (here π is the usual cotangent fibration). Let g^+, resp. μ^+, be the holomorphic tensor obtained by analytic continuation of g, resp. the Riemannian volume μ, and let g^Y, resp. μ^Y be the complex valued tensor field obtained pulling back to Y. For ϵ sufficiently small, it is possible to define div^Y and grad^Y with respect to g^Y and μ^Y, and to form the corresponding "Laplacian," $\Delta^Y = -\operatorname{div}^Y \operatorname{grad}^Y$. (We emphasize that generically g^Y is not real valued.) We again let $u_k(x,y)$ denote the analytic continuation of the coefficients in the formal solution of the heat equation, and let

$$H_k(t,x,y) = (4\pi t)^{-n/2} e^{d^2(x,y)/4t}\big(u_0(x,y) - tu_1(x,y) + \cdots + (-t)^k u_k(x,y)\big).$$

We may now state our first inversion formula for $e^{-t\Delta}$.

THEOREM 0.2. – *Let k be a nonnegative integer. There exists $\epsilon' > 0$ such that, for all $0 < \epsilon < \epsilon'$ and all real analytic functions f on X such that $\int_X f\mu = 0$,*

(0.3)

$$(\sqrt{-1})^n f(q) = \int_0^{+\infty} dt \int_Y Tf(t,\cdot)\, t^k\, \Delta^Y u_k(q,\cdot)\mu^Y + \int_0^{+\infty} dt \int_{\partial Y} \Big[Tf(t,\cdot)\, i^Y_{\operatorname{grad}^Y H_k(t,q,\cdot)}\mu^Y - H_k(t,q,\cdot) i^Y_{\operatorname{grad}^Y Tf(t,\cdot)}\mu^Y\Big]$$

with $Y = \pi^{-1}(q) \subset \overline{M}_\epsilon$.

The condition $\int_X f\mu = 0$ means that f is orthogonal to the space of harmonic functions (that is, the constants since X is connected and compact). The integral over ∂Y in (0.3) makes sense if f is "only" C^∞ (*see* Lemma 5.7), but we do not know if Theorem 0.2 is true in this case.

We will see that for $t > 0$, $(t, m) \to H_k(t, q, m)$ is an approximate parametrix for the operator $\partial_t - \Delta^Y$ on Y (Proposition 3.0), whereas $(t, m') \to H_k(-t, m, m')$ is (up to the constant factor $(\sqrt{-1}\,)^n$) an approximate parametrix for the heat equation $\partial_t + \Delta$ on X (*see* [B-G-M], page 208).

In Theorem 3.4 we will prove that, for ϵ sufficiently small, there exists a "pseudo-heat kernel" in Y for $\partial_t - \Delta^Y$: a function $K(t, p, m) \in C^0(\mathbf{R}_+^* \times Y \times Y)$, C^1 with respect to $t > 0$ and C^2 with respect to $m \in Y$ such that for all $p \in Y$, $(\partial_t - \Delta_2^Y)K(t, p, m) \equiv 0$, and for all continuous complex-valued functions $u \in C^0([0, 1] \times \overline{Y})$,

$$\lim_{t \to 0^+} \int_Y u(t, m) K(t, q, m) \mu^Y(m) = u(0, q)\,.$$

Moreover we will show that K satisfies certain growth estimates as $t \to 0^+$. These will allow us to prove our second inversion theorem:

THEOREM 0.3. – *For ϵ sufficiently small, we can find a pseudo-heat kernel K such that:*
1] $K(t, q, m) \sim (-4\pi t)^{-n/2} e^{d^2(q,m)/4t}$ as $t \to 0^+$, *uniformly with respect to $m \in Y$.*
2] *For all $t_2 > 0$ and all $f \in C^\infty(X)$ such that $\int_X f\mu = 0$,*

$$\begin{aligned} f(q) = & \int_0^{t_2} dt \int_{\partial Y} \left[Tf(t, \cdot) i^Y_{\mathrm{grad}^Y K(t,q,\cdot)} \mu^Y \; - \; K(t, q, \cdot) i^Y_{\mathrm{grad}^Y Tf(t,\cdot)} \mu^Y\right] \\ & + \int_Y Tf(t_2, \cdot) K(t_2, q, \cdot) \mu^Y\,. \end{aligned}$$

Note. – Unlike in Theorems 0.2 and 0.4 (below), we do not know whether $K(t, q, m)$ is bounded for $t \geq 1$, hence we cannot let $t_2 \to +\infty$ in the formula above. Furthermore, Lemma 7.2 and Proposition 2.4 show that this formula still holds if f belongs to $H^{6n+4}(X)$.

The idea of the proof of theorem 0.3. is the following. Since we have:

$$(\Delta_2^Y - \partial_t) K(t, q, m) \equiv 0$$

an integration by parts and a Green's formula on Y show that for $0 < t_1 < t_2$:

$$0 = \int_{t_1}^{t_2} dt \int_Y K(t, q, m)(\Delta_2^Y + \partial_t) Tf(t, m) \mu^Y(m)$$

$$\begin{aligned} = & \int_Y Tf(t_2, m) K(t_2, q, m) \mu^Y(m) - \int_Y Tf(t_1, m) K(t_1, q, m) \mu^Y(m) \\ & + \int_{t_1}^{t_2} dt \int_{\partial Y} \left[Tf(t, \cdot) i^Y_{\mathrm{grad}^Y K(t,q,\cdot)} \mu^Y \; - \; K(t, q, \cdot) i^Y_{\mathrm{grad}^Y Tf(t,\cdot)} \mu^Y\right] \end{aligned}$$

then we let t_1 goes to zero.

The definition of the pseudo-heat kernel K (*see* (3.37) and the proof of Theorem 3.4) combined with the estimate (3.37) shows that K has the following asymptotic expansion as $t \to 0^+$:

$$K(t,x,y) \sim (-4\pi t)^{-n/2} e^{d^2(x,y)/4t} \sum_{0 \leq k \leq 1/Ct} (-t)^k u_k(x,y) + O\big(e^{-2\eta/t}\big) \tag{0.4}$$

for some $\eta > 0$, uniformly for x, y in Y. A comparison between (0.1) and (0.4) suggests, heuristically, that K can be thought of as the heat kernel of X at points (t,x,y) in $]-\infty,0[\times Y \times Y$. Thus we obtain a kernel for the "inverse" of the heat operator, at the expense of working in the fiber Y in complexified manifold M_ϵ (essentially because for $q \in X$, $\xi \in T^\epsilon X$, $d^2(q, \mathrm{Exp}_q(\sqrt{-1}\,\xi)) = -|\xi|^2 \leq 0$). In general, it makes no sense to write $f = e^{t\Delta}[e^{-t\Delta} f]$ because it is not possible to define the heat kernel associated to X at a point $(-t,x,y)$ in $\mathbf{R}^*_- \times X \times X$. For instance, in the case of $X = S^1$ endowed with the usual metric (by embedding $S^1 \subset \mathbf{R}^2$), one has:

$$\mathrm{Trace}\,(e^{-t\Delta}) = \int_X E(t,m,m)d\mu(m) = \sum_{n\in\mathbf{Z}} e^{-tn^2}, \quad t > 0$$

but this function cannot be extended for $t < 0$ because the imaginary axis $\sqrt{-1}\,\mathbf{R}$ is a barrier for the analytic continuation (*see* [D-G] p. 45). The same is true for $X = S^1 \times S^1 \times \ldots \times S^1$ (we think that it would be nice to have a proof that for any compact X the singular support of Trace $e^{-\sqrt{-1}\,t\Delta}$ fills the real line).

Finally we study the case when X is locally symmetric. We show that this is true if and only if $-g^Y$ is a field of real, positive definite quadratic forms on all of the fibers Y (Proposition 1.17). In this case we can simplify our inversion formula.

THEOREM 0.4. – *Assume that $-g^Y$ is a field of real, positive definite quadratic forms on Y. Then:*

1] *$-\Delta^Y$ is the Laplacian of the Riemannian manifold $(Y, -g^Y)$.*

2] *There exists a solution $K(t,m) \in C^\infty(\mathbf{R}^*_+ \times Y)$ of the "heat" equation $(\partial_t - \Delta^Y)K \equiv 0$ which is bounded on $[1,+\infty[\times Y$ and such that:*

a] *For all $m \in Y$, $K(t,m) \sim (-4\pi t)^{-n/2} e^{d^2(q,m)/4t}$ as $t \to 0^+$.*

b] *For all $f \in C^\infty(X)$ such that $\int_X f\mu = 0$,*

$$f(q) = \int_0^{+\infty} dt \int_{\partial Y} \big[Tf(t,\cdot)\, i^Y_{\mathrm{grad}^Y K(t,\cdot)}\, \mu^Y \;-\; K(t,\cdot)\, i^Y_{\mathrm{grad}^Y Tf(t,\cdot)} \mu^Y\big]. \tag{0.5}$$

In §8 we show that if X is a complete (not necessarily compact, but still connected and orientable) locally symmetric space, then, for ϵ small enough, the Riemannian manifold $(Y, -g^Y)$ is isometric to a neighborhood of the identity coset in a symmetric space dual to the universal cover of X, and the restriction of the analytic continuation of $-d^2$ $(= -d_X^2)$ to $Y \times Y$ is equal to the square of the distance function of $(Y, -g^Y)$. This allows us to show in Theorem 8.8 that if X is a compact locally symmetric space and $K_1(t,x,y)$ is *any* good heat kernel for $(Y, -g^Y)$ (*see* Definition 8.7), then, after possibly shrinking Y, K_1 has the following asymptotic expansion as $t \to 0^+$:

$$(\sqrt{-1}\,)^{-n} K_1(t,x,y) = (-4\pi\, t)^{-n/2} e^{-d^2(x,y)/4t} \sum_{0 \leq k \leq 1/Ct} (-t)^k u_k(x,y) + O\big(e^{-\delta/t}\big)$$

for some $\delta > 0$, uniformly with respect to x, y in Y. We note that if ϵ is small enough, then a good heat kernel for $(Y, -g^Y)$ exists. A comparison with the equation (0.1) shows that, heuristically, up to an exponentially decreasing term, $(\sqrt{-1}\,)^{-n} K_1(t,x,y)$ may be considered as the value at $(-t,x,y) \in]-\infty, 0[\times Y \times Y$ of the heat kernel of X.

In §9 we assume that X is a compact Riemannian globally symmetric space of rank one. We show that both the heat kernel and the formal solution of the heat equation depend only on $(t, d^2(x,y))$, and we give a simple and constructive proof of Proposition 3.1 in this situation. Our proof provides an algorithm which allows us to compute inductively the coefficients of the formal solution from three invariants of the root system associated with X.

The outline of this article is as follows. In §1 we review some facts about totally real submanifolds, and show that the complex structure on $T^\epsilon M$ induced by the analytic continuation of the exponential map is the same as the adapted complex structure of [L-S] and [G-S1]. We construct the tensors g^Y and μ^Y, the differential operators grad^Y and div^Y, and discuss the relationship between Δ^Y and the Laplacian of X (Theorem 1.16). We show that g^Y is real valued if and only if the geodesic symmetry about Y is a local isometry (Proposition 1.17).

In §2 we give some estimates on the growth of eigenfunctions of Δ in the complex domain (Proposition 2.1), and prove some preliminary estimates on the growth of the F.B.I. transform (Proposition 2.3). In §3 we construct the "pseudo-heat kernel" in Y (Theorem 3.4), and prove a crucial estimate on the growth of the coefficients u_k in the formal solution of the heat equation in the complex domain (Proposition 3.1).

§4 gives the proof of Theorem 0.1, §5 the proof of Theorem 0.2, §6 the proof of Theorem 0.4, and §7 the proof of Theorem 0.3. §8 considers in more detail the case where X is locally symmetric, and §9 deals with the rank one case.

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Finally, we have gathered in the following table some nonstandard notations adapted to the problem considered in this article: for each symbol we refer to the place in the article where it first appears.

Table of Notations

g^+	Proposition and Definition 1.13
μ^+	Proposition and Definition 1.13
Δ^+	Proof of Theorem 1.16
g^Y	Proposition and Definition 1.13
grad^Y	Definition 1.15
μ^Y	Proposition and Definition 1.13
div^Y	Definition 1.15
Δ^Y	Definition 1.15
Δ_2, Δ_2^Y	Notation 0.0

1. Geometric Constructions

Let M be a connected complex manifold of (real) dimension $2n$ with complex structure denoted by J and X be a C^ω submanifold of M of dimension n. The complexified tangent space of X at $q \in X$ is denoted by $T_q^{\mathbf{C}} X$; $T_m^{1,0} M$ (resp. $T_m^{0,1} M$) is the space of holomorphic (resp. antiholomorphic) tangent vectors of M at $m \in M$. Similarly, $T_m^{*(1,0)} M$ (resp. $T_m^{*(0,1)} M$) is the space of holomorphic (resp. antiholomorphic) tangent covectors of M at $m \in M$.

We first recall some elementary facts concerning totally real submanifolds of M which will be used constantly in the sequel.

Definition 1.1. – *X is said to be totally real in M if and only if*

$$T_q^{\mathbf{C}} X \cap T_q^{1,0} M = T_q^{\mathbf{C}} X \cap T_q^{0,1} M = \{0\} \text{ for all } q \in X.$$

The following lemma is classical and can be found for example in Guillemin's paper on Toeplitz operators [G].

Lemma 1.2. – *The two following conditions are equivalent:*

[a] X is a totally real C^ω submanifold of M;

[b] for all $q \in X$ there exists an open neighborhood W of q in M and a holomorphic coordinate system on W, $(z^1, ..., z^n)$ such that

$$X \cap W = \{m \in W \text{ s.t. } \Im z^1(m) = ... = \Im z^n(m) = 0\} \tag{1.1}$$

and $(x^1, ..., x^n)$ is a local coordinate system on $W \cap X$.

Lemma 1.2 means that in a complex manifold, totally real submanifolds play the same rôle as $\mathbf{R}^n$ or $\sqrt{-1}\,\mathbf{R}^n$ in $\mathbf{C}^n$. In particular, Lemma 1.2 b) shows that the following analogue of the analytic continuation principle holds.

COROLLARY 1.3. – *Let $X \subset M$ be a totally real submanifold of the complex manifold M, M' be a complex manifold and $f : X \to M'$ a C^ω mapping. Then, there exists a connected open neighborhood W of X in M and a unique holomorphic mapping $f^+ : W \to M'$ such that $f^+_{|X} = f$.*

Remark 1.4. – That X is totally real is necessary to ensure that the extension f^+ is unique.

Any compact C^ω manifold can be viewed as a totally real submanifold in some complex manifold, as shown by the

THEOREM *(Bruhat-Whitney [B-W]).*

1) *Let X be a compact C^ω manifold of dimension n. There exists a complex manifold M of dimension $2n$ and a C^ω embedding $j : X \to M$ such that $j(X)$ is a totally real submanifold of M.*

2) *Let $j_1 : X \to M_1$ and $j_2 : X \to M_2$ two such embeddings. There exists an open neighborhood W_1 of $j_1(X)$ in M_1, a neighborhood W_2 of $j_2(X)$ in M_2 and a biholomorphic one-to-one mapping $\phi : W_1 \to W_2$ such that $j_2 = \phi \circ j_1$.*

3) *There exists an open neighborhood W of X in M and a unique antiholomorphic involution $\sigma : W \to W$ such that $X = \{m \in W \text{ s.t. } \sigma(m) = m\}$.*

Let now X be a compact connected C^ω manifold of dimension n endowed with a C^ω Riemannian metric g. Denoting by $B(0, \rho)$ the ball centered at 0 with radius ρ in T_qX equipped with the metric g_q, for all $q \in X$ there exists $\rho_0(q) > 0$ such that

$$\mathrm{Exp}_q : B(0, \rho_0(q)) \subset T_qX \to X$$

is a C^ω diffeomorphism onto its image. Moreover the function $q \mapsto \rho_0(q)$ can be chosen lower semicontinuous on X. Let (M, j) be a Bruhat-Whitney complexification of X (we shall identify X and $j(X)$ from now on). It follows from Corollary 1.3 that, for all $q \in X$, there exists a connected open neighborhood W_q of 0 in $T_q^{\mathbf{C}}X$ and a unique holomorphic extension of Exp_q (still denoted by Exp_q) as a map $W_q \subset T_q^{\mathbf{C}}X \to M$. Hence one can define the C^ω map

$$\Phi : \Omega \to M\,, \quad \Phi(q, \xi) = \mathrm{Exp}_q(\sqrt{-1}\,\xi)\,; \tag{1.2}$$

on $\Omega = \{(q, \xi) \text{ s.t. } \xi \in T_qX\,,\ |\xi|_q < \rho_1(q)\}$ where the function ρ_1 can also be chosen lower semicontinuous on X. (For $\xi \in T_qX$, we shall use the notation $|\xi|_q = \sqrt{g_q(\xi, \xi)}$).

THEOREM 1.5. – *There exists $0 < \epsilon_0 \leq \inf_{q \in X} \rho_1(q)$ such that, for all $0 < \epsilon < \epsilon_0$,*

1)
$$\Phi : T^\epsilon X \to M\,, \quad \Phi(q, \xi) = \mathrm{Exp}_q(\sqrt{-1}\,\xi)$$

with $T^\epsilon X = \{(q, \xi) \text{ s.t. } \xi \in T_qX\,,\ |\xi|_q < \epsilon\}$ is a C^ω diffeomorphism onto its image.

2) *the map*

$$\pi : M_\epsilon = \Phi(T^\epsilon X) \to X, \quad \mathrm{Exp}_q(\sqrt{-1}\,\xi) \mapsto q$$

is a C^ω fibration with totally real fibers.

Proof. – 1) We first compute the differential of Φ on the zero section of TX. Let J^0 be the complex structure of $T_q^{\mathbf{C}}X$, *i.e.* the multiplication by $\sqrt{-1}$; for all $q \in X$, the following identification is understood: $T_{(q,0)}(TX) \sim T_qX \oplus T_0(T_qX) \sim T_qX \oplus T_qX$ and

$$(d\Phi)_{(q,0)} : T_{(q,0)}(TX) \to T_qM, \quad (d\Phi)_{(q,0)}(\xi + \eta) = \xi + J_q\eta .$$

The submanifold X being totally real in M, $T_qM = T_qX \oplus J_q(T_qX)$ and hence $d\Phi$ has rank $2n$ on the zero section of TX. Statement 1) for some small enough ϵ_0 follows since X is compact.

Remark 1.6. – The relation $(d\,\mathrm{Exp}_q)_0(\xi + J^0\xi) = \xi + J_q\xi$, $\forall \xi \in T_qX$ (identified with $T_0(T_qX)$) shows that $T_q\pi^{-1}(q) = J_qT_qX$.

2) That π is a C^ω fibration follows from 1). Let $q \in X$ and $Y = \pi^{-1}(q)$. One has $T_q^{\mathbf{C}}Y \cap T_q^{1,0}M = J_q(T_qX) \cap T_q^{1,0}M = \{0\}$ since X is totally real in M. But $T_p^{\mathbf{C}}Y \cap T_p^{1,0}M = (\ker d\pi_p)^{\mathbf{C}} \cap (\ker(J_p - \sqrt{-1}\,Id))^{\mathbf{C}}$ for all $p \in Y$ and this intersection is $\{0\}$ at q and hence in some neighborhood of q by continuity. Likewise $T_p^{\mathbf{C}}Y \cap T_p^{0,1}M = \{0\}$ for p in some neighborhood of q. Using again the compactness of X and reducing ϵ_0 if necessary shows that for all $q \in X$, $\pi^{-1}(q)$ is a totally real submanifold of M. □

At this point, we digress a little in order to discuss the relation between our constructions, those of Guillemin-Stenzel [GS] and the adapted complex structures of Lempert-Szöke [LS]. Statement 1] of Theorem 1.5 associates to a C^ω Riemannian metric on X a canonical complex structure on $T^\epsilon X$ which does not depend on the choice of M. We will show that this complex structure is nothing but the "adapted" complex structure of Lempert and Szöke [L-S]. As γ runs over all geodesics in X, the images of the maps

$$(t, s) \mapsto (\gamma(t), s\dot\gamma(t))$$

for $s \neq 0$ define a smooth foliation of $TX\backslash 0_X$, called the *Riemannian foliation*. The leaves of the Riemannian foliation carry a natural complex structure: one simply identifies $\mathbf{R}^2$ with $\mathbf{C}$ in the usual way. A complex structure on $T^\epsilon X$ is said to be *adapted* if the leaves of the Riemannian foliation, together with their natural complex structure, are (immersed) complex submanifolds of $T^\epsilon X$. One of the main results of [L-S] is that for any compact Riemannian manifold and any ϵ, $0 < \epsilon \leq \infty$, there is *at most one* adapted complex structure on $T^\epsilon X$ (*see* [L-S], Theorem 4.2).

Proposition 1.7. – *The adapted complex structure on $T^\epsilon X$ is the only complex structure for which the complexified exponential map defined in Theorem 1.5*

$$T^\epsilon X \ni (q, \xi) \mapsto \mathrm{Exp}_q \sqrt{-1}\,\xi \in M_\epsilon$$

is a biholomorphism for all $0 < \epsilon \leq \epsilon_0$.

Proof. – Fix $(q, \xi) \in T^\epsilon X$ and let L^ϵ denote the intersection of the leaf of the Riemannian foliation through (q, ξ) with $T^\epsilon X$. Let $\gamma(t)$ denote the geodesic with initial conditions $(\gamma(0), \dot\gamma(0)) = (q, \xi)$. We must show that the map

$$t + \sqrt{-1}\,s \mapsto \mathrm{Exp}_{\gamma(t)} \sqrt{-1}\,s\dot\gamma(t) \in M$$

is holomorphic; then for $|s|$ sufficiently small this will parametrize an (immersed) complex submanifold of M. Let $\beta(r)$ be the geodesic with initial conditions $(\beta(0), \dot{\beta}(0)) = (\gamma(t), s\dot{\gamma}(t))$. We have defined $\mathrm{Exp}_{\gamma(t)}\sqrt{-1}\,s\dot{\gamma}(t)$ to be the analytic continuation of the map $r \to \beta(r) \in X \subset M$ at $r = \sqrt{-1}$. Since $\beta(r) = \gamma(t+rs)$ for all real r, by uniqueness of analytic continuation we conclude that $\beta(\sqrt{-1}) = \gamma(t+\sqrt{-1}\,s)$, *i.e.*, $\mathrm{Exp}_{\gamma(t)}\sqrt{-1}\,s\gamma(t) = \gamma(t+\sqrt{-1}\,s)$. Since the map $t+\sqrt{-1}\,s \to \gamma(t+\sqrt{-1}\,s)$ is holomorphic we are done. □

We can identify TX with T^*X by the map $L_g : TX \to T^*X$, where $L_g(q,\xi)$ is the linear form on T_xX, $L_g(q,\xi)(\eta) = g_q(\xi,\eta)$. Via L_g the adapted complex structure induces an integrable complex structure on $T^{*\epsilon}X$, where $T^{*\epsilon}X$ is the set of covectors of length less than ϵ. Lempert and Szöke prove the following facts about the adapted complex structure on $T^\epsilon X$ ([L-S], Theorem 5.7 and Corollary 5.5):

1. The involution $(q,\xi) \mapsto (q,-\xi)$ is antiholomorphic.

2. $\Im\bar{\partial}(L_g^*\sigma) = L_g^*\alpha$, where α is the canonical one-form on T^*X and σ is the principal symbol of the Laplacian.

It follows immediately from the uniqueness part of the theorem on p. 568 of [G-S1] that the pushforward of the adapted complex structure by L_g is the complex structure described in [G-S1] (which we will refer to as the "adapted" complex structure on $T^{*\epsilon}X$). It is easy to see that the embedding of X in $T^{*\epsilon}X$ as the zero section is totally real.

THEOREM 1.8. – *Consider in the definition of Φ (1.2) the Bruhat-Whitney embedding of X as the zero section in $T^{*\epsilon}X = M$ (with the adapted complex structure). Then $\Phi(q,\xi) = Exp_q\sqrt{-1}\,\xi = L_g(q,\xi)$.*

Proof. – Proposition 1.7 (with $M = T^{*\epsilon}X$) shows that L_g and Φ are holomorphic maps from $T^\epsilon X$ to $T^{*\epsilon}X$ (with their respective adapted complex structures). Then $L_g^{-1}\circ\Phi$ is holomorphic from $T^\epsilon X$ to itself, and is equal to the identity on 0_X. Since 0_X is totally real in the connected complex manifold $T^{*\epsilon}X$, it follows from the uniqueness of the analytic continuation as in Corollary 1.3 that $L_g^{-1}\circ\Phi$ must be the identity on all of $T^\epsilon X$. □

Remark. – This result shows that the fiber one integrates over in the inversion formula for the heat equation (0.3) is *the same* as the one in the Toeplitz correspondence (0.2).

In the sequel, we shall construct on each fiber $\pi^{-1}(q)$ various objects corresponding to analogous objects defined on X. The first step in this direction is an analytic continuation principle for C^ω covariant tensors on X analogous to Corollary 1.3.

PROPOSITION 1.9. – *Let τ be a C^ω section of $(T^*X)^{\otimes m}$. There exists an open connected neighborhood $W \subset M$ of X and a unique holomorphic section τ^+ of $(T^{*+}W)^{\otimes m}$ such that*

$$(1.3) \qquad \forall q \in X \text{ and } v_1,\dots,v_m \in T_q^{1,0}M\,, \quad \tau_q^+(v_1,\dots,v_m) = \tau_q(P_q^X v_1,\dots,P_q^X v_m)\,,$$

where P_q^X is the projection on $T_q^{\mathbf{C}}X$ in the decomposition $T_q^{\mathbf{C}}M = T_q^{\mathbf{C}}X \oplus T_q^- M$. (In (1.3), τ_q has been extended in the natural way to a $\mathbf{C}$−multilinear form on $(T^{\mathbf{C}}X)^{\otimes m}$).

Proof. – Let $(W_1; z_1^1,\dots,z_1^n)$ a local holomorphic coordinate system on M at q as in b) of Lemma 1.2, with $x_1^i = \Re z_1^i$ and $y_1^i = \Im z_1^i$ $(1 \le i \le n)$. $(W_1 \cap X; x_1^1,\dots,x_1^n)$ is a local coordinate system on X, in which τ can be expressed as

$$(1.4) \qquad \sum_{i_1,\ldots,i_m} \tau_{1;i_1,\ldots,i_m}(x(p))dx_1^{i_1} \otimes \cdots \otimes dx_1^{i_m}\,, \quad p \in X \cap W_1$$

Let W_1' be a connected open neighborhood of $X \cap W_1$ where all the C^ω functions $\tau_{1;i_1,\ldots,i_m}(x)$ have holomorphic continuations denoted by $\tau^+_{1;i_1,\ldots,i_m}(x)$ and consider the holomorphic tensor τ_1 defined on W_1' by

$$(1.5) \qquad \tau^+_{1|p'} = \sum_{i_1,\ldots,i_m} \tau^+_{1;i_1,\ldots,i_m}(z(p'))dz_1^{i_1} \otimes \ldots \otimes dz_1^{i_m}\,, \quad p' \in W_1\,.$$

The tensor τ_1^+ defined in this way verifies property (1.3) on $W_1 \cap X$. Since X is compact and since a system of coordinates as in Lemma 1.2 b) can be associated to each point of X, there exists a finite family of connected open sets of M covering X with connected pairwise intersections, $W_1', \ldots, W_l'$ constructed as above with local holomorphic coordinates system $z_j = (z_j^1, \ldots, z_j^n)$ defined on W_j' for all $1 \leq j \leq l$ as in Lemma 1.2 b). For all $1 \leq j \leq l$ a holomorphic tensor τ_j^+ is defined on W_j' in the z_j coordinates as in (1.3)-(1.5). The local components of τ in each intersection $W_i' \cap W_j' \cap X$ satisfy C^ω compatibility relations. It is easy to check that, by analytic continuation of these compatibility relations, one can paste all the local holomorphic tensors τ_j^+ for $1 \leq j \leq l$ into a holomorphic covariant tensor τ^+ on the connected open neighborhood $W = W_1' \cup \ldots \cup W_l'$ of X in M and which satisfies (1.3). Uniqueness follows easily by the same arguments. $\square$

Remark 1.10. – Let $\xi_1, \ldots, \xi_m \in T_qX$. The vectors $\xi_1 - \sqrt{-1}\, J_q\xi_1, \ldots, \xi_m - \sqrt{-1}\, J_q\xi_m$ are holomorphic by construction and one has $P_q^X(\xi_1 - \sqrt{-1}\, J_q\xi_1) = 2\xi_1, \ldots, P_q^X(\xi_m - \sqrt{-1}\, J_q\xi_m) = 2\xi_m$ (*see* Remark 1.6). Therefore

$$\tau_q^+(\xi_1 - \sqrt{-1}\, J_q\xi_1, \ldots, \xi_m - \sqrt{-1}\, J_q\xi_m) = \tau_q(2\xi_1, \ldots, 2\xi_m).$$

Remark 1.11. – It follows from the definition of τ^+ that τ^+ is alternate on W (that is, a holomorphic $m-$form on W) whenever τ is alternate on X. In the same way, τ^+ is symmetric whenever τ is.

Assume from now on that (X, g) is orientable, and denote by μ the Riemannian volume form defining the orientation of X. For any (real or complex) vector bundle $\mathcal{E}$ we shall denote by $BS(\mathcal{E})$ the bundle of symmetric bilinear forms on the fibers of $\mathcal{E}$. Proposition 1.9 can be applied to the C^ω covariant tensors g and μ on X: there exists $0 < \epsilon_1 \leq \epsilon_0$, a unique holomorphic section g^+ of $BS(T^{1,0}M_{\epsilon_1})$ and a unique holomorphic section μ^+ of $\Lambda^n(T^{1,0}M_{\epsilon_1})$ defined in terms of g and μ respectively by condition (1.3). For any complex manifold M, we shall denote by $BS^{2,0}(T^{\mathbf{C}}M)$ the vector bundle of bilinear symmetric forms on complex tangent vectors to M such that

$$\forall p \in M\,,\ b \in BS^{2,0}(T^{\mathbf{C}}M)_p\,,\ v, w \in T_p^{\mathbf{C}}M\,, \quad v \in T_p^{0,1}M \Rightarrow b(v, w) = 0\,.$$

We identify the tensors g^+ and μ^+ with C^ω sections of $BS^{2,0}(T^{\mathbf{C}}M_{\epsilon_1})$ and of $\Lambda^{n,0}(T^{\mathbf{C}}M_{\epsilon_1})$ respectively, in the following natural way:

$$\forall p \in M_{\epsilon_1},\ v_1, \ldots, v_n \in T_p^{\mathbf{C}}M_{\epsilon_1},\ g^+(v_1, v_2) := g^+(v_1^+, v_2^+),\ \mu^+(v_1, \ldots, v_n) := \mu^+(v_1^+, \ldots, v_n^+)$$

where
$$v_i = v_i^+ + v_i^- \text{ , in the decomposition } T_p^{\mathbf{C}} M_{\epsilon_1} = T_p^+ M_{\epsilon_1} \oplus T_p^- M_{\epsilon_1}.$$

Remark 1.12. – Observe that, if j_X denotes the embedding of X into M_{ϵ_1}, the definitions of g^+ and μ^+ with the identifications above show that g^+ and μ^+ are the only C^ω sections of $BS^{2,0}(T^{\mathbf{C}} M_{\epsilon_1})$ and of $\Lambda^{n,0}(T^{\mathbf{C}} M_{\epsilon_1})$ respectively to verify $j_X^* g^+ = g$ and $j_X^* \mu^+ = \mu$.

