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The Hochschild cohomology ring of regular maximal primitive quotients of enveloping algebras of semisimple Lie algebras


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THE HOCHSCHILD COHOMOLOGY RING
OF REGULAR MAXIMAL PRIMITIVE QUOTIENTS
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OF SEMISIMPLE LIE ALGEBRAS

BY WOLFGANG SOERGEL

ABSTRACT. – Let $U$ be the enveloping algebra of a semisimple complex Lie algebra, $\chi$ a regular maximal ideal of its center. We show that the Hochschild cohomology ring of $U/\chi U$ is just the coinvariant algebra of the Weyl group. This turns out to be almost immediate if one uses localization.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, $U = U(\mathfrak{g}) \supset Z$ the enveloping algebra with its center. Let $S = S(\mathfrak{h}^*)$ be the regular functions on $\mathfrak{h}$ and $S^+ \subset S$ the maximal ideal of all functions vanishing at $0 \in \mathfrak{h}$.

The Weyl group $W$ acts on $S$ and we consider the coinvariant algebra $C = S/(S^+)^W S$, the quotient of $S$ by the ideal generated by all $W$-invariant functions vanishing at zero.

THEOREM 1. – Let $\chi \subset Z$ be a regular maximal ideal, i.e. the kernel of a regular central character. Then the Hochschild cohomology ring of $U/\chi U$ is the coinvariant algebra with its obvious grading doubled,

$$HH^*(U/\chi U) = C^{2\bullet}.$$  

Remark. – I thank Patrick Polo for telling me this problem along with the expected answer, and for his helpful comments on a first version.

Proof. – By the Hochschild cohomology ring of an associative $k$-algebra $A$ we mean just the ring $HH^*(A) = \text{Ext}^*_{A \otimes_k A^{\text{opp}}}(A, A)$ of self-extensions of $A$ considered as an $A \otimes_k A^{\text{opp}}$-module.

I now give an outline of the proof. Let $X$ be the variety of all Borel subalgebras of $\mathfrak{g}$. Under localization [BB81] the bimodule $U/\chi U$ becomes a $D$-module $\mathcal{M}$ for some

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sheaf of twisted differential operators $\mathcal{D}$ on $X \times X$ such that $X \times X$ is $\mathcal{D}$-affine, and we just need to compute the selfextensions of this $\mathcal{D}$-module $\mathcal{M}$. Now choosing $\mathcal{D}$ correctly we can assume that

1. the restriction of $\mathcal{D}$ to the open orbit $Y \subset X \times X$ for the diagonal action of $G$ is just the standard sheaf of differential operators $\mathcal{D}_Y$ on $Y$

2. the $\mathcal{D}$-module $\mathcal{M}$ is just the standard module (with unique simple quotient) $i_!\mathcal{O}_Y$, where $i$ is the inclusion of $Y$ into $X \times X$.

Once we know that, we find quickly

$$\text{Ext}^\bullet_{\mathcal{D}}(\mathcal{M}, \mathcal{M}) = \text{Ext}^\bullet_{\mathcal{D}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) = H^\bullet(Y, \mathbb{C}),$$

the first step since $i_!$ is fully faithful even on derived categories, the last step by the Riemann-Hilbert-correspondence. Since $Y$ is fibered over $X$ with fibers just affine spaces, the cohomology ring of $Y$ coincides with the cohomology ring of the flag manifold $X$ which is known to be the ring $C$ of coinvariants (with its obvious grading doubled).

Certainly 1 and 2 are known in much greater generality, they are just a special case of the correspondence between the classifications of Beilinson-Bernstein and Langlands. However, I want to avoid these heavy arguments and provide a more simple minded approach. So let us now start the work and find the sheaf of twisted differential operators (tdo for short) $\mathcal{D}$. We fix some notations concerning homogeneous twisted differential operators. I will follow [HMSW87], Appendix A.

For a complex algebraic group $G$ and a closed subgroup $B \subset G$ and a $B$-invariant linear form $A$ on $\text{Lie}B$ we have a sheaf $\mathcal{D}_A$ of homogeneous twisted differential operators on $G/B$. Let $\mathfrak{g} \supset \mathfrak{b}$ be the Lie algebras of $G \supset B$. The geometric stalk of $\mathcal{D}_A$ at the point $x = B$ of $G/B$ is given as

$$\mathcal{D}_A \mid_{x = B} = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_\lambda \otimes \Lambda^{\max}(\mathfrak{g}/\mathfrak{b})).$$

The ordinary differential operators are $\mathcal{D}_0$. Suppose now $H \subset G$ is a closed subgroup and let $f : H/H \cap B \hookrightarrow G/B$ be the inclusion. Pulling back the tdo $\mathcal{D}_\lambda$ via $f$ we find $\mathcal{D}_\lambda^f = \mathcal{D}_\mu$ where $\mu$ is the restriction of $f$ to $\text{Lie}(H \cap B)$.