The considerations above show the existence of some $\epsilon_1 \in]0, \epsilon_0]$ such that g^+ and μ^+ are defined as C^ω covariant tensors on M_ϵ, for all $0 < \epsilon \le \epsilon_1$. We will denote by Y the fiber $Y := \pi^{-1}(q) \subset M_\epsilon$ for some arbitrary q in X. Equivalently, $Y = \{\mathrm{Exp}_q(\sqrt{-1}\,\xi) \text{ s.t. } |\xi|_q < \epsilon, \, \xi \in T_q X\}$. Although ϵ_1 will be reduced again in the sequel, we shall keep the same notation Y for the fiber $\pi^{-1}(q)$, $0 < \epsilon < \epsilon_1$. Y and ∂Y are orientable, and equipped with compatible orientations so as to apply Stokes' formula. Denoting by j_Y the embedding of Y into M_{ϵ_1}, one has the following

Proposition and Definition 1.13. – *There exists $\epsilon_1 \in]0, \epsilon_0]$ such that*

a] $g^+ \in \Gamma(M_{\epsilon_1}, BS(T^{1,0} M_{\epsilon_1}))$ is a holomorphic section such that for all $p \in M_{\epsilon_1}$, g_p^+ is non-degenerate;

b] $\mu^+ \in \Gamma(M_{\epsilon_1}, \Lambda^n(T^{1,0} M_{\epsilon_1}))$ is a holomorphic section such that for all $p \in M_{\epsilon_1}$, $\mu_p^+ \neq 0$.

Moreover, for all $q \in X$ and $Y = \pi^{-1}(q) \subset M_{\epsilon_1}$ the following properties hold:

c] $g^Y := j_Y^*(g^+) \in \Gamma(Y, BS(T^{\mathbf{C}} Y))$ is a C^ω section such that for all $p \in Y$, $\Re g_p^Y < 0$ (is negative definite);

d] $\mu^Y := j_Y^*(\mu^+) \in \Gamma(Y, \Lambda^n(T^{*\mathbf{C}} Y))$ is a C^ω section such that for all $p \in Y$, $\mu_p^Y \neq 0$.

e] Let $(\xi_1, ..., \xi_n)$ be a system of normal coordinates at q on X, $(\zeta_1, ..., \zeta_n)$ its holomorphic extension to some connected neighborhood of q in M_{ϵ_1} and $\eta_i = \Im \zeta_i$. In the coordinates $(\eta_1, ..., \eta_n)$, $\mu^Y_{Exp_q(\sqrt{-1}\,\eta)} = \theta(q, Exp_q(\sqrt{-1}\,\eta))(\sqrt{-1}\,)^n d\eta_1 \wedge ... \wedge d\eta_n$ where $\zeta \mapsto \theta(q, Exp_q(\zeta))$ is the (unique) holomorphic extension of $\xi \mapsto |det\,(dExp_q)_\xi|$ to some (connected) open neighborhood of 0 in $T_q^{\mathbf{C}} X$. In particular, $\theta(q, q) = 1$.

For the convenience of the reader, let us describe the result of these constructions in the case where $X = \mathbf{R}^n$ equipped with its usual Euclidean metric – all of this actually works in this case although $\mathbf{R}^n$ is not compact. As recalled in the introduction $\mathrm{Exp}_q \xi = q + \xi$ for $q \in \mathbf{R}^n$ and $\xi \in T_q \mathbf{R}^n \sim \mathbf{R}^n$; $M_\epsilon = \mathbf{R}^n + \sqrt{-1}\,B(0, \epsilon) \subset \mathbf{C}^n$ (with the complex structure induced by that of $\mathbf{C}^n$, *i.e.*, the multiplication by $\sqrt{-1}$ in the complexified tangent space at any point of $\mathbf{C}^n$). Hence, the complexification of the exponential map is defined by $\mathrm{Exp}_q(\xi + \sqrt{-1}\,\eta) = q + \xi + \sqrt{-1}\,\eta$ for all $\eta \in T_q^{\mathbf{C}} X$ and, the point q being fixed, $Y = q + \sqrt{-1}\,B(0, \epsilon) \subset q + \sqrt{-1}\,\mathbf{R}^n$. Without loss of generality, we assume $q = 0$. The current point of $\mathbf{R}^n$ is denoted by $(x^1, ..., x^n)$, that of $\mathbf{C}^n$ by $(z^1, ..., z^n)$ with $\Re z^i = x^i$ and $\Im z^i = y^i$; hence the current point of Y is denoted $(y^1, ..., y^n)$, with $\sum_{1 \le i \le n} (y^i)^2 < \epsilon^2$. The Euclidean metric on $\mathbf{R}^n$ is $g = \sum_{1 \le i \le n} dx^i \otimes dx^i$, the corresponding $g^+ = \sum_{1 \le i \le n} dz^i \otimes dz^i$ and therefore $g^Y = -\sum_{1 \le i \le n} dy^i \otimes dy^i$. Similarly, $\mu = dx^1 \wedge ... \wedge dx^n$, the corresponding $\mu^+ = dz^1 \wedge ... \wedge dz^n$ and $\mu^Y = (\sqrt{-1}\,)^n dy^1 \wedge ... \wedge dy^n$.

As the proof of the statements in Proposition-Definition 1.13 is fairly direct, we shall not give it. The most important point is statement c] of 1.13 which results from the elementary computation stated in the next remark.

Remark 1.14. – Using remarks 1.6, 1.10, 1.12 one sees that, for ξ_1, $\xi_2 \in T_qX$,

$$g_q^+(\xi_1 - \sqrt{-1}\, J_q\xi_1, \xi_2 - \sqrt{-1}\, J_q\xi_2) = g_q(2\xi_1, 2\xi_2) = -g_q^Y(2J_q\xi_1, 2J_q\xi_2).$$

Assuming $0 < \epsilon \leq \epsilon_2$ from now on, one can now define gradients and divergences on Y with respect to the tensors g^Y and μ^Y.

DEFINITION 1.15. – *For complex-valued $f \in C^\infty(Y)$ we denote by* $\mathrm{grad}^Y f$ *the unique complex vector field on Y such that $g^Y(\mathrm{grad}^Y f, \cdot) = d^Y f$ (where d^Y denotes the exterior derivative on the manifold Y).*

For all complex vector fields V on Y, we denote by $\mathrm{div}^Y V$ *the unique scalar (complex-valued) function on Y such that $(\mathrm{div}^Y V)\mu^Y = d^Y(\iota_V^Y \mu^Y)$, where ι^Y denotes the interior product on the manifold Y.*

For all complex-valued $f \in C^\infty(Y)$ we define $\Delta^Y f := -\mathrm{div}^Y(\mathrm{grad}^Y f)$.

For instance, in the case of $X = \mathbf{R}^n$ with the usual Euclidean metric described before the proof of Proposition 1.13, $\Delta^Y = \sum_{1 \leq i \leq n} \partial_{y^i}^2$ is the opposite of the usual Laplacian.

The following analogue of the classical Green's formula will be used in the proofs of the inversion formulas.

GREEN'S FORMULA FOR Δ^Y. – *For all f and $f' \in C^\infty(\overline{Y})$ one has*

$$\int_Y (f\Delta^Y f' - f'\Delta^Y f)\mu^Y = \int_{\partial Y} \left[f' \iota_{\mathrm{grad}^Y f}^Y \mu^Y - f\, \iota_{\mathrm{grad}^Y f'}^Y \mu^Y \right]. \tag{1.6}$$

In particular, if f and f' have compact support in Y, one has

$$\int_Y (f\Delta^Y f' - f'\Delta^Y f)\mu^Y = 0.$$

The proof is identical to that of Green's formula for the Laplacian of a compact Riemannian manifold.

The following theorem explains the relation between the Laplace-Beltrami operator on X, Δ, and the operator Δ^Y defined above.

THEOREM 1.16. – *There exists $\epsilon_1 \in]0, \epsilon_0]$ such that, for all $0 < \epsilon \leq \epsilon_1$, for all connected open neighborhoods $\tilde{W}$ of q in M_ϵ and all f holomorphic on $\tilde{W}$, the functions $\Delta(f_{|X})$ and $\Delta^Y(f_{|Y})$ have the same holomorphic extension to $\tilde{W}$.*

It is again instructive to look at the case of $X = \mathbf{R}^n$. Indeed, Theorem 1.16 means exactly that, for all f holomorphic on $\mathbf{R}^n + \sqrt{-1}\, B(0,\epsilon)$, $\sum_{1 \leq i \leq n} \partial_{y^i}^2 f = -\sum_{1 \leq i \leq n} \partial_{x^i}^2 f$, which is obvious since $\sum_{1 \leq i \leq n} \partial_{z^i}\partial_{\overline{z}^i} f = 0$.

Proof. – X being compact, it is possible to cover X with a finite number of holomorphic coordinates patches having the property b) of Lemma 1.2. Let $(z^1, ..., z^n)$ be one such local holomorphic coordinate system with $x^i = \Re z^i$ $(1 \leq i \leq n)$, and denote by g^{ij} and $a = \sqrt{\det(g_{ij})_{1 \leq i,j \leq n}}$ the coefficients of the inverse metric tensor and of the volume element in the local coordinates $(x^1, ..., x^n)$, that is $g_{ij} = g(\partial_{x^i}, \partial_{x^j})$. Then, there exists $0 < \epsilon_1 \leq \epsilon_0$ such that, for all the (finite collection of) coordinate patches covering X, all

$1 \leq i, j \leq n$, the real analytic functions g_{ij}, g^{ij} and $a = (\det(g_{ij})_{1\leq i,j\leq n})^{1/2}$ and a^{-1} on X have holomorphic extensions to M_{ϵ_1} (denoted by the same letters).

Let f be holomorphic on some open set $\tilde{W} \subset M_{\epsilon_1}$. The holomorphic extension of $\Delta(f_{|X})$ to $\tilde{W}$ is, in the z-coordinates:

$$\Delta^+ f = -\sum_{1\leq i,j\leq n} a^{-1}\partial_{z^i}\left(a\, g^{ij}\partial_{z^j} f\right)$$

Let $q \in X$ belong to the domain of the coordinates system $(z^1, ..., z^n)$ and set $Y = \pi^{-1}(\{q\}) \subset M_{\epsilon_1}$. Y being totally real in M (*see* Remark 1.6), there exists a local holomorphic coordinates system $(w^1, ..., w^n)$ (on an open neighborhood of q in M) such that, denoting $\Re w^i = u^i$, $\Im w^i = v^i$ $(1 \leq i \leq n)$, Y is defined locally near q by the equations $v^1 = ... = v^n = 0$. In the w-coordinates

$$\Delta^+ f = -\sum_{1\leq i,j,k,l\leq n} a^{-1}P_i^k\partial_{w^k}\left(a\, g^{ij}P_j^l\partial_{w^l} f\right) = -\sum_{1\leq i,r,k,l\leq n} a^{-1}P_i^k\partial_{w^k}\left(Q_r^i\, a\, \tilde{g}^{rl}\partial_{w^l} f\right),$$

denoting by P the Jacobian matrix $P_i^k = (\partial_{z^i} w^k)$ $(1 \leq i, k \leq n)$, by $Q = (\partial_{w^i} z^k)$ its inverse, and letting $\tilde{g}^{rl} = \sum_{1\leq i,j\leq n} P_i^r g^{ij} P_j^l$. Observing that $\partial_{w^k} Q_r^i = (\partial_{w^k}\partial_{w^r} z^i)$ is symmetric in the indices k, r for all $1 \leq i \leq n$,

$$\sum_{1\leq i,k\leq n} P_i^k\partial_{w^k} Q_r^i = \sum_{1\leq i,k\leq n} P_i^k\partial_{w^r} Q_k^i = (\det Q)^{-1}\partial_{w^r}\det Q$$

(this last equality is based on the fact that the logarithmic differential of the determinant at Q is the linear form $R \mapsto \mathrm{Trace}(PR)$ with $PQ = I$). Hence

$$\Delta^+ f = -\sum_{1\leq k,l\leq n} b^{-1}\partial_{w^k}\left(b\,\tilde{g}^{kl}\partial_{w^l} f\right)$$

with

$$b = a\det Q \quad \text{and} \quad b^2 = a^2(\det Q)^2 = (\det(\tilde{g}_{ij})_{1\leq i,j\leq n}). \tag{1.7}$$

Since the coordinate system w has the property b) of Lemma 1.2 with respect to the totally real submanifold Y, one sees that

$$(\Delta^+ f)_{|Y} = -\sum_{1\leq k,l\leq n} b^{-1}\partial_{u^k}\left(b\,\tilde{g}^{kl}\partial_{u^l} f_{|Y}\right). \tag{1.8}$$

The expression for $(\Delta^+ f)_{|Y}$ above is exactly equal to that of $\Delta^Y(f_{|Y})$ in the local coordinates on Y defined by $(u^1, ..., u^n)$: to see this, it suffices to compute the expressions of g^Y and μ^Y in the u-coordinates, using in particular (1.7) above. □

In the case where $-g^Y$ is a metric on the fiber Y, the following result is easily derived from the proof above. This case is however not generic, as shown by the next proposition.

First, we recall that the geodesic symmetry about q is the map defined on a normal coordinate neighborhood of q by

$$s_q : \mathrm{Exp}_q(v) \to \mathrm{Exp}_q(-v)\,.$$

PROPOSITION 1.17. – *The tensor g^Y is real if and only if the geodesic symmetry on (X, g) about q is an isometry on a neighborhood of q. In that case we can shrink Y so that $-g^Y$ is positive definite.*

Proof. – Let exponential coordinates centered at q be given by

$$r : p = \mathrm{Exp}_q v \to r(p) = v\,.$$

In these coordinates the geodesic symmetry is given by $s_q(r) = -r$, and the metric by

$$g(r) = g_{ij}(r)\, dr^i\, dr^j\,.$$

The geodesic symmetry is an isometry if and only if the matrix entries g_{ij} are even functions of r: $g_{ij}(-r) = g_{ij}(r)$. Identify X with its image in the zero section of TX in the usual way and give $T^\epsilon X$ the "adapted" complex structure. We will need the following lemma.

LEMMA 1.18. – *There is a holomorphic coordinate system $z(p, v)$ on a neighborhood of q in the complex manifold $T^\epsilon X$ such that*

[1] $z(p, 0) = r(p)$,

[2] $z(q, v) = \sqrt{-1}\, v$.

Remark. – Condition 2] holds only on the fiber over q. In general $r(p) + \sqrt{-1}\, v$ does not provide holomorphic coordinates for the adapted complex structure on $T^\epsilon X$ (unless g is flat).

Proof. – The map

$$v = r(p) \to \mathrm{Exp}_q v = p \in X \cong 0_X$$

is a real analytic diffeomorphism near the origin. Analytically continuing into the complex manifold $T^\epsilon X$ we get holomorphic coordinates near q by

$$v + \sqrt{-1}\, u = z(p, w) \to \mathrm{Exp}_q(v + \sqrt{-1}\, u) = (p, w)\,.$$

Property 1 is clear. To check property 2 we recall that the adapted complex structure on $T^\epsilon X$ is constructed by embedding X as a totally real submanifold of a complex manifold, analytically continuing the Riemannian exponential map, and identifying (p, w) with $\mathrm{Exp}_p(\sqrt{-1}\, w)$ (*see* Proposition 1.7). So $\mathrm{Exp}_q(\sqrt{-1}\, v) = (q, v)$, which is property 2.

Analytically continuing g we obtain the holomorphic metric given in these coordinates by

$$g^+ = g_{ij}(z)\, dz^i\, dz^j\,.$$

Restricting to Y we get $z(q, v) = \sqrt{-1}\, v$ and

$$g^Y = -g_{ij}(\sqrt{-1}\, v)\, dy^i\, dy^j\,.$$

Since the coefficients in the (convergent) Taylor series of $g_{ij}(z)$ are real, to say that g^Y is real means that the odd order terms in the Taylor series of g_{ij} are zero. This is true if and only if $g_{ij}(r)$ is an even function of r. $\square$

In the sequel we shall analyze the situation where $-g^Y$ is a metric on Y.

PROPOSITION 1.19. – *Assume that $-g^Y$ is (real) positive definite on Y. Then*

i] *μ^Y is $(\sqrt{-1})^n$ times a Riemannian volume form associated to the metric $-g^Y$; this Riemannian volume form defines an orientation of Y;*

ii] *$-\Delta^Y$ is the Laplacian associated to the metric $-g^Y$.*

Proof. – We keep the notation of the proof of Theorem 1.16. In the local coordinates system $(u^1, ..., u^n)$ on Y one has $\mu^Y = bdu^1 \wedge ... \wedge du^n$ with $b = a \det Q$, (*see* 1.7) whence $b^2 = (-1)^n \det(-\tilde{g}_{kl})$ as already seen. The function b is continuous and does not vanish on Y; since Y is connected, there exists a constant $C = \pm(\sqrt{-1})^n$ such that $b = C\sqrt{\det(-\tilde{g}_{kl})}$, which proves i). To prove ii), it suffices to insert the above formula for b in the expression (1.8) for Δ^Y in the local coordinate system $(u^1, ..., u^n)$ on Y. $\square$

Actually, more is true under this assumption: the exponential map at q on Y for the metric $-g^Y$ is given in terms of the complexified exponential map of X restricted to Y. To show this, we need some preliminary material concerning affine connections on Y and M_{ϵ_1}.

PROPOSITION AND DEFINITION 1.20. – *Let $p \in M_{\epsilon_1}$, V and W two holomorphic vector fields on some open neighborhood of p in M_{ϵ_1}. There exists a unique vector in $T_p^{1,0}M$ denoted by $(\nabla_V^+ W)_p$ such that, for all holomorphic vector fields U defined on an open neighborhood of p in M_{ϵ_1}*

$$2g_p^+(U_p, (\nabla_V^+ W)_p) = V \cdot g^+(U, W)(p) + g^+(V, [U, W])(p)$$

$$+W \cdot g^+(U, V)(p) + g^+(W, [U, V])(p) - U \cdot g^+(V, W)(p) - g^+(U, [W, V])(p).$$

$(\nabla_V^+ W)_p$ depends only on V_p and the germ of W at p; moreover ∇^+ is an affine holomorphic connection.

The existence and uniqueness of the vector $(\nabla_V^+ W)_p$ follows from the fact that g_p^+ is nondegenerate on $T_p^{1,0}M$ (*see* Proposition 1.13 above). That ∇^+ is an affine holomorphic connection can be proved as in the usual Riemannian case (*see* [He1] p. 48).

The following result explains the relation between the connection ∇^+ and the Levi-Civita connections of (X, g) and $(Y, -g^Y)$. We denote P_p^X, $p \in X$ (resp. P_p^Y, $p \in Y$) the projection on $T_p^{\mathbf{C}}X$ (resp. $T_p^{\mathbf{C}}Y$) in the decomposition $T_p^{\mathbf{C}}M = T_p^{\mathbf{C}}X \oplus T_p^{0,1}M$ (resp. $T_p^{\mathbf{C}}M = T_p^{\mathbf{C}}Y \oplus T_p^{0,1}M$). (These definitions make sense since X and Y are totally real submanifolds of M).

PROPOSITION 1.21. – *Let U and V be two holomorphic vector fields defined on an open neighborhood Ω of q in M_{ϵ_1}. Then*

(i) $[P^X U, P^X V] = P^X[U, V]$ and $[P^Y U, P^Y V] = P^Y[U, V]$ on $\Omega \cap X$ *and* $\Omega \cap Y$ *respectively;*

(ii) $P^X(\nabla_U^+ V) = \nabla_{P^X U}(P^X V)$ *and* $P^Y(\nabla_U^+ V) = \nabla_{P^Y U}^Y(P^Y V)$ on $\Omega \cap X$ and $\Omega \cap Y$ *respectively, where ∇ is the Levi-Civita connection of (X, g) and ∇^Y* that *of $(Y, -g^Y)$.*

Proof. – The proofs of the statements relative to X and Y are similar.

i) Let $p \in \Omega \cap Y$ and $(z^1, ..., z^n)$ be local holomorphic coordinates on an open neighborhood of p in M_{ϵ_1} verifying property (1.1) relatively to the totally real submanifold $Y \subset M_{\epsilon_1}$, $x^i = \Re z^i$ and $y^i = \Im z^i$. In these coordinates, Y is defined by the system of equations $y^i = 0$ $(1 \leq i \leq n)$, and

$$U = \sum_{1 \leq i \leq n} U^i \partial_{z^i}\,, \qquad P^Y U = \sum_{1 \leq i \leq n} U^i_{|Y} \partial_{x^i}\,,$$

with analogous expressions for V and $[U, V]$. Statement i) then follows from a direct computation.

ii) Let U, V and W be holomorphic vector fields in an open neighborhood of p in M_{ϵ_1}. Then $p \mapsto P^Y_p U_p$ defines a complex tangent vector field on Y and one has:

$$2g^+_p(U_p, (\nabla^+_V W)_p) = 2g^+_p(P^Y_p U_p, P^Y_p (\nabla_V W)_p) = 2g^Y_p(P^Y_p U_p, P^Y_p(\nabla^+_V W)_p)$$

where the first equality holds because g^+_p vanishes whenever one of its argument is anti-holomorphic and the second equality holds by definition of g^Y. In the same way, using assertion i) shows that

$$g^+(V, [U, W])(p) = g^+_p(P^Y_p V_p, P^Y_p [U, W]_p) = g^Y_p(P^Y_p V_p, [P^Y U, P^Y W]_p)\,.$$

Now, using that U and W are holomorphic vector fields shows that $g^+(U, W)$ is a holomorphic function near p and hence, denoting d^M the exterior derivative on the manifold M:

$$V \cdot g^+(U, W)(p) = \langle P^Y_p V_p, d^M_p g^+(U, W)\rangle$$

$$= \langle P^Y_p V_p, d^M_p g^Y(P^Y U, P^Y W)\rangle = P^Y V \cdot g^Y(P^Y U, P^Y W)(p)\,.$$

Using the characterization of the Levi-Civita connection of $-g^Y$ as in [He1] p. 48 and the three equalities above leads to

$$-2g^Y_p(P^Y_p U_p, (\nabla^+_{P^Y_p V} P^Y W)_p)) = -2g^Y_p(P^Y_p U_p, P^Y_p(\nabla^+_V W)_p)$$

whence assertion ii) follows since g^Y_p is non-degenerate for $p \in M_{\epsilon_1}$. □

PROPOSITION AND PROPOSITION 1.22. – *Let $\tau \mapsto \phi(\tau)$ be a simple holomorphic curve in M_{ϵ_1} defined for $\tau \in \mathbf{C}$ close to 0 and such that $\phi(0) = p$ and $\partial_\tau \phi(\tau) \neq 0$. Let U and V be two holomorphic vector fields defined on an open neighborhood of p such that $U(\phi(\tau)) = V(\phi(\tau)) = \partial_\tau \phi(\tau)$. The vector $(\nabla^+_U V)(\phi(0)) \in T^{1,0}_p M$ does not depend on the choice of U or V (the proof being the same as in the case of Riemannian geometry); it is denoted by $(\nabla^+_{\phi'} \phi')_{|\tau=0}$.*

PROPOSITION 1.23. – *Let $q \in X$ and $\xi \in T_q X$ such that $|\xi|_q = 1$. For $|t| < \epsilon_1$, the map $t \mapsto Exp_q(\sqrt{-1}t\xi)$ is a geodesic curve on Y associated to the metric $-g^Y$ with velocity* *vector of norm* 1. *(In this statement, Y is the fiber at q for the fibration $\pi : M_{\epsilon_1} \to X$ defined in Theorem 1.5 and $-g^Y$ is a Riemannian metric on Y.)*

Proof. – Let $\phi(\tau) = \mathrm{Exp}_q(\tau\xi)$, with the notation $\tau = t + \sqrt{-1}\, s$. Since ϕ is holomorphic and lies on X for real values of τ, one has for all t real near 0, $P^X_{\phi(t)}\partial_\tau\phi(t) = \partial_t\phi(t)$. In the same way, for all s real close to zero, one has $(P^Y_{\phi(\sqrt{-1}\,s)}\partial_\tau\phi)(\sqrt{-1}\,s) = \partial_s\phi(\sqrt{-1}\,s)$. We apply now Proposition 1.21 ii). For τ complex close to zero, one has $(\nabla^+_{\partial_\tau\phi}\partial_\tau\phi)(\tau) = 0$. Indeed, let f be a holomorphic function defined in some open neighborhood of q; one has, for t real near 0:

$$\langle(\nabla^+_{\partial_\tau\phi}\partial_\tau\phi)(t), d^M_{\phi(t)}f\rangle = \langle P^X_{\phi(t)}(\nabla^+_{\partial_\tau\phi}\partial_\tau\phi)(t), d^M_{\phi(t)}f\rangle = \langle(\nabla_{\partial_t\phi}\partial_t\phi)(t), d^M_{\phi(t)}f\rangle = 0$$

whence $\langle(\nabla^+_{\partial_\tau\phi}\partial_\tau\phi)(\tau), d^M_{\phi(\tau)}f\rangle = 0$ for τ complex near 0 by analytic continuation. Therefore

$$\nabla^Y_{\partial_s\phi(\sqrt{-1}\,s)}\partial_s\phi(\sqrt{-1}\,s) = P^Y_{\phi(\sqrt{-1}\,s)}(\nabla^+_{\partial_s\phi}\partial_\tau\phi)(\sqrt{-1}\,s) = 0$$

for s real sufficiently close to 0. By (real-)analytic continuation, $\nabla^Y_{\partial_s\phi(\sqrt{-1}\,s)}\partial_s\phi(\sqrt{-1}\,s) = 0$ for all real $s \in]-\epsilon_1, \epsilon_1[$, which shows that the map $s \mapsto \phi(\sqrt{-1}\,s) = \mathrm{Exp}_q(s\sqrt{-1}\,\xi)$ of the real variable s is a geodesic curve on Y for the metric $-g^Y$. Then, the map $\tau = t + \sqrt{-1}\,s \mapsto \phi(\tau)$ being holomorphic near 0, one has $d_0\phi(\partial_{s|0}) = J_q d_0\phi(\partial_{t|0}) = J_q\xi$. Then, $-g^Y_q(J_q\xi, J_q\xi) = g_q(\xi,\xi) = 1$ according to remark 1.14. The conclusion follows from the fact that the norm of the velocity along a geodesic curve for a Riemannian metric remains constant. □

COROLLARY 1.24. – *Let d_1 be the geodesic distance on Y associated to the metric $-g^Y$. For $0 < \epsilon_1 \le \epsilon_0$ small enough, the squared geodesic distance $d(\cdot,\cdot)^2$ on X associated to the metric g can be extended as a holomorphic function on some connected open neighborhood Ω of the diagonal in $M_{\epsilon_1} \times M_{\epsilon_1}$, still denoted by $d(\cdot,\cdot)^2$. For $p \in Y \subset M_{\epsilon_1}$, one has $d_1(q,p)^2 = -d(q,p)^2$.*

Proof. – The squared geodesic distance $d(\cdot,\cdot)^2$ is a C^ω map on some open neighborhood of the diagonal in $X \times X$. Since X is totally real in M, $X \times X$ is totally real in $M \times M$ whence the existence and uniqueness of the holomorphic extension on Ω for small enough ϵ_1. Let $p \in Y \subset M_{\epsilon_1}$; there exists $\xi \in T_qX$ with $|\xi|_q < \epsilon_1$ such that $p = \mathrm{Exp}_q(\sqrt{-1}\,\xi)$ and it follows from Proposition 1.23 that $d_1(q,p)^2 = |\xi|^2_q$. Then, for all $\zeta \in T_qX$, one has, by definition, $d^2(q, \mathrm{Exp}_q(\zeta)) = |\zeta|^2_q$, and hence, by analytic continuation, if $|\xi|_q < \epsilon_1$, one has $d^2(q, \mathrm{Exp}_q(\sqrt{-1}\,\xi)) = -|\xi|^2_q$. □

2. Growth of the F.B.I. transform

The Hermitian scalar product of $L^2(X)$ is defined as usual by $\langle f, g\rangle = \int_X f\bar{g}\mu$. The Laplacian is an unbounded self-adjoint nonnegative operator on $L^2(X)$ with discrete spectrum. We denote by $0 = \lambda_0 < \lambda_1 \le \lambda_2 \cdots \le \lambda_k \le \cdots$ the nondecreasing sequence of eigenvalues of Δ counted with their multiplicities. Let $\{\phi_k\}_{k\ge 0}$ be an orthonormal basis of $L^2(X)$ such that $\forall k \in \mathbf{N}^*\ \Delta\phi_k = \lambda_k\phi_k$. For all $k \in \mathbf{N}^*$, ϕ_k is real analytic on X

and $\phi_0 \equiv \pm(\mathrm{vol}(X))^{-1/2}$ (indeed Stokes's formula shows that any harmonic function is constant on the connected compact manifold X).

The following proposition controls the growth of these eigenfunctions in the complex tuboids M_ϵ.

PROPOSITION 2.1. – *For $\epsilon_1 > 0$ small enough, there exists a locally bounded positive function $]0, \epsilon_1] \ni \epsilon \to C(\epsilon)$ such that for all $\epsilon \in]0, \epsilon_1]$ and all integers $k \geq 1$:*

1] *The function ϕ_k has a unique holomorphic extension to M_ϵ (again denoted by ϕ_k) satisfying:*

$$\sup_{m \in M_\epsilon} |\phi_k(m)| \leq C(\epsilon)\lambda_k^{n+1} e^{\epsilon\sqrt{\lambda_k}}.$$

2] *For all $\epsilon \in]0, \epsilon_1]$, all continuous functions f on $\overline{Y} \bigcap \partial M_\epsilon \, (= \partial Y)$:*

$$\left| \int_{\partial Y} f \, i^Y_{\mathrm{grad}^Y \phi_k} \, \mu^Y \right| \leq \sup_{\partial Y} |f| \, C(\epsilon)\lambda_k^{n+1} e^{\epsilon\sqrt{\lambda_k}}.$$

Remark. – We do not know whether $C(\epsilon)$ is bounded as $\epsilon \to 0^+$.

Proof. – 1] We first recall a result of L. Boutet de Monvel (*see* [BdM]). Let $\pi_0 : T^*X \to X$ be the canonical projection and ψ_t the Hamiltonian flow on T^*X of the principal symbol of $\sqrt{\Delta}$. For ϵ small consider the flow-out to imaginary time,

$$\Omega_\epsilon = \{\pi_0\big(\psi_{\sqrt{-1}\,t}(x, \zeta)\big) \in M \text{ where } 0 \leq t < \epsilon \,, \ \zeta \in T^*_x X \,, \ |\zeta|_x = 1\}.$$

THEOREM *(L. Boutet de Monvel [BdM]). – If $\epsilon_1 > 0$ is small enough, then for all $\epsilon \in]0, \epsilon_1]$ and all $s \in \mathbf{R}$ the operator $e^{-\epsilon\sqrt{\Delta}}$ defines a continuous linear one-one mapping from the Sobolev space $H^s(X)$ onto $O^{s+\frac{n-1}{4}}(\partial\Omega_\epsilon)$, where $O^{s+\frac{n-1}{4}}(\partial\Omega_\epsilon)$ is the subspace of $H^{s+\frac{n-1}{4}}(\partial\Omega_\epsilon)$ consisting of the functions which are restrictions on $\partial\Omega_\epsilon$ (in the distribution sense) of functions holomorphic in the open set Ω_ϵ. For each fixed s, the norm of this operator is a function of ϵ locally bounded on $]0, \epsilon_1]$. Moreover, each eigenfunction ϕ_k has a holomorphic extension to $\Omega_{2\epsilon_1}$.*

Let such an ϵ_1 be fixed and consider an arbitrary $\epsilon \in]0, \epsilon_1]$. Let $\zeta \in T^*_x X$ with $|\zeta|_x = 1$ and ξ the tangent vector which defines ζ via the scalar product of $T_x X$. It is well known that for real t close enough to 0,

$$\mathrm{Exp}_x(t\xi) = \pi_0\big(\psi_t(x, \zeta)\big).$$

By analytic continuation with respect to t, the domain Ω_ϵ in [BdM] coincides with M_ϵ. Hence the Sobolev injection theorem ($\dim \partial\Omega_\epsilon = 2n-1$) and Boutet de Monvel's theorem with $s = 2n + 2 > \frac{1}{2}\dim \partial M_\epsilon$ show that, for all $k \geq 1$:

$$\begin{aligned} \left\| e^{-\epsilon\sqrt{\Delta}}\phi_k \, = \, e^{-\epsilon\sqrt{\lambda_k}}\phi_k \right\|_{L^\infty(\partial\Omega_\epsilon)} &\leq C(\partial\Omega_\epsilon, n) \left\| e^{-\epsilon\sqrt{\lambda_k}}\phi_k \right\|_{H^{2n+2+n-\frac{1}{4}}(\partial\Omega_\epsilon)} \\ &\leq C'(\epsilon) \sum_{p=0}^{n+1} \|\Delta^p \phi_k\|_{L^2(X)} \leq C''(\epsilon)\lambda_k^{n+1} \end{aligned}$$

where $C(\partial\Omega_\epsilon, n)$ is the norm of the embedding of $H^{2n+2+n-\frac{1}{4}}(\partial\Omega_\epsilon)$ into $L^\infty(\partial\Omega_\epsilon)$. In the last inequality, we used that $0 < \lambda_1 \leq \lambda_k$ for $k \geq 1$. It is obvious from its definition in the above inequality that $C''(\epsilon)$ is locally bounded. This establishes the estimate in 1].

To prove 2], observe that the compact manifold $\partial Y \subset \partial M_{\epsilon_1}$ admits a finite open cover by real analytic local charts of $M_{2\epsilon_1}$, say $\bigcup_{l\in L} V_l$. In V_l we choose coordinates $(y^1, ..., y^{2n})$ such that $Y \cap V_l$ is defined by the equations $y^{n+1} = \cdots = y^{2n} = 0$ and $\partial Y \cap V_l$ by the equations $y^n = y^{n+1} = \cdots = y^{2n} = 0$. Hence, on each V_l and for any smooth function φ defined on some open neighborhood of ∂Y in $M_{2\epsilon_1}$:

$$\text{grad}^Y \varphi = \sum_{j=1}^{n} (A_j \cdot \varphi)\partial_{y^j}\,, \tag{2.1}$$

where A_j are complex vector fields defined on an open neighborhood of ∂Y in $M_{2\epsilon_1}$.

Now let $p \in \partial M_\epsilon$, which is a real-analytic hypersurface of $M_{2\epsilon_1}$; near p, ∂M_ϵ is defined by a real-analytic equation $f(m) = 0$ with nowhere vanishing differential df. Since f is real-valued, its antiholomorphic differential $\overline{\partial} f$ also satisfies $\overline{\partial} f \neq 0$ near p, and hence $T_p^{\mathbf{C}} M_{2\epsilon_1} = \text{Ker}(df_p)^{\mathbf{C}} + T_p^{0,1} M_{2\epsilon_1} = T_p^{\mathbf{C}} \partial M_\epsilon + T_p^{0,1} M_{2\epsilon_1}$.

Hence there exists a finite covering of ∂Y finer than (V_l) (but still denoted by (V_l)) such that, on each V_l, and with the notations of equation (2.1):

$$A_j \cdot \varphi = U_j \cdot \varphi + V_j \cdot \varphi, \quad 1 \leq j \leq n \tag{2.2}$$

where, for all $m \in V_l$, $U_j(m)$ is a (complex) tangent vector to ∂M_ϵ and $V_j(m)$ is an antiholomorphic vector. In particular, for each eigenfunction ϕ_k: $V_j \cdot \phi_k \equiv 0$. Equations (2.1), (2.2) and the Sobolev injection theorem ($\dim \partial M_\epsilon = 2n - 1$) show that

$$\left| \int_{\partial Y} f\, i^Y_{\text{grad}^Y \phi_k} \mu^Y \right| \leq D(\epsilon) \sup_{\partial Y} |f|\, \|\phi_k\|_{H^{\frac{2n-1}{2}+2}(\partial M_\epsilon)}.$$

Using Boutet de Monvel's theorem with $s = 2n + 2$, we obtain:

$$\left\| e^{-\epsilon\sqrt{\Delta}} \phi_k \right\|_{H^{\frac{2n-1}{2}+2}(\partial M_\epsilon)} \leq D'(\epsilon) \|\phi_k\|_{H^{2n+2}(X)} \leq D''(\epsilon) \lambda_k^{n+1}$$

where $D(\epsilon)$ and $D''(\epsilon)$ are locally bounded. Proposition 2.1 immediately follows. □

PROPOSITION 2.2.

1] *Let u belong to the Sobolev space $H^{2p}(X)$ where p is a nonnegative integer. The sequence $(\langle u, \phi_k\rangle \lambda_k^p)_{k\geq 0}$ is bounded by the Sobolev norm of u. Moreover, if $u \in C^\infty(X)$, then for all $m > 0$, the sequence $(\langle u, \phi_k\rangle k^m)_{k\geq 0}$ is bounded.*

2] *Let u belong to $L^2(X)$. u is real-analytic on X if and only if there exists $\eta > 0$ such that the sequence $(\langle u, \phi_k\rangle \exp(\eta\sqrt{\lambda_k}))_{k\geq 0}$ is bounded.*

3] *u is a hyperfunction [resp. a distribution] on X if and only if $u = \sum_{k\geq 0} a_k \phi_k$ where for all $\eta > 0$:*

$$a_k = O(e^{\eta\sqrt{\lambda_k}}) \quad [\text{resp. } \exists m > 0,\ a_k = O(k^m)] \quad \text{as } k \to +\infty$$

Proof. – Weyl's asymptotic estimate shows that $\lambda_k \sim C(X)k^{2/n}$ as $k \to +\infty$. For each nonnegative integer p, one can write:

$$\Delta^p u = \sum_{k\geq 0} \langle u, \phi_k \rangle \lambda_k^p \phi_k.$$

Computing the L^2-norm of $\Delta^p u$ yields immediately the estimate in 1].

2]. Let us assume that u is real-analytic on X. Since X is totally real in M_{ϵ_1}, there exists $\eta > 0$ small enough such that u has a holomorphic extension to $M_{2\eta} = \Omega_{2\eta}$. Then u restricted to ∂M_η is smooth; Boutet de Monvel's result (applied with $s \to +\infty$) recalled in the proof of Proposition 2.1 shows the existence of $v \in C^\infty(X)$ such that:

$$u = e^{-\eta\sqrt{\Delta}}\, v = \sum_{k\geq 0} \langle v, \phi_k \rangle e^{-\eta\sqrt{\lambda_k}}\, \phi_k\,;$$

then, statement 1] shows that $\langle u, \phi_k \rangle e^{\eta\sqrt{\lambda_k}} = \langle v, \phi_k \rangle$ remains bounded as $k \to +\infty$. The converse is an easy consequence of Proposition 2.1.1]

3] We will deal only with the case of hyperfunctions. First of all, since $\partial X = \emptyset$, observe that a hyperfunction on X is nothing but an analytic functional on X. Let u be a hyperfunction on X, then Proposition 2.1.1] shows that for any $\epsilon > 0$:

$$\forall k \in N, \quad |\langle u, \phi_k \rangle| \leq C_1(\epsilon) \sup_{M_\epsilon} |\phi_k| \leq C_2(\epsilon)\, \lambda_k^{2n+2} \exp\left(\epsilon\sqrt{\lambda_k}\right).$$

Therefore, for any $\eta > 0$, $a_k = \langle u, \phi_k \rangle$ is a $O(e^{\eta\sqrt{\lambda_k}})$ as $k \to +\infty$.

Conversely, if the sequence (a_k) is $O(e^{\eta\sqrt{\lambda_k}})$ for any $\eta > 0$, it is clear (by 2]) that $u = \sum_{k\geq 0} a_k \phi_k$ defines an analytic functional on X. □

The heat kernel $E(t, x, y)$ of X can be written in the so-called Sturm form:

$$E(t, x, y) = \sum_{k\geq 0} \exp(-t\lambda_k)\, \phi_k(x)\overline{\phi}_k(y) \quad \text{for} \quad t > 0.$$

DEFINITION 2.3. – *For all functions $f \in C^\infty(X)$ (or with Sobolev regularity), the Fourier-Bros-Iagolnitzer transform of f is the function $Tf(t, m)$ defined on $\mathbf{R}^{*+} \times X$ by:*

$$Tf(t, m) = \int_X f(m') E(t, m, m') \mu(m')\,.$$

Tf is smooth on $\mathbf{R}^{*+} \times X$. The following proposition allows us to control the growth of $Tf(t, m)$ in the complex domain, as $t \to 0^+$.

PROPOSITION 2.4. – *Let $f \in C^\infty(X)$. For each $t > 0$, $m \to Tf(t, m)$ has a holomorphic extension to the tuboid M_{ϵ_1} and for all $p \in \mathbf{N}^*$ there exists a positive locally bounded function $]0, \epsilon_1] \ni \epsilon \to C(p, \epsilon)$ such that, for all $\epsilon \in]0, \epsilon_1]$:*

1] *For all $t > 0$ and all $m \in \partial M_\epsilon$ i.e. $m = \mathrm{Exp}_q(\sqrt{-1}\xi)$ with $q \in X$ and $|\xi|_q = \epsilon$,*

$$|\, Tf(t, m) \,| \leq C(p, \epsilon)\, t^p\, e^{\epsilon^2/4t}.$$

2] *For any family* $\{h_t\}_{t>0}$ *of continuous functions on* $\partial Y (= \overline{Y} \bigcap \partial M_\epsilon)$ *and all* $t > 0$,

$$\left| \int_{\partial Y} h_t \, i^Y_{\mathrm{grad}^Y Tf(t,\cdot)} \mu^Y \right| \leq \sup_{\partial Y} |h_t| \; C(p, \epsilon) t^p \, e^{\epsilon^2/4t}.$$

3] *Let* $t > 0$ *and* u *be a distribution on* X. *Then,* $(x, y) \to E(t, x, y)$ *[resp.* $x \to Tu(t, x)$*] admits a holomorphic extension to* $M_{\epsilon_1} \times M_{\epsilon_1}$ *[resp.* M_{ϵ_1}*]. Moreover for all* $\epsilon \in]0, \epsilon_1]$, *for all* x, $y \in M_\epsilon$ *and all* $t \in]0, 1]$, $E(t, x, y)$ *satisfies the bound*

$$|E(t, \; x, \; y)| \leq C(\epsilon) e^{2\epsilon^2/t}.$$

Remark. – We can assume that $C(p, \epsilon)$ depends only on the Sobolev norm of f in $H^{2(2n+p+1)}(X)$, p, ϵ and X.

Proof – We shall only prove 1]; the proof of 2] is very similar and is left to the reader. We begin with the following

LEMMA 2.5. – *For all* $\epsilon > 0$ *and* $p \in \mathbf{N}$*:*

$$\sup_{k \geq 1, v > 0} \left[F_k(v) = e^{-(v - \epsilon\sqrt{\lambda_k}/2v)^2} \, v^{-2p} \right] = D(\epsilon, p) < +\infty$$

with $\epsilon \to D(\epsilon, p)$ locally bounded on R^{+*}.

Proof of Lemma 2.5. – For any integer $k \geq 1$ we have $\lambda_k \geq \lambda_1$. Thus:

$$\forall v \in]0, \left(\epsilon\sqrt{\lambda_1}/4\right)^{1/2}], \qquad F_k(v) \leq \sup_{0 < v < (\epsilon\sqrt{\lambda_1}/4)^{1/2}} v^{-2p} e^{-\epsilon^2 \lambda_1 / 16 v^2}.$$

If $v > (\epsilon\sqrt{\lambda_1}/4)^{1/2}$ then $F_k(v)$ is less than $(\epsilon\sqrt{\lambda_1}/4)^{-p}$, which completes the proof of Lemma 2.5. □

Assume that $f \in H^{2(2n+p+1)}(X)$ and denote by C_1 its Sobolev norm. Proposition 2.1 shows that:

$$|\langle f, \phi_1 \rangle| \leq C_1 \quad \forall k \geq 1, \; |\langle f, \phi_k \rangle| \leq C_1 \lambda_k^{-(2n+p+1)}.$$

Moreover

$$Tf(t, m) = \sum_{k \geq 0} e^{-t\lambda_k} \, \langle f, \phi_k \rangle \phi_k(m).$$

According to Weyl's asymptotic estimate (*i.e.* $\lambda_k \sim C(X) k^{2/n}$) and Proposition 2.1, for all $t > 0$, $m \mapsto Tf(t, m)$ has a holomorphic extension to M_{ϵ_1} and for all $\epsilon \in]0, \epsilon_1]$, all $m \in \partial M_\epsilon$ and all $t > 0$

$$|Tf(t, m)| \leq C_1 \big(1 + C(\epsilon) \sum_{k \geq 1} e^{-t\lambda_k} \, \lambda_k^{n+1-n-1-n-p} \, e^{\epsilon\sqrt{\lambda_k}}\big)$$

Next we apply Lemma 2.5 with $\sqrt{t\lambda_k}$ in place of v, which shows that

$$|Tf(t, m)| \leq \; C_1 \big(1 + e^{\epsilon^2/4t} \, t^p \, C(\epsilon) \, D(\epsilon, p) \sum_{k \geq 1} \lambda_k^{-n}\big) \, .$$

Weyl's formula shows that the series $\sum \lambda_k^{-n}$ converges.

We complete the proof of Proposition 2.4 with a short proof of 3]. One can choose the eigenfunctions ϕ_k so that they are real-valued on X:

$$E(t, x, y) = \sum_{k \geq 0} \exp\left(-t\lambda_k\right) \phi_k(x) \phi_k(y)$$

Point 3] follows directly from Weyl's formula, and from Propositions 2.1.1] and 2.2. □

3. Construction of a "Pseudo-Heat Kernel" in Y

In this section, we fix a point $q \in X$ and denote by Y the fiber $\pi^{-1}(q) \subset M_{\epsilon_1}$, with the definitions and notations of Section 1; ϵ_1 will be decreased several times in the sequel. We shall apply Minakshisundaram's method to construct a function $K : \mathbf{R}_+^* \times Y \times Y \to \mathbf{C}$ such that for all $p \in Y$,

$$(\partial_t - \Delta_2^Y)K(t,p,m) = 0 \text{ and } K(t,q,\cdot)\mu^Y \to \delta_q \text{ weakly.}$$

Such a function will be referred to as a "pseudo-heat kernel", for in general $-\Delta^Y$ is not the Laplacian associated to a Riemannian metric on Y, except in the case where $-g^Y$ is a Riemannian metric on Y (*see* Propositions 1.17 and 1.19). However, we insist on the fact that $\partial_t - \Delta_2^Y$ is a kind of heat operator, and not a backwards heat operator (the main difference between $\partial_t - \Delta_2^Y$ and a standard heat operator being the fact that $-g^Y$ is not in general a metric on Y). We use the notation

$$\begin{aligned}\theta(m, \mathrm{Exp}_m(\xi)) &= |\det((d\mathrm{Exp}_m)_\xi)| \\ &= \Big(\det(g_{\mathrm{Exp}_m \xi}((d\mathrm{Exp}_m)_\xi \partial_{\xi^i}, (d\mathrm{Exp}_m)_\xi \partial_{\xi^j}))_{1\le i,j\le n}\Big)^{1/2}\end{aligned} \tag{3.1}$$

for all $m \in X$, $\xi \in B(0, \mathrm{inj}_m(X)) \subset T_xX$. Clearly, θ is a C^ω, positive valued function defined on some open neighborhood of the diagonal in $X \times X$.

As in [B-G-M], pp. 204 and ff., consider the following sequence of C^ω functions, defined by induction on $U = \{(m,m') \in X \times X \text{ s.t. } d(m,m') < \mathrm{inj}(X)\}$ which is an open neighborhood of the diagonal in $X \times X$:

$$u_0(m,m') = \theta^{-1/2}(m,m') \tag{3.2}$$

and, for all $k \geq 1$:
(3.3)

$$u_k(m,m') = -\theta^{-1/2}(m,m') \int_0^1 \theta^{1/2}(m, \mathrm{Exp}_m(s\xi))(\Delta_2 u_{k-1})(m, \mathrm{Exp}_m(s\xi)) s^{k-1}\, ds\,,$$

with ξ defined by $\mathrm{Exp}_m(\xi) = m'$.

The behavior of pseudo-heat kernels K as $t \to 0^+$ is fundamental to prove inversion formulae as stated in Theorems 0.2–0.3, and more generally to define microlocal singularities intrinsically. In particular, the dominant exponential factor in K as $t \to 0^+$compensates exactly the exponential factors in Proposition 2.4. These facts are the subject matter of the present section, which is organized as follows:

– in Propositions 3.0 to 3.3, we construct the Minakshisundaram parametrix for the pseudo-heat operator $(\partial_t - \Delta_2^Y)$, which is given by the expansion (3.4) in powers of the time variable t; we emphasize the growth estimate (3.8) bearing on the coefficients of the expansion (3.4) which is essential in obtaining the behavior of the pseudo-heat kernel constructed as $t \to 0^+$.

– Theorem 3.4 states the main result of this section; its proof is given immediately after Proposition 3.10. The remaining Propositions and Lemmas in the section are intermediate steps used in the proof of Theorem 3.4 but of independent interest. We start with an expansion (3.14) analogous to (3.4) but with a floating truncation depending on the time variable "à la Sjöstrand" [Sj §1]. Lemma 3.6 shows that this expansion with the floating truncation almost satisfies the pseudo-heat equation. Proposition 3.7 studies the structure of the phase of the Gaussian factor in the parametrix (3.4) below. Lemma 3.8 studies the effect of removing the "floating truncation" in (3.14), while Proposition 3.9 studies the behavior of the approximate solution in (3.14) as $t \to 0^+$. Proposition 3.10 studies a particular quantity Q entering into the Duhamel formula when one passes from the approximate solution (3.14) to the pseudo-heat kernel K; it says essentially that, from the point of view of singularities as $t \to 0^+$, the kernel K behaves like the expansion (3.4). (In the following we denote the interval $[0, \infty)$ by $\mathbf{R}_+$ and $(0, \infty)$ by $\mathbf{R}_+^*$.)

PROPOSITION 3.0. – *For all $k \geq 0$, u_k defines a holomorphic germ at $(q, q) \in X \times X$ in $M_{\epsilon_1} \times M_{\epsilon_1}$ (still denoted u_k in the sequel). Consider, for all $k \geq 0$*

$$H_k(t, m, m') = (4\pi t)^{-n/2} e^{d^2(m,m')/4t} \sum_{l=0}^{k} (-t)^l u_l(m, m')\,; \tag{3.4}$$

H_k is C^ω on $\mathbf{R}_+^* \times U$ and defines, for all $t > 0$, a germ of holomorphic function at $(t, q, q) \in \mathbf{R}_+^* \times X \times X$ in $\mathbf{C} \times M_{\epsilon_1} \times M_{\epsilon_1}$ (still denoted by H_k). In particular, the restriction $H_k(t, \cdot, \cdot)|_{\mathbf{R}_+^* \times Y \times Y}$ induces, for all $t > 0$, a real-analytic germ at (q, q) in $Y \times Y$ which satisfies:

(3.5)
$$(\Delta_2^Y - \partial_t) H_k|_{\mathbf{R}_+^* \times Y \times Y}(t, m, m') = (4\pi t)^{-n/2} e^{d^2(m,m')/4t} (-t)^k \Delta_2^Y u_k|_{Y \times Y}(m, m')\,.$$

N.B. – Proposition 3.1 will show that we can shrink the connected manifold Y so that *all* the u_k are defined and C^ω on $Y \times Y$. Therefore, (3.5) will be valid on $\mathbf{R}_+^* \times Y \times Y$.

Proof. – In [B-G-M], pp. 204 and ff., it is proved that

$$(\Delta_2^X + \partial_t) \left[(4\pi t)^{-n/2} e^{-d^2(m,m')/4t} \sum_{l=0}^{k} t^l u_l(m, m') \right] = (4\pi t)^{-n/2} e^{-d^2(m,m')/4t} t^k \Delta_2^X u_k(m, m'), \tag{3.6}$$

for $t > 0$ and $(m, m') \in U$. By analytic continuation with respect to t in $\mathbf{C} \setminus \sqrt{-1}\,\mathbf{R}_-$ for (m, m') fixed, one can change t into $-t$ in (3.6) to get

$$(\Delta_2^X - \partial_t) \left[(4\pi t)^{-n/2} e^{d^2(m,m')/4t} \sum_{l=0}^{k} (-t)^l u_l(m, m') \right] = (4\pi t)^{-n/2} e^{d^2(m,m')/4t} (-t)^k \Delta_2^X u_k(m, m')\,, \tag{3.7}$$

for $t > 0$ and $(m, m') \in U$. Relation (3.5) follows from (3.7) and Theorem 1.16 by analytic continuation with respect to the variable m', (t, m) being fixed. $\square$

The key to controlling the growth of the pseudo-heat kernel as $t \to 0^+$ is the following estimate.

PROPOSITION 3.1. – *Let $\{u_k\}_{k=0}^{\infty}$ be the sequence of functions defined recursively by (3.2)–(3.3). There exists an open neighborhood V of q in M_{ϵ_1} and a positive constant $L > 0$ such that for all k, u_k is holomorphic on $V \times V$ and*

$$\sup_{V\times V} |u_k| \leq L^{k+1}k!. \tag{3.8}$$

The proof of (3.8) relies on some majorizing series techniques recalled below.

DEFINITION 3.2. – *Let $u(x) = \sum_{\alpha\in\mathbf{N}^n} u_\alpha x^\alpha$ and $v(x) = \sum_{\alpha\in\mathbf{N}^n} v_\alpha x^\alpha$ be two formal series with u_α complex and v_α non-negative real numbers. We shall denote by $u(x) << v(x)$ the following relation: for all $k \in \mathbf{N}^n$, $|u_\alpha| \leq v_\alpha$.*

The proof uses the following formal series: for $k \in \mathbf{N}$ and $R > 0$,

$$\phi_k^R(x) = \frac{R^{-k}k!}{(1-\tilde{x}/R)^{k+1}}\,, \text{ with } \tilde{x} = x_1 + \cdots + x_n\,.$$

We shall use the following properties of these series:

PROPOSITION 3.3. – *For all $k \in \mathbf{N}$ and $R > 0$,*

1) $\phi_k^R(x) << R\phi_{k+1}^R(x)$ and $\partial_{x_j}\phi_k^R(x) = \phi_{k+1}^R(x)$.

2) Let $x \mapsto b(x)$ be a holomorphic function on $P(0,R)$ (the open polydisk centered at 0 with radius $R > 0$) such that $\sup_{P(0,R)} |b| = M$. Then $b(x) << M\phi_0^R(x)$.

3) Let $R_0 > 2R$ and a a holomorphic function on $P(0,R_0)$ such that $\sup_{P(0,R)} |a| = M$. Let $C > 0$, $k \in \mathbf{N}$ and $u(x)$ be a formal series such that $u(x) << C\phi_k^R(x)$. Then $a(x)u(x) << 2MC\phi_k^R(x)$.

Proposition 3.3 is essentially the same as Proposition 5.3 in [Le], to which we refer for a complete proof.

Proof of Proposition 3.1. – Since the metric g is C^ω on X, the function θ defined in (3.1) is positive and C^ω on U. In particular, there exists a relatively compact open neighborhood W of q in M_{ϵ_1} and a constant $A > 0$ such that for all $m, m' \in W$, $A^{-1} \leq |\theta(m,m')| \leq A$ and θ is holomorphic on $W \times W$. Pick $R_0 \in]0,1[$ small enough, let $0 < 2R \leq R_0$ and $W_0 \subset W$ be a relatively compact open neighborhood of q in M_{ϵ_1} so that, for all $m \in W_0$, $\mathrm{Exp}_m(P(0,R_0)) \subset W$ and $W_0 \subset \mathrm{Exp}_m(P(0,R/2n))$.

Proposition 3.3 2) shows that for all $m \in W_0$ the formal series $u_0(m, \mathrm{Exp}_m(\xi))$ (in the unknown ξ) satisfies:

$$u_0(m, \mathrm{Exp}_m(\xi)) = \theta^{1/2}(m, \mathrm{Exp}_m(\xi)) << A^{1/2}\phi_0^R(\xi)\,.$$

Observe that $\theta^{1/2}(m, \mathrm{Exp}_m(\xi))(\Delta_2^X u_{i-1})(m, \mathrm{Exp}_m(\xi))$ is given by an expression of the form:

$$\theta^{1/2}(m, \mathrm{Exp}_m(\xi))(\Delta_2^X u_{i-1})(m, \mathrm{Exp}_m(\xi)) = \sum_{|\alpha|\leq 2} a_\alpha(m, \mathrm{Exp}_m(\xi))\partial_\xi^\alpha(u_{i-1}(m, \mathrm{Exp}_m(\xi))) \tag{3.9}$$

where we can assume that the coefficients a_α are holomorphic on $W \times W$, do not depend on u_{i-1}, and satisfy $|a_\alpha(m, m')| \le C'$ for all m, $m' \in W$, all α such that $|\alpha| \le 2$ and some constant $C' > 1$.

We now proceed by induction: assume that the induction hypothesis holds at order $i - 1$, *i.e.*

$$u_{i-1}(m, \mathrm{Exp}_m(\xi)) << A^{1/2} C^{i-1} \phi^R_{i-1}(\xi)\,, \tag{3.10}$$

where $C = 4A^{1/2}C''/R$, C'' being defined by $C'' = 2(n^2 + n + 1)C'$.

There are at most $n^2 + n + 1$ terms in the sum (3.9). Then, the induction hypothesis, Proposition 3.3 1) and 3) (using in particular that $0 < R < 1$) show that

$$\theta^{1/2}(m, \mathrm{Exp}_m(\xi))(\Delta^X_2 u_{i-1})(m, \mathrm{Exp}_m(\xi)) << A^{1/2} C^{i-1} C'' \phi^R_{i+1}(\xi)\,.$$

Hence

$$\begin{aligned}\int_0^1 \theta^{1/2}(m, \mathrm{Exp}_m(s\xi))\Delta_2 u_{i-1}(m, \mathrm{Exp}_m(s\xi)) s^{i-1}\, ds \\ << A^{1/2} C^{i-1} C'' \int_0^1 \phi^R_{i+1}(s\xi) s^{i-1}\, ds.\end{aligned} \tag{3.11}$$

We compute

$$\begin{aligned}&\int_0^1 \phi^R_{i+1}(s\xi) s^{i-1}\, ds \\ &= R^{-(i+1)} \sum_{p=0}^{\infty} \frac{(i+1+p)!}{p!} \left(\frac{\tilde{\xi}}{R}\right)^p \int_0^1 s^{i-1+p}\, ds \\ &= R^{-(i+1)} \sum_{p=0}^{\infty} \frac{(i+p)!}{p!} \frac{(i+1+p)}{(i+p)} \left(\frac{\tilde{\xi}}{R}\right)^p << \frac{2}{R}\phi^R_i(\xi).\end{aligned} \tag{3.12}$$

Therefore, (3.11) and (3.12) show that

$$\int_0^1 \theta^{1/2}(m, \mathrm{Exp}_m(s\xi))\Delta_2 u_{i-1}(m, \mathrm{Exp}_m(s\xi)) s^{i-1}\, ds << \frac{2A^{1/2}C^{i-1}C''}{R} \phi^R_i(\xi)\,. \tag{3.13}$$

We now apply Proposition 3.3 3) with $a(\xi) = -\theta^{-1/2}(m, \mathrm{Exp}_m(\xi))$ which is holomorphic on $P(0, R_0)$ and satisfies $|\theta^{-1/2}(m, \mathrm{Exp}_m(\xi))| \le A^{1/2}$ for all $\xi \in P(0, R_0)$. Therefore, according to (3.13),

$$\begin{aligned}&u_i(m, \mathrm{Exp}_m(\xi)) \\ &= -\theta^{-1/2}(m, \mathrm{Exp}_m(\xi)) \int_0^1 \theta^{1/2}(m, \mathrm{Exp}_m(s\xi))\Delta_2 u_{i-1}(m, \mathrm{Exp}_m(s\xi)) s^{i-1}\, ds \\ &<< 2A^{1/2} \frac{2A^{1/2}C^{i-1}C''}{R} \phi^R_i(\xi) = A^{1/2} C^i \phi^R_i(\xi),\end{aligned}$$

and hence (3.10) holds at order i. This shows that (3.10) holds for all $i \ge 1$.

By assumption, for all $m \in W_0$ we have: $W_0 \subset \mathrm{Exp}_m(P(0, R/2n))$. Thus (3.10) shows that, for all $m,\ m' \in W_0$ and $i \geq 1$,

$$\begin{aligned}|u_{i-1}(m, m')| &\leq A^{1/2} C^{i-1} \frac{R^{-(i-1)}(i-1)!}{(1-1/2)^i} \\ &= 2A^{1/2}\left(\frac{2C}{R}\right)^{i-1}(i-1)! \leq \sup\left(2A^{1/2}, \frac{2C}{R}\right)^i (i-1)!\end{aligned}$$

which proves (3.8) with $L = \sup\left(2A^{1/2}, \frac{2C}{R}\right)$.

The main result of this section is the following theorem, which states the existence of a pseudo-heat kernel $K(t, p, m)$ for $\Delta_2^Y - \partial_t$ on Y satisfying certain estimates. These estimates will be fundamental later, especially in the proof of the inversion formula in Theorem 0.3.