Let us return now to our situation. We fix a semisimple connected complex algebraic group $G$ with $\text{Lie}G = \mathfrak{g}$, let $T \subset G$ be the maximal torus with $\text{Lie}T = \mathfrak{t}$ and fix two Borel subgroups $B^+, B^- \subset G$ containing $T$ which are opposite, i.e. $B^+ \cap B^- = T$. Their Lie algebras will be denoted $\mathfrak{b}^+$ and $\mathfrak{b}^-$ respectively. Any $\lambda \in \mathfrak{h}^*$ determines a homogeneous tdo $\mathcal{D}_\lambda^+$ on $X = G/B^+$ whose stalk at $B^+$ is the Verma module $M^+(\lambda) = U \otimes_{U(\mathfrak{b}^+)} (\mathbb{C}_\lambda \otimes \Lambda^{\max}(\mathfrak{g}/\mathfrak{b}^+))$. If we put $\xi^+(\lambda) = \Lambda_{\mathfrak{h}^+} M^+(\lambda)$, then $\Gamma(\mathcal{D}_\lambda^+) = U/\xi^+(\lambda)U$. The same statements hold with + replaced by −, and since the $\mathfrak{h}$-finite vectors in the (ordinary) dual of $M^+(\lambda)$ have the same $\mathfrak{h}$-weights as $M^-(-\lambda)$, we see directly that the principal antiautomorphism of $\mathfrak{g}$ induces an isomorphism $U/\xi^+(\lambda)U \to (U/\xi^-(-\lambda)U)^{op}$. This way we associate to $\lambda \in \mathfrak{h}^*$ a tdo $\mathcal{D}(\lambda) = \mathcal{D}_\lambda^+ \boxtimes \mathcal{D}_\lambda^-$ on $X \times X$ with global sections $\Gamma(\mathcal{D}(\lambda)) = U/\xi^+(\lambda)U \otimes (U/\xi^+(\lambda)U)^{op}$. We leave it to the interested reader to verify that $\mathcal{D}(\lambda)$ and this identification of the global sections depend only on the linear form $\lambda$ on the "abstract Cartan subgroup".
The localization theorem [BB81] says that for a homogeneous tdo $D$ on $X$ the functor of global sections is an equivalence of categories

$$\Gamma : \mathcal{D}\text{-mod} \to \Gamma(\mathcal{D})\text{-mod}$$

if and only if for one (equivalently any) point $x \in X$ the geometric stalk $D/\mathcal{D}m_x$ at $x$ is a simple $g$-module with regular central character. In this case $X$ is called $D$-affine. It is clear that for any regular maximal ideal $\chi \subset Z$ we can find $\lambda \in h^*$ such that $\xi^+(\lambda) = \chi$ and that $X \times X$ is $\mathcal{D}(\lambda)$-affine. This homogeneous tdo $\mathcal{D}(\lambda)$ is the $D$ we were looking for.

To see this, remark first that by definition of $\lambda$ the functor

$$\Gamma : \mathcal{D}(\lambda)\text{-mod} \to U/\chi U\text{-mod}-U/\chi U$$

is an equivalence of categories. Furthermore, if $i : Y \hookrightarrow X \times X$ denotes the inclusion of the open $G$-orbit $Y = G(B^+, B^-) = G/T$, it is plain that $\mathcal{D}(\lambda)$ pulls back to the ordinary differential operators $\mathcal{D}^i(\lambda) = \mathcal{D}_Y$ on $Y$.

We have to check that $\Gamma(i_*\mathcal{O}_Y) = U/\chi U$. Remark first that since $i$ is an open embedding, the direct and inverse images in the category of sheaves, $\mathcal{O}$-modules and $\mathcal{D}$-modules coincide. We thus have an adjoint pair of functors $(i^*, i_*)$ between sheaves, $\mathcal{O}$-modules and $\mathcal{D}$-modules for any tdo $\mathcal{D}$ on $X \times X$. Certainly $i^*$ is exact, and since $i$ is affine $i_*$ is exact, too. Now $\Gamma(i_*\mathcal{O}_Y)$ coincides with $\Gamma(\mathcal{O}_Y)$ and it is clear that every nonzero $\mathcal{D}(\lambda)$-submodule of $i_*\mathcal{O}_Y$ contains the global section $1_Y$ on $Y$. Thus $i_*\mathcal{O}_Y$ has a simple socle. Let us define the adjoint $g$-action on a $U$-bimodule by the formula

$$(\text{ad}A)(m) = Am - mA.$$ 

It is then clear that the socle of the $U$-bimodule $\Gamma(i_*\mathcal{O}_Y)$ is simple and generated by an $(\text{ad}g)$-invariant line.

Now we study $i_*\mathcal{O}_Y$. Recall its definition from [Bei83]. If $\mathcal{D}$ is a tdo on a smooth variety of dimension $n$ and $\mathcal{M}$ a holonomic $\mathcal{D}$-module, one defines its dual $*\mathcal{M} = R\text{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$. This is a holonomic $\mathcal{D}^{opp}$-module, and $**\mathcal{M} = \mathcal{M}$ naturally. By definition $i_*\mathcal{O}_Y = *i_* * \mathcal{O}_Y$. It is clear that in our case ($i$ an open immersion) the duality commutes with $i^*$, whence an adjoint pair $(i^!, i^*)$ of functors between holonomic modules. It is known in general (and clear in our case) that $i_*\mathcal{O}_Y$ has as its unique simple quotient the socle of $i_*\mathcal{O}_Y$. Passing to global sections we see that $\Gamma(i_*\mathcal{O}_Y)$ is generated by an $(\text{ad}g)$-invariant line. From this we already find a surjection of bimodules

$$U/\chi U \twoheadrightarrow \Gamma(i_*\mathcal{O}_Y).$$

Now let us go to the category $\chi\mathcal{H}_\chi$ of all locally $(\text{ad}g)$-finite $U/\chi U$-bimodules. Under localization this corresponds to the category of all $G$-equivariant $\mathcal{D}(\lambda)$-modules. All objects of this category are mapped under $i^*$ to a finite direct sum of some copies of $\mathcal{O}_Y$. Hence $i_*\mathcal{O}_Y$ is a projective object in our category, whence $\Gamma(i_*\mathcal{O}_Y)$ is a projective object of $\chi\mathcal{H}_\chi$. But this means that the surjection $U/\chi U \twoheadrightarrow \Gamma(i_*\mathcal{O}_Y)$ is split, and since $U/\chi U$ is indecomposable it has to be an isomorphism. \hfill $\Box$
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