THEOREM 3.4. – *If ϵ_1 is small enough then, for all $0 \leq \epsilon < \epsilon_1$, there exists a complex-valued function $K \equiv K(t, p, m) \in C^0(\mathbf{R}_+^* \times Y \times Y)$ which is C^1 with respect to $t > 0$ and C^2 with respect to $m \in Y$ such that:*

1) *For any $p \in Y$, $(\partial_t - \Delta_2^Y)K(t, p, m) \equiv 0$;*

2) *For all continuous complex-valued function $u \equiv u(t, m) \in C^0([0, 1] \times \overline{Y})$*

$$\lim_{t \to 0^+} \int_Y u(t, m) K(t, q, m) \mu^Y(m) = u(0, q)\,;$$

3) *There exists $R > 0$ such that for any $t \in]0, 1]$ and $m \in Y$*

$$|K(t, q, m)| \leq R(4\pi t)^{-n/2} e^{d^2(q,m)/4t}\,;$$

4) *For all smooth vector fields U on $\overline{Y}$, all compact subsets $\mathcal{K}_0$ of Y and all $0 < \epsilon' < \epsilon$, there exists a positive constant $D(\mathcal{K}_0, \epsilon', U)$ such that for all $t \in]0, 1]$ and $m \in \mathcal{K}_0$ such that $d^2(q, m) = -\epsilon'^2$,*

$$|U \cdot K(t, q, m)| \leq D(\mathcal{K}_0, \epsilon', U) t^{-1-n/2} e^{-\epsilon'^2/4t}\,.$$

The proof of this theorem is somewhat involved and requires some technical preliminary results. Before going into this, we observe that in the case where $-g^Y$ is a Riemannian metric on Y, we can think of Y with the metric $-g^Y$ as isometrically embedded in some compact Riemannian manifold so that $(\sqrt{-1})^n K$ can be constructed as the restriction to Y of the heat kernel on that manifold, with one point equal to q (*see* comments following definition 8.7). In this very particular case, the estimate 3) above has been obtained by Molchanov [Mo] and Kannai [K] for C^∞ Riemannian manifolds. However, both proofs involve the geometrical objects attached to the Laplacian of a metric (and in particular geodesic curves) and does not seem to extend to the general case considered here, where $-\Delta^Y$ is a differential operator with complex coefficients. Our proof for this theorem is based on Proposition 3.1 instead (and hence requires analyticity of the metric).

CONVENTION 3.5. – In what follows, we shall assume that ϵ_1 is small enough that there is a holomorphic chart $(\Omega; z = x + \sqrt{-1}\,y)$ for M_{ϵ_1} at q such that Y is defined by $\Re z = x = 0$ and $(\Omega \cap Y; y)$ is a C^ω chart for Y at q.

Pick $C > 1$ such that $0 < \frac{L}{eC} < \frac{1}{2}$, where L is the constant in Proposition 3.1. Then we choose a real-valued function $\chi \in C^\infty(\mathbf{R})$ such that $\chi(t) = 0$ on $]-\infty, 0]$, $\chi(t) \in]0,1]$ on $]0, \frac{1}{2}]$, and $\chi(t) = 1$ on $[\frac{1}{2}, \infty[$. Finally, for $(x,y) \in V \times V$ (with the same V as in Proposition 3.1), we set

$$A(t,x,y) = M(t,x,y) e^{d^2(x,y)/4t} \tag{3.14}$$
$$\text{where} \quad M(t,x,y) = (-4\pi t)^{-n/2} \sum_{k\geq 0} (-t)^k u_k(x,y) \chi\left(\frac{1}{t} - kC\right)$$

This type of "floating" truncation was introduced by Sjöstrand [Sj §1]. The role of such a truncation is to stop the summation at $k \leq \frac{1}{Ct}$, for in general the series $\sum_{k\geq 0} t^k u_k(x,y)$ diverges–see estimate (3.8). That such a truncation is useful can be seen as follows. Denoting by $\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt$ the usual Eulerian function, we recall the Stirling formula: $\Gamma(1+x) \sim \sqrt{2\pi x}\,(x/e)^x$ as $x \to +\infty$. For $t \to 0^+$, the last term in the summation (3.14) can be estimated by

$$t^{-\frac{n}{2}+1/Ct} L^{1/Ct} \Gamma(1 + \tfrac{1}{Ct}) \sim t^{-\frac{n}{2}} \left(\frac{2\pi}{Ct}\right)^{1/2} \left(\frac{L}{eC}\right)^{1/Ct} = O(e^{-\eta/t}) \quad \text{as } t \to 0^+$$

for some $\eta > 0$.

The next lemma shows that $A(t,x,y)$ is nearly a solution of $(\partial_t - \Delta_2^Y) f(t,x,y) \equiv 0$.

LEMMA 3.6.

1) *Let* $k_t = \left[\frac{1}{C}\left(\frac{1}{t} - \frac{1}{2}\right)\right]$ *for all* $t > 0$ *(where* $[\cdot]$ *denotes the integer part). Then for all* $(x,y) \in Y \times Y$ *and* $t > 0$*:*

$$(\Delta_2^Y - \partial_t) A(t,x,y) = (-4\pi t)^{-n/2} (-t)^{k_t} e^{d^2(x,y)/4t} \Delta_2 u_{k_t}(x,y) + P(t,x,y) \tag{3.15}$$

where $P(t,x,y) = 0$ if $1 + k_t \geq \frac{1}{Ct}$, *and if* $k = 1 + k_t < \frac{1}{Ct}$,

$$P(t,x,y) = (-4\pi)^{-n/2} (\Delta_2^Y - \partial_t) \left[(-1)^k t^{k-n/2} e^{d^2(x,y)/4t} u_k(x,y) \chi\left(\frac{1}{t} - kC\right)\right].$$

2) *For* ϵ_1 *small enough, there exists* $\eta > 0$ *and a positive constant* D *such that for all* $\epsilon \in]0, \epsilon_1]$, *all multi-indices* $(\alpha, \beta) \in \mathbf{N}^n \times \mathbf{N}^n$ *of length* ≤ 4, *all integers* $a \in \{0,1\}$ *and all* $(t,x,y) \in \mathbf{R}_+^* \times Y \times Y$ *satisfying* $|d^2(x,y)| \leq \eta$ *the following estimate holds:*

$$\left|\partial_t^a \partial_x^\alpha \partial_y^\beta (\Delta_2^Y - \partial_t) A(t,x,y)\right| \leq D e^{-2\eta/t}. \tag{3.16}$$

N.B. – Estimate (3.16) holds only for $(x,y) \in Y \times Y$ such that $|d^2(x,y)| \leq \eta$ and not in general.

Proof of 1). – Observe that

$$M(t,x,y) = (-4\pi t)^{-n/2} \sum_{l=0}^{k_t} (-t)^l u_l(x,y) + (-4\pi t)^{-n/2} (-t)^{1+k_t} u_{1+k_t}(x,y) \chi(\tfrac{1}{t} - C(1+k_t)).$$

Indeed, according to the definition of χ: $\chi(\frac{1}{t} - Ck) = 1$ if and only if $\frac{1}{t} - Ck \geq \frac{1}{2}$

so that if $k \leq k_t$ then $\chi(\frac{1}{t} - Ck) = 1$ and if $k \geq 2 + k_t$, then $Ck > \frac{1}{t} - \frac{1}{2} + C > \frac{1}{t}$ and $\chi(\frac{1}{t} - Ck) = 0$. Moreover, if $1 + k_t \geq \frac{1}{Ct}$, then $\chi(\frac{1}{t} - C(1 + k_t)) = 0$ so that $M(t,x,y) = (-4\pi t)^{-n/2} \sum_{l=0}^{k_t} (-t)^l u_l(x,y)$ and the first formula in 1) results directly from (3.5). If $k = 1 + k_t < \frac{1}{Ct}$, the extra term in the formula above is not zero and the second formula in 1) follows again from (3.5) with the extra term $P(t,x,y)$.

Proof of 2). – In view of 1), $\partial_t^a \partial_x^\alpha \partial_y^\beta (\Delta_2^Y - \partial_t) A(t,x,y)$ will be a linear combination of terms of the form

$$e^{d^2(x,y)/4t} t^{\lambda+k-n/2} \partial_x^{\alpha_1} \partial_y^{\alpha_2} u_k(x,y)$$

with coefficients bounded on $\overline{Y}$ and $-8 \leq \lambda \leq 0$, $k = k_t$ or $k_t + 1$ and $|\alpha_1| + |\alpha_2| \leq 6$. Moreover, all k's appearing in the expression of $\partial_t^a \partial_x^\alpha \partial_y^\beta (\Delta_2^Y - \partial_t) A(t,x,y)$ should also satisfy $k \leq \frac{1}{Ct}$ because of the truncation $\chi(\frac{1}{t} - kC)$ in (3.14). Choosing $\epsilon_1 > 0$ small enough, one can apply Cauchy's estimate and (3.8) to bound $\partial_x^{\alpha_1} \partial_y^{\alpha_2} u_k$ uniformly on $\overline{Y} \times \overline{Y}$:

$$(3.17) \quad |\partial_x^{\alpha_1} \partial_y^{\alpha_2} u_k(x,y)| \leq C_{\alpha_1,\alpha_2} L^{k+1} k!\,, \quad \forall (\alpha_1, \alpha_2) \in \mathbf{N}^n \times \mathbf{N}^n\,,\ \forall (x,y) \in \overline{Y} \times \overline{Y}$$

(where C_{α_1,α_2} denotes some positive constant).

Let $C_0 = \sup\{C_{\alpha_1,\alpha_2} \text{ s.t. } |\alpha_1| + |\alpha_2| \leq 4\}$. Since $C > 1$, $k_t \geq \frac{1}{Ct} - 2$; hence all k's appearing in the expression of $\partial_t^a \partial_x^\alpha \partial_y^\beta (\Delta_2^Y - \partial_t) A(t,x,y)$ satisfy $k \in [\frac{1}{Ct} - 2, \frac{1}{Ct}]$. Then (3.17) shows that, for all $0 < t < 1$,

$$\forall -8 \leq \lambda \leq 0\,,\ \forall (\alpha_1, \alpha_2) \in \mathbf{N}^n \times \mathbf{N}^n \text{ s.t. } |\alpha_1| + |\alpha_2| \leq 4\,,\ \forall (x,y) \in \overline{Y} \times \overline{Y}$$

one has

$$\left| e^{d^2(x,y)/4t} t^{\lambda+k-\frac{n}{2}} \partial_x^{\alpha_1} \partial_y^{\alpha_2} u_k(x,y) \right| \leq \left| e^{d^2(x,y)/4t} \right| t^{\lambda - \frac{n}{2} + \frac{1}{Ct} - 2} C_0 L^{\frac{1}{Ct}+1} \Gamma(1 + \tfrac{1}{Ct})\,.$$

Applying Stirling's asymptotic equivalent recalled above as $t \to 0^+$ leads to

$$(3.18)$$

$$\begin{aligned} \left| e^{d^2(x,y)/4t} t^{\lambda+k-\frac{n}{2}} \partial_x^{\alpha_1} \partial_y^{\alpha_2} u_k(x,y) \right| &\leq C_1 \left| e^{d^2(x,y)/4t} \right| t^{\lambda-2-\frac{n}{2}} C^{-\frac{1}{Ct}} C_0 L^{\frac{1}{Ct}+1} \sqrt{\frac{2\pi}{Ct}}\, e^{-\frac{1}{Ct}} \\ &\leq C_2 L t^{\lambda-2-\frac{n+1}{2}} \left| e^{d^2(x,y)/4t} \right| \left(\frac{L}{eC} \right)^{\frac{1}{Ct}}, \end{aligned}$$

where C_1 and C_2 are positive constants. Now, $C > 1$ has been chosen so that $\frac{L}{eC} < \frac{1}{2}$; hence there exists a positive constant C_3 such that

$$\forall -8 \leq \lambda \leq 0\,,\ \forall (\alpha_1, \alpha_2) \in \mathbf{N}^n \times \mathbf{N}^n \text{ s.t. } |\alpha_1| + |\alpha_2| \leq 4\,,$$

$$\forall (x,y) \in \overline{Y} \times \overline{Y} \text{ s.t. } |d^2(x,y)| \leq \eta\,,$$

$$\begin{aligned} \left| e^{d^2(x,y)/4t} t^{\lambda+k-\frac{n}{2}} \partial_x^{\alpha_1} \partial_y^{\alpha_2} u_k(x,y) \right| &\leq C_3 t^{\lambda+k-\frac{n}{2}} \left| e^{d^2(x,y)/4t} \right| e^{-\frac{\log 2}{Ct}} \\ &\leq C_3 e^{\frac{\eta}{4t}} e^{-\frac{\log 2}{Ct}} \leq C_3 e^{-\frac{2\eta}{t}} \end{aligned}$$

whenever η is sufficiently small, e.g. $100\eta \leq \frac{\log 2}{C}$. Statement 2) follows easily from this last estimate. $\square$

The following proposition discusses the structure of $d^2(x,y)$ on $Y \times Y$; it will be used in particular to compute the limit of the pseudo-heat kernel as $t \to 0^+$.

PROPOSITION 3.7. – *For $\epsilon_1 > 0$ small enough, there exists a C^∞ function $h : Y \times Y \to \mathbf{R}^n$ such that*

1) $\forall y \in Y$, *$h(\cdot, y)$ is a diffeomorphism of Y onto its image. Moreover:*

$$\forall \xi \in T_q X, \text{ such that } |\xi| < \epsilon_1, \quad h(Exp_q(\sqrt{-1}\xi), q) = \xi ;$$

2) *For any $y \in Y$, $h(y,y) = 0 (= h(q,q))$;*

3) $\forall x, y \in Y$, *denoting $h = (h_1, ..., h_n)$, one has*

$$\Re d^2(x,y) = -\|h(x,y)\|^2 = -\sum_{i=1}^n h_i(x,y)^2 ;$$

4) *There exists C^∞ functions $c_{ij} \equiv c_{ij}(h,y)$ defined for all $1 \leq i,j \leq n$ on some open neighborhood of $h(Y \times Y) \times Y$ such that $c_{ij}(0,q) = 0$ for all $1 \leq i,j \leq n$ and*

$$\Im d^2(x,y) = \sum_{i,j=1}^n c_{ij}(h(x,y),y) h_i(x,y) h_j(x,y) .$$

Proof. – First observe that $\Re d^2 \in C^\infty(Y \times Y)$. For all $m \in X$ and $m' \in X$ close enough to m, one has $d^2(m',m) = g_m(\mathrm{Exp}_m^{-1}(m'), \mathrm{Exp}_m^{-1}(m'))$; therefore, by analytic continuation one has, for m, $m' \in Y$ both close enough to q,

$$d^2(m',m) = g_m(\mathrm{Exp}_m^{-1}(m'), \mathrm{Exp}_m^{-1}(m')) ; \text{ hence } d(d^2(m,\cdot))_m = 0 . \tag{3.19}$$

Moreover, when $m = q$ and ϵ_1 is small enough we can find for each m' in Y a unique $\xi \in T_q X$ near 0 such that $m' = \mathrm{Exp}_q(\sqrt{-1}\,\xi)$ and

$$d^2(\mathrm{Exp}_q(\sqrt{-1}\,\xi), q) = \Re d^2(\mathrm{Exp}_q(\sqrt{-1}\,\xi), q) = -g_q(\xi, \xi) . \tag{3.20}$$

Hence, for ξ, $\xi' \in T_q^{\mathbf{C}} X$ close enough to 0, $\partial^2_{\xi'} \Re d^2(\mathrm{Exp}_q(\sqrt{-1}\,\xi'), \mathrm{Exp}_q(\sqrt{-1}\,\xi))\big|_{\xi'=\xi}$ is negative definite. Applying the Morse lemma (with parameters) as stated in [C-P, p. 155] and reducing further, if necessary, the size of Y by choosing $\epsilon_1 > 0$ small enough leads to a map $(x,y) \to h(x,y)$ satisfying 1), 2) and 3).

As for point 4), for all $y \in Y$, $h(Y,y)$ is an open neighborhood of 0 in $\mathbf{R}^n$ and there exists by 1) and 2) a function $\psi(\cdot, y) \in C^\infty(h(Y,y))$ such that for all $x \in Y$, $\psi(h(x,y),y) = \Im d^2(x,y)$. One has, by (3.19) and (3.20)

$$\psi(0,y) = 0, \ \textit{see} \text{ points 1) and 2)}, \quad d(\psi(\cdot,y))_0 = 0 , \text{ see (3.19)}, \quad \psi(\cdot, q) \equiv 0 , \tag{3.21}$$

this last equality following from (3.20). Hence, Taylor's formula shows that, for all $y \in Y$ fixed,

$$\psi(z,y) = \tfrac{1}{2} \sum_{i,j=1}^n z^i z^j \int_0^1 (1-t) \partial_{z^i} \partial_{z^j} \psi(tz,y) dt .$$

Point 4) follows upon defining

$$c_{ij}(z,y) = \tfrac{1}{2}\int_0^1 (1-t)\partial_{z^i}\partial_{z^j}\psi(tz,y)dt\,, \quad 1 \le i,j \le n\,;$$

and observing that $\partial_{z^i}\partial_{z^j}\psi(0,q) = 0$ since $\psi(\cdot,q) \equiv 0$ (*see* the last equality in (3.21)). This shows that $c_{ij}(0,q) = 0$ for all $1 \le i,j \le n$. □

We shall also need the following estimate, which analyzes the singularity of $\partial_y A(t,x,y)$ as $t \to 0^+$.

LEMMA 3.8. – *For $\epsilon_1 > 0$ small enough, for all $t \in]0,1]$, $(x,y) \in Y \times Y$ and all multi-indices α, $\beta \in \mathbf{N}^n$:*

$$\left[\partial_x^\alpha \partial_y^\beta M(t,x,y) - (-4\pi t)^{-n/2}\sum_{k=0}^{n}\partial_x^\alpha\partial_y^\beta u_k(x,y)(-t)^k\right] \le D_{\alpha,\beta}t \tag{3.22}$$

(with M defined in (3.14)) where $D_{\alpha,\beta}$ is a positive constant.

Proof. – Pick $t^* = 1/((1+n)C)$ – we recall that $C > 1$ has been chosen so that $\frac{L}{eC} < \frac{1}{2}$; hence, for all $0 < t < t^*$ we have: $n+1 \le 1 + [\frac{1}{Ct}]$. For $0 < t < t^*$, one has
(3.23)

$$\left|\partial_x^\alpha\partial_y^\beta M(t,x,y) - (-4\pi t)^{-n/2}\sum_{k=0}^{n}\partial_x^\alpha\partial_y^\beta u_k(x,y)(-t)^k\right| \le t\sum_{k=n+1}^{1+[\frac{1}{Ct}]} t^{k-1-n/2}\left|\partial_x^\alpha\partial_y^\beta u_k(x,y)\right|.$$

By taking $\epsilon_1 > 0$ small enough as in the proof of Lemma 3.6, one can assume that (3.17) holds. Hence, for $n+1 \le k \le 1 + [\frac{1}{Ct}]$, one has $kt \le \frac{n+2}{C(n+1)}$ and

$$t^{k-1-n/2}|\partial_x^\alpha\partial_y^\beta u_k(x,y)| \le C_{\alpha,\beta}t^{k-1-n/2}L^{1+k}k!\,.$$

Using Stirling's asymptotic equivalent $k! \sim \sqrt{2\pi k}\,(k/e)^k$ as $k \to +\infty$ and the definition of C recalled above leads to

$$\begin{aligned} t^{k-1-n/2}\left|\partial_x^\alpha\partial_y^\beta u_k(x,y)\right| &\le C'_{\alpha,\beta}k^{\frac{n+3}{2}}(kt)^{k-1-n/2}\left(\frac{L}{e}\right)^k \\ &\le C'_{\alpha,\beta}k^{\frac{n+3}{2}}\left(\frac{n+2}{C(n+1)}\right)^{-\frac{n+2}{2}}\left(\frac{n+2}{2(n+1)}\right)^k \le C''_{\alpha,\beta}\left(\frac{3}{4}\right)^k \end{aligned}$$

for some constants $C'_{\alpha,\beta}$ and $C''_{\alpha,\beta}$. Combining this estimate with (3.23) shows that the series in the right side of (3.23) converges, which proves (3.22). □

PROPOSITION 3.9. – *For $\epsilon_1 > 0$ small enough and for any locally compact (topological) space Λ*

1) *There exists a positive constant D_1 such that for all $t \in \mathbf{R}_+^*$ and all $x \in Y$,*

$$\int_Y t^{-n/2}\left|e^{d^2(x,y)/4t}\mu^Y(y)\right| \le D_1\,.$$

2) *For every multi-index* $(\alpha, \beta) \in \mathbf{N}^n \times \mathbf{N}^n$ *there exists a complex-valued function* $c_{\alpha,\beta} \in C^0(\overline{Y})$ *(independent of the topological space* Λ*) such that for any* (λ, p') *in* $\Lambda \times \overline{Y}$*:*

$$\lim_{t \to 0^+} \int_Y Q(\lambda, y) e^{d^2(x,y)/4t} \partial_y^\alpha \partial_{p'}^\beta M(t, y, p') \mu^Y(y) = c_{\alpha,\beta}(p') Q(\lambda, p') \tag{3.24}$$

for all complex-valued $Q \in C_b^0(\Lambda \times \overline{Y})$ *(with* M *defined in (3.14), here* C_b^0 *denotes the space of continuous functions with bounded modulus), the convergence being locally uniform in* $(\lambda, p') \in \Lambda \times \overline{Y}$. Moreover, $c = c_{0,0} \in C^0(\overline{Y}) \cap C^2(Y)$ *and* $1/c$ *is bounded on* $\overline{Y}$*. Finally* $c_{0,0}(q) = 1$.

3) *For all* $p \in Y$ *and all complex-valued functions* $f \in C_b^0([0,1] \times Y)$

$$\lim_{t \to 0^+} \int_Y f(t, m) A(t, p, m) \mu^Y(m) = c(p) f(0, p). \tag{3.25}$$

Proof. – We begin with 1). Using Proposition 3.7, one has

$$\begin{aligned} \int_Y t^{-n/2} \left| e^{d^2(x,y)/4t} \mu^Y(y) \right| &= \int_Y t^{-n/2} e^{\Re d^2(x,y)/4t} |\mu^Y(y)| \\ &= \int_Y t^{-n/2} e^{-\|h(y,x)\|^2/4t} |\mu^Y(y)| . \end{aligned}$$

Again by Proposition 3.7, for all $x \in Y$, $y \mapsto h(y,x)$ induces a diffeomorphism of Y onto its image denoted by $h(Y,x)$; changing variables in the last integral above leads to

$$\begin{aligned} \int_Y t^{-n/2} \left| e^{d^2(x,y)/4t} \mu^Y(y) \right| &= \int_{h(Y,x)} t^{-n/2} e^{-\|z\|^2/4t} \left| h(\cdot, x)^* \mu^Y(z) \right| \\ &= \int_{h(Y,x)} t^{-n/2} e^{-\|z\|^2/4t} |\phi(z,x)| dz \end{aligned}$$

where the function $\phi(\cdot, x) : h(Y,x) \to \mathbf{C}$ is defined by $h(\cdot, x)^* \mu^Y = \phi(\cdot, x) dz^1 \wedge \ldots \wedge dz^n$; $\forall x \in Y$. Moreover, using Propositions 1.13 e] and 3.7 1] one sees easily that $\phi(0, q) = (\sqrt{-1})^n$.

One can see that ϕ is a C^∞ nowhere vanishing function on $\overline{Y} \times \overline{Y}$, after reducing the size of Y if necessary (*see* Proposition 3.7). In particular, ϕ is uniformly bounded on $\overline{Y} \times \overline{Y}$. Hence, the inequality claimed in 1) holds with $D_1 = (4\pi)^{n/2} \|\phi\|_\infty$.

Proof of 2). – Lemma 3.8 and the dominated convergence theorem show that one can replace $M(t,x,y)$ by $(-4\pi t)^{-n/2} \sum_{k=0}^n u_k(x,y)(-t)^k$ without affecting the result. Therefore, (3.24) follows from studying the limit as $t \to 0^+$ of a finite sum of terms of the form $(-t)^k$ times

$$\int_Y Q(\lambda, y) \, (-4\pi t)^{-n/2} e^{d^2(x,y)/4t} u(x,y) \mu^Y(y) \tag{3.26}$$

with $u \in C^0(\overline{Y} \times \overline{Y})$. Changing variables as in the proof of 1), keeping the same notations and using Proposition 3.7 1] one transforms the integral (3.26) into

$$\int_{h(Y,x)} Q(\lambda, h^{-1}(z,x)) \, (-4\pi t)^{-n/2} e^{\left(-\|z\|^2 + \sqrt{-1} \sum c_{ij}(z,x) z_i z_j\right)/4t} u(x, h^{-1}(z,x)) \phi(z,x) \, dz.$$

Let $\delta > 0$ be small enough so that $B(0,\delta) \subset \bigcap_{x\in Y} h(Y,x)$. We split the above integral as $I_1(t) + I_2(t)$ with I_1 the integral over $B(0,\delta)$ and I_2 the integral over the complementary region in $h(Y,x)$. One has obviously

(3.27)
$$|I_2(t)| \leq (4\pi t)^{-n/2} e^{-\delta^2/4t} \int_{h(Y,x)\cap B(0,\delta)^c} \left|Q(\lambda, h^{-1}(z,x))u(x,h^{-1}(z,x))\phi(z,x)\right| dz \to 0$$

as $t \to 0^+$. Then, in I_1 we perform the change of variables $z = \sqrt{t}\,\zeta$ leading to

$$(\sqrt{-1})^n\, I_1 = \int_{B(0,\delta/\sqrt{t})} (4\pi)^{-n/2} Q(\lambda, h^{-1}(\sqrt{t}\,\zeta, x)) e^{\left(-\|\zeta\|^2 + \sqrt{-1}\sum c_{ij}(\sqrt{t}\,\zeta,x)\zeta_i\zeta_j\right)/4} u(x, h^{-1}(\sqrt{t}\,\zeta, x))\phi(\sqrt{t}\,\zeta, x)\, d\zeta\,.$$

Let $\mathcal{K} \subset \Lambda$ be any compact set; for all $R > 0$, the function

$$(\lambda, t, x, \zeta) \mapsto e^{\sqrt{-1}\sum c_{ij}(\sqrt{t}\,\zeta,x)\zeta_i\zeta_j/4} Q(\lambda, h^{-1}(\sqrt{t}\,\zeta,x))u(x,h^{-1}(\sqrt{t}\,\zeta,x))\phi(\sqrt{t}\,\zeta,x)$$

is continuous on the compact set $\mathcal{K}\times[0,1]\times\overline{Y}\times\overline{B}(0,R)$ and hence uniformly continuous. Hence, as $t \to 0^+$, $I_1(t) \to$

$$\begin{aligned}(3.28)\qquad &(\sqrt{-1})^{-n}\int_{\mathbf{R}^n}(4\pi)^{-n/2}Q(\lambda,h^{-1}(0,x))\\ &\quad\times e^{\left(-\|\zeta\|^2+\sqrt{-1}\sum c_{ij}(0,x)\zeta_i\zeta_j\right)/4}u(x,h^{-1}(0,x))\phi(0,x)\,d\zeta\\ &=(\sqrt{-1})^{-n}\,Q(\lambda,x)u(x,x)\phi(0,x)\\ &\quad\times\int_{\mathbf{R}^n}(4\pi)^{-n/2}e^{\left(-\|\zeta\|^2+\sqrt{-1}\sum c_{ij}(0,x)\zeta_i\zeta_j\right)/4}\,dz,\end{aligned}$$

uniformly on $\mathcal{K}\times\overline{Y}$, since $h^{-1}(0,x) = x$. Hence

$$(3.29)\qquad I_1(t) + I_2(t) \to Q(\lambda,x)u(x,x)c_{0,0}(x)$$

uniformly on $\mathcal{K}\times\overline{Y}$ as $t \to 0^+$, where $c_{0,0}(x)$ is defined by:

$$(3.30)\qquad c_{0,0}(x) = (\sqrt{-1})^{-n}\,\phi(0,x)\int_{\mathbf{R}^n}(4\pi)^{-n/2}e^{\left(-\|\zeta\|^2+\sqrt{-1}\sum c_{ij}(0,x)\zeta_i\zeta_j\right)/4}\,d\zeta\,.$$

The convergence of (3.24) follows immediately and is locally uniform on $\Lambda\times\overline{Y}$ since Λ is locally compact. Let us recall that $u_0(x,x) = 1$, where $u_0(\cdot,\cdot)$ is the first coefficient of the formal solution of the heat equation. The formula (3.30) shows that $c_{0,0} \in C^0(\overline{Y}) \cap C^2(Y)$. According to Proposition 3.7 4), one has $c_{ij}(0,q) = 0$; hence $c_{0,0}(q) = (\sqrt{-1})^{-n}\,\phi(0,q) = 1$ since we have seen in the beginning of the proof that $\phi(0,q) = (\sqrt{-1})^n$. Therefore, reducing $\epsilon_1 > 0$ if necessary leads to $c_{0,0}(x) \neq 0$ for all $x \in \overline{Y}$ and this completes the proof of 2).

Proof of 3). – Applying (3.29) to the situation where $\Lambda = [0,1]$ shows that

$$\int_Y f(\lambda, y)(4\pi t)^{-n/2} e^{d^2(x,y)/4t} u(x,y)\, d\mu^Y(y) \to f(\lambda, x) c_{0,0}(x) u(x,x)$$

as $t \to 0^+$, uniformly with respect to $(\lambda, x) \in [0,1] \times \overline{Y}$. Therefore

$$\lim_{t\to 0^+} \int_Y f(t,y)(4\pi t)^{-n/2} e^{d^2(x,y)/4t} u(x,y)\, d\mu^Y(y) = f(0,x) c_{0,0}(x) u(x,x). \tag{3.31}$$

The conclusion follows again from (3.31), Lemma 3.8 and the dominated convergence theorem which show that one can replace $A(t,x,y)$ by

$$(-4\pi t)^{-n/2} e^{d^2(x,y)/4t} \sum_0^n u_k(x,y)(-t)^k,$$

which leads to a finite sum of terms as in the left side of (3.31). This completes the proof of Proposition 3.9. □

Let $f,\, g \in C_b^0([0,T] \times Y \times Y)$ for all $T > 0$; we define the following (noncommutative) convolution product

$$f * g(t,x,y) = \int_Y \int_0^t f(\theta, x, z) g(t-\theta, z, y)\, d\theta\, d\mu^Y(z)\,,$$

for all $(t,x,y) \in \mathbf{R}_+^* \times Y \times Y$, which is associative. Then we set

$$B(t,x,y) = (\partial_t - \Delta_2^Y) A(t,x,y)\,.$$

PROPOSITION 3.10. – *Let η be the positive constant defined in Lemma 3.6 2. There exists a positive constant C_1 such that, for small enough, for all multi-indices $(\alpha, \beta) \in \mathbf{N}^n \times \mathbf{N}^n$ of length ≤ 4 and any integer $a \in \{0,1\}$*

1] $\forall k \in \mathbf{N}^*$, $\forall (t,x,y) \in \mathbf{R}_+^* \times Y \times Y$

$$\left| \partial_t^a \partial_x^\alpha \partial_y^\beta \left(\frac{B}{1 \otimes c} \right)^{*k} \right| (t,x,y) \leq C_1^k \frac{t^{k-1}}{(k-1)!} e^{-2\eta/t};$$

2] $\forall (t,x,y) \in \mathbf{R}_+^* \times Y \times Y$, $|\partial_t^a \partial_x^\alpha \partial_y^\beta Q|(t,x,y) \leq C_1 e^{C_1 t - 2\frac{\eta}{t}}$, *where*

$$Q = \sum_{k\geq 1} (-1)^{k+1} \left(\frac{B}{1 \otimes c} \right)^{*k}.$$

In particular, $\partial_t^a \partial_x^\alpha \partial_y^\beta Q$ extends as a continuous function on $\mathbf{R}^+ \times \overline{Y} \times \overline{Y}$.

Proof. – Point 2) obviously follows from point 1). To prove point 1), we observe that the case $k = 1$ is a consequence of Lemma 3.6 2) and Proposition 3.9 (which states that $1/c$ is of class C^2 on $\overline{Y}$, reducing $\epsilon_1 > 0$ if necessary). In particular Lemma 3.6 shows that $\partial_t^a \partial_x^\alpha \partial_y^\beta B$ can be continuously extended by 0 for $t = 0$. Hence, point 1) follows by induction on k from taking derivatives under the integral sign in the definition of $(B/1\otimes c)^{*(k+1)}$ as $(B/1\otimes c)^{*k} * (B/1\otimes c)$ and using the estimate of Lemma 3.6 2) and the regularity of $1/c$ stated in Proposition 3.9 2). □

With all these preliminary results (Lemmas and Propositions 3.6 to 3.10) at our disposal, we can now give the

Proof of Theorem. 3.4 – We define a function K by

$$K(t,p,m) = (A - Q * A)(t,p,m)\,, \quad \forall t>0\,,\ p\,,\ m\in \overline{Y}$$

with A and Q defined respectively in (3.14) and Proposition 3.10 above, and $\epsilon_1 > 0$ small enough so that all Lemmas and Propositions 3.6 to 3.10 hold and $|d^2(m',m)| \le \eta$ for all $m, m' \in Y$, where $\eta > 0$ is the (real) constant defined in Lemma 3.6. Now we show that K satisfies all the conditions stated in Theorem 3.4.

First we show that $K \in C^0(\mathbf{R}_+^* \times \overline{Y} \times \overline{Y})$. Let $\delta > 0$ and consider, for $t > \delta$

$$u_\delta(t,p,p') = \int_0^{t-\delta} d\theta \int_Y Q(\theta,p,y)A(t-\theta,y,p')\mu^Y(y) \tag{3.32}$$

which is continuous with respect to $(t,p,p') \in]\delta,+\infty[\times Y\times Y$. Using Proposition 3.10 2] (for Q) and Proposition 3.9 1], one can see that $u_\delta \to Q*A$ as $\delta \to 0^+$ uniformly on every compact subset of $\mathbf{R}_+^* \times Y \times Y$. Hence $K \in C^0(\mathbf{R}_+^* \times \overline{Y} \times \overline{Y})$.

Then we show that $K(t,p,m)$ is C^1 with respect to $t>0$. Indeed, changing the variable θ in the integral (3.32) into $t-\tau$ and using the identity $Q(0,p,y)\equiv 0$ (*see* Proposition 3.10 2]) gives

$$\partial_t u_\delta(t,p,p') = \int_\delta^t d\tau \int_Y \partial_t Q(t-\tau,p,y)A(\tau,y,p')\mu^Y(y)\,.$$

Propositions 3.9 1] and 3.10 2] show again that $t\mapsto \partial_t u_\delta(t,p,p')$ converges uniformly on every compact subset of $\mathbf{R}_+^*$ as $\delta\to 0^+$. Hence K is C^1 with respect to $t>0$.

Next we prove that $K(t,p,m)$ is C^1 with respect to m. Consider a fixed index $1\le j\le n$;

$$\partial_{p'_j} u_\delta(t,p,p') = \int_0^{t-\delta} d\theta \int_Y Q(\theta,p,y)\partial_{p'_j}A(t-\theta,y,p')\mu^Y(y) \tag{3.33}$$

(we recall Convention 3.5). By symmetry of d^2 in the p' and y variables, one has

$$\partial_{p'_j} e^{d^2(y,p')/4(t-\theta)} = \partial_{y_j} e^{d^2(y,p')/4(t-\theta)}\,.$$

Integrating by parts and using the definition of M in (3.14) shows that

$$\begin{aligned}\partial_{p'_j} u_\delta(t,p,p') = &\int_0^{t-\delta} d\theta \int_Y Q(\theta,p,y)e^{d^2(y,p')/4(t-\theta)}\partial_{p'_j}M(t-\theta,y,p')\mu^Y(y)\\ &-\int_0^{t-\delta} d\theta \int_Y \partial_{y_j}[Q(\theta,p,y)M(t-\theta,y,p')]e^{d^2(y,p')/4(t-\theta)}\mu^Y(y)\\ &+\int_0^{t-\delta} d\theta \int_{\partial Y} Q(\theta,p,y)M(t-\theta,y,p')e^{d^2(y,p')/4(t-\theta)}\iota_{\partial y_j}\mu^Y(y)\\ &-\int_0^{t-\delta} d\theta \int_Y Q(\theta,p,y)M(t-\theta,y,p')L_{\partial_{y_j}}\mu^Y\end{aligned} \tag{3.34}$$

where $L_{\partial_{y_j}}$ denotes the Lie derivative. Let $\mathcal{K}_0$ be a compact subset of Y; then, Proposition 3.7 shows the existence of a real number $\kappa > 0$ such that for all $(y, p') \in \partial Y \times \mathcal{K}_0$, $\Re d^2(y, p') \leq -\kappa < 0$. Therefore, Propositions 3.7 and 3.8 demonstrate that, for all $t > 0$, $\partial_{p'_j} u_\delta(t, \cdot, \cdot)$ converges uniformly on $\mathcal{K}_0 \times \mathcal{K}_0$ as $\delta \to 0^+$. In particular, $p' \mapsto Q * A(t, p, p')$ is C^1 and so is $m \mapsto K(t, p, m)$. In the same way, one shows that $p' \mapsto Q * A(t, p, p')$ and $m \mapsto K(t, p, m)$ are C^2 on Y.

Now we prove that $(\partial_t - \Delta_2^Y)K(t, p, m) \equiv 0$. For $\delta > 0$, we compute $\partial_t u_\delta$ by (3.32):

$$(3.35) \quad (\partial_t - \Delta_2^Y)u_\delta(t, p, p') = \int_0^{t-\delta} d\theta \int_Y Q(\theta, p, y)(\partial_t - \Delta_2^Y)A(t - \theta, y, p')\mu^Y(y) + \int_Y Q(t - \delta, p, y)A(\delta, y, p')\mu^Y(y).$$

By Lemma 3.6 2), one has

$$\left|(\partial_t - \Delta_2^Y)A(t - \theta, y, p')\right| \leq De^{-2\eta/(t-\theta)};$$

this estimate shows that the first integral in the right side of (3.35) converges uniformly on compact sets of $\mathbf{R}_+^* \times Y \times Y$ as $\delta \to 0^+$. Proposition 3.9 2) shows that the second integral in the right side of (3.35) converges to $c(p')Q(t, p, p')$ as $\delta \to 0^+$, uniformly on compact sets of $\mathbf{R}_+^* \times Y \times Y$. This shows in particular that $(-\Delta_2^Y + \partial_t)u_\delta(t, p, p')$ converges as $\delta \to 0^+$ uniformly on compact sets of $\mathbf{R}_+^* \times Y \times Y$. Also, $u_\delta \to Q * A$ as $\delta \to 0^+$ uniformly on compact sets of $\mathbf{R}_+^* \times Y \times Y$; then, using (3.35) again with Propositions 3.9 2) with $\delta \to 0^+$ in place of t and $\lambda = (t, p)$ and Proposition 3.10 with $B = (\partial_t - \Delta_2^Y)A$ leads to

$$\begin{aligned} \left[(\partial_t - \Delta_2^Y)(Q * A)\right](t, p, p') &= Q * (\partial_t - \Delta_2^Y)A(t, p, p') + c(p')Q(t, p, p') \\ &= \sum_{k \geq 1}(-1)^{k+1}\left(\frac{B}{1 \otimes c}\right)^{*k} * B(t, p, p') + c(p')\sum_{k \geq 1}(-1)^{k+1}\left(\frac{B}{1 \otimes c}\right)^{*k}(t, p, p') \\ &= B(t, p, p'). \end{aligned}$$

Then

$$(\partial_t - \Delta_2^Y)(A - Q * A) \equiv (\partial_t - \Delta_2^Y)K \equiv 0.$$

Next we prove point 4) in Theorem 3.4. In view of Lemma 3.8,

$$\sup_{\substack{t \in]0,1],\, m \in Y \\ d^2(q,m) = -\epsilon'^2}} t^{1+n/2}e^{\epsilon'^2/4t}|U \cdot A(t, p, m)| < +\infty.$$

Therefore, it suffices to prove that for all compact $\mathcal{K}_0 \subset Y$,

$$(3.36) \quad \sup_{\substack{t \in]0,1],\, p' \in \mathcal{K}_0 \\ d^2(q,p') = -\epsilon'^2}} e^{2\eta/t}|\partial_{p'_j}(Q * A)(t, p, p')| < +\infty$$

for all $\epsilon' > 0$ such that $\epsilon'^2 \leq \eta$.

To do this, we use first (3.34) with $\delta = 0$ and estimate Q and its derivatives by Proposition 3.10 2], while M and its derivatives are estimated by Lemma 3.8. It suffices then to apply the estimate in Proposition 3.9 1] to control the two first integrals in the right side of (3.34) with $\delta = 0$. To deal with the third integral in (3.34), observe as we did just after (3.34) that Proposition 3.7 shows the existence of a real number $\kappa > 0$ such that for all $(y, p') \in \partial Y \times \mathcal{K}_0$, $\Re d^2(y, p') \leq -\kappa < 0$. This third integral in (3.34) is then estimated using Lemma 3.8 and Proposition 3.10; these estimates altogether prove (3.36).

The proof of point 3) follows the same lines: it suffices to show that

$$\sup_{t\in]0,1]\,,\, m,m'\in Y} e^{2\eta/t}|(Q * A)(t, m, m')| < +\infty \tag{3.37}$$

observing that $|d^2(q, m)| \leq \eta$ for all $m \in Y$ (*see* the N.B. after Lemma 3.6 and remember that $\epsilon_1 > 0$ has been chosen small enough that for all m, $m' \in Y$, one has $|d^2(q, m)| \leq \eta$). Hence (3.37) follows from Propositions 3.10, Lemma 3.8 and Proposition 3.9 1].

Finally, point 2) follows from Proposition 3.9 3], the definition of K and the inequality (3.37) above.

The proof of Theorem 3.4 is complete. □

4. Characterization of the Analytic Wave Front Set

In this section we will give the proof of Theorem 0.1. The strategy of the proof is to use the estimates given in Proposition 3.1 to show that $e^{-t\Delta}u$ defines a F.B.I. transformation in the sense of Sjöstrand ([Sj]) near the point $(\mathrm{Exp}_{x_0}(\sqrt{-1}\,\xi_0), x_0)$ modulo an exponentially decreasing term (*see* Prop. 4.11 and 4.12). With the help of Gauss's lemma we will show that $\frac{1}{2}\sqrt{-1}\,d^2(x, y)$ is a F.B.I. phase and that the value at $\mathrm{Exp}_{x_0}(\sqrt{-1}\,\xi_0)$ of the associated weight is $\frac{1}{2}|\xi_0|^2 = -d^2(\mathrm{Exp}_{x_0}(\sqrt{-1}\,\xi_0), x_0)$.

Let us recall that (X, g) is a connected compact Riemannian real-analytic orientable manifold of dimension n with volume form μ, and that $L^2(X)$ is naturally endowed with a scalar product: $\langle f, g\rangle = \int_X f\bar{g}\mu$. Let $\{\phi_j(y)\}_{j\geq 0}$ be an orthonormal basis of $L^2(X)$ consisting of eigenfunctions of the (nonnegative) Laplace Beltrami operator, Δ, with corresponding eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$. We can assume that all the ϕ_j are real valued. We will choose $\epsilon > 0$ small enough that for any distribution u and any $t > 0$, $e^{-t\Delta}u(x)$ admits a holomorphic extension to the complex tube

$$M_\epsilon = \{\mathrm{Exp}_x(\sqrt{-1}\,\xi),\ \xi \in T_xX,\ |\xi| < \epsilon\}$$

(*see* Proposition 2.4), and small enough that the square of the geodesic distance, $d^2(\cdot,\cdot)$, admits a holomorphic extension, still denoted $d^2(\cdot,\cdot)$, to an open neighborhood in $M_\epsilon \times M_\epsilon$ of the diagonal of $X \times X$.

We recall that the formal solution of the heat equation is a formal series of the form

$$(4\pi t)^{-n/2} e^{-d^2(x,y)/4t} \left(\sum_{j\geq 0} u_j(x, y)t^j\right)$$

where the coefficients u_j, given by (3.2), (3.3), are analytic in an open neighborhood in $X \times X$ of the diagonal and satisfy, for any nonnegative integer k, the following equation (*see* [B-G-M], page 208):

$$(4.0)_k \qquad (\partial_t + \Delta_2)\left\{(4\pi t)^{-n/2} e^{-d^2(x,y)/4t}\left(\sum_{j=0}^{k} u_j(x,y)t^j\right)\right\} = t^k (4\pi t)^{-n/2} e^{-d^2(x,y)/4t} \Delta_2 u_k(x,y)$$

In the sequel we will use $\widetilde{C}$ to denote various positive constants.

We make the following remarks concerning Theorem 0.1:

a) Part i) means that for $y \in X$ close to x_0 the maximum of $-\Re d^2(\mathrm{Exp}_{x_0}(\sqrt{-1}\,\xi_0), y)$ is attained for $y = x_0$ so that for $y \in X$ close to x_0 and any $t \in]0,1]$ we have:

$$|E(t, \mathrm{Exp}_{x_0}(\sqrt{-1}\,\xi_0), y)| \leq \widetilde{C}\, t^{-n/2} \exp\left(\frac{|\xi_0|^2}{4t}\right).$$

b) Part (ii) is essentially a consequence of i). Let us recall (*see* Proposition 2.4), that without any hypothesis on the wave front set of u we have "only" the following estimate: for $t > 0$ small and x close to $\mathrm{Exp}_{x_0}(\sqrt{-1}\,\xi_0)$,

$$|e^{-t\Delta}u(x)| \leq \widetilde{C} \exp\left(\frac{|\xi_0|^2 + \delta}{4t}\right).$$

LEMMA 4.1. – *It suffices to prove Theorem 0.1 ii) in the case that u is a bounded measurable function on X.*

Proof. – It follows easily from Proposition 2.2 that if v is a distribution on X there is a nonnegative integer p such that $\Delta^{-p}\big(v - \langle v, 1\rangle\big) = u$ defines a bounded *function* on X. Note that u and v have the same analytic wave front set. If $Tu(t,m)$ satisfies $(\star)$ then, using Cauchy's inequalities for m in a suitable polydisc of Z, one sees easily that $T(v - \langle v, 1\rangle)$ and Tv satisfy an estimate like (*). In order to prove the converse, we introduce the following notation: for any function $f(t) \in S(R_+^*)$ with rapid decay near $+\infty$ we set for $t \in \mathbf{R}_+^*$

$$D^{-1}f(t) = \int_t^{+\infty} f(s)\,ds \text{ and } D^{-p}f(t) = D^{-1} \circ \cdots \circ D^{-1} f(t).$$

Conversely, if Tv satisfies $(\star)$ then so does $T(v - \langle v, 1\rangle)$, and one easily shows that $Tu = D^{-p}\big(T(v - \langle v, 1\rangle)\big)$ satisfies an estimate like $(\star)$. □

Now we introduce a few geometrical constants. Let L be the constant introduced in Proposition 3.1.

GEOMETRICAL. DEFINITION 4.2. – Let $C > 1$ be a real number such that $L/Ce < \frac{1}{2}$. Let us denote $\nu = \frac{1}{2C}\log(eC/L)$, which is positive. Let β be a smooth function of compact support whose (small) support is included in both $V \cap X$ and in the domain of a real analytic normal geodesic chart, and such that $\beta \equiv 1$ on an open neighborhood Ω of x_o in X. Then we can choose $\alpha \in C_0^\infty(M_\epsilon)$ whose (very small) support is included in V such that $(\operatorname{supp}\alpha) \cap X \subset \Omega$, $\alpha \equiv 1$ on an open connected neighborhood W of x_0 in $V \subset M_\epsilon$ and such that:

$$(x,y) \in \operatorname{supp}\alpha \times \operatorname{supp}\beta \Rightarrow \Re d^2(x,y) \geq -\nu \tag{4.1}$$

$$\inf_{(x,y)\in \operatorname{supp}\alpha\times(\operatorname{supp}\beta\setminus\Omega)} \Re d^2(x,y) := \tilde{\nu} > 0. \tag{4.2}$$

Moreover we can assume that for any bounded continuous function u with support in $V \cap X$ and any $x \in V \cap X$ we have (*see* [B-G-M], pages 208-210):

$$\lim_{t\to 0^+} \int_X u(y)(4\pi t)^{-n/2} e^{-d^2(x,y)/4t} \sum_{k=0}^{n+1} u_k(x,y) t^k \mu(y) \;=\; u(x). \tag{4.3}$$

Note. – The support of α is very small with respect to ν and Ω.

Now we define an analytic symbol $N(t,x,y)$ (the one of Theorem 0.1 i)!) with respect to the great parameter $\frac{1}{t}$.

DEFINITION 4.3. – *Let $\chi : R \to R$ a real C^∞ function of the real variable s such that $\chi(s) \equiv 0$ on $]-\infty, 0]$, $\chi(s) \in]0,1[$ on $]0,\frac{1}{2}[$, and $\chi(s) \equiv 1$ on $]\frac{1}{2}, +\infty[$ (as in §3). For real positive t and $(x,y) \in V \times V$ we define*

$$N(t,x,y) = (4\pi t)^{-n/2} \sum_{k\geq 0} t^k u_k(x,y) \chi(\frac{1}{t} - kC).$$

N is of course very similar to the M in equation (3.18). For each $t > 0$ the sum above is finite since the terms in it vanish if k is not smaller than $\frac{1}{Ct}$, and $(x,y) \to N(t,x,y)$ is holomorphic on $V \times V$. Moreover, using Proposition 3.1 and Stirling's formula we see easily that the nonvanishing terms (those for which $t < \frac{1}{Ck}$) can be estimated with the help of the following inequality:

$$t^k L^{k+1} k! \leq (Ck)^{-k} L^{k+1} k! \leq \widetilde{C}\sqrt{k}\,(L/Ce)^k.$$

The **Crucial Fact** behind this is that for $k = \frac{1}{Ct} \to +\infty$ we have (with the ν of the Definition 4.2):

$$t^k L^{k+1} k! \sim \widetilde{C}\, t^{-\frac{1}{2}} \exp\left(-\frac{2\nu}{t}\right).$$

Using the fact that $\frac{L}{Ce} < \frac{1}{2}$ one proves easily the following lemma (*see* also the end of Prop. 4.9).

LEMMA 4.4. – *There is a positive constant C' such that for any $t \in]0,1]$ and any $(x,y) \in V \cap X \times V \cap X$ we have:*

$$\left| \left\{ N(t,x,y) - (4\pi t)^{-n/2} \sum_{k=0}^{n+1} u_k(x,y) t^k \right\} e^{-d^2(x,y)/4t} \right| \leq C't.$$

Philosophically the following lemma analyzes the second member of the equation $(4.0)_k$ **for** $k \sim \frac{1}{Ct}$ (associated with the formal solution of the heat equation); later we will see that this second member is exponentially decreasing when $t \to 0^+$. The proof is similar to the proof of Lemma 3.6.

LEMMA 4.5. – *For each real $t > 0$, and each $(x,y) \in V \times (V \cap X)$ we put $k_t = [(\frac{1}{t} - \frac{1}{2})C^{-1}]$ (the integer part) and we have:*

$$\begin{aligned}(\partial_t + \Delta_2)\big(N(t,x,y)e^{-d^2(x,y)/4t}\big) \\ = (4\pi t)^{-n/2} t^{k_t}\, e^{-d^2(x,y)/4t} \Delta_2 u_{k_t}(x,y) + R(t,x,y)\end{aligned}$$

where $R(t,x,y)$ equals 0 if $1 + k_t \geq \frac{1}{Ct}$, and if $k = 1 + k_t < \frac{1}{Ct}$ then:

$$R(t,x,y) = (4\pi)^{-n/2}\big(\partial_t + \Delta_2\big)\left\{ u_k(x,y) t^{k-\frac{n}{2}} e^{-d^2(x,y)/4t} \chi(\frac{1}{t} - kC)\right\}.$$

Now we write a decomposition of the "analytic symbol" $N(t,x,y)$ with respect to the orthonormal basis of eigenfunctions $\{\phi_j\}_{j\geq 0}$:

$$\alpha(x)\beta(y)N(t,x,y)e^{-d^2(x,y)/4t} = \sum_{j\geq 0} f_j(t,x)\phi_j(y)$$

where we have:

$$f_j(t,x) = \alpha(x) \int_X \beta(y) N(t,x,y) e^{-d^2(x,y)/4t} \phi_j(y)\mu(y). \tag{4.4}$$

The idea of the proof of Theorem 0.1 i) is to analyze $e^{-t\lambda_j}\phi_j(x) - f_j(t,x)$. To this aim, we will use the following Lemma 4.6 and Proposition 4.8.

LEMMA 4.6. – *There exists n smooth vector fields U_h, $1 \leq h \leq n$, on X and n differential operators of the first order V_l, $1 \leq l \leq n$, (operating on the variable y) such that for each $x \in M_\epsilon$, $j \in N$, and $t > 0$ we have:*

$$\begin{aligned}-2\alpha(x) \int_X \langle \text{grad } \beta(y), \text{grad } \phi_j(y)\rangle N(t,x,y) e^{-d^2(x,y)/4t}\mu(y) \\ = \alpha(x) \int_X \phi_j(y) \Big(\sum_{1\leq h,l \leq n} V_l \cdot [N(t,x,y)e^{-d^2(x,y)/4t} U_h \cdot \beta(y)]\Big)\mu(y).\end{aligned}$$

Proof. – Since the (compact) support of β is included in the domain of a real analytic chart, one gets the result by a simple integration by parts. □

DEFINITION 4.7. – *Let us denote*

$$F(t,x,y) = \alpha(x)(N(t,x,y)e^{-d^2(x,y)/4t}\Delta_2\beta(y) + \sum_{1\le h,l\le n} V_l\cdot[N(t,x,y)e^{-d^2(x,y)/4t}U_h\cdot\beta(y)]).$$

The map $(t,x,y)\to F(t,x,y)$ is of class C^∞ on $R_+^*\times V\times X$. For each $(t,y)\in R_+^*\times X$, $x\to F(t,x,y)$ is holomorphic on W.

PROPOSITION 4.8. – *For all $j\in N$, all x in X, and all real $t>0$ we have:*

a) $f_j(0,x) = \alpha(x)\phi_j(x)$

b) $\partial_t f_j(t,x) = -\lambda_j f_j(t,x) - \int_X F(t,x,y)\phi_j(y)\mu(y)$
$$+\alpha(x)\int_X \beta(y)\phi_j(y)(\partial_t+\Delta_2)\big(N(t,x,y)e^{-d^2(x,y)/4t}\big)\mu(y).$$

Proof. – Let us prove part a). The Geometrical Definition 4.2 states that $\beta\equiv 1$ on $\operatorname{supp}\alpha\cap X$. We recall the equation (4.4), and the definition 4.3. Part a) is therefore an immediate consequence of equation (4.3) and of Lemma 4.4. Let us prove part b). Since Δ_2 is self-adjoint we can write:

$$\begin{aligned}(4.5)\qquad \partial_t f_j(t,x) =&\alpha(x)\int_X \beta(y)\partial_t\big(N(t,x,y)e^{-d^2(x,y)/4t}\big)\phi_j(y)\mu(y)\\ =&\alpha(x)\int_X \beta(y)(\partial_t+\Delta_2)\big(N(t,x,y)e^{-d^2(x,y)/4t}\big)\phi_j(y)\mu(y)\\ &-\alpha(x)\int_X N(t,x,y)e^{-d^2(x,y)/4t}\Delta_2(\beta(y)\phi_j(y))\mu(y).\end{aligned}$$

Let us recall that (*see* [B-G-M], page 127):

$$\Delta_2(\phi_j\beta) = \phi_j\Delta_2\beta + \beta\Delta_2\phi_j - 2\langle\operatorname{grad}\phi_j,\operatorname{grad}\beta\rangle.$$

Inserting this identity in the last integral of equation (4.5), using on one hand Lemma 4.6 and the Definition 4.7 of $F(t,x,y)$ and on the other hand the fact that $\Delta_2\phi_j = \lambda_j\phi_j$ and the equation (4.4), we prove easily the Proposition 4.8. □

In order to integrate with respect to t the equation in Proposition 4.8 b) (for x complex) we will need the following lemma.

LEMMA 4.9. – *Let ν and $\tilde\nu$ be the two constants introduced in the Geometrical Definition 4.2. Then there exists two positive constants N and D such that for all (complex) $x\in W$ and for all $t\in]0,1]$ we have:*

a) $$\|\beta(\cdot)(\partial_t+\Delta_2)(N(t,x,\cdot)e^{-d^2(x,\cdot)/4t})\|_{H^{2+2n}(X)} \le Dt^{-N}\exp\left(-\frac{\nu}{t}\right)$$

b) $$\|F(t,x,\cdot)\|_{H^{2+2n}(X)} \le Dt^{-N}\exp\left(-\frac{\tilde\nu}{4t}\right).$$

c) For any $t \in]0,1]$ and any $y \in X$, the two maps $x \to F(t,x,y)$ and

$$x \to \beta(y)(\partial_t + \Delta_2)(N(t,x,y)e^{-d^2(x,y)/4t})$$

are holomorphic on W.

Proof. – *a*) Let us use the notations of Lemma 4.5. Proposition 3.1 (and the Sobolev's injection theorem) shows that for every $p \in \{0,\dots,n+1\}$, for all $(x,y) \in V\times$ suppβ, and all $t \in]0,1]$ we have:

$$\left|\Delta_y^p\left[\beta(y)e^{-d^2(x,y)/4t}\,t^{k_t}\Delta_2 u_{k_t}(x,y)\right]\right| \leq \widetilde{C}\,t^{k_t}L^{1+k_t}k_t!t^{-2n-4}\left|e^{-d^2(x,y)/4t}\right|.$$

Since $C > 1$ we have $\frac{1}{Ct} - 2 \leq k_t \leq \frac{1}{Ct}$, so using Stirling's formula and the definition of ν we get:

$$\text{(4.6)}\qquad \begin{aligned}\left|t^{k_t}L^{1+k_t}k_t!\right| &\leq \left(\frac{tLk_t}{e}\right)^{k_t}Lk_t\widetilde{C}\\ &\leq \frac{\widetilde{C}}{t}\left(\frac{L}{eC}\right)^{\frac{1}{Ct}} = \frac{\widetilde{C}}{t}\exp\left(-\frac{2\nu}{t}\right).\end{aligned}$$

Now let us denote $k = 1 + k_t$. The inequality (4.6) then shows that for $0 < t < 1$:

$$\left|t^kL^{1+k}k!\right| = \left|t^{k_t}L^{1+k_t}k_t!t(k_t+1)L\right| \leq \frac{\widetilde{C}}{t}\exp\left(-\frac{2\nu}{t}\right)$$

On the other hand, since the support of $\beta(y)$ is compact and included in $X \cap V$, Lemma 4.5 (which defines $R(t,\cdot,\cdot)$) and Proposition 3.1 show that for every $p \in \{0,\dots,n+1\}$, $x \in W$, and $y \in X$ we have:

$$\left|\Delta_2^p(\beta(y)R(t,x,y))\right| \leq \widetilde{C}\,t^{-2n-4-\frac{n}{2}}\left|\exp\left(-\frac{d^2(x,y)}{4t}\right)\right|t^kL^{1+k}k!.$$

Moreover, since $W \subset \operatorname{supp}\alpha$, the inequality (4.1) shows that:

$$\forall (x,y) \in W \times \operatorname{supp}\beta,\quad \left|\exp\left(-\frac{d^2(x,y)}{4t}\right)\right| \exp\left(-\frac{2\nu}{t}\right) \leq \exp\left(-\frac{\nu}{t}\right).$$

Now we deduce easily the Lemma 4.9 a) with $N = \frac{n}{2} + 2n + 5$ from inequality (4.6) and Lemma 4.5. Let us prove part b). Let us recall that $\beta \equiv 1$ on the open set Ω, so for all $y \in \Omega$ and $h \in \{1,\dots,n\}$, $U_h \cdot \beta(y) = 0$ (*see* Lemma 4.6) and $\Delta_2\beta(y) = 0$. The inequality (4.2) shows that for any $t \in]0,1]$:

$$\text{(4.7)}\qquad \forall (x,y) \in \operatorname{supp}\alpha \times (\operatorname{supp}\beta \setminus \Omega),\ \left|\exp\left(-\frac{d^2(x,y)}{4t}\right)\right| \leq \exp\left(-\frac{\tilde{\nu}}{4t}\right).$$

Moreover using Proposition 3.1, Stirling's formula, and the definition of $N(t,\cdot,\cdot)$ we easily show that for any $t \in]0,1]$, $x \in W$, and any test function $\varphi(y)$ with compact support in

$X \cap V$ belonging to the following finite set of functions $\{\Delta_2\beta(y),\ U_h \cdot \beta(y),\ 1 \leq h \leq n\}$ we have:

$$\|\varphi(\cdot)N(t,x,\cdot)\|_{H^{4+2n}(X)} \leq \widetilde{C}\, t^{-n/2} \sum_{k \leq \frac{1}{Ct}} \sqrt{k}\left(\frac{tkL}{e}\right)^k \leq \widetilde{C}\, t^{-n/2} \sum_{k \geq 0} \sqrt{k}\left(\frac{L}{Ce}\right)^k.$$

Since $\frac{L}{Ce} < \frac{1}{2}$ we can now use the inequality (4.7) and the Definition 4.7 of $F(t,x,\cdot)$ to obtain easily the Lemma 4.9 b) with $N = \frac{n}{2} + 2n + 5$. Since the coefficients $u_k(x,y)$ of the formal solution are holomorphic on $V \times V$ we prove easily part c). □

Let us denote by $E(t,x,y) = \sum_{j \geq 0} e^{-t\lambda_j}\phi_j(x)\phi_j(y)$ the heat kernel of (X,g).

PROPOSITION 4.10. – *For every real t in $]0,1]$, for every $(x,y) \in W \times X$ we have:*

$$\alpha(x)\beta(y)N(t,x,y)e^{-d^2(x,y)/4t} = \alpha(x)E(t,x,y) - \int_0^t e^{(s-t)\Delta_2}F(s,x,y)ds$$
$$+\alpha(x)\int_0^t e^{(s-t)\Delta_2}\left[\beta(y)(\partial_s + \Delta_2)\left(N(s,x,y)e^{-d^2(x,y)/4s}\right)\right]ds.$$

Proof. – We begin with a few preliminary remarks. Let us denote $\rho = \inf(\nu,\tilde{\nu})$. We easily prove the following inequality (with the constant N of Lemma 4.9):

$$\forall t \in]0,1],\ \int_0^t s^{-N}\exp\left(-\frac{\rho}{4s}\right)ds \leq \widetilde{C}\exp\left(-\frac{\rho}{8t}\right). \tag{4.8}$$

Using Lemma 4.9, the fact that the $e^{(s-t)\Delta}$, $0 \leq s \leq t \leq 1$ define a uniformly bounded family of endomorphisms of $H^{2+2n}(X)$ and the inequality (4.8), one obtains easily for each $t \in]0,1]$ and $x \in W$ the two following estimates:

$$\left\|\int_0^t e^{(s-t)\Delta_2}\left[\beta(\cdot)(\partial_s + \Delta_2)\left(N(s,x,\cdot)e^{-d^2(x,\cdot)/4s}\right)\right]ds\right\|_{H^{2n+2}(X)} \leq \widetilde{C}\exp\left(-\frac{\rho}{8t}\right) \tag{4.9}$$

$$\left\|\int_0^t e^{(s-t)\Delta_2}F(s,x,\cdot)\,ds\right\|_{H^{2n+2}(X)} \leq \widetilde{C}\exp\left(-\frac{\rho}{8t}\right). \tag{4.10}$$

Now as a first step we assume that x is "real" and belongs to X. For each real $t \in]0,1]$ and each nonnegative integer j, Lemma 4.9 and Sobolev's injection theorem allow us to integrate from $s = 0$ to t the second member of the equality of Proposition 4.8 b). We therefore obtain:

(4.11)

$$f_j(t,x) = e^{-t\lambda_j}\alpha(x)\phi_j(x) - e^{-t\lambda_j}\int_X\int_0^t e^{s\lambda_j}F(s,x,z)\phi_j(z)\mu(z)\,ds$$
$$+\, e^{-t\lambda_j}\int_X\int_0^t e^{s\lambda_j}\alpha(x)\beta(z)(\partial_s + \Delta_2)(N(s,x,z)e^{-d^2(x,z)/4s})\ \phi_j(z)\mu(z)\,ds.$$

Now we multiply each member of the equality (4.11) by $\phi_j(y)$ and sum for $0 \leq j \leq +\infty$. Using the estimates (4.9), (4.10) and equation (4.4) we then obtain easily the equality of

Proposition 4.10 for $x \in X$. But, Lemma 4.9 c) and the estimates (4.9) and (4.10) allow us to use an analytic continuation argument to show that this equality is still true for all $x \in W$ (which is connected). The Proposition 4.10 is proven. □

PROPOSITION 4.11.

(i) *Let us denote* $\rho = \inf(\nu, \tilde{\nu})$. *Let* $u(y)$ *be a bounded measurable function on* X. *Then there exists a positive constant* D_1 *such that for every* $x \in W$ *and every* $t \in]0,1]$ *we have:*

$$\left| \int_X \beta(y) N(t,x,y) e^{-d^2(x,y)/4t} u(y)\mu(y) - e^{-t\Delta}u(x) \right| \leq D_1 \exp\left(-\frac{\rho}{8t}\right).$$

(ii) *We can find an open neighborhood* W_1*(in* W*) of* x_0 *such that for* $(x,y) \in W_1^2$ *and* $0 < t < 1$ *the decomposition of* $E(t,x,y)$ *in Theorem 0.1 (i) is valid. (For the assertion relevant to the FBI phase, see the next proposition.)*

Note. – Therefore, the values of $u(y)$ for $y \notin \operatorname{supp}\beta$ "do not contribute" to the growth of $e^{-t\Delta}u(x)$ when $t \to 0^+$ and $x \in W_1$.

Proof. – (i)Let us recall that $\alpha \equiv 1$ on W. Let us fix t and x, we multiply each member of the equality of Proposition 4.10 by $u(y)$ and we integrate over X with respect to $\mu(y)$. Using the estimates (4.9), (4.10), and Sobolev's injection theorem ($n = \dim X$) we obtain easily the estimate of Proposition 4.11 i). In the same way (since α [resp. β] $\equiv 1$ on W [resp. Ω]) we prove that the decomposition of $E(t,x,y)$ in Theorem 0.1 i) is valid for $(x,y) \in W \times \Omega$, but y is only "real." By Proposition 2.4 we have:

$$\forall t \in]0,1[, \quad \sup_{(x,y)\in M_\epsilon^2} |E(t,x,y)| \leq C(\epsilon) \exp\left(\frac{2\epsilon^2}{t}\right).$$

The existence of the complex neighborhood W_1 of x_0 in W such that the decomposition of $E(t,x,y)$ is valid on $W_1 \times W_1$ is therefore an easy consequence of 2^n successive applications of the following lemma:

LEMMA. – *Let* δ *and* μ *be two reals in* $]0,1[$. *Let* $\{g_t(z)\}_{0<t<1}$ *be a family of functions of one complex variable* z *holomorphic on a neighborhood of a lozenge* K *of* C *whose corners are* $-\mu$, μ, $\sqrt{-1}\mu^2$, $-\sqrt{-1}\mu^2$. *Let us assume that for any* t *in* $]0,1[$ *we have:*

$$\forall z \in K, \quad |g_t(z)| \leq \exp\left(\frac{\delta}{t}\right)$$

$$\text{for any real number } z \in K, \quad |g_t(z)| \leq \exp\left(-\frac{\delta}{t}\right).$$

Then we can find an open neighborhood D (depending on δ and K but not on the family $\{g_t\}_{0<t<1}$) of 0 in C such that:

$$\forall t \in]0,1[, \quad \sup_{z\in D\cap K} |g_t(z)| \leq \exp\left(-\frac{\delta}{2t}\right).$$

Proof. – We just have to apply the maximum modulus principle on the rectangle K with the following subharmonic functions:

$$t\log|g_t(z)| - \alpha\,\Im z - \beta\,\Re z^2$$

where α and β are two real constants such that: $\alpha >> \beta >> 1$.

Proof of Theorem 0.1. – We have assumed in the Geometrical Definition 4.2 that supp β is contained in the domain of a normal real analytic geodesic chart at x_0 : $\xi_1 \to \mathrm{Exp}_{x_0}\xi_1$. We will use the holomorphic chart of M_ϵ at x_0 given by: $\xi_1 + \sqrt{-1}\,\xi_2 \to \mathrm{Exp}_{x_0}(\xi_1 + \sqrt{-1}\xi_2)$, where ξ_1 and ξ_2 belong to the real vector space $T_{x_0}X$. The next proposition completes the proof of Theorem 0.1 i) and shows that the holomorphic function defined by:

$$\phi(x, \xi_1 + \sqrt{-1}\,\xi_2) = \frac{\sqrt{-1}}{2} d^2(x, \mathrm{Exp}_{x_0}(\xi_1 + \sqrt{-1}\xi_2))$$

is a F.B.I. phase (in the sense of [Sj]) near the point $(x, \xi_1 + \sqrt{-1}\,\xi_2) = (\mathrm{Exp}_{x_0}(\sqrt{-1}\,\xi_0), 0)$.

PROPOSITION 4.12. – *There exists a positive real ϵ_2 such that if ξ_0 belongs to $T_{x_0}X$ and satisfies $|\xi_0| < \epsilon_2$ then $Exp_{x_0}(\sqrt{-1}\,\xi_0)$ belongs to the W_1 of Proposition 4.11 and we have:*

(*a*) $\Im\big[\frac{\partial^2}{\partial\xi_1^2}\phi(Exp_{x_0}(\sqrt{-1}\,\xi_0), 0)\big]$ *is a positive definite quadratic form.*

(*b*) $\mathrm{Det}\big[\frac{\partial^2}{\partial x\partial\xi_1}\phi(\mathrm{Exp}_{x_0}(\sqrt{-1}\,\xi_0), 0)\big]$ *is not zero.*

(*c*) $\frac{1}{2}\big(\partial_{\xi_1} - \sqrt{-1}\,\partial_{\xi_2}\big)\phi(\mathrm{Exp}_{x_0}(\sqrt{-1}\,\xi_0), 0) = \xi_0$ *(real vector!).*

Remark 4.13. – Of course these three results are obvious in the case of $\mathbf{R}^n$ endowed with the usual (flat) Riemannian metric. Recall that $d^2(\cdot,\cdot)$ is holomorphic on $V \times V$, so using the Cauchy-Riemann equations we easily see from a) and c) that $\xi_1 \to -\Im\phi(\mathrm{Exp}_{x_0}(\sqrt{-1}\,\xi_0), \xi_1)$ admits a non-degenerate critical point for $\xi_1 = 0$ which is a local maximum. If $x_0' \in X$ and $\xi_0' \in T_{x_0'}X$ are close to x_0 and 0 respectively then $y\ (\in X) \to -\frac{1}{2}\Im[\sqrt{-1}\,d^2(\mathrm{Exp}_{x_0'}(\sqrt{-1}\,\xi_0'), y)]$ admits a non-degenerate critical point at $y = x_0'$, which is a local maximum. The value at $\mathrm{Exp}_{x_0'}(\sqrt{-1}\,\xi_0')$ of the associated strictly plurisubharmonic weight is $-\frac{1}{2}\Im[\sqrt{-1}\,d^2(\mathrm{Exp}_{x_0'}(\sqrt{-1}\,\xi_0'), x_0')] = \frac{1}{2}|\xi_0'|^2$ (*see* Delort [De] page 17 formula (2.22)). Moreover the Morse lemma with parameters shows that we can find an open neighborhood Ω_1 of x_0 in X and can shrink W_1 (at the beginning of Proposition 4.12) so that there exists $\delta_1 > 0$ such that for all $(x, y) \in W_1 \times (\mathrm{supp}\beta \setminus \Omega_1)$,

$$\Re d^2(x, y) > \delta_1$$

$$\beta(y) \equiv 1 \quad \text{on} \quad \Omega_1$$

and for all $(\mathrm{Exp}_{x_0'}(\sqrt{-1}\,\xi_0'), y) \in W_1 \times \Omega_1$,

$$\frac{1}{2}|\xi_0'|^2 + \Im[\frac{1}{2}\sqrt{-1}\,d^2(\mathrm{Exp}_{x_0'}(\sqrt{-1}\,\xi_0'), y)] \geq \delta_1 d^2(x_0', y).$$

Therefore part i) of Theorem 0.1 (or Proposition 4.11) shows that the points $y \in \mathrm{supp}\beta \setminus \Omega_1$ do not contribute to the growth of $e^{-t\Delta}u(x)$ when $t \to 0^+$ for $x \in W_1$.

Proof. – Part (*a*) and (*b*) are obvious since they are true for $\xi_0 = 0$. Let us prove c). We see from the definition of $\phi(\cdot,\cdot)$ and the fact that $D\text{Exp}_{x_0}(0) = \text{Identity}$ that it is enough to show that: $\partial_{\xi_1}\phi(\text{Exp}_{x_0}(\sqrt{-1}\,\xi_0), 0) = \xi_0$. Let ϵ_2 be a positive real small enough so that $\text{Exp}_{x_0}\big(B(0,\epsilon_2)\big)$ defines an open geodesically convex neighborhood U of x_0 in X and all the Riemannian exponential maps to be used soon will induce diffeomorphisms onto U. Let us fix $\xi \in T_{x_0}X \setminus 0$ with $|\xi| < \epsilon_2$. We put $m = \text{Exp}_{x_0}\xi$ and consider the function:

$$F(y) = \frac{\sqrt{-1}}{2} d^2(m, y).$$

There exists a unique $\nu_0 \in T_mX$ such that $|\nu_0| < \epsilon_2$ and $x_0 = \text{Exp}_m(\nu_0)$. Of course we have for any (small) $\nu \in T_mX$, $F(\text{Exp}_m(\nu)) = \frac{\sqrt{-1}}{2}|\nu|^2$ (the norm induced by the Riemannian metric.) So a differentiation at ν_0 gives:

$$\text{(4.12)} \qquad \forall h \in T_{\nu_0}X, \quad (DF(x_0) \circ D\text{Exp}_m(\nu_0)).h = \sqrt{-1}\,\langle \nu_0, h\rangle.$$

According to the Gauss lemma (*see* [B-G-M], page 50) $D\text{Exp}_m(\nu_0)$ sends the orthogonal subspace of the real line $\mathbf{R}\nu_0$ in T_mX onto the orthogonal subspace of $\mathbf{R}(D\text{Exp}_m(\nu_0).\nu_0)$, and moreover the vector $D\text{Exp}_m(\nu_0).\nu_0$ is tangent at $x_0 = \text{Exp}_m(\nu_0)$ to the geodesic curve joining x_0 and $\text{Exp}_{x_0}\nu_0 = m$, so this vector is colinear to ξ. Therefore the equality (4.12) shows that $DF(x_0)$ vanishes on $(\mathbf{R}\xi)^\perp$.

Moreover $DF(x_0).\xi$ is the derivative for $s = 0$ of the numerical function $F(\text{Exp}_{x_0}(s\xi)) = \frac{\sqrt{-1}}{2}(1-s)^2|\xi|^2$. So it is clear that $DF(x_0).\xi = -\sqrt{-1}\,|\xi|^2$, and that $\text{grad}DF(x_0) = \partial_{\xi_1}\phi(\text{Exp}_{x_0}\xi, 0) = -\sqrt{-1}\,\xi$.

Now we observe that the function $\xi \to \partial_{\xi_1}\phi(\text{Exp}_{x_0}\xi, 0)$ is holomorphic so, replacing ξ by $\sqrt{-1}\,\xi_0$ yields the Proposition 4.12 (*c*). $\square$

Now we shrink W_1 as in the Remark 4.13, we fix $\epsilon_2 > 0$ as in the Proposition 4.12, and we consider $\xi_0 \in T_{x_0}X \setminus 0$ such that $|\xi_0| < \epsilon_2$. Since the coefficient $u_0(\cdot,\cdot)$ of the formal solution of the heat equation never vanishes on $V \times V$, we see easily, putting $\lambda = \frac{1}{2t}$, that $N(t,x,y)$ is an analytic symbol in the sense of Sjöstrand (*see* [Sj]) of order $n/2$, which is elliptic at the point $(\text{Exp}_{x_0}(\sqrt{-1}\,\xi_0), x_0)$. Now let $u(y)$ be a bounded measurable function on X (which we may assume by Lemma 4.1). Using Sjöstrand's result [Sj] page 46 (*see* also Delort [D] Cor. 4.4 page 27), the two inequalities of remark 4.13, and the fact that $\beta \equiv 1$ on Ω_1 we see that $(x_0, -\zeta_0)$ does not belong to the analytic wave front set of u if and only if we can find an open neighborhood Z' of $\text{Exp}_{x_0}(\sqrt{-1}\,\xi_0)$ and a positive real δ' such that:

(4.13)

$$\forall x \in Z', \quad \forall t \in]0,1], \qquad \left|\int_X \beta(y)N(t,x,y)e^{-d^2(x,y)/4t}u(y)\mu(y)\right| \le \widetilde{C}\exp\left(\frac{|\xi_0|^2 - \delta'}{4t}\right).$$

Using Proposition 4.11, we see that $\int_X \beta(y)N(t,x,y)e^{-d^2(x,y)/4t}u(y)\mu(y)$ satisfies an estimate such as (4.13) if and only if $(e^{-t\Delta}u)(x)$ satisfies an estimate such as (*) in Theorem 4.1(ii). Theorem 0.1 is therefore proven.

5. Proof of Theorem 0.2

In this section, we consider a fixed point $q \in X$ and denote by Y the fiber $\pi^{-1}(q)$ in M_ϵ for $\epsilon > 0$ small enough (*see* Theorem 1.5 for the definition of the fibration π; we also keep the same notation Y for different values of ϵ whenever this is unambiguous). We recall the notation

$$(5.0) \qquad G(t,m,m') = (4\pi t)^{-n/2} e^{d^2(m,m')/4t}$$

and we take $k > n/2$ some fixed nonnegative integer. Throughout the present section, we shall assume that $\epsilon_1 > 0$ is small enough so that the functions $u_j(m,m')$ defined by (3.3) are holomorphic on some open neighborhood of $\overline{Y} \times \overline{Y}$ in $M_{\epsilon_1} \times M_{\epsilon_1}$ for all $0 \le j \le k$. We take a fixed $\epsilon \in]0,\epsilon_1]$ and assume that the function f in Theorem 0.2 is C^ω on X and satisfies $\int_X f\mu = 0$. In other words f is orthogonal to the space of harmonic functions on X, that is the constants since X is connected and compact. Applying Proposition 2.2 shows the existence of a sequence of complex numbers $\{a_j\}_{j\ge 1}$ and of two constants $C > 0$ and $\eta \in]0,\epsilon]$ such that:

$$(5.1) \qquad \forall j \ge 1 \quad |a_j| \le C e^{-\eta\sqrt{\lambda_j}}, \qquad f(m) \equiv \sum_{j\ge 1} a_j \phi_j(m)\,.$$

By definition of Tf, $(\Delta_2 + \partial_t)Tf \equiv 0$ on $\mathbf{R}_+^* \times X$. Since for every $t > 0$, $m \mapsto Tf(t,\cdot)$ admits a holomorphic extension on the tuboid M_ϵ, Theorem 1.16 shows that:

$$(\Delta_2^Y + \partial_t)Tf(t,m) = 0 \quad \text{on } \mathbf{R}_+^* \times Y\,.$$

The idea for the proof of all inversion formulas in this article is to start from the following equality (with $0 < t_1 < t_2$):

$$\int_{t_1}^{t_2} dt \int_Y H_k(t,q,m)(\Delta_2 + \partial_t)Tf(t,m)\mu^Y(m) = 0$$

(with the H_k as in Proposition 3.0) and integrate by parts. The following lemma is an immediate consequence of the Green formula (*see* (1.6)) for Δ^Y:

LEMMA 5.1. – *For all* $t \in [t_1,t_2]$

$$\int_Y H_k(t,q,m)\Delta_2^Y Tf(t,m)\mu^Y(m) = \int_Y Tf(t,m)\Delta_2^Y H_k(t,q,m)\mu^Y(m) + \int_{\partial Y} \big[Tf(t,\cdot)i^Y_{\mathrm{grad}^Y H_k(t,q,\cdot)}\,\mu^Y - H_k(t,q,\cdot)i^Y_{\mathrm{grad}^Y Tf(t,\cdot)}\,\mu^Y\big].$$

Integrating by parts with respect to t shows that

$$\begin{aligned}(5.2)\quad 0 &= \int_{t_1}^{t_2} dt \int_Y H_k(t,q,m)(\Delta_2^Y + \partial_t)Tf(t,m)\mu^Y(m)\\
&= \int_Y Tf(t_2,m)\,H_k(t_2,q,m)\mu^Y(m) - \int_Y Tf(t_1,m)\,H_k(t_1,q,m)\mu^Y(m)\\
&\quad + \int_{t_1}^{t_2} dt \int_Y Tf(t,m)(\Delta_2^Y - \partial_t)H_k(t,q,m)\mu^Y(m)\\
&\quad + \int_{t_1}^{t_2} dt \int_{\partial Y} \big[Tf(t,\cdot)i^Y_{\mathrm{grad}^Y H_k(t,q,\cdot)}\,\mu^Y - H_k(t,q,\cdot)i^Y_{\mathrm{grad}^Y Tf(t,\cdot)}\,\mu^Y\big].\end{aligned}$$

In the following proposition are gathered several asymptotic estimates satisfied by the function $(t,m) \mapsto H_k(t,q,m)$.

PROPOSITION 5.2.

a) There exists a positive constant D *such that* $\forall t \geq 1$, $\forall m \in Y$, $|H_k(t,q,m)| \leq Dt^k$.

b) For all smooth vector fields U *with compact support in the manifold with boundary* $\overline{Y}_\epsilon$ *there exists a positive constant* $D(U)$ *such that:*

$$\forall t \geq 1, \quad \sup_{m \in \partial Y} \left| U \cdot H_k(t,q,m) \right| \leq D(U) t^k .$$

c) For all smooth vector fields U *with compact support in the manifold with boundary* $\overline{Y}_\epsilon$*, there exists a positive constant* $D(U)$ *such that*

$$\forall t \in]0,1], \quad \sup_{m \in \partial Y} \left| U \cdot H_k(t,q,m) \right| \leq D(U)\, t^{-1-n/2}\, e^{-\epsilon^2/4t}.$$

(Recall that for all m *in* ∂Y*,* $\epsilon^2 = -d^2(q,m)$*).*

d) There exists a positive constant R *such that for all* m *in* Y *with* $\epsilon'^2 = -d^2(q,m)$

$$\forall t \in]0,1], \quad |H_k(t,q,m)| \leq R\, G(t,q,m) = R\, (4\pi t)^{-n/2}\, e^{-\epsilon'^2/4t}.$$

e) For any function $(t,m) \mapsto u(t,m)$ *continuous and bounded on* $[0,1] \times Y$*,*

$$\lim_{t \to 0^+} \int_Y u(t,m) H_k(t,q,m) \mu^Y(m) \;=\; (\sqrt{-1})^n\, u(0,q).$$

The formula of Theorem 0.2 will be a consequence of (5.2) and the four following lemmas.

Remark 5.3. – The proofs of Theorem 0.2 and especially those of Lemmas 5.4, 5.5 and 5.7 below, are still valid if $(\sqrt{-1})^{-n}\, H_k(t,q,m)$ is replaced by any function $K(t,m)$ for which equation (5.2) and the properties listed in Proposition 5.2 are satisfied. This observation will be used in the next two sections.

LEMMA 5.4. – *Under the above assumptions*

$$\lim_{t_2 \to +\infty} \int_Y Tf(t_2,m) H_k(t_2,q,m) \mu^Y(m) \;=0.$$

Proof. – For all integers $j \geq 1$ introduce $b_j = CC(\epsilon) e^{(-\eta+\epsilon)\sqrt{\lambda_j} - \lambda_j}\, \lambda_j^{n+1}$ where $C(\epsilon)$, η and C have been defined in Proposition 2.1 and relation (5.1) respectively. The series $\sum_{j \geq 1} b_j$ converges. Proposition 2.1 and (5.1) show that, for all $j \geq 1$, $m \in Y$, $t \geq 1$:

$$\left| e^{-t\lambda_j} a_j \phi_j(m) \right| \leq b_j e^{-(t-1)\lambda_j} \leq b_j e^{-(t-1)\lambda_1}$$

Since $\lambda_j \geq \lambda_1 > 0$ this inequality implies

$$(5.3) \quad \forall t \geq 1, \quad \forall m \in Y \qquad \left| Tf(t,m) = \sum_{j \geq 1} a_j \phi_k(m) e^{-t\lambda_j} \right| \leq \left(\sum_{j \geq 1} b_j \right) e^{-(t-1)\lambda_1} .$$

Then, Proposition 5.2 a) shows that, for any $m \in Y,\ t \geq 1$ we have:

$$\big|Tf(t,m)\,H_k(t,q,m)\big| \leq D\,t^k e^{-(t-1)\lambda_1} \sum_{j\geq 1} b_j\,.$$

Letting t tend to $+\infty$ gives immediately the result claimed in Lemma 5.4.

LEMMA5.5. – *Under the above assumptions*

$$\lim_{t_1\to 0^+} \int_Y Tf(t_1,m)H_k(t_1,q,m)\,\mu^Y(m) = (\sqrt{-1})^n\, f(q)\,.$$

Proof. – We shall use the constant η defined in (5.1). Proposition 2.4 shows that for all $t_1 \in]0,1]$, for all $m \in Y \setminus \overline{M}_{\frac{\eta}{2}}$ such that $\epsilon'^2 = -d^2(q,m)$,

$$\big|Tf(t_1,m)\big| \leq t_1^n C(n,\epsilon')\mathrm{e}^{\epsilon'^2/4t_1}\,.$$

Hence, according to Proposition 5.2 d), for all $t_1 \in]0,1]$

(5.4)

$$\left|\int_{Y\setminus M_{\frac{\eta}{2}}} Tf(t_1,m)H_k(t_1,q,m)\,\mu^Y(m)\right| \leq (4\pi)^{-n/2}R\,t_1^{n/2} \sup_{\frac{\eta}{2}\leq\epsilon'\leq\epsilon} C(n,\epsilon') \int_Y \big|\mu^Y(m)\big|.$$

and therefore the left side of the previous inequality tends to 0 as $t_1 \to 0^+$. Proposition 2.1 with ϵ replaced by $\eta/2$ shows that for all $j \geq 1$,

$$\sup_{m\in Y\cap\overline{M}_{\frac{\eta}{2}}} |\phi_j(m)| \leq C(\tfrac{\eta}{2})\lambda_j^{n+1} e^{\frac{\eta}{2}\sqrt{\lambda_j}}.$$

Property (5.1) and Proposition 5.2 d) show that for all $j \geq 1$ and all $t_1 \in (0,1]$,

$$(5.5)_j \qquad \int_{Y\cap M_{\frac{\eta}{2}}} \big|a_j\big[e^{-t_1\lambda_j}\phi_j(m) - \phi_j(q)\big]\,H_k(t_1,q,m)\mu^Y(m)\big|$$

$$\leq 2C(\tfrac{\eta}{2})Ce^{\sqrt{\lambda_j}(-\eta+\frac{\eta}{2})}\lambda_j^{n+1}R \sup_{0<t<1} \int_{Y\cap M_{\frac{\eta}{2}}} G(t,q,m)\big|\mu^Y\big|.$$

According to Proposition 5.2 e) with $u(t,m)$ replaced by $e^{-t\lambda_j}\phi_j(m) - \phi_j(q)$, the left side of $(5.5)_j$ tends to 0 as $t \to 0^+$. Since $f(q) = \sum_{j\geq 1} a_j\phi_j(q)$ a normal convergence argument based on the family of inequalities $(5.5)_j$ shows that

$$(5.6) \qquad \lim_{t_1\to 0^+} \int_{Y\cap M_{\frac{\eta}{2}}} \big[Tf(t_1,m) - f(q)\big]\,H_k(t_1,q,m)\mu^Y = 0\,.$$

Proposition 5.2 e) with $u(t,m) \equiv 1$ shows that

$$\lim_{t_1\to 0^+} \int_{Y\cap M_{\frac{\eta}{2}}} H_k(t_1,q,m)\mu^Y = (\sqrt{-1})^n$$

and Lemma 5.5 follows from (5.4) and (5.6). □

LEMMA 5.6. – *The following integral converges:*

$$\int_0^{+\infty} dt \left| \int_Y Tf(t,m)(\Delta_2 - \partial_t)H_k(t,q,m)\mu^Y(m) \right| < +\infty\,.$$

Proof. – Let us recall that $k > n/2$ and

$$(\Delta_2^Y - \partial_t)H_k(t,q,m) = t^k(G\Delta_2^Y u_k)(t,q,m) \tag{5.7}$$

(*see* Proposition 3.0 and equation (5.0)).

FIRST STEP. – *Estimates on the integrand as* $t \to +\infty$, $t > 1$.

Inequality (5.3) used in the proof of Lemma 5.4 shows the existence of a positive constant C_2 such that for all $t \geq 1$ for all $m \in Y$:

$$\left|Tf(t,m)t^k(G\Delta_2^Y u_k)(t,q,m)\right| \leq C_2 e^{-(t-1)\lambda_1}\, t^{k-n/2}\left|e^{d^2(q,m)/4t}\right|.$$

Thus for every $A > 0$ the integral

$$\int_A^{+\infty} dt \left| \int_Y Tf(t,m)(\Delta_2 - \partial_t)H_k(t,q,m)\mu^Y(m) \right|$$

is convergent.

SECOND STEP. – *Estimates on the integrand as* $t \to 0^+$, $t \in]0,1]$.

Working as in the proof of (5.4) (*see* the proof of Lemma 5.5) we demonstrate the existence of a positive constant C_3 such that for all t in $]0,1]$:

$$\left| \int_{Y\setminus M_{\frac{\eta}{2}}} Tf(t,m)t^k(G\Delta_2^Y u_k)(t,q,m)\mu^Y(m) \right| \leq C_3 t^{k+n/2} \sup_{\frac{\eta}{2}\leq\epsilon'\leq\epsilon} C(n,\epsilon')\,. \tag{5.8}$$

Moreover, according to Proposition 2.1:

$$\forall j \geq 1 \qquad \sup_{m\in Y\cap M_{\frac{\eta}{2}}} |\phi_j(m)| \leq C(\tfrac{\eta}{2})\lambda_j^{n+1} e^{\frac{\eta}{2}\sqrt{\lambda_j}}.$$

Property (5.1) shows that, for all t in $]0,1]$ and all m in $Y \cap M_{\frac{\eta}{2}}$,

$$\begin{aligned} |Tf(t,m)| &\leq \sum_{j\geq 1} C(\tfrac{\eta}{2})\lambda_j^{n+1} e^{\frac{\eta}{2}\sqrt{\lambda_j}}\, |a_j| \\ &\leq CC(\tfrac{\eta}{2}) \sum_{j\geq 1} \lambda_j^{n+1} e^{-\frac{\eta}{2}\sqrt{\lambda_j}}. \end{aligned}$$

Since $k > \frac{n}{2}$ and for all $m \in Y$, $d^2(q,m) \leq 0$, there exists a positive constant C_4 such that for all $t \in]0,1]$:

$$\left| \int_{Y\cap M_{\frac{\eta}{2}}} Tf(t,m)t^k(G\Delta_2^Y u_k)(t,q,m)\mu^Y(m) \right| < C_4. \tag{5.9}$$

Lemma 5.6 follows therefore from the equality (5.7), the first step above and both inequalities (5.8) and (5.9). □

LEMMA 5.7. – *With the same assumptions as above except that $f \in H^{2(3n+2)}(X)$ "only", one has*

$$\int_0^{+\infty} dt \left| \int_{\partial Y} \left[Tf(t,\cdot) i^Y_{\operatorname{grad}^Y H_k(t,q,\cdot)} \mu^Y - H_k(t,q,\cdot) i^Y_{\operatorname{grad}^Y Tf(t,\cdot)} \mu^Y \right] \right| < +\infty .$$

Proof. – As in the proof of Lemma 4.6, we proceed in two steps.

FIRST STEP. – *Estimate of the integral as $t \to +\infty$, $t \geq 1$.*

We can write $Tf = \sum_{j\geq 1} a_j \phi_j e^{-t\lambda_j}$, where the sequence $\{|a_j| \lambda_j^{2n+1+n+1}\}_{j\geq 1}$ is bounded. Working as in the proof of (5.3), using Propositions 2.1 and 5.2.b), one can check directly the existence of a positive constant C_1' such that for all $t \geq 1$

$$\left| \int_{\partial Y} Tf(t,\cdot) i^Y_{\operatorname{grad}^Y H_k(t,q,\cdot)} \mu^Y \right| \leq C_1' t^k C(\epsilon) e^{-\lambda_1(t-1)} \sum_{j\geq 1} \lambda_j^{n+1} e^{\epsilon\sqrt{\lambda_j}} |a_j| \, e^{-\lambda_j}.$$

Moreover Propositions 2.1 2) and 5.2 a) applied with $f(m) = H_k(t,q,m)$ show that for all $j \geq 1$, for all $t \geq 1$

$$\left| \int_{\partial Y} H_k(t,q,\cdot) i^Y_{\operatorname{grad}^Y \phi_j} \mu^Y \right| \leq D t^k C(\epsilon) \lambda_j^{n+1} e^{\epsilon\sqrt{\lambda_j}}.$$

From this, we infer that for all $t \geq 1$

$$\left| \int_{\partial Y} H_k(t,q,\cdot) i^Y_{\operatorname{grad}^Y Tf(t,\cdot)} \mu^Y \right| \leq C_2' t^k C(\epsilon) e^{-\lambda_1(t-1)} \sum_{j\geq 1} \lambda_j^{n+1} e^{\epsilon\sqrt{\lambda_j}} |a_j| \, e^{-\lambda_j} .$$

Hence we conclude that for all $A > 0$ the integral

$$\int_A^{+\infty} dt \left| \int_{\partial Y} \left[Tf(t,\cdot) i^Y_{\operatorname{grad}^Y H_k(t,q,\cdot)} \mu^Y - H_k(t,q,\cdot) i^Y_{\operatorname{grad}^Y Tf(t,\cdot)} \mu^Y \right] \right| < +\infty.$$

SECOND STEP. – *Estimate of the integral as $t \to 0^+$, $t \in]0,1]$.*

Using Proposition 2.4 1] with $p = n+1$ and Proposition 5.2 c), one can check the existence of a positive constant C_3' such that for all $t \in]0,1]$

$$\left| \int_{\partial Y} Tf(t,\cdot) i^Y_{\operatorname{grad}^Y H_k(t,q,\cdot)} \mu^Y \right| \leq C_3' C(n+1,\epsilon) t^{n/2} .$$

Applying Proposition 2.4 2] with $p = n+1$, $g_t = H_k(t,q,\cdot)|_Y$ and Proposition 5.2 d), one sees that for all $t \in]0,1]$

$$\left| \int_{\partial Y} H_k(t,q,\cdot) i^Y_{\operatorname{grad}^Y Tf(t,\cdot)} \mu^Y \right| \leq (4\pi)^{-n/2} R C(n+1,\epsilon) t^{1+n/2} ,$$

which completes the proof of Lemma 5.7. □

Taking into account the formula (5.2) and applying Lemma 5.4 to 5.7 proves the inversion formula stated as Theorem 0.2.

6. Proof of Theorem 0.4

In this section we assume that $-g^Y$ is a field of real, positive definite quadratic forms on Y. The following statement has been established in Proposition 1.19 and Corollary 1.24.

THEOREM 6.0.

a) $-\Delta^Y$ *is the Laplace-Beltrami operator of the Riemannian manifold* $(Y, -g^Y)$.

b) Let $d_1(\cdot,\cdot)$ *be the geodesic distance for* $(Y, -g^Y)$. *Then for all* $m \in Y$, $d_1^2(q,m) = -d^2(q,m)$.

c) $(\sqrt{-1}\,)^{-n}\mu^Y$ *is a Riemannian volume form for* $(Y, -g^Y)$.

For ϵ' small enough, $Y_{\epsilon'} = \{\mathrm{Exp}_q(\sqrt{-1}\xi) \text{ s. t. } |\xi|_q < \epsilon'\}$ can be isometrically embedded into some compact orientable Riemannian manifold $\widetilde{Y}$. Let us denote by $K_1(t,x,y)$ the heat kernel of $\widetilde{Y}$. Let ϵ_1 be a positive real, if ϵ_1/ϵ' is small enough then, for all $m \in Y_{\epsilon_1}$ $d_1(q,m)$ is the distance (for $\widetilde{Y}$) from q to m. In the next proposition are gathered several estimates satisfied by K_1.

PROPOSITION 6.1. – *Let* $K(t,m) = (\sqrt{-1}\,)^{-n} K_1(t,q,m)$ *for all* $(t,m) \in]0,+\infty[\times\widetilde{Y}$. *For* $0 < \epsilon_1/\epsilon'$ *small enough, one has, for all* $\epsilon \in]0,\epsilon_1]$:

a) There exists a positive constant D *such that for all* $t \geq 1$, *for all* $m \in \overline{Y}_\epsilon$, $|K(t,m)| \leq D$.

b) For all smooth vector fields U *on the manifold with boundary* $\overline{Y}_\epsilon$ *there exists a positive constant* $D(U)$ *such that:*

$$\forall t \geq 1, \qquad \sup_{m\in\partial Y_\epsilon} \big|U\cdot K(t,m)\big| \leq D(U)\,.$$

c) For all smooth vector fields U *on the manifold with boundary* $\overline{Y}_\epsilon$ *there exists a positive constant* $D(U)$ *such that:*

$$\forall t \in]0,1], \qquad \sup_{m\in\partial Y_\epsilon} \big|U\cdot K(t,m)\big| \leq D(U)\, t^{-1-n/2}\, e^{-\epsilon^2/4t}$$

(we recall that for all $m \in \partial Y_\epsilon$, $\epsilon^2 = -d^2(q,m)$*).*

d) There exists a positive constant R *such that for any* $m \in \overline{Y}_\epsilon$

$$\forall t \in]0,1], \quad |K(t,m)| \leq RG(t,q,m) = R(4\pi t)^{-n/2} e^{d^2(q,m)/4t}$$

e) For any function $(t,m) \mapsto u(t,m)$ *continuous and bounded on* $[0,1]\times\overline{Y}_\epsilon$,

$$\lim_{t\to 0^+} \int_Y u(t,m) K(t,m) \mu^Y(m) = u(0,q)\,.$$

Proof. – Statement a) follows at once from the results of Section 2 applied to $\widetilde{Y}$ instead of X. The other points follow from Theorem 6.0 and the fact that the heat kernel $K_1(t,x,y)$ of $\widetilde{Y}$ is "almost Euclidean" (*see* [K]). □

Now we go back to the proof of Theorem 0.4 and assume, to begin with, that the function f is C^ω on X and orthogonal to the space of harmonic functions (that is, the constants): $\int_X f\mu = 0$. We recall that

$$(\Delta_2^Y + \partial_t)Tf(t,m) = 0 \quad \text{on } \mathbf{R}_+^* \times Y .$$

Then, for all $0 < t_1 < t_2$:

$$\int_{t_1}^{t_2}\int_Y K(t,m)(\Delta_2 + \partial_t)Tf(t,m)\,dt\,\mu^Y(m) = 0\,,$$

in which we integrate by parts. The following lemma is based on the Green's formula (1.6) for Δ^Y.

LEMMA 6.2. – *For all* $t \in [t_1, t_2]$

$$\int_Y K(t,m)\Delta_2^Y Tf(t,m)\mu^Y(m) = \int_Y Tf(t,m)\Delta_2^Y K(t,m)\mu^Y(m)$$
$$+ \int_{\partial Y}\big[Tf(t,\cdot)i^Y_{\mathrm{grad}^Y K(t,\cdot)}\,\mu^Y - K(t,\cdot)i^Y_{\mathrm{grad}^Y Tf(t,\cdot)}\,\mu^Y\big].$$

According to Theorem 6.0 a) $(\Delta_2^Y - \partial_t)K(t,m) \equiv 0$. Hence, integrating by parts with respect to t in the equality above shows that

$$(6.1)\qquad \begin{aligned} 0 &= \int_{t_1}^{t_2}dt\int_Y K(t,m)(\Delta_2^Y + \partial_t)Tf(t,m)\mu^Y(m)\\ &= \int_Y Tf(t_2,m)\,K(t_2,m)\mu^Y(m) - \int_Y Tf(t_1,m)\,K(t_1,m)\mu^Y(m)\\ &\quad+ \int_{t_1}^{t_2}dt\int_{\partial Y}\big[Tf(t,\cdot)i^Y_{\mathrm{grad}^Y K(t,\cdot)}\,\mu^Y - K(t,\cdot)i^Y_{\mathrm{grad}^Y Tf(t,\cdot)}\,\mu^Y\big].\end{aligned}$$

Since the kernel $K(t,m)$ satisfies equation (6.1) and all the estimates of Proposition 6.1, we proceed as in Section 5 (*see* in particular Remark 5.3) to prove

LEMMA 6.3.

$$\lim_{t_2\to+\infty}\int_Y Tf(t_2,m)K(t_2,m)\mu^Y(m) = 0$$

LEMMA 6.4.

$$\lim_{t_1\to 0^+}\int_Y Tf(t_1,m)K(t_1,m)\,\mu^Y(m) = f(q)$$

LEMMA 6.5. – *Assume that f "only" has Sobolev regularity: precisely $f \in H^{6n+4}(X)$. Then the following integral converges*

$$\int_0^{+\infty}dt\left|\int_{\partial Y}\big[Tf(t,\cdot)i^Y_{\mathrm{grad}^Y K(t,\cdot)}\,\mu^Y - K(t,\cdot)i^Y_{\mathrm{grad}^Y Tf(t,\cdot)}\,\mu^Y\big]\right| < +\infty.$$

Taking into account formula (6.1) and applying Lemma 6.3, 6.4 and 6.5 leads immediately to the inversion formula of Theorem 0.4. Assume next that the function f is not C^ω but "only" C^∞ on X. We apply Theorem 0.4 to $f_h = \exp(-\frac{1}{h}\Delta)f \in C^\omega(X)$ for all $h \in \mathbf{N}^*$. Observing that

$$f - f_h = \sum_{j\geq 1} a_j(1 - \exp(-\lambda_j/h))\phi_j$$

and proceeding as in the proofs of Lemmas 5.7 and 6.5 shows that dominated convergence arguments apply to prove that

$$\int_0^{+\infty} dt \int_{\partial Y} [Tf_h(t,\cdot) i^Y_{\mathrm{grad}^Y K(t,\cdot)}\,\mu^Y - K(t,\cdot) i^Y_{\mathrm{grad}^Y Tf_h(t,\cdot)}\,\mu^Y]$$

tends to

$$\int_0^{+\infty} dt \int_{\partial Y} [Tf(t,\cdot) i^Y_{\mathrm{grad}^Y K(t,\cdot)}\,\mu^Y - K(t,\cdot) i^Y_{\mathrm{grad}^Y Tf(t,\cdot)}\,\mu^Y]$$

as $h \to +\infty$. This completes the proof of Theorem 0.4.

7. Proof of Theorem 0.3

In this section we make no particular assumption on $-g^Y$. In Theorem 3.4 of Section 3 we have constructed a pseudo-heat kernel $K(t,p,m)$ on $]0,+\infty[\times Y \times Y$ for $\partial_t - \Delta^Y$ such that $K(t,m) = K(t,q,m)$ satisfies the estimates gathered in the following proposition (for any $\epsilon \in]0,\epsilon_1]$).

PROPOSITION 7.0.

a) For all smooth vector fields U on the manifold with boundary $\overline{Y}$ there exists a positive constant $D(U)$ such that:

$$\forall t \in]0,1], \quad \sup_{m\in\partial Y} |U \cdot K(t,m)| \leq D(U) t^{-1-n/2} e^{-\epsilon^2/4t}$$

(we recall that for all $m \in \partial Y$ $\epsilon^2 = -d^2(q,m)$).

b) There exists a positive constant R such that for all m in Y

$$\forall t \in]0,1], \quad |K(t,m)| \leq RG(t,q,m) = R\,(4\pi t)^{-\frac{n}{2}} e^{d^2(q,m)/4t}$$

c) For any function $(t,m) \to u(t,m)$ bounded and continuous on $[0,1] \times Y$

$$\lim_{t\to 0^+} \int_Y u(t,m) K(t,m) \mu^Y(m) = u(0,q).$$

We do not know whether $K(t,m)$ is bounded as $t \to +\infty$.

Next we go back to the proof of Theorem 0.3 and assume, to begin with, that the function f is real-analytic on X and orthogonal to the space of harmonic functions on X, that is, the constants: $\int_X f\mu = 0$. It was proved in Theorem 3.4 that $(\Delta_2^Y - \partial_t)K(t,m) \equiv 0$.

Therefore (proceeding as in the case of the equation (5.1)) integrating by parts with respect to t shows that for $0 < t_1 < t_2$,

$$\begin{aligned} (7.1) \qquad 0 &= \int_{t_1}^{t_2} dt \int_Y K(t,m)(\Delta_2^Y + \partial_t)Tf(t,m)\mu^Y(m) \\ &= \int_Y Tf(t_2,m)\,K(t_2,m)\mu^Y(m) - \int_Y Tf(t_1,m)\,K(t_1,m)\mu^Y(m) \\ &+ \int_{t_1}^{t_2} dt \int_{\partial Y} \left[Tf(t,\cdot) i^Y_{\mathrm{grad}^Y K(t,\cdot)}\,\mu^Y - K(t,\cdot) i^Y_{\mathrm{grad}^Y Tf(t,\cdot)}\,\mu^Y\right]. \end{aligned}$$

In the sequel we shall be working with some fixed $t_2 > 0$. Since the kernel $K(t,m)$ satisfies (7.1) and all the estimates of Proposition 7.0, one can proceed as in Section 5 (*see* Remark 5.3) to prove the two following lemmas.

LEMMA 7.1.

$$\lim_{t_1 \to 0^+} \int_Y Tf(t_1,m)K(t_1,m)\,\mu^Y(m) = f(q)\,.$$

LEMMA 7.2. – *Assume that f "only" has Sobolev regularity: $f \in H^{6n+4}(X)$. Then the following integral converges:*

$$\int_0^{t_2} dt \left| \int_{\partial Y} \left[Tf(t,\cdot) i^Y_{\mathrm{grad}^Y K(t,\cdot)}\,\mu^Y - K(t,\cdot) i^Y_{\mathrm{grad}^Y Tf(t,\cdot)}\,\mu^Y\right] \right| < +\infty$$

Taking into account the formula (7.1) and applying Lemma 7.1 and 7.2 leads immediately to the inversion formula of Theorem 0.3. Assuming that the function f is no longer real-analytic but only in $H^{6n+4}(X)$, one can apply Theorem 0.3 to the real-analytic functions on X $f_h = \exp(-\frac{1}{h}\Delta)f$, $h \in \mathbf{N}^*$. The proofs of Lemma 5.7, 6.5 and 7.2 show that dominated convergence arguments apply to show that

$$\int_0^{t_2} dt \int_{\partial Y} \left[Tf_h(t,\cdot) i^Y_{\mathrm{grad}^Y K(t,\cdot)}\,\mu^Y - K(t,\cdot) i^Y_{\mathrm{grad}^Y Tf_h(t,\cdot)}\,\mu^Y\right]$$

tends to

$$\int_0^{t_2} dt \int_{\partial Y} \left[Tf(t,\cdot) i^Y_{\mathrm{grad}^Y K(t,\cdot)}\,\mu^Y - K(t,\cdot) i^Y_{\mathrm{grad}^Y Tf(t,\cdot)}\,\mu^Y\right]$$

as $h \to +\infty$. This completes the proof of Theorem 0.3. □

8. The Case of the Symmetric Spaces

In this section we examine in more detail the consequences of the assumption that X is a locally symmetric space. Recall (Proposition 1.17) that this is the case precisely when $-g^Y$ is a field of real, positive definite quadratic forms on Y, for each fiber Y. Our main technical result in this section is Theorem 8.6, which shows (in the locally symmetric case) that the analytic continuation of the square of the distance function is negative one times the square of the distance function in the Riemannian manifold $(Y, -g^Y)$. We will use this in Theorem 8.9 to give a nice connection between the heat kernel of X and any "good" heat kernel of $(Y, -g^Y)$ (*see* Definition 8.8). This combined with the remarks following Definition 8.8, shows that there is a natural choice for the kernel $K(t, m)$ in Theorem 0.4, up to a term which is exponentially decreasing as $t \to 0^+$.

The square of the Riemannian distance function, d_X^2, is C^ω near the diagonal in $X \times X$. Identifying X with the zero section in $T^\epsilon X$, we can analytically continue d_X^2 to a holomorphic function on a neighborhood of the diagonal in $T^\epsilon X \times T^\epsilon X$ and restrict to the fiber $Y \times Y$ (after perhaps shrinking ϵ further). Let h be minus one times the restriction of d_X^2 to $Y \times Y$. It is not hard to show, as in the proof of Proposition 1.17, that if X is locally symmetric then h is real valued. To show that the distance function associated with $-g^Y$ is h we will need to understand more explicitly the holomorphic extension of X

Let X be a complete (not necessarily compact) locally Riemannian symmetric space (we will omit the adjective "Riemannian" from now on). The universal cover $\widehat{X}$ of X is isometric to a product:

$$\widehat{X} = M_o \times U/K_1 \times G_o/K_2 \tag{8.1}$$

where M_o is a Euclidean space and U/K_1, G_o/K_2 are globally symmetric spaces of the compact and non-compact type, respectively. By the proof of Proposition 4.2 in [He1], chap. V, we may assume that U, G_o, and K_i are connected. Furthermore we may assume that the center of G_o is trivial (since the center is contained in K_2), so that G_o is a linear group. Let $\widehat{G}$ be the product group, $U \times G_o$, K the product $K_1 \times K_2$ and $\widehat{\mathbf{g}}$ and $\mathbf{k}$ their respective Lie algebras, K is compact (*see* [He1], chap. VI Theorem 1.1, Chap VII Prop 1.1). We observe that $\widehat{G}$ is a connected, real, semisimple, closed (*) subgroup of $GL(N, \mathbf{C})$ for some N.

For any Lie group L there exists a unique (up to biholomorphism) complex Lie group $L_{\mathbf{C}}$, called the universal complexification of L, and a C^ω homomorphism ι from L into $L_{\mathbf{C}}$ with the following universal property: for every continuous homomorphism η from L into a complex Lie group H there exists a unique homomorphism $\eta_{\mathbf{C}}$ from $L_{\mathbf{C}}$ to H such that $\eta_{\mathbf{C}} \circ \iota = \eta$. The map ι need not be injective: for example, the complexification of the universal cover of $SL(2, \mathbf{R})$ is $SL(2, \mathbf{C})$, and the kernel of ι is the fundamental group of $SL(2, \mathbf{R})$. However for the linear group $\widehat{G}$ above ι is injective and embeds $\widehat{G}$ as a closed subgroup of $\widehat{G}_{\mathbf{C}}$ (*see* [Ho], chap. XVII.5). The inclusion of K in $\widehat{G}$ induces an isomorphism of $K_{\mathbf{C}}$ with the connected subgroup of $\widehat{G}_{\mathbf{C}}$ with Lie algebra $\mathbf{k}_{\mathbf{C}} := \mathbf{k} + \sqrt{-1}\,\mathbf{k}$, which we will identify with $K_{\mathbf{C}}$ (to *see* this note that the complex semisimple Lie group $\widehat{G}_{\mathbf{C}}$ is necessarily linear, so the induced homomorphism of $K_{\mathbf{C}}$ to $\widehat{G}_{\mathbf{C}}$ can be thought of as a linear representation of $K_{\mathbf{C}}$, faithful on K. By [Ho], chap. XVII.5, Theorem 5.2, it is faithful on $K_{\mathbf{C}}$).

(*) A connected semisimple subgroup of $GL(N, \mathbf{C})$ is necessarily closed (*see* [He1], chap. II, exercise D.4(iv)).

PROPOSITION 8.1. – *$K_{\mathbf{C}}$ is a closed subgroup of $\widehat{G}_{\mathbf{C}}$ and the natural map, $gK \to gK_{\mathbf{C}}$,* embeds *$\widehat{G}/K$ as a closed, totally real submanifold of $\widehat{G}_{\mathbf{C}}/K_{\mathbf{C}}$.*

Proof. – Since $\widehat{G}$ is linear, this is corollary 1 in section 3 of [Hz].

Let $G = \mathbf{R}^n \times \widehat{G}$ where $\mathbf{R}^n$ is the vector group appearing in the decomposition (8.1). The universal complexification of $\mathbf{R}^n$ is simply $\mathbf{C}^n$, and the dual of $\mathbf{R}^n$ is the complex axis $\sqrt{-1}\,\mathbf{R}^n$. The universal complexification of G is $G_{\mathbf{C}} := \mathbf{C}^n \times \widehat{G}_{\mathbf{C}}$, and the holomorphic $G_{\mathbf{C}}$ extension of $\widehat{X}$ (*see* [Hz], section 3) is the product homogeneous space $G_{\mathbf{C}}/K_{\mathbf{C}}$ (identifying K with the subgroup $\{0\} \times K$ of G and similarly for $K_{\mathbf{C}}$). The natural inclusion is a closed embedding. We will use $G_{\mathbf{C}}/K_{\mathbf{C}}$ to describe the adapted complex structure on $T^\epsilon \widehat{X}$.

The local symmetry on X induces an involution of $\mathbf{g}$ (the Lie algebra of G) whose $+1$ eigenspace is $\mathbf{k}$. Letting $\mathbf{e}$ denote the -1 eigenspace, we obtain a direct sum decomposition $\mathbf{g} = \mathbf{k} + \mathbf{e}$. The derivative of the natural projection $\widehat{\pi}$ from G to $\widehat{X}$ identifies $\mathbf{e}$ with the tangent space to the identity coset. Let $\tau(g)$ be the natural action of G on $\widehat{X}$. Every $v \in T\widehat{X}$ can be written as $V = d\tau(g)d\widehat{\pi}_e(V)$ for some (not unique) $g \in G$, $V \in \mathbf{e}$, and $d\tau(g)d\widehat{\pi}_e(V) = d\tau(g')d\widehat{\pi}_e(V')$ if and only if for some $k \in K$, $g' = gk$ and $V = \mathrm{Ad}(k)V'$.

PROPOSITION 8.2. – *There exists an $\epsilon > 0$ such that the map*

$$\Phi : T^\epsilon \widehat{X} \to G_c/K_c$$

$$d\tau(g)d\widehat{\pi}_e(V) \to g\exp\sqrt{-1}\,V \cdot K_{\mathbf{C}} \quad (V \in \mathbf{e})$$

is a G-equivariant diffeomorphism onto its image, and induces the adapted complex structure on $T^\epsilon \widehat{X}$.

Proof. – We recall that the geodesics on the symmetric space $\widehat{X}$ are the images of one–parameter subgroups of G under π (*see* [He1], chap. IV, §3 (3)). Thus $d\tau(g)d\pi(V) \to g\exp\sqrt{-1}\,V \cdot K_{\mathbf{C}}$ is the analytic continuation of the exponential map of $\widehat{X}$ into $G_{\mathbf{C}}/K_{\mathbf{C}}$ as in (8.1). Since $\widehat{X}$ covers a compact quotient, we can find a uniform ϵ as in the proposition. The map is clearly equivariant.

Remark 8.3. – Let $\mathbf{e}_* = \sqrt{-1}\,\mathbf{e}$ and let $\mathbf{g}_*$ be the subalgebra of $\mathbf{g}_{\mathbf{C}}$ given by $\mathbf{k} + \mathbf{e}_*$. The Lie algebras $\mathbf{g}$ and $\mathbf{g}_*$ are said to be dual to each other (*see* [He1], chap. V, §2). Let G_* denote the connected subgroup of $G_{\mathbf{C}}$ with Lie algebra $\mathbf{g}_*$. Since G_* is the product of $\sqrt{-1}\,\mathbf{R}^n$ and a semisimple group it is closed in $G_{\mathbf{C}}$. The natural map from G_*/K to $G_{\mathbf{C}}/K_{\mathbf{C}}$, $gK \to gK_{\mathbf{C}}$, is an embedding in a small neighborhood of the identity coset. After perhaps shrinking ϵ, the map Φ identifies the fiber over the identity coset in $T^\epsilon \widehat{X}$ with a neighborhood of the origin in the image of the dual symmetric space, G_*/K, in $G_{\mathbf{C}}/K_{\mathbf{C}}$. $\Phi(Y)$ is locally homogeneous, in the sense that every point has a neighborhood which is mapped to a neighborhood of the origin by an element of $G_* \subset G_{\mathbf{C}}$.

Remark 8.4. – If the negatively curved factor G_o/K_2 does not appear in the decomposition (8.1), then Φ extends to a global identification of $T\widehat{X}$ with $G_{\mathbf{C}}/K_{\mathbf{C}}$, and identifies $T_{eK}\widehat{X}$ with the dual symmetric space G_*/K.

Fixing $\epsilon > 0$ so that Proposition 8.2 and the Remark 8.3 following it are true, we can now prove the main result of this section.

THEOREM 8.5. – *Let X be a complete, connected, orientable locally symmetric space. Let $q \in X$ and Y, g^Y as in the beginning of this section. Then:*

1. *The Riemannian manifold $(Y, -g^Y)$ is isometric to a neighborhood of the identity coset in a symmetric space dual to the universal cover of X.*

2. *The restriction of the analytic continuation of $-d_X^2$ to $Y \times Y$ is equal to the square of the distance function of the Riemannian manifold $(Y, -g^Y)$.*

Proof. – The conclusions of the theorem are local assertions about (X, g). In fact, if an open set W in (X, g) is isometric to W' in (X', g'), then the lift of the isometry to the tangent bundles is a biholomorphic identification of the adapted complex structures of $T^\epsilon W$ and $T^\epsilon W'$, which will identify the analytic continuation of the respective distance functions and metric tensors. So we may replace X by its Riemannian universal cover and assume that $X = G/K$ equipped with a G–invariant symmetric metric.

Since G acts transitively by isometries, by the preceding remarks it suffices to prove the theorem for $q = eK$. We embed X in $G_{\mathbf{C}}/K_{\mathbf{C}}$, identify $T^\epsilon X$ with its image in $G_{\mathbf{C}}/K_{\mathbf{C}}$ as in Proposition 8.2, and Y with a neighborhood of the origin in the dual symmetric space.

Proof of 1. – We first show that g^+ extends to a $G_{\mathbf{C}}$–invariant holomorphic metric on $G_{\mathbf{C}}/K_{\mathbf{C}}$. By the uniqueness of analytic continuation, it suffices to show that there *exists* an invariant holomorphic metric whose pull-back to X is the given metric g. Let $Q_{\mathbf{e}}$ denote the $Ad(K)$–invariant inner product on $\mathbf{e}$ induced by the metric on X and $Q_{\mathbf{e_C}}$ its complex bilinear extension to $\mathbf{e_C} := \mathbf{e} + \sqrt{-1}\,\mathbf{e}$. It suffices to show that $Q_{\mathbf{e_C}}$ is $Ad(K_{\mathbf{C}})$–invariant, for then it induces a $G_{\mathbf{C}}$–invariant complex bilinear form on $T^{(1,0)}G_{\mathbf{C}}/K_{\mathbf{C}}$, which is necessarily holomorphic, and whose pullback to X is the given metric. But the adjoint representation is holomorphic and $Q_{\mathbf{e_C}}$ is invariant under the compact real form K, so it must be invariant under $K_{\mathbf{C}}$.

Let $Q_{\mathbf{e}_*}$ be the restriction of $Q_{\mathbf{e_C}}$ to $\mathbf{e}_*$. Then $-Q_{\mathbf{e}_*}$ is a positive definite, $Ad(K)$–invariant inner product and induces a G_*–invariant Riemannian symmetric metric on G_*/K. This in turn induces a metric on Y, invariant by the local G_* action, which agrees with $-g^Y$ at the origin. Thus $(Y, -g^Y)$ is isometric to a neighborhood of the origin in G_*/K, proving 1.

Proof of 2. – Recall that the group operations and the exponential mapping of a complex Lie group are holomorphic, and that the geodesics through the origin eK in G/K (resp. G_*/K) are the curves

$$t \mapsto \exp tV \cdot K$$

where $V \in \mathbf{e}$ (resp. $\mathbf{e}_*$) (*see* [He1], chap. V, §3 (3)). We will write down an explicit local expression for the analytic continuation of d_X^2 near $eK_{\mathbf{C}}$ and use the local homogeneity to show that it has the desired properties.

Choose a connected neighborhood $\mathcal{W}_{\mathbf{e_C}}$ of zero in $\mathbf{e_C}$ such that the map

$$A \in \mathcal{W}_{\mathbf{e_C}} \mapsto \exp A \cdot K_{\mathbf{C}}$$

is a biholomorphic diffeomorphism onto a neighborhood of $eK_{\mathbf{C}}$ in $G_{\mathbf{C}}/K_{\mathbf{C}}$, and such that the canonical projection $\pi_{\mathbf{C}}$ is a biholomorphism from $\exp \mathcal{W}_{\mathbf{e_C}}$ onto a neighborhood of

$eK_{\mathbf{c}}$ (for the existence of such a local cross section see [He1], chap. II, Lemma 4.1 and the remarks following). Choose a connected subneighborhood $\mathcal{V}_{\mathbf{e_c}} \subset \mathcal{W}_{\mathbf{e_c}}$ of zero with the property that

$$\exp - \mathcal{V}_{\mathbf{e_c}} \exp \mathcal{V}_{\mathbf{e_c}} \cdot K_{\mathbf{c}} \subset \exp \mathcal{W}_{\mathbf{e_c}} \cdot K_{\mathbf{c}}.$$

We can do this because if $\exp A \cdot K_{\mathrm{c}}$ and $\exp B \cdot K_{\mathbf{c}}$ are in $\exp \mathcal{W}_{\mathbf{e_c}} \cdot K_{\mathbf{c}}$, the map

$$(\exp A \cdot K_{\mathbf{c}},\ \exp B \cdot K_{\mathbf{c}}) \mapsto \exp -A \exp B \cdot K_{\mathbf{c}}$$

is well defined and continuous. If A, B are in $\mathcal{V}_{\mathbf{e_c}}$, there is a unique V in $\mathcal{W}_{\mathbf{e_c}}$ such that

$$\exp -A \exp B \cdot K_{\mathbf{c}} = \exp V \cdot K_{\mathbf{c}}.$$

Define a function H on the Cartesian product of $\exp \mathcal{V}_{\mathbf{e_c}} \cdot K_{\mathbf{c}}$ with itself by

$$H(\exp A \cdot K_{\mathbf{c}}, \exp B \cdot K_{\mathbf{c}}) = Q(V, V)_{\mathbf{e_c}}$$

where V is as above and $Q_{\mathbf{e_c}}$ is the complex bilinear extension of $Q_{\mathbf{e}}$ to $\mathbf{e_c}$. We will show that H is the analytic continuation of d_X^2.

We first claim that H is a holomorphic function. The map $A \mapsto \exp A \cdot K_{\mathbf{c}}$ provides holomorphic coordinates near $eK_{\mathbf{c}}$, and the form $Q_{\mathbf{e_c}}$ is complex bilinear; so the map

$$\exp V \cdot K_{\mathbf{c}} \mapsto Q(V, V)_{\mathbf{e_c}}$$

is holomorphic. To prove that H is holomorphic we must show that the map

$$(8.2) \qquad (\exp A \cdot K_{\mathbf{c}}, \exp B \cdot K_{\mathbf{c}}) \mapsto \exp -A \exp B \cdot K_{\mathbf{c}}$$

is holomorphic (as a map from $\exp \mathcal{V}_{\mathbf{e_c}} \cdot K_{\mathbf{c}} \times \exp \mathcal{V}_{\mathbf{e_c}} \cdot K_{\mathbf{c}}$ to $G_{\mathbf{c}}/K_{\mathbf{c}}$). Since the group operations on $G_{\mathbf{c}}$ are holomorphic and $\exp \mathcal{V}_{\mathbf{e_c}}$ is a complex submanifold of $G_{\mathbf{c}}$, the map

$$(\exp A, \exp B) \mapsto \exp -A \exp B$$

is holomorphic, as a map from $\exp \mathcal{V}_{\mathbf{e_c}} \times \exp \mathcal{V}_{\mathbf{e_c}}$ to $G_{\mathbf{c}}$. Since the projection $\pi_{\mathbf{c}}$ is a biholomorphic identification of $\exp \mathcal{W}_{\mathbf{e_c}}$ and $\exp \mathcal{W}_{\mathbf{e_c}} \cdot K_{\mathbf{c}}$, we can write the map (8.2) as the composition of holomorphic maps, verifying our claim that H is holomorphic. Since $\exp \mathcal{V}_{\mathbf{e_c}} \cdot K_{\mathbf{c}}$ is a neighborhood of $eK_{\mathbf{c}}$ in $G_{\mathbf{c}}/K_{\mathbf{c}}$, it contains the image of a neighborhood of eK in G/K and G_*/K. We will show that there is a neighborhood of eK in G/K (resp. G_*/K) such that the restriction of H to this neighborhood (*) is d_X^2 (resp. $-d_{X_*}^2$, the square of the distance function on $X_* := G_*/K$). We can find a neighborhood $\mathcal{U}_{\mathbf{e}}$ of zero in $\mathcal{V}_{\mathbf{e_c}} \cap \mathbf{e}$ and a neighborhood $\mathcal{U}_{\mathbf{k}}$ of zero in $\mathbf{k}$, the Lie algebra of K, such that $(V, W) \mapsto \exp V \exp W$ is a C^ω diffeomorphism from $\mathcal{U}_{\mathbf{e}} \times \mathcal{U}_{\mathbf{k}}$ onto a neighborhood of e in G, and such that the canonical projection maps $\exp \mathcal{U}_{\mathbf{e}}$ diffeomorphically onto a

(*) (*i.e.*, the image of this neighborhood in $\exp \mathcal{V}_{\mathbf{e_c}} \cdot K_{\mathrm{c}}$)

neighborhood of eK in G/K. We may further assume (shrinking $\mathcal{U}_{\mathbf{e}}$) that for each $V \in \mathcal{U}_{\mathbf{e}}$, the distance minimizing geodesic path from eK to $\exp V \cdot K$ is

$$t \mapsto \exp tV \cdot K, \quad 0 \leq t \leq 1.$$

Finally let $\mathcal{V}_{\mathbf{e}}$ be a subneighborhood of zero in $\mathcal{U}_{\mathbf{e}}$ such that

$$\exp -\mathcal{V}_{\mathbf{e}} \exp \mathcal{V}_{\mathbf{e}} \subset \exp \mathcal{U}_{\mathbf{e}} \exp \mathcal{U}_{\mathbf{k}}$$

We have now arranged that for A, B in $\mathcal{V}_{\mathbf{e}}$, the unique V in $\mathcal{W}_{\mathbf{e_c}}$ satisfying

$$\exp -A \exp B \cdot K_{\mathbf{c}} = \exp V \cdot K_{\mathbf{c}}$$

is in $\mathbf{e}$. Since the restriction of $Q_{\mathbf{e_c}}$ to $\mathbf{e}$ is the original inner product $Q_{\mathbf{e}}$, we conclude that for A, B in $\mathcal{V}_{\mathbf{e}}$,

$$H(\exp A \cdot K_{\mathbf{c}}, \exp B \cdot K_{\mathbf{c}}) = Q(V, V)_{\mathbf{e}}.$$

We can also find neighborhoods $\mathcal{U}_{\mathbf{e}_*}$, $\mathcal{V}_{\mathbf{e}_*}$ of zero in $\mathcal{V}_{\mathbf{e_c}} \cap \mathbf{e}_*$ with the corresponding properties, since we have only used the local structure of homogeneous spaces. For A, B in $\mathcal{V}_{\mathbf{e}_*}$,

$$H(\exp A \cdot K_{\mathbf{c}}, \exp B \cdot K_{\mathbf{c}}) = Q(V, V)_{\mathbf{e}_*}, \tag{8.3}$$

where $V \in \mathbf{e}_*$ and the right hand side is the restriction of $Q_{\mathbf{e_c}}$ to $\mathbf{e}_*$.

The action of G on X preserves distances so for A, B in $\mathcal{V}_{\mathbf{e}}$,

$$\begin{aligned} d_X^2(\exp A \cdot K, \exp B \cdot K) &= d_X^2(eK, \exp -A \exp B \cdot K) \\ &= d_X^2(eK, \exp V \cdot K) \\ &= \text{square of the Euclidean length of } V \text{ in } \mathbf{e} \\ &= Q(V, V)_{\mathbf{e}}, \end{aligned}$$

proving that H is the analytic continuation of d_X^2. Under the identification Φ of Y with G_*/K, $\exp \mathcal{V}_{\mathbf{e}_*} \cdot K_{\mathbf{c}}$ corresponds to a fiber neighborhood of zero in Y. Restricting H to $\exp \mathcal{V}_{\mathbf{e}_*} \cdot K_{\mathbf{c}}$ we obtain the expression (8.3). On the other hand, the distance between points in G_*/K can be computed as above, except that the square of the Euclidean length of V in $\mathbf{e}_*$ is $-Q(V, V)_{\mathbf{e}_*}$. □

Exemple 8.6. – $X = S^n = \{x \in \mathbf{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1\}$ endowed with the usual metric. The complexification of S^n is the complex quadric

$$Q_{\mathbf{C}}^n = \{z \in \mathbf{C}^{n+1} : z_0^2 + \cdots + z_n^2 = 1\}.$$

The identification of $TS^n = \{(x, v) \in S^n \times \mathbf{R}^{n+1} : x \cdot v = 0\}$ with $Q_{\mathbf{C}}^n$ is

$$(x, v) \mapsto (\cosh |v|)x + \sqrt{-1}\,(\sinh |v|)\frac{v}{|v|}.$$

Let $e_0, \ldots, e_n$ be the standard basis vectors of $\mathbf{R}^{n+1}$. The fiber $T_{e_o} S^n$ can be identified with the hyperbolic space form

$$Q^{(1,n)} = \{-x_o^2 + x_1^2 + \cdots + x_n^2 = -1\} \cap \{x_o \geq 1\}$$

by the map

$$(x_o, x_1, \ldots, x_n) \mapsto (x_o, \sqrt{-1}\,x_1, \ldots, \sqrt{-1}\,x_n).$$

The analytic continuation of the distance function on S^n is

$$d^2_{S^n}(z, w) = 4\left\{\sin^{-1} \frac{1}{2}\sqrt{\sum (z_i - w_i)^2}\right\}^2.$$

Since $(\sin^{-1} t)^2$ is an *even* analytic function of t, $d^2_{U/K}$ is an analytic function of $\sum(z_i - w_i)^2$. It is invariant under the diagonal action of the complexified group $SO(n+1, \mathbf{C})$, and so it is invariant under the action of the subgroup isomorphic to $SO(1, n)$ preserving $Q^{(1,n)}$. The geodesic parameterized by arclength through e_o in $Q^{(1,n)}$ with tangent vector e_1 is

$$t \mapsto (\cosh t, \sinh t, 0, \ldots, 0).$$

Since $Q^{(1,n)}$ is a rank one symmetric space, all other unit speed geodesics are obtained by the action of $SO(1, n)$. Restricting $d^2_{S^n}$ to the image geodesic $w = (\cosh t)e_o + \sqrt{-1}\,(\sinh t)e_1$ in $Q^n_{\mathbf{C}}$ we obtain

$$d^2_{S^n}(e_o, w) = 4(\sin^{-1} \frac{1}{2}\sqrt{2 - 2\cosh t})^2.$$

Since $\sqrt{2 - 2\cosh t} = -2\sqrt{-1}\,\sinh \frac{t}{2}$,

$$d^2_{S^n}(e_o, w) = -t^2,$$

where t is the hyperbolic distance from e_o to w.

Now, let (X, g) be a compact locally symmetric space, and let $(Y, -g^Y)$ be "the dual space" considered in Theorem 8.5. We recall that

$$Y = Y_\epsilon = \{\mathrm{Exp}_q \sqrt{-1}\xi : \ \xi \in T_q X, \ |\xi| < \epsilon\} \tag{8.4}$$

We are going to use Theorem 8.5 to establish a nice connexion between the heat kernel of X and any good heat kernel (*see* Definition 8.7 below) of $(Y, -g^Y)$.

Definition 8.7. – We say that $K_1(t, x, y) \in C^0(R^{+*} \times Y \times Y)$ is a good heat kernel of $(Y, -g^Y)$ if the four following properties hold:

1) $K_1(t, x, y)$ is of class C^1[resp. C^2] with respect to $t > 0$ [resp. $y \in Y$].

2) For any $x \in Y$, $(t, y) \to K_1(t, x, y)$ is a solution of the heat equation of $(Y, -g^Y)$ and $K_1(t, x, \cdot) \rightharpoonup \delta_x$ as $t \to 0^+$ (weak convergence on the vector space of continuous functions on $\overline{Y}$: i.e up to the boundary).

3) $K_1(\cdot,\cdot,\cdot)$ is bounded on $[1,+\infty[\times Y\times Y$.

4) For any smooth partial differential operator P on $\overline{Y}$ of the first order we have:

$$P\big[K_1(t,x,\cdot)\big](y)\sim P\big[(4\pi t)^{-n/2}e^{-d^2(x,\cdot)/4t}\big](y)\quad \text{as } t\to 0^+,$$

uniformly with respect to $(x,y)\in Y\times Y$.

It is well known that the heat kernel of a compact Riemannian manifold satisfies the four properties of Definition 8.7. Thanks to Theorem 8.5, we can isometrically identify Y with a neighborhood of the "dual" symmetric space, whose universal cover is a product of symmetric spaces of compact and non-compact type. The noncompact ones cover a compact quotient (*see* [Bo]), so if Y is small enough, it can be isometrically embedded in a compact Riemannian manifold. Therefore, if Y is small enough, it has a good heat kernel.

The next theorem combined with Theorem 0.1 and Definition 4.3 shows, roughly speaking, that one goes from the heat kernel $E(t,x,y)$ of X to a good heat kernel $K_1(t,x,y)$ of $(Y,-g^Y)$ modulo an exponentially decreasing term by replacing t by minus t. Moreover, $K'=(\sqrt{-1})^{-n}K_1$ may be used in the inversion formula (0.5).

THEOREM 8.8. – *Let $K_1(t,x,y)$ be a good heat kernel of $(Y,-g^Y)$. With the notations of Theorem 0.1, there is $\delta>0$ and an open neighborhood Y' of q in Y such that for all $(x,y)\in Y'^2$:*

$$K_1(t,x,y)=(4\pi t)^{-n/2}e^{d_X^2(x,y)/4t}\sum_{k\le\frac{1}{Ct}}u_k(x,y)\,(-t)^k\;+O\big(\exp(-\delta/t)\big)$$

as $t\to 0^+$, the $O(\cdot)$ being uniform with respect to (x,y).

Note. – With the notation of formula (8.4) we can assume that $Y'=Y_{\epsilon'}$ for some $\epsilon'\in]0,\epsilon[$.

Proof. – By Theorem 0.3 and equation (0.4), there exists a real $\rho>0$ and of a function $K(t,x,y)\in C^o(]0,+\infty[\times Y\times Y)$ which is of class C^1 [resp. C^2] with respect to t [resp. y] such that $(\partial_t-\Delta_2^Y)K(t,x,\cdot)\equiv 0$ and:

$$\begin{aligned}(8.5)\quad K_2(t,x,y) &:= (\sqrt{-1})^nK(t,x,y)\\ &= (4\pi t)^{-n/2}e^{d_X^2(x,y)/4t}\sum_{k\le\frac{1}{Ct}}u_k(x,y)\,(-t)^k\;+O\big(\exp(-\rho/t)\big)\end{aligned}$$

uniformly with respect to $(x,y)\in Y\times Y$ as $t\to 0^+$. But, Theorem 8.6 shows that for $(x,y)\in Y\times Y$, $d_X^2(x,y)=-d_Y^2(x,y)$; moreover $u_0(x,x)\equiv 1$. Therefore for each $x\in Y$ we can use the exponential normal coordinate system centered at x of $(Y,-g^Y)$ to *see* (as in [B-G-M] page 208) that $K_2(t,x,\cdot)\rightharpoonup\delta_x$ as $t\to 0^+$. So $K_2(t,x,y)$ is a "fundamental solution" of the heat equation of $(Y,-g^Y)$. Therefore for each $x\in Y$, $(t,y)\to\big(K_1-K_2\big)(t,x,y)$ is a solution of the heat equation with initial data equal to 0. We can shrink Y so that K_1-K_2 is continuous on $]0,+\infty[\times\overline{Y}\times\overline{Y}$. Using the equation (8.5) and the properties (*see* Definition 8.7) of $K_1(t,x,y)$ one checks easily the existence of a real $\delta \in]0, \rho]$ and an open (relatively compact in Y) neighborhood Y' of q in Y and of a positive constant C such that:

$$\forall t \in]0,1], \quad \forall (x,y) \in Y' \times \partial Y, \quad \sup_{0<s<t} \big|K_1(s,x,y) - K_2(s,x,y)\big| \leq C\, e^{-\delta/t}.$$

Using the maximum principle for the heat equation one sees easily that:

$$\forall (x,y) \in Y'^2, \quad \big|K_1(t,x,y) - K_2(t,x,y)\big| = O\big(\exp\left(-\rho/t\right)\big)$$

as $t \to 0^+$. The Theorem 8.9 then follows from the equation (8.5). □

9. Appendix. The Rank One Case

In this section we assume that the n-dimensional Riemannian manifold (X, g) is a compact globally symmetric space G/K of rank one. As usual we write $\mathbf{g} = \mathbf{k} + \mathbf{p}$ where $\mathbf{p}$ is the orthogonal complement to $\mathbf{k}$ with respect to the Killing form on $\mathbf{g}$ (the Killing form B is negative definite on $\mathbf{g}$). We identify the inner product space $(T_o G/K, g_o)$ with $(\mathbf{p}, -B|_{\mathbf{p}})$ via the natural projection $\pi : G \to G/K$. Since $-B|_{\mathbf{p}}$ is invariant under $\mathrm{Ad}\,(K)$ (in fact B is invariant under $\mathrm{Ad}\,(G)$), this identification gives rise to a G-invariant metric on G/K which, under suitable assumptions on the pair (G, K), turns G/K into a compact globally symmetric space of rank one. We recall that the geodesics on G/K are the projections by π of the one parameter subgroups of G.

The following theorem will be proved at the end of this section and shows explicitly that, in this case, the "formal solution" of the heat equation satisfies the estimate of Proposition 3.1 and has nice geometric properties.

THEOREM 9.1. – *There exists two positive constants R, P and for each nonnegative integer j a function $z \to a_j(z)$ holomorphic and bounded by $P^{j+1} j!$ on the open disk $D(0,R)$ in* $\mathbf{C}$ *such that $a_0(0) \neq 0$ and $F(t,x,y) = (4\pi t)^{-n/2} e^{-d^2(x,y)/4t} \sum_{j\geq 0} a_j(d^2(x,y)) t^j$ is the formal solution of the heat equation on the domain $\mathbf{R}_+^* \times \{(x,y) \in X \times X; d^2(x,y) < R\}$:* i.e., *the equations (4.3) and $(4.0)_k$ are satisfied for each nonnegative integer k.*

Remark. – The proof will provide an algorithm which allows us to compute recursively the functions $a_j(z)$ and will show that these functions are determined by three invariants: λ, p, q, (*see* [He2] page 164) of the root system associated with X. In general this expansion does not converge and the equation (9.3) of the algorithm shows that the estimation of theorem 9.1 for the a_j is probably the best possible.

Parts 1) and 2) of the next proposition give crucial geometric information for the proof of Theorem 9.1, moreover its part 3) improves (in the case of a compact symmetric space of rank one!) part i) of Theorem 0.1.

PROPOSITION 9.2:

1. *Let L be the diameter of X. There exists an entire holomorphic function on* $\mathbf{C}$, *$z \to b(z)$, such that for any $x \in X$ the normal exponential system coordinate centered at*

x is defined on $\{y \in X : d(x,y) < L\}$ and the Riemannian volume (in this coordinate system) is defined by:

$$\sqrt{\det g_{ij}(y)} = b(d^2(x,y))$$

Moreover we have $b(r^2) = \left(\frac{\sin r\lambda}{r\lambda}\right)^p \left(\frac{\sin 2r\lambda}{2r\lambda}\right)^q$, *where the real λ and the nonnegative integers p and q are determined from the root system associated with X* (see *[He2] page 164).*

2. *There exists a positive real R_1 and a numerical function $(t,s) \to K(t,s)$ defined on $\mathbf{R}_+^* \times \mathbf{R}_+$ such that the heat kernel $E(t,x,y)$ of X is given by $K(t, d^2(x,y))$, and for any $t > 0$ $K(t,\cdot)$ may be extended as a holomorphic function on the open disk $D(0,R_1)$.*

Remark. – In the proof of 3) (and of 3) only) we use the result of Theorem 9.1. The fact that $\sqrt{\det g_{ij}(y)}$ is a holomorphic function of the square of the distance from x to y will be crucial in the sequel.

Proof of 1). – Fix $x \in X$ and let f be an element of K_x^o, the identity component of the isotropy group at x. For y in any geodesic ball about x where normal coordinates are defined, let $\overline{g}(y) = \det g_{ij}(y)$ where the matrix g_{ij} is expressed in normal coordinates determined by an orthonormal frame $e_1, \ldots, e_n$ for T_xX. We claim that $\overline{g} \circ f = \overline{g}$. Since f is in K_x^o, it preserves the normal coordinate system and its orientation.

Let $x^i(q) = g((\mathrm{Exp}_x)^{-1}(q), e_i)_x$ be the normal coordinates and let $\omega = \sqrt{\overline{g}}\, dx^1 \wedge \cdots \wedge dx^n$ be the Riemannian volume form. Since f is an orientation preserving isometry, $f^*\omega = \omega$. *i.e.*,

$$\sqrt{\overline{g}} \circ f\, d(x^1 \circ f) \cdots \wedge d(x^n \circ f) = \omega.$$

Let $y^i = x^i \circ f$. Since $f \circ \mathrm{Exp}_x = \mathrm{Exp}_x \circ df_x$,

$$y^i(q) = g(df_x \circ (\mathrm{Exp}_x)^{-1}(q), e_i)_x = g((\mathrm{Exp}_x)^{-1}(q), (df_x)^t e_i)_x.$$

Since df_x is an orientation preserving linear isometry of T_xX, the $y^i(q)$ are related to the $x^i(q)$ by a matrix in $SO(n)$. Thus $dy^1 \wedge \cdots dy^n = dx^1 \wedge \cdots dx^n$, and $\overline{g} \circ f = \overline{g}$.

If X is a rank one symmetric space, the linear action of K_x^o is transitive on the unit sphere (*See* [W], Theorem 8.12.2). So K_x^o acts transitively on the geodesic spheres centered at x in the normal coordinate neighborhood. Since $\overline{g}$ is constant on each such sphere, it is a function of the distance $d(x,y)$ only.

It is well known that for any $x \in X$, the exponential map is a diffeomorphism from the ball of radius $L =$ diameter of X in T_xX to its image (*see* [He2], chapter I, §4.2). As we will see, it is easy to understand how $\overline{g}$ depends on $d(x,y)$. Since $d(x,y)$ is invariant under the diagonal action of G on $X \times X$, we may assume that $x = o\,(= K)$. We recall Helgason's formula for the derivative of the exponential map (*see* [He1], chapter IV, §4): for $V \in \mathbf{p}$,

$$d\,(\mathrm{Exp}_o)_{d\,\pi(V)} = d\,\tau(\exp V) \circ d\,\pi\left(\sum_0^\infty \frac{1}{(2n+1)!} T_V^n\right) := d\,\tau(\exp V) \circ d\,\pi(A_V)$$

where T_V is the linear operator on $\mathbf{p}$, $\mathrm{ad}\,(V)^2|_{\mathbf{p}}$,

$$A_V = \sum_0^\infty \frac{1}{(2n+1)!} T_V^n,$$

and $\tau(\exp V)$ is the map: $x \to \exp Vx$. Then setting $x = o$, $y = \mathrm{Exp}_o(d\,\pi(V))$, $\partial/\partial x_i|_y = d\,(\mathrm{Exp}_o)_{d\,\pi(V)}(e_i)$, and E_i the basis of $\mathbf{p}$ corresponding to e_i,

$$\begin{aligned} g_{ij}(y) &= g(d\,\tau(\exp V) \circ d\,\pi(A_V E_i), d\,\tau(\exp V) \circ d\,\pi(A_V E_j))_{\tau(\exp V)\cdot o} \\ &= g(d\,\pi(A_V E_i), d\,\pi(A_V E_j))_o \\ &= -B(A_V E_i, A_V E_j). \end{aligned}$$

Since B is $\mathrm{ad}\,(\mathbf{g})$ invariant, A is symmetric. Thus

$$\sqrt{\det g_{ij}}(\mathrm{Exp}_o d\,\pi(V)) = |\det A_V|.$$

We evaluate this determinant as in [He2], loc. cit., and find that there is a real number λ and integers p and q, with $p + q = n - 1$, such that

$$\sqrt{\det g_{ij}} = \Big(\frac{\sin r\lambda}{r\lambda}\Big)^p \Big(\frac{\sin 2r\lambda}{2r\lambda}\Big)^q$$

where $r = d(x, y) < L$. The parameters p, q, and λ are determined from the root system associated with the X. From this it is clear that $\sqrt{\det g_{ij}}$ is a function of $d^2(x, y)$ and 1) is proven.

Proof of 2). – It is well known that $E(t, x, y)$ is invariant under the diagonal action of the isometry group on $X \times X$. Fixing x and t, we get a function $E_{x,t}(y)$ on X which is invariant under the action of the isotropy group at x. By the argument above, $E_{x,t}(y)$ depends only on $d(x, y)$, so there is a numerical function $K(t, s)$ defined on $]0, +\infty[\times[0, +\infty[$ such that $E(t, x, y) = K(t, d^2(x, y))$. Therefore for $\xi \in T_x X$ small enough, we have: $E(t, x, \mathrm{Exp}_x \xi) = K(t, |\xi|^2)$. Proposition 2.4 shows that for any $(t, x) \in \mathbf{R}_+^* \times X$, $\xi \to E(t, x, \mathrm{Exp}_x \xi)$ is holomorphic on a complex neighborhood (which does not depend on (t, x)) of the origin of $T_x^{\mathbf{C}} X$; using the expansion in power series in (real) $\xi \in T_x X$, we easily see that this function is even with respect to ξ. Let $(e_1, \ldots, e_n)$ be an orthonormal basis of $T_x X$. Then, considering vectors of the form $\xi = \xi_1 e_1 + \xi_1 e_2 \cdots + \xi_1 e_n$ we see easily that there exists $R_1 > 0$ such that for any $t > 0$ $K(t, \cdot)$ may be extended as a holomorphic function on $D(0, R_1)$. □

Now we use the construction of the formal solution of the heat equation given in [B-G-M], page 208. Let us fix $x \in X$ and consider a normal exponential system of coordinates centered at x. Their function θ satisfies the relation $\theta(y) \equiv b(d^2(x, y))$ where $b(z)$ has been introduced in Proposition 9.2 (*see* [B-G-M], page 55).

X being compact globally symmetric of rank one, we look for a formal solution of the form:

$$F(t, x, y) = (4\pi t)^{-n/2} e^{-d^2(x,y)/4t} \sum_{j \geq 0} a_j(d^2(x, y)) t^j.$$

The equations of [B-G-M], page 208, then show that:

$$a_0(d^2(x, y)) = b^{-\frac{1}{2}}(d^2(x, y)) \tag{9.0}$$

and that for each integer $k \geq 1$:

$$(9.1) \qquad r\partial_r\big[a_k(d^2(x,y))\big] + \left(\frac{r}{2}\frac{\partial_r\theta}{\theta} + k\right)a_k + \Delta_2\big[a_{k-1}(d^2(x,y))\big] \equiv 0.$$

Proposition 9.2 allows us to give the following definition:

DEFINITION 9.3. – Let us fix $R_2 > 0$ such that there exists a function $U(z) = \sum_{p=1}^{+\infty} U_p z^p$ holomorphic and bounded by M on the disk $D(0, R_2)$ so that for $y \in X$ close to x we have:

$$U(d^2(x,y)) = d(x,y)\frac{\partial_r\theta}{\theta(y)}.$$

Let us recall that when applied to radial functions (which depend only on $d(x,y)$) the Laplacian is equal to $\Delta_2 = -\frac{\partial^2}{\partial r^2} - (\frac{\partial_r\theta}{\theta} + \frac{n-1}{r})\frac{\partial}{\partial r}$. So equation (9.1) is equivalent to the following one (with $k \geq 1$):

$$(9.2) \qquad \begin{aligned} &2z^2\partial_z a_k(z^2) + \left(\frac{1}{2}U(z^2) + k\right)a_k(z^2) \\ &= 2(n + U(z^2))\partial_z a_{k-1}(z^2) + 4z^2\partial_z^2 a_{k-1}(z^2). \end{aligned}$$

Now for each nonnegative integer k let us write: $a_k(z) = \sum_{p\geq 0} a_{k,p}z^p$. Then, using the definition 9.3 we see that for each integer $k \geq 1$ the equation (9.2) is equivalent to:

$$\begin{aligned} &ka_{k,0} = 2na_{k-1,1}, \quad \text{and for each p} \geq 1: \\ &(2p+k)a_{k,p} + \frac{1}{2}\sum_{j=0}^{p-1} U_{p-j}a_{k,j} \\ &\quad = 2n(p+1)a_{k-1,p+1} + 2\sum_{j=0}^{p-1}(j+1)U_{p-j}a_{k-1,j+1} + 4(p+1)pa_{k-1,p+1}. \end{aligned}$$

It is clear that these last equations (for $k \geq 1$) are equivalent to the following ones:

$$(9.3) \qquad \begin{aligned} a_{k,0} &= \frac{2n}{k}a_{k-1,1}, \quad \text{and for each p} \geq 1: \\ a_{k,p} &= \frac{-1}{2(2p+k)}\sum_{j=0}^{p-1} U_{p-j}a_{k,j} \\ &+ \frac{2n(p+1)}{2p+k}a_{k-1,p+1} + \frac{2}{2p+k}\sum_{j=0}^{p-1}(j+1)U_{p-j}a_{k-1,j+1} \\ &+ \frac{4(p+1)p}{2p+k}a_{k-1,p+1}. \end{aligned}$$

Since $a_0(z)$ is given by the equation (9.0), we see that the equations (9.3) define by induction a unique sequence of formal power series $\{a_k(z)\}_{k\geq 1}$.

Now we are going to prove Theorem 9.1 by using the classical method of the majorizing series. We use the notation $<<$ as in Definition 3.2. Let us fix $R_1 > 0$ such that $a_0(z) \equiv b^{\frac{-1}{2}}(z)$ (*see* (9.0)) is holomorphic and bounded on the open disk $D(0, R_1)$ of C of center 0 and such that:

$$M \sum_{i=1}^{+\infty} \left(\frac{R_1}{R_2} \right)^i \leq \frac{1}{2} \tag{9.4}$$

where R_2 and M have already been introduced in the definition 9.3.

DEFINITION 9.4. – For each nonnegative integer k, we define $\Phi_k(z) =$ by

$$\Phi_k(z) := \frac{k! R_1^{-k}}{\left(1 - \frac{z}{R_1}\right)^{k+1}} = R_1^{-k} \sum_{p \geq 0} (p+k) \ldots (p+1) \left(\frac{z}{R_1} \right)^p.$$

We have $\partial_z \Phi_k(z) = \Phi_{k+1}(z)$. Since $a_0(z)$ is bounded on $D(0, R_1)$ we can fix a constant $C > 0$ such that $a_0(z) << C\Phi_0(z)$ and $\frac{1}{2}C > 2n + 4 + \frac{1}{2}$. Theorem 9.1 is a consequence of the following proposition the easy proof of which is left to the reader.

PROPOSITION 9.5. – *The formal power series $a_0(z)$, defined by the equation (9.0) and $a_k(z)$, $k \geq 1$ defined recursively by the equations (9.3) satisfy for each $k \geq 0$ the inequality: $a_k(z) << C^{k+1} \Phi_k(z)$.*

Note. – We get Theorem 9.1 by letting $R = \frac{R_1}{2}$ and $P = 2CR_1^{-1}$.

REFERENCES

[B-L-R] C. BARDOS, G. LEBEAU and J. Rauch, *Scattering Frequencies and Gevrey 3 Singularities* (*Inv. Math.*, Vol. 90, 1987, pp. 77-114).

[B-G-M] M. BERGER, P. GAUDUCHON and E.MAZET, *Le Spectre d'une Variété Riemannienne* (*Lecture Notes in Mathematics*, 194, Springer Verlag).

[BdM] L. BOUTET DE MONVEL, *Convergence dans le domaine complexe des séries de fonctions propres* (*C. R. Acad. Sc. Paris,* T. 287, Series A, 1978, pp. 855-856).

[Bo] A. BOREL, *Compact Clifford-Klein forms of symmetric spaces* (*Topology*, Vol. 2, 1963, pp. 111-122).

[B-W] F. BRUHAT and H. WHITNEY, *Quelques propriétés fondamentales des ensembles analytiques-réels* (*Commentarii Math. Helvet.*, Vol. 33, 1959, pp. 132-160).

[C-P] CHAZARAIN-PIRIOU, *Introduction à la Théorie des Équations aux Dérivées Partielles Linéaires*, Gauthier-Villars.

[C] A. CONNES, *Noncommutative Geometry* (*Academic Press*, New York 1994).

[D] J-M. DELORT, *FBI Transformation, Second Microlocalization and Semilinear Caustics*, (*Lecture Notes in Mathematics*, 1522, Springer Verlag).

[D-G] J. DUISTERMAAT and V. GUILLEMIN, *The Spectrum of Positive Elliptic Operators and Periodic Bicharacteristics* (Inv. Math., Vol. 29, 1975, pp. 39-79).

[[E-M] C. EPSTEIN and R. MELROSE, *Shrinking tubes and the ∂-Neumann problem*, preprint 1991.

[G] V. GUILLEMIN, Toeplitz *Operators in n-dimensions, Integral Equations and Operator Theory*, Vol. 7, 1984, pp. 145-205.

[G-S1] V. GUILLEMIN and M. STENZEL, *Grauert tubes and the homogeneous Monge-Ampère equation. I* (*J. Differential Geometry*, Vol. 34, 1991, pp. 561-570).

[G-S2] V. GUILLEMIN and M. STENZEL, *Grauert tubes and the homogeneous Monge-Ampère equation. II* (*J. Differential Geometry*, Vol. 35, 1992, pp. 627-641).

[He1] S. HELGASON, *Differential Geometry, Lie Groups, and Symmetric Spaces*, 2nd ed. (*Academic Press*, 1978).

[He2] S. HELGASON, *Groups and Geometric Analysis* (*Academic Press*, 1984).

[Hz] P. HEINZNER, *Equivariant holomorphic extensions of real analytic manifolds*, (*Bull. Soc. Math. France*, Vol. 121, 1993, pp. 445-463).

[Ho] G. HOCHSCHILD, *The Structure of Lie Groups* Holden-Day, 1965.

[K] Y. KANNAI, *Off diagonal short time asymptotics for fundamental solutions of diffusion equations* (*Communication in P.D.E.*, Vol. 2, 1977, pp. 781-830).

[L1] G. LEBEAU, *Contrôle analytique I: estimations* a priori (*Duke Math. J.*, Vol. 68, 1992, pp. 1-30).

[L2] G. LEBEAU , *Deuxième microlocalisation sur les sous-variétés isotropes* (*Annales de l'Institut Fourier*, Vol. 35, 1985, pp. 145-217).

[Le] E. LEICHTNAM, *Le problème de Cauchy ramifié linéaire pour des données à singularités algébriques* (*Mémoire de la SMF* n^0 53, 1993).

[L-S] L. LEMPERT and R. SZÖKE, *Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundle of Riemannian manifolds* (*Math. Ann.*, Vol. 290, 1991, pp. 689-712).

[Sj] SJÖSTRAND, *Singularités analytiques microlocales* (*Astérisque* 95, 1982.)

[W] J. WOLF, *Spaces of Constant Curvature*, 2nd ed. Publish or Perish 1972.

